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Approximate Controllability of Second-Order Nonlocal Impulsive Partial Functional Integro-Differential Evolution Systems in Banach Spaces

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Abstract. This manuscript is involved with a class of second-order impulsive partial functional integro-differential evolution equations with nonlocal conditions in Banach spaces. Sufficient conditions ensuring the existence and approximate controllability of mild solutions are established. Theory of cosine family, Banach contraction principle and Leray-Schauder nonlinear alternative fixed point theorem are employed for achieving the required results. An example is analyzed to illustrate the effectiveness of the outcome.

1. Introduction

In this manuscript, we initially look at the next second order nonlocal impulsive partial functional integro-differential evolution systems of the model

$$u''(t) = \mathscr{A}(t)u(t) + \mathscr{F}\left(t, u(\xi_1(t)), \dots, u(\xi_n(t)), \int_0^t k_1(t,s,u(\xi_{n+1}(s)))ds\right) + \mathscr{G}\left(t, u(\zeta_1(t)), \dots, u(\zeta_p(t)), \int_0^t k_2(t,s,u(\zeta_{p+1}(s)))ds\right), \quad (1.1)$$

$$t \in \mathscr{J} = [0,b],\ t \neq t_k,\ k = 1,2,\dots,m,$$

$$u(0) = u_0 + q(u), \quad u'(0) = \widetilde{u}_0 + \widetilde{q}(u), \quad (1.2)$$

$$\Delta u(t_k) = I_k(u(t_k)), \quad \Delta u'(t_k) = \bar{I}_k(u(t_k)),\ k = 1,2,\dots,m, \quad (1.3)$$

where the unknown $u(\cdot)$ takes values in the Banach space $\mathbb{X}$, and $\mathscr{A}(t) : \mathscr{D}(\mathscr{A}(t)) \subseteq \mathbb{X} \to \mathbb{X}$ is a closed linear operator on a Banach space $\mathbb{X}$; $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$, are prefixed points and the symbol

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$\Delta u(t_k) = u(t_k^+) - u(t_k^-), \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, where $u(t_k^+), u(t_k^-)$ and $u'(t_k^+), u'(t_k^-)$ are represent the right and left limits of $u(t)$ and $u'(t)$ at $t = t_k$, respectively. $\mathscr{F}(\cdot), \mathscr{G}(\cdot), k_1(\cdot), k_2(\cdot), q(\cdot), \widetilde{q}(\cdot), I_k(\cdot), \bar{I}_k(\cdot), \xi_i$, $i = 1, 2, \ldots, n+1$ and ζ_l, $l = 1, \ldots, p+1$, are apposite functions to be identified afterwards.

The principle of impulsive differential equations (IDEs) in the discipline of current applied mathematics has become an dynamic area of research in the past few years, for the reason that several physical systems and realistic mathematical models are subjects to abrupt change at certain moments. IDEs occur naturally from a vast range of applications, such as spacecraft control, electrical engineering, medicine biology, echoing, and so on. For additional aspect on this theory and its applications, we refer to [1, 7, 10, 11, 16, 18, 22, 30, 32, 36, 42, 43] and references therein. In the past few years, impulsive integro-differential equations have grown to be an important area of research simply because of their applications to diverse problems coming up in communications, control technology, impact mechanics and electrical engineering. However, the corresponding theory of impulsive integro-differential equations in abstract spaces is still in its developing stage and many aspects of the theory remain to be addressed.

The notion of controllability brings to some crucial findings concerning the behavior of linear and nonlinear dynamical systems. Almost all of the practical systems are nonlinear in nature and for this reason the study of nonlinear systems is significant. For the fundamental concept on evolution system, the reader is referred to Tanabe's book (see [37]). The controllability problem for evolution system consists in driving the state of the system (the solution of the controlled equation under consideration) to a prescribed final target in finite time (exactly or in some approximate way). Hence this is a stronger notion of controllability. For classical nonlinear control system the fixed point methods are widely used as a tool to study the controllability problem [6, 12, 15, 19, 23, 26, 28, 34, 40, 41]. In the mathematical perspective, the issues of exact and approximate controllability are to be distinguished. Exact controllability allows to steer the system to arbitrary final state while approximate controllability signifies that the system is usually steered to arbitrary small neighborhood of final state. Especially, approximate controllable systems are more common and frequently approximate controllability is fully acceptable in applications. There are actually a lot of papers on the exact and approximate controllability of the different kinds of nonlinear systems under various hypotheses (see for instance [9, 10, 25, 27, 31, 33, 44] and references cited therein). Second-order differential and integro-differential equations provide as an theoretical formulation of several integro-differential equations which occur in problems linked with the transverse motion of an extensible beam, the vibration of hinged bars and various other physical phenomena. So it is very huge to concentrate the controllability issue for such systems in Banach spaces.

