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RESEARCH PAPER

NUMERICAL SOLUTIONS OF THE INITIAL VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS BY MODIFICATION OF THE ADOMIAN DECOMPOSITION METHOD

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and Dumitru Baleanu [2,3,4]

Abstract

In this paper, we extend a reliable modification of the Adomian decomposition method presented in [34] for solving initial value problem for fractional differential equations.

In order to confirm the applicability and the advantages of our approach, we consider some illustrative examples.

MSC 2010: 34K28, 34K37

Key Words and Phrases: fractional derivative, modified decomposition method, Adomian polynomials

1. Introduction

Recently, fractional differential calculus has attracted a lot of attention by many researchers of different fields, such as physics, chemistry, biology, economics, control theory and biophysics, etc. [27, 29, 33]. Since most fractional differential equations do not have exact analytic solutions, approximate and numerical techniques, are used extensively, such as homotopy analysis method [10, 16, 31], homotopy perturbation method [20, 21], variational iteration method [15, 17, 18, 19], Chebyshev spectral method [12, 13],

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new iterative method [9, 22, 23, 24], orthogonal polynomial method [33, 36], Oldham-Spanier L1 method [32], Grunwald-Letnikov method [33, 14], fractional Adams method [11], and several other methods [33, 27, 28, 46].

The Adomian decomposition method (ADM) [1, 2, 3, 4, 5, 6, 7, 42, 43, 44, 45, 38, 39, 40] is a powerful tool for solving both linear and nonlinear functional equations.

Consider the equation

$$Lu + Ru + Nu = g, \tag{1.1}$$

where L is an invertible operator, which is taken as the highest order derivative, R is the remainder of the linear operator, N represents the nonlinear terms and g is the specified analytic input function. Applying the inverse operator L^{-1} on both sides of equation (1.1) yields

$$u = \phi + L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu), \tag{1.2}$$

where ϕ is determined by using the given initial values. This method decomposes the solution $u(x)$ into a rapidly convergent series of solution components, and then decomposes the analytic nonlinearity Nu into the series of the Adomian polynomials [1, 2, 3]

$$u(x) = \sum_{n=0}^{\infty} u_n, \tag{1.3}$$

$$Nu(x) = \sum_{n=0}^{\infty} A_n, \tag{1.4}$$

where $A_n = A_n(u_0, u_1, ..., u_n)$ are the well-known Adomian polynomials, whose definitional formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\big(\sum_{k=0}^{\infty} u_k \lambda^k\big)|_{\lambda=0},\ n \geq 0, \tag{1.5}$$

was first published by Adomian and Rach in 1983 [1]. Then the standard Adomian recursion scheme:

$$u_0(x) = \phi + L^{-1}g,$$
$$u_{n+1} = -L^{-1}(Ru_n + A_n),$$

is given.

Shawagfeh [41], Daftardar-Gejji and Jafari [8, 26, 25] have employed ADM for solving nonlinear fractional differential equations and a system of fractional differential equations, respectively. Momani [30] presented an algorithm for the numerical solution of linear and nonlinear multi-order fractional differential equations. The algorithm is based on Adomian's decomposition approach.

Rach et al [34] proposed a new modification of the Adomian decomposition method for resolution of higher-order inhomogeneous nonlinear differential equations. This new modified decomposition method provides a significant advantage for computing the solution's Taylor expansion series, both systematically and rapidly.

The purpose of this paper is to extend this reliable modification of the Adomian decomposition method for different type of initial value problem for fractional differential equations. The advantage of this new modified method is that first we compute the Taylor expansion series for g in equation (1.1) and next we apply the operator L^{-1} on it. This trick allows us to admit various type of analytic functions as g, since sometimes it is difficult to compute $L^{-1}(g)$ when L^{-1} is a fractional integral and g is a trigonometric or exponential function.

This paper is organized as follows: in Section **2** some facts and results about fractional calculus and related properties are given, while in spire of [34] we clarify the steps of the new modification for solving fractional differential equations in Section **3** and we conclude this paper by considering some examples to illustrate a possible application of this new method in Section **4**.