The literary works relevant to existence and controllability of second-order systems with impulses continues to be restricted. Chang et al. [11] analyzed the existence of mild solutions for a second order impulsive neutral functional differential equations with state-dependent delay by using a fixed point theorem for condensing maps combined with theories of a strongly continuous cosine family of bounded linear operators. Zhang et al. [45] established the sufficient conditions for the controllability of second-order semilinear impulsive stochastic neutral functional evolution equations by using Sadovskii's fixed point theorem. Sakthivel et al. [34] studied the controllability of second-order impulsive systems in Banach spaces without imposing the compactness condition on the cosine family of operators under Banach contraction mapping principle. In [1–5], the authors discussed the different types of second-order impulsive differential systems with different conditions on the given functions. The results are obtained by using the classical fixed point theorems. Dimplekumar N. Chalishajar [8] analyzed the controllability of a partial neutral functional differential inclusion of second order with impulse effect and infinite delay without assuming the compactness conditions of the family of cosine operators and also author introduced a new phase space axioms to derive the results. Lately, Meili Li and Junling Ma [23] studied the approximate controllability of second order impulsive functional differential systems with infinite delay in Banach spaces. Sufficient conditions are formulated and proved for the approximate controllability of such system under the assumption that the associated linear part of system is approximately controllable. However, it needs to be pointed out, to the best of our knowledge, the existence and approximate controllability results for second-order impulsive partial functional integro-differential evolution equations with nonlocal conditions of the form (1.1)-(1.3) has not been examined yet. According to fixed point techniques, the proposed work

in this manuscript on the second-order functional integro-differential evolution systems with nonlocal and impulsive conditions is new in the literature. This fact is the important objective of this work.

The structure of this manuscript is as per the following. In Section 2, some fundamental certainties are reviewed. Section 3 is dedicated to the existence of mild solutions to problem (1.1)-(1.3). The approximate controllability result is shown in Section 4. In Section 5, a case is given to delineate our outcomes.

2. Preliminaries

In this section, we review some fundamental concepts, notations, and properties required to establish our main results.

Nowadays there has been an increasing interest in studying the abstract non-autonomous second order initial value problem

$$x''(t) = \mathscr{A}(t)x(t) + f(t), \quad 0 \le s, t \le b, \tag{1}$$

$$x(s) = x_0, \quad x'(s) = y_0, \tag{2}$$

where $\mathscr{A}(t) : \mathscr{D}(\mathscr{A}(t)) \subseteq X \to X$, $t \in \mathscr{J} = [0, b]$ is a closed densely defined operator and $f : \mathscr{J} \to X$ is an appropriate function. Equations of this type have been considered in many papers. The reader is referred to [24, 29] and the references mentioned in these works. In the most of works, the existence of solutions to the problem (2.1)-(2.2) is related to the existence of an evolution operator $S(t, s)$ for the homogeneous equation

$$x''(t) = \mathscr{A}(t)x(t), \quad 0 \le s, t \le b, \tag{3}$$

Let as assume that the domain of $\mathscr{A}(t)$ is a subspace $\mathscr{D}$ dense in $\mathbb{X}$ and independent of t, and for each $x \in \mathscr{D}$ the function $t \mapsto \mathscr{A}(t)x$ is continuous.

Following Kozak [21], in this work we will use the following concept of evolution operator.

Definition 2.1. *A family S of bounded linear operators $S(t, s) : \mathscr{J} \times \mathscr{J} \to \mathcal{L}(\mathbb{X})$ is called an evolution operator for (2.3) if the following conditions are satisfied:*

(Z1) *For each $x \in \mathbb{X}$, the mapping $[0, b] \times [0, b] \ni (t, s) \to S(t, s)x \in \mathbb{X}$ is of class C^1 and*

- (i) *for each $t \in [0, b]$, $S(t, t) = 0$,*
- (ii) *for all $t, s \in [0, b]$, and for each $x \in \mathbb{X}$,*

$$\frac{\partial}{\partial t}S(t, s)x\Big|_{t=s} = x, \qquad \frac{\partial}{\partial s}S(t, s)x\Big|_{t=s} = -x.$$

The equalities (ii) cannot be true unless $x = 0$.