2. Preliminaries

In this section, we recall some definitions and facts about fractional calculus, which will be needed later. For more details see [29, 33, 27].

A real function $f(x)$ is said to be of class C, if $f(x)$ is piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $(0, \infty)$.

DEFINITION **2.1**. Let $f(x)$ be a function of class C, then the Riemann-Liouville fractional integral of order $\alpha > 0$, is defined as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) ds, \ \ \alpha > 0, \ t > 0,$$
$$I^0 f(x) = f(x).$$

DEFINITION **2.2**. Let α be a positive real number, such that $m-1 < \alpha \leq m$, $m \in \mathbb{N}$ and let $f^{(m)}(x)$ exist and be a function of class C. Then the Caputo fractional derivative of f is defined as

$$D^{\alpha} f(x) = I^{m-\alpha} \left(\frac{d^m}{dt^m} f(x) \right).$$

We have the following properties of fractional integrals and derivatives:

(1) $I^\alpha I^\beta f = I^{\alpha+\beta} f, \quad \alpha, \beta \geq 0.$

(2) $I^\alpha x^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\gamma+\alpha}, \quad \alpha > 0, \gamma > -1.$

(3) $I^\alpha D^\alpha f(x) = f(x) - \displaystyle\sum_{k=0}^{m-1} f^{(k)}(0) \frac{x^k}{k!}, \quad m-1 < \alpha \leq m.$

3. Method Analysis

In this section we propose the modified Adomian decomposition method for solving different type of inhomogeneous nonlinear fractional differential equations.

3.1. Type (I)

We consider the following inhomogeneous nonlinear fractional differential equation:

$$D^\alpha u(x) + \alpha_0 u(x) + \beta f(u(x)) = g(x), \tag{3.1}$$
$$u(0) = c_0, \ 0 < \alpha \leq 1,$$

or equivalently:

$$Lu(x) = g(x) - Ru(x) - N(u(x)),$$

where

$$L = D^\alpha, \ Ru(x) = \alpha_0 u(x), \ N(u(x)) = \beta f(u(x)).$$

We suppose that $g(x)$ is analytic, so has Taylor expansion series:

$$g(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}.$$

Notice that by properties of the fractional integral and derivatives we have:

$$I^\alpha D^\alpha u(x) = u(x) - c_0,$$

$$I^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds.$$

Now, applying the integral operator $L^{-1} = I^\alpha$ to both sides of equation (3.1) we obtain:

$$u(x) = c_0 + \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds - \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds$$
$$- \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(u(s)) ds. \quad (3.2)$$

By equations (1.3), (1.4), equation (3.2) becomes:

$$\sum_{n=0}^{\infty} u(x) = c_0 + \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds$$
$$- \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \sum_{n=0}^{\infty} u(s) ds - \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \sum_{n=0}^{\infty} A_n(s) ds.$$

Now, we set the following recursion scheme:

$$u_0(x) = c_0,$$
$$u_{n+1}(x) = \frac{g_n}{\Gamma(\alpha) n!} \int_0^x (x-s)^{\alpha-1} s^n ds - \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_n(s) ds$$
$$- \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} A_n(s) ds. \quad (3.3)$$

Hence, we can approximate the solution $u^*(x)$ by:

$$\phi_{m+1} = \sum_{n=0}^{m} u_n(x),$$

that gives:

$$\lim_{m \to \infty} \phi_{m+1}(x) = \sum_{n=0}^{\infty} u_n(x) = u^*(x).$$

3.2. Type (II)

We now consider the following inhomogeneous nonlinear fractional differential equation:

$$D^\alpha u(x) + \lambda_1 D^{\alpha-1} u(x) + \alpha_0 u(x) + \beta f(u(x), D^{\alpha-1} u(x)) = g(x), \quad (3.4)$$
$$u(0) = c_0, \ u'(0) = c_1, \ 1 < \alpha \leq 2,$$

or equivalently:

$$Lu(x) = g(x) - Ru(x) - N(u(x)),$$

where

$$L = D^\alpha, \ Ru(x) = \lambda_1 D^{\alpha-1} u(x) + \alpha_0 u(x), \ N(u(x)) = \beta f(u(x), D^{\alpha-1} u(x)).$$

Now, applying the integral operator $L^{-1} = I^{\alpha}$ to both sides of equation (3.4), we obtain:

$$u(x) - (c_0 + c_1 x) = I^{\alpha} g(x) - I^{\alpha}(\lambda_1 D^{\alpha-1} u(x) + \alpha_0 u(x)) - I^{\alpha}(N(u(x))).$$

By the properties of fractional integrals and derivatives, we have

$$I^{\alpha} D^{\alpha-1} = I^1 I^{\alpha-1} D^{\alpha-1}, \qquad \text{and so,}$$

$$u(x) = c_0 + c_1 x + \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds - \lambda_1 I(u(x) - c_0)$$
$$- \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds - \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(u(s)) ds.$$

Similarly to the type (I), we set the following recursion scheme:

$$u_0(x) = c_0,$$
$$u_1(x) = c_1 x,$$
$$u_{n+2}(x) = \frac{g_n}{\Gamma(\alpha) n!} \int_0^x (x-s)^{\alpha-1} s^n ds - \lambda_1 \int_0^x u_{n+1}(s) ds \tag{3.5}$$
$$- \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_n(s) ds - \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} A_n(s) ds.$$

Hence, we can approximate the solution $u^*(x)$ by:

$$\phi_{m+1} = \sum_{n=0}^{m} u_n(x),$$

that gives:

$$\lim_{m\to\infty} \phi_{m+1}(x) = \sum_{n=0}^{\infty} u_n(x) = u^*(x).$$

3.3. Type (III)

We consider the following inhomogeneous nonlinear fractional differential equation:

$$D^{\alpha} u(x) + \lambda_1 D^{\alpha-1} u(x) + \lambda_2 D^{\alpha-2} u(x) + \alpha_0 u(x)$$
$$+ \beta f(u(x), D^{\alpha-2} u(x), D^{\alpha-1} u(x)) = g(x), \tag{3.6}$$
$$u(0) = c_0,\ u'(0) = c_1,\ u''(0) = c_2,\ 2 < \alpha \leq 3,$$

or equivalently:

$$Lu(x) = g(x) - Ru(x) - N(u(x)),$$

where

$$L = D^{\alpha},\ Ru(x) = \lambda_1 D^{\alpha-1} u(x) + \lambda_2 D^{\alpha-2} u(x) + \alpha_0 u(x),$$
$$N(u(x)) = \beta f(u(x), D^{\alpha-2} u(x), D^{\alpha-1} u(x)).$$

Now, applying the integral operator $L^{-1} = I^{\alpha}$ to both sides of equation (3.6), we obtain:

$$u(x) - (c_0 + c_1 x + c_2 \frac{x^2}{2!}) = I^{\alpha} g(x) - I^{\alpha}(\lambda_1 D^{\alpha-1} u(x) + \lambda_2 D^{\alpha-2} u(x) + \alpha_0 u(x)) - I^{\alpha}(N(u(x))).$$

Similarly to the type (II), we set the following recursion scheme:

$$\begin{aligned} u_0(x) &= c_0, \\ u_1(x) &= c_1 x, \\ u_2(x) &= c_2 \frac{x^2}{2!}, \\ u_{n+3}(x) &= \frac{g_n}{\Gamma(\alpha) n!} \int_0^x (x-s)^{\alpha-1} s^n ds - \lambda_1 \int_0^x u_{n+2}(s) ds \\ &- \lambda_2 \int_0^x (x-s) u_{n+1}(s) ds - \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_n(s) ds \\ &- \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} A_n(s) ds. \end{aligned} \tag{3.7}$$