(Z2) *For all $t, s \in [0, b]$, if $x \in \mathscr{D}(\mathscr{A})$, then $S(t, s)x \in \mathscr{D}(\mathscr{A})$, the mapping $[0, b] \times [0, b] \ni (t, s) \to S(t, s)x \in \mathbb{X}$ is of class C^2 and*

- (i) $\frac{\partial^2}{\partial t^2}S(t, x)x = \mathscr{A}(t)S(t, s)x$,
- (ii) $\frac{\partial^2}{\partial s^2}S(t, x)x = S(t, s)\mathscr{A}(s)x$,
- (iii) $\frac{\partial}{\partial s}\frac{\partial}{\partial t}S(t, x)x\Big|_{t=s} = 0$.

(Z3) *For all $t, s \in [0, b]$, if $x \in \mathscr{D}(\mathscr{A})$, then $\frac{\partial}{\partial s}S(t, s)x \in \mathscr{D}(\mathscr{A})$, then $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}S(t, s)x$, $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}S(t, s)x$ and*

- (i) $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}S(t, s)x = \mathscr{A}(t)\frac{\partial}{\partial s}S(t, s)x$,
- (ii) $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t}S(t, s)x = \frac{\partial}{\partial t}S(t, s)\mathscr{A}(s)x$,

and the mapping $[0, b] \times [0, b] \ni (t, s) \to \mathscr{A}(t)\frac{\partial}{\partial s}S(t, s)x$ is continuous.

Throughout this work we assume that there exists an evolution operator $S(t,s)$ associated to the operator $\mathscr{A}(t)$. To abbreviate the text, we introduce the operator $C(t,s) = -\frac{\partial S(t,s)}{\partial s}$. In addition, we set $\widehat{M}_1$ and $\widehat{M}_2$ for positive constants such that $\sup_{0\leq t,s\leq b} \|S(t,s)\| \leq \widehat{M}_2$ and $\sup_{0\leq t,s\leq b} \|C(t,s)\| \leq \widehat{M}_1$. Furthermore, we denote by $\overline{N}_1$ a positive constant such that

$$\|S(t+h,s) - S(t,s)\| \leq \overline{N}_1|h|,$$

for all $s, t, t+h \in [0,b]$. Assuming that $f : \mathscr{J} \to \mathbb{X}$ is an integrable function, the mild solution $x : [0,b] \to X$ of the problem (2.1)-(2.2) is given by

$$x(t) = C(t,s)x_0 + S(t,s)y_0 + \int_0^t S(t,\tau)h(\tau)d\tau.$$

In the literature several techniques have been discussed to establish the existence of the evolution operator $S(\cdot,\cdot)$. In particular, a very studied situation is that $\mathscr{A}(t)$ is the perturbation of an operator $\mathscr{A}$ that generates a cosine operator function. For this reason, below we briefly review some essential properties of the theory of cosine functions. Let $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subseteq \mathbb{X} \to \mathbb{X}$ be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on a Banach space $\mathbb{X}$. We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by

$$S(t)x = \int_0^t C(s)xds, \quad x \in \mathbb{X}, \quad t \in \mathbb{R}.$$

We refer the reader to [14, 38, 39] for the necessary concepts about cosine functions. Next we only mention a few results and notations about this matter needed to establish our results. It is immediate that

$$C(t)x - x = \mathscr{A}\int_0^t S(s)xds,$$

for all $\mathbb{X}$. The notation $[\mathscr{D}(\mathscr{A})]$ stands for the domain of the operator $\mathscr{A}$ endowed with the graph norm $\|x\|_{\mathscr{A}} = \|x\| + \|\mathscr{A}x\|$, $x \in \mathscr{D}(\mathscr{A})$. Moreover, in this paper the notation E stands for the space formed by the vectors $x \in \mathbb{X}$ for which the function $C(\cdot)x$ is a class C^1 on $\mathbb{R}$. It was proved by Kisyński [20] that the space E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0\leq t\leq 1} \|\mathscr{A}S(t)x\|, \quad x \in E,$$