Hence, we can approximate the solution $u^*(x)$ by:

$$\phi_{m+1} = \sum_{n=0}^{m} u_n(x),$$

that gives:

$$\lim_{m \to \infty} \phi_{m+1}(x) = \sum_{n=0}^{\infty} u_n(x) = u^*(x).$$

3.4. Generalized form of inhomogeneous nonlinear fractional differential equation

We consider the following generalized form of inhomogeneous nonlinear fractional differential equation:

$$D^{\alpha} u(x) + \alpha_0 u(x) + \sum_{i=1}^{m-1} \lambda_i D^{\alpha-i} u(x) + \beta f(u(x), D^{\alpha-1} u(x), ..., D^{\alpha-(m-1)} u(x)) = g(x), \tag{3.8}$$

$$u(0) = c_0,\ u'(0) = c_1, ...,\ u^{(m-1)}(0) = c_{m-1}, \quad m-1 < \alpha \le m,\ 2 \le m.$$

Proceeding as before, we set the following recursion scheme:

$$u_0(x) = c_0,$$
$$u_1(x) = c_1 x,$$
$$u_2(x) = c_2 \frac{x^2}{2!},$$
$$\vdots$$
$$u_{m-1}(x) = c_{m-1} \frac{x^{m-1}}{(m-1)!}, \tag{3.9}$$
$$u_{n+m}(x) = \frac{g_n}{\Gamma(\alpha) n!} \int_0^x (x-s)^{\alpha-1} s^n ds \tag{3.10}$$
$$- \sum_{i=1}^{m-1} \frac{\lambda_i}{(i-1)!} \int_0^x (x-s)^{(i-1)} u_{n+m-i}(s) ds$$
$$- \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_n(s) ds - \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} A_n(s) ds.$$

Hence, we can approximate the solution $u^*(x)$ by:

$$\phi_{m+1} = \sum_{n=0}^{m} u_n(x),$$

that gives:

$$\lim_{m \to \infty} \phi_{m+1}(x) = \sum_{n=0}^{\infty} u_n(x) = u^*(x).$$

4. Applications and numerical results

To demonstrate the effectiveness of the new modified method, we consider some nonlinear fractional differential equations.

EXAMPLE **4.1**. Consider the following type (I) inhomogeneous nonlinear fractional differential equation:

$$D^\alpha u(x) + u^2(x) = 1, \ u(0) = c_0 = 0, \ 0 < \alpha \leq 1.$$

The exact solution, when $\alpha = 1$ is $\dfrac{e^{2x}-1}{e^{2x}+1}$.

Note that:

$$\alpha_0 = 0, \ \beta = 1,$$

and the inhomogeneous term:

$$g(x) = 1 \ (g_0 = 1, \ g_n = 0, \ n \geq 1),$$

and the first few Adomian polynomials for $f(u(x)) = u^2(x)$ are:

$$\begin{aligned} A_0(x) &= u_0^2(x), \\ A_1(x) &= 2u_0(x)u_1(x), \\ A_2(x) &= 2u_0(x)u_2(x) + u_1^2(x), \\ &..., \end{aligned}$$

by the new modified recursion scheme for $\alpha = 0.98$,

$$\begin{aligned} u_0(x) &= c_0 = 0, \\ u_1(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds = 1.008x^{0.98}, \\ u_2(x) &= 0, \\ u_3(x) &= \frac{-1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_1^2(s) ds = -0.350x^{2.94}, \\ &..., \end{aligned}$$

so

$$u(x) = 1.008x^{0.98} - 0.350x^{2.94} + ... \ .$$

Comparing the above results with the obtained results by the iterative method which is introduced in [24], affirms the simplicity and accuracy of the given method.

In Figure 1, approximate solution of the new modified recursion scheme and the iterative method [24] for $\alpha = 0.98$ and the exact solution have been plotted.

EXAMPLE **4.2**. Consider the following type (I) inhomogeneous nonlinear fractional differential equation:

$$D^\alpha u(x) + u(x) = \sin(x), \quad u(0) = c_0 = 0, \ 0 < \alpha \leq 1.$$

The exact solution, when $\alpha = 1$ is $\frac{-1}{2}\cos(x) + \frac{1}{2}\sin(x) + \frac{1}{2}e^{-x}$.