is a Banach space. The operator valued function

$$G(t) = \begin{bmatrix} C(t) & S(t) \\ \mathscr{A}S(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of linear operators on the space $E\times\mathbb{X}$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $\mathscr{D}(\mathscr{A}) \times E$. It follows from this that $\mathscr{A}S(t) : E \to \mathbb{X}$ is a bounded linear operator such that $\mathscr{A}S(t)x \to 0$ as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0,\infty) \to \mathbb{X}$ is a locally integrable function, then $z(t) = \int_0^t S(t,s)x(s)ds$ defines an E-valued continuous function.

The existence of solutions for the second order abstract Cauchy problem

$$x''(t) = Ax(t) + h(t), \quad 0 \leq t \leq b, \tag{4}$$

$$x(0) = x_0, \quad x'(0) = y_0, \tag{5}$$

where $h : [0, b] \to X$ is an integrable function, has been discussed in [38]. Similarly, the existence of solutions of the semilinear second order Cauchy problem it has been treated in [39]. We only mention here that the function $x(\cdot)$ given by

$$x(t) = C(t-s)x_0 + S(t-s)y_0 + \int_s^t S(t-\tau)h(\tau)d\tau, \quad 0 \le t \le b, \tag{6}$$

is called the mild solution of (2.4)-(2.5) and that when $x_0 \in E$, $x(\cdot)$ is continuously differentiable and

$$x'(t) = \mathscr{A}S(t-s)x_0 + C(t-s)y_0 + \int_s^t C(t-\tau)h(\tau)d\tau, \quad 0 \le t \le b.$$

In addition, if $x_0 \in \mathscr{D}(\mathscr{A})$, $y_0 \in E$ and f is a continuously differentiable function, then the function $x(\cdot)$ is a solution of the initial value problem (2.4)-(2.5).

Assume now that $\mathscr{A}(t) = \mathscr{A} + \widetilde{B}(t)$ where $\widetilde{B}(\cdot) : \mathbb{R} \to \mathcal{L}(E, \mathbb{X})$ is a map such that the function $t \to \widetilde{B}(t)x$ is continuously differentiable in $\mathbb{X}$ for each $x \in E$. It has been established by Serizawa [35] that for each $(x_0, y_0) \in \mathscr{D}(\mathscr{A}) \times E$ the nonautonomous abstract Cauchy problem

$$x''(t) = (\mathscr{A} + \widetilde{B}(t))x(t), \quad t \in \mathbb{R}, \tag{7}$$

$$x(0) = x_0, \quad x'(0) = y_0, \tag{8}$$

has a unique solution $x(\cdot)$ such that the function $t \mapsto x(t)$ is continuously differentiable in E. It is clear that the same argument allows us to conclude that equation (2.7) with the initial condition (2.5) has a unique solution $x(\cdot, s)$ such that the function $t \mapsto x(t, s)$ is continuously differentiable in E. It follows from (2.6) that

$$x(t,s) = C(t-s)x_0 + S(t-s)y_0 + \int_s^t S(t-\tau)\widetilde{B}(\tau)x(\tau,s)d\tau.$$

In particular, for $x_0 = 0$ we have

$$x(t,s) = S(t-s)y_0 + \int_s^t S(t-\tau)\widetilde{B}(\tau)x(\tau,s)d\tau.$$

Consequently,

$$\|x(t,s)\|_1 \le \|S(t-s)\|_{\mathcal{L}(X,E)}\|y_0\| + \int_s^t \|S(t-s)\|_{\mathcal{L}(X,E)}\|\widetilde{B}(\tau)\|_{\mathcal{L}(X,E)}\|x(s,\tau)\|_1 d\tau.$$

and, applying the Gronwall-Bellman lemma we infer that

$$\|x(t,s)\|_1 \le \widetilde{M}\|y_0\|, \quad s, t \in I.$$

We define the operator $S(t,s)y_0 = x(t,s)$. It follows from the previous estimate that $S(t,s)$ is a bounded linear map on E. Since E is dense in $\mathbb{X}$, we can extend $S(t,s)$ to $\mathbb{X}$. We keep the notation $S(t,s)$ for this extension. It is well known that, except in the case $\dim(\mathbb{X}) < \infty$, the cosine function $C(t)$ cannot be compact for all $t \in \mathbb{R}$. By contrast, for the cosine functions that arise in specific applications, the sine function $S(t)$ is very often a compact operator for all $t \in \mathbb{R}$.

Theorem 2.2. *[17, Theorem 1.2]. Under the preceding conditions, $S(\cdot,\cdot)$ is an evolution operator for (7) − (8). Moreover, if $S(t)$ is compact for all $t \in \mathbb{R}$, then $S(t,s)$ is also compact for all $s \le t$.*

To consider the impulsive conditions (1.1)-(1.3), it is convenient to introduce some additional concepts and notations.

A function $u : [\sigma, \tau] \to \mathbb{X}$ is said to be a normalized piecewise continuous function on $[\sigma, \tau]$ if u is piecewise continuous and left continuous on $(\sigma, \tau]$. We denote by $\mathcal{PC}([\sigma, \tau], \mathbb{X})$ the space of normalized

piecewise continuous functions from $[\sigma, \tau]$ into $\mathbb{X}$. In particular, we introduce the space $\mathcal{PC}$ formed by all normalized piecewise continuous functions $u : [0, b] \to \mathbb{X}$ such that u is continuous at $t \neq t_k$, $k = 1, \ldots, m$. It is clear that $\mathcal{PC}$ endowed with the norm $\| u \|_{\mathcal{PC}} = \sup_{s \in \mathscr{J}} \| u(s) \|$ is a Banach space. Likewise, $\mathcal{PC}^1$ will be the space of the functions $u(\cdot) \in \mathcal{PC}$ such that $u(\cdot)$ is continuously differentiable on $\mathscr{J} - \{t_k : k = 1, 2, \ldots, m\}$ and the lateral derivatives $u'_R(t) = \lim_{s \to 0^+} \frac{u(t+s) - u(t^+)}{s}$, $u'_L(t) = \lim_{s \to 0^-} \frac{u(t+s) - u(t^-)}{s}$ are continuous functions on $[t_k, t_{k+1})$ and $(t_k, t_{k+1}]$, respectively. Next, for $u \in \mathcal{PC}^1$ we represent by $u'(t)$ the left derivative at $t \in (0, b]$ and by $u'(0)$ the right derivative at zero.

In what follows, we set $t_0 = 0$, $t_{m+1} = b$, and for $u \in \mathcal{PC}$ we denote by $\tilde{u}_k$, for $k = 0, 1, ..., m$, the function $\tilde{u}_k \in C([t_k, t_{k+1}]; \mathbb{X})$ given by $\widetilde{u}_k(t) = u(t)$ for $t \in (t_k, t_{k+1}]$ and $\widetilde{u}_k(t_k) = \lim_{t \to t_k^+} u(t)$. Moreover, for a set $B \subseteq \mathcal{PC}$, we denote by $\widetilde{B}_k$, for $k = 0, 1, ..., m$, the set $\widetilde{B}_k = \{\tilde{u}_k : u \in B\}$.

Lemma 2.3. *A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, each set $\widetilde{B}_k$, $k = 0, 1, \ldots, m$, is relatively compact in $C([t_i, t_{i+1}], \mathbb{X})$.*

Now, we are in a position to present the mild solution for the system (1.1)-(1.3).

Definition 2.4. *A function $u(\cdot) \in \mathcal{PC}(\mathscr{J}, \mathbb{X})$ is said to be a mild solution to the problem* (1.1) – (1.3) *if it satisfies the following integral equation*

$$\begin{aligned} u(t) &= C(t,0)[u_0 + q(u)] + S(t,0)[\widetilde{u}_0 + \widetilde{q}(u)] \\ &+ \int_0^t S(t,s)\Big[\mathscr{F}\Big(s, u(\xi_1(s)), \ldots, u(\xi_n(s)), \int_0^s k_1(s,\tau,u(\xi_{n+1}(\tau)))d\tau\Big) \\ &+ \mathscr{G}\Big(s, u(\zeta_1(s)), \ldots, u(\zeta_p(s)), \int_0^s k_2(s,\tau,u(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds \\ &+ \sum_{0<t_k<t} C(t,t_k)I_k(u(t_k)) + \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(u(t_k)), \quad t \in \mathscr{J}. \end{aligned}$$

The key tool in our approach is the following fixed point theorem.