Note that:

$$\alpha_0 = 1, \ \beta = 0,$$

and the inhomogeneous term:

$$g(x) = \sin(x),$$

where the first few coefficients $g_n = g^{(n)}(0)$ are given by:

$$g_0 = 0, \ g_1 = 1, \ g_2 = 0, \ g_3 = -1, ... \ .$$

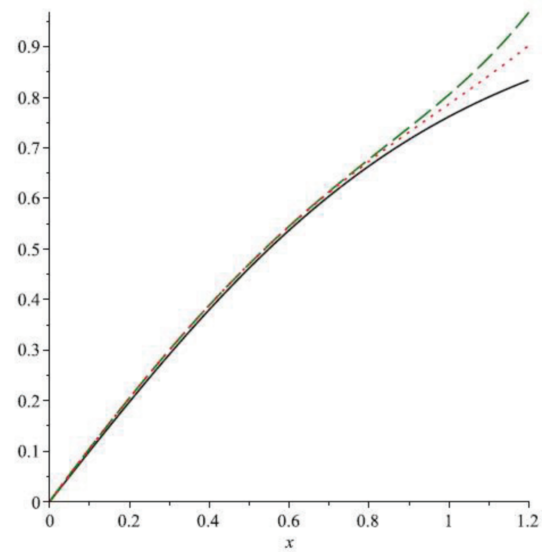

FIGURE 1. Solid Line: Analytic solution; Dot Line: Approximate solution for $\alpha = 0.98$; Dashed Line: Approximate solution [24] for $\alpha = 0.98$.

by the new modified recursion scheme for $\alpha = 0.98$,

$$\begin{aligned} u_0(x) &= c_0 = 0, \\ u_1(x) &= 0, \\ u_2(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} s ds = 0.509x^{1.98}, \\ u_3(x) &= \frac{-0.504}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} s^{1.98} ds = -0.175x^{2.96}, \\ &\ldots, \end{aligned}$$

so

$$u(x) = 0.509x^{1.98} - 0.175x^{2.96} + \ldots .$$

EXAMPLE **4.3**. Consider the following type (II) inhomogeneous nonlinear fractional differential equation:

$$D^\alpha u(x) + e^{-2u(x)} = 0, \; u(0) = c_0 = 0, \; u'(0) = c_1 = 1, \quad 1 < \alpha \leq 2.$$

The exact solution, when $\alpha = 2$ is $ln(1+x)$, which has been solved in [34].

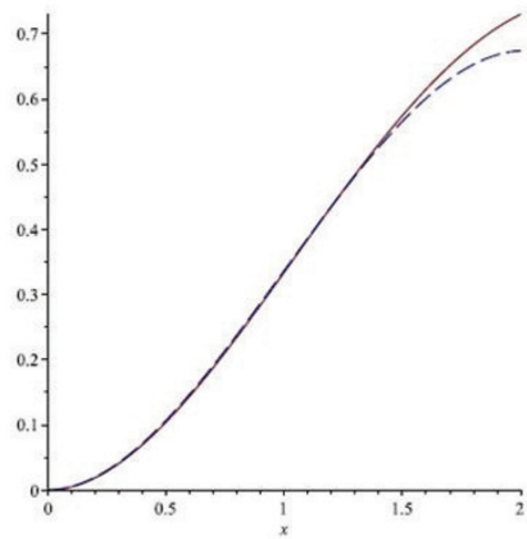

FIGURE 2. Solid Line: Analytic solution; Dashed Line: Approximate solution for $\alpha = 0.98$.