Lemma 2.5. *(Leray-Schauder Nonlinear Alternative [13]) Let $\mathbb{X}$ be a Banach space with $Z \subset \mathbb{X}$ closed and convex. Assume that U is a relatively open subset of Z with $0 \in U$ and $\Upsilon : \overline{U} \to Z$ is a compact map. Then either*

(i) Υ has a fixed point in $\overline{U}$, or

(ii) there is a point $v \in \partial U$ such that $v \in \lambda\Upsilon(v)$ for some $\lambda \in (0, 1)$.

3. Existence Results

In this section, we present and prove the existence results for the problem (1.1) – (1.3). In order to utilize the Lemma 2.5, we need to list the subsequent hypotheses:

(H1) The functions $\mathscr{F} : \mathscr{J} \times \mathbb{X}^{n+1} \to \mathbb{X}$ and $\mathscr{G} : \mathscr{J} \times \mathbb{X}^{p+1} \to \mathbb{X}$ are continuous and there exist constants $\mathscr{L} > 0, \widetilde{\mathscr{L}} > 0, \mathscr{L}_1 \geq 0, \widetilde{\mathscr{L}_1} \geq 0$ such that for all $x_i, y_i \in \mathbb{X}$, $i = 1, \ldots, n+1$ and $x_l, y_l \in \mathbb{X}$, $l = 1, \ldots, p+1$, we have

$$\| \mathscr{F}(t, x_1, x_2, \ldots, x_{n+1}) - \mathscr{F}(t, y_1, y_2, \cdots, y_{n+1}) \| \leq \mathscr{L}\Big(\sum_{i=1}^{n+1} \|x_i - y_i\|\Big)$$

and

$$\| \mathscr{G}(t, x_1, x_2, \ldots, x_{p+1}) - \mathscr{G}(t, y_1, y_2, \cdots, y_{p+1}) \| \leq \widetilde{\mathscr{L}}\Big(\sum_{l=1}^{p+1} \|x_l - y_l\|\Big)$$

with $\mathscr{L}_1 = \max_{t\in\mathscr{J}} \|\mathscr{F}(t,0,\ldots,0)\|$ and $\widetilde{\mathscr{L}_1} = \max_{t\in\mathscr{J}} \|\mathscr{G}(t,0,\ldots,0)\|$.

(H2) The functions $k_1, k_2 : \mathscr{J}\times\mathscr{J}\times\mathbb{X}\to\mathbb{X}$ are continuous and we can find constants $\mathscr{N}>0, \widetilde{\mathscr{N}}>0, \mathscr{N}_1\geq 0, \widetilde{\mathscr{N}_1}\geq 0$ such that for all $x, y\in\mathbb{X}$,

$$\|k_1(t,s,x)-k_1(t,s,y)\|\leq \mathscr{N}\|x-y\|$$

and

$$\|k_2(t,s,x)-k_2(t,s,y)\|\leq \widetilde{\mathscr{N}}\|x-y\|$$

with $\mathscr{N}_1 = \max_{0\leq s\leq t\leq b}\|k_1(t,s,0)\|$ and $\widetilde{\mathscr{N}_1} = \max_{0\leq s\leq t\leq b}\|k_2(t,s,0)\|$.

(H3) The functions $\xi_i : \mathscr{J}\to\mathscr{J}$, $i=1,\ldots,n+1$ and $\zeta_l : \mathscr{J}\to\mathscr{J}$, $l=1,\ldots,p+1$ are continuous functions such that $\xi_i(t)\leq t$, $i=1,\ldots,n+1$ and $\zeta_l(t)\leq t$, $l=1,\ldots,p+1$.