Note that:

$$\lambda_1 = \alpha_0 = 0, \ \beta = 1,$$

and the inhomogeneous term:

$$g(x) = 0,$$

and the first few Adomian polynomials for $f(u(x), D^{\alpha-1}u(x)) = e^{-2u(x)}$ is:

$$\begin{aligned}
A_0(x) &= e^{-2u_0(x)},\\
A_1(x) &= -2u_1(x)e^{-2u_0(x)},\\
A_2(x) &= -2u_2(x)e^{-2u_0(x)} + 2u_1^2(x)e^{-2u_0(x)},\\
&\dots,
\end{aligned}$$

by the new modified recursion scheme for $\alpha = 1.98$,

$$\begin{aligned}
u_0(x) &= c_0 = 0,\\
u_1(x) &= c_1 x = x,
\end{aligned}$$

$$u_2(x) = \frac{-1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds = -0.509x^{1.98},$$
$$u_3(x) = \frac{2}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} s ds = 0.341x^{2.98},$$
$$...,$$

so

$$u(x) = x - 0.509x^{1.98} + 0.341x^{2.98} + ... \, .$$

In Figure 3, the approximate solution of the new modified recursion scheme for 1.98 and the exact solution have been plotted.

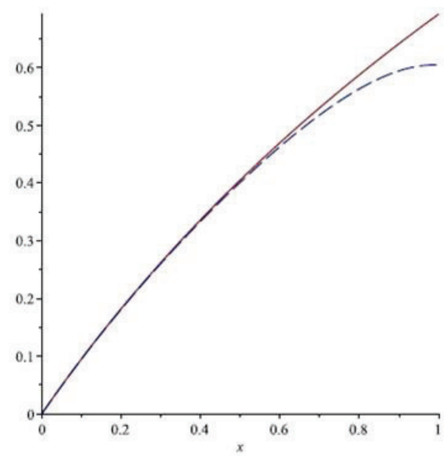

FIGURE 3. Solid Line: Analytic solution; Dashed Line: Approximate solution for $\alpha = 1.98$.

EXAMPLE **4.4**. Consider the following type (III) inhomogeneous nonlinear fractional differential equation:

$$D^{\alpha}u(x) + D^{\alpha-2}u(x) + u^4(x) = (1 + \sin(2x))^2, \tag{4.1}$$
$$u(0) = c_0 = 1, \; u'(0) = c_1 = 1, \; u''(0) = c_2 = -1, \; 2 < \alpha \leq 3.$$

The exact solution, when $\alpha = 3$ is $\sin(x)+\cos(x)$, which has been considered in [34].

Note that:

$$\lambda_2 = 1, \; \lambda_1 = 0, \; \alpha_0 = 0, \; \beta = 1,$$

and the inhomogeneous term:

$$g(x) = (1 + \sin(2x))^2,$$

where the first few coefficients $g_n = g^{(n)}(0)$ are given by:

$$g_0 = 1,\ g_1 = 4,\ g_2 = 8,\ g_3 = -16, ...,$$

and the first few Adomian polynomials for $f(u(x), D^{\alpha-1}u(x), D^{\alpha-2}u(x)) = u^4(x)$ is:

$$\begin{aligned} A_0(x) &= u_0^4(x), \\ A_1(x) &= 4u_0^3(x)u_1(x), \\ A_2(x) &= 4u_0^3(x)u_2(x) + 6u_0^2(x)u_1^2(x), \\ &..., \end{aligned}$$

by the new modified recursion scheme for $\alpha = 2.98$,

$$\begin{aligned} u_0(x) &= c_0 = 1, \\ u_1(x) &= c_1 x = x, \\ u_2(x) &= \frac{-x^2}{2}, \\ u_3(x) &= -\int_0^x (x-s) s ds = \frac{-x^3}{6}, \\ u_4(x) &= \frac{1}{2}\int_0^x (x-s)^{\alpha-1} s^2 ds = \frac{x^4}{24}, \\ &..., \end{aligned}$$

so

$$u(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + ... \ .$$

EXAMPLE **4.5**. Consider the following inhomogeneous nonlinear fractional differential equation:

$$\begin{aligned} &D^\alpha u(x) - 24e^{-5u(x)} = 0, \qquad (4.2) \\ &u(0) = c_0 = 1,\ u'(0) = c_1 = e^{-1},\ u''(0) = c_2 = -e^{-2}, \\ &u'''(0) = c_3 = 2e^{-3},\ u^{(iv)}(0) = c_4 = -6e^{-4},\ 4 < \alpha \le 5. \end{aligned}$$

The exact solution, when $\alpha = 5$ is $ln(e + x)$, which has been considered in [34].

Note that:

$$\alpha_0 = 0,\ \beta = -24,\ g(x) = 0,\ \lambda_i = 0\ for\ 1 \le i \le 4,$$

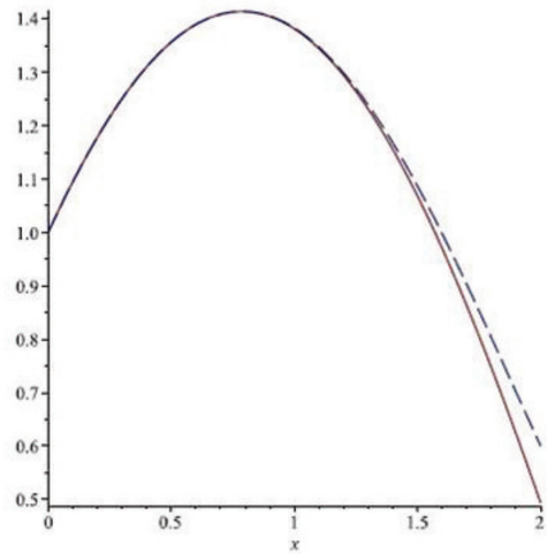

FIGURE 4. Solid Line: Analytic solution, Dashed Line: Approximate solution for $\alpha = 2.98$.

by the new modified recursion scheme for $\alpha = 4.98$,

$$
\begin{aligned}
u_0(x) &= 1,\\
u_1(x) &= e^{-1}x,\\
u_2(x) &= -e^{-2}\frac{x^2}{2},\\
u_3(x) &= e^{-3}\frac{x^3}{3},\\
u_4(x) &= -e^{-4}\frac{x^4}{4},\\
u_5(x) &= e^{-5}(0.206)x^{4.98},\\
&\ldots,
\end{aligned}
$$

so

$$
u(x) = 1 + e^{-1}x - e^{-2}\frac{x^2}{2} + e^{-3}\frac{x^3}{3} - e^{-4}\frac{x^4}{4} + e^{-5}(0.206)x^{4.98} + \ldots .
$$

In Figure 5, approximate solution of the new modified recursion scheme for $\alpha = 4.98$ and the exact solution have been plotted.

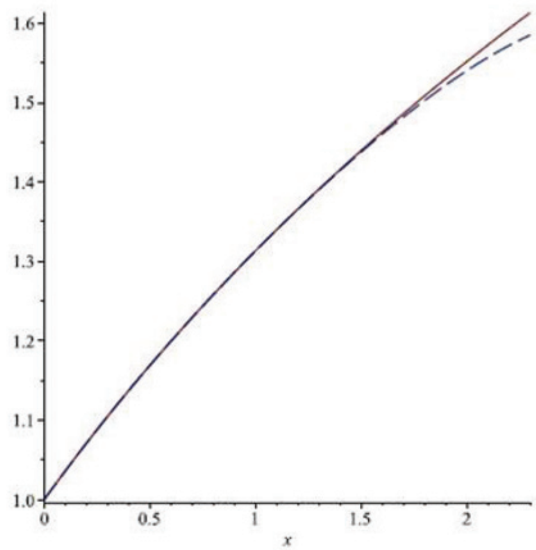

FIGURE 5. Solid Line: Analytic solution; Dashed Line: Approximate solution for $\alpha = 4.98$.

5. Conclusion

In this paper, we have developed a reliable modification of the Adomian decomposition method presented in [34] for solving initial value problem for fractional differential equation. The great advantage of this new modified method is that, we apply the operator L^{-1} on Taylor expansion series of function g in (1.1) and this technique allows us to admit various type of analytic functions as g, since sometimes it is difficult to compute $L^{-1}(g)$ when L^{-1} is a fractional integral and g is a trigonometric or exponential function. The results for numerical examples demonstrate that the present method can give a more accurate approximation.

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