(H4) (i) $I_k\in C(\mathbb{X},\mathbb{X})$, $k=1,\cdots,m$ are all compact operators, and there exist continuous nondecreasing functions $\Psi_k : [0,\infty)\to(0,\infty)$, $k=1,\cdots,m$, such that

$$\|I_k(u)\|\leq \Psi_k(\|u\|), \quad \text{for each } u\in\mathbb{X}.$$

(ii) $\bar{I}_k\in C(\mathbb{X},\mathbb{X})$, $k=1,\cdots,m$ are all compact operators, and there exist continuous nondecreasing functions $\widetilde{\Psi}_k : [0,\infty)\to(0,\infty)$, $k=1,\cdots,m$, such that

$$\|\bar{I}_k(u)\|\leq \widetilde{\Psi}_k(\|u\|), \quad \text{for each } u\in\mathbb{X}.$$

(H5) (i) The function $q(\cdot) : \mathcal{PC}(\mathscr{J},\mathbb{X})\to\mathbb{X}$ is continuous and there exists a $\delta\in(0,t_1)$ such that $q(\phi)=q(\widetilde{\phi})$ for any $\phi,\widetilde{\phi}\in\mathcal{PC}(\mathscr{J},\mathbb{X})$ with $\phi=\widetilde{\phi}$ on $[\delta,b]$.

(ii) The function $\widetilde{q}(\cdot) : \mathcal{PC}(\mathscr{J},\mathbb{X})\to\mathbb{X}$ is continuous and there exists a $\delta\in(0,t_1)$ such that $\widetilde{q}(\phi)=\widetilde{q}(\widetilde{\phi})$ for any $\phi,\widetilde{\phi}\in\mathcal{PC}(\mathscr{J},\mathbb{X})$ with $\phi=\widetilde{\phi}$ on $[\delta,b]$.

(iii) There is a continuous nondecreasing function $\Lambda : [0,\infty)\to(0,\infty)$ such that

$$\|q(\phi)\|\leq\Lambda(\|\phi\|_{\mathcal{PC}}),\ \phi\in\mathcal{PC}(\mathscr{J},\mathbb{X}).$$

(iv) There is a continuous nondecreasing function $\widetilde{\Lambda} : [0,\infty)\to(0,\infty)$ such that

$$\|\widetilde{q}(\phi)\|\leq\widetilde{\Lambda}(\|\phi\|_{\mathcal{PC}}),\ \phi\in\mathcal{PC}(\mathscr{J},\mathbb{X}).$$

(H6) There exists a constant $\widehat{M}^*>0$ such that

$$\frac{\widehat{M}^*}{e^{\eta}\left[\widehat{M}_* + \widehat{M}_1\left[\Lambda(\widehat{M}^*)+\sum_{k=1}^{m}\Psi_k(\widehat{M}^*)\right]+\widehat{M}_2\left[\widetilde{\Lambda}(\widehat{M}^*)+\sum_{k=1}^{m}\widetilde{\Psi}_k(\widehat{M}^*)\right]\right]}>1, \tag{3.1}$$

where $\eta=\widehat{M}_2 b\Big[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+b(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big]$, $\widehat{M}_* = \widehat{M}_1\|u_0\|+\widehat{M}_2\|\widetilde{u}_0\|+\widehat{M}_2 b\left[b(\mathscr{L}\mathscr{N}_1+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1})+(\mathscr{L}_1+\widetilde{\mathscr{L}_1})\right]$.

Theorem 3.1. *Let $u(0),u'(0)\in\mathbb{X}$. If assumptions (H1) − (H6) are fulfilled, then the impulsive nonlocal Cauchy problem* (1.1) − (1.3) *has at least one mild solution on $\mathscr{J}$.*

Proof. Let $\mathscr{L}_0 = 2\widehat{M}_2\Big[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})b\Big]$ and we introduce in the space $\mathcal{PC}(\mathscr{J},\mathbb{X})$ the equivalent norm defined as

$$\|\phi\|_{\mathcal{V}} := \sup_{t\in\mathscr{J}} e^{-\mathscr{L}_0 t}\|\phi(t)\|.$$

Then, it is easy to see that $\mathcal{V} := (\mathcal{PC}(\mathscr{J},\mathbb{X}),\|\cdot\|_{\mathcal{V}})$ is a Banach space. Fix $v \in \mathcal{PC}(\mathscr{J},\mathbb{X})$ and for $t \in \mathscr{J}, \phi \in \mathcal{V}$, we now define an operator

$$\begin{aligned}(\Upsilon_v\phi)(t) &= C(t,0)[u_0+q(v)] + S(t,0)[\widetilde{u}_0+\widetilde{q}(v)]\\ &\quad+\int_0^t S(t,s)\Big[\mathscr{F}\Big(s,\phi(\xi_1(s)),\dots,\phi(\xi_n(s)),\int_0^s k_1(s,\tau,\phi(\xi_{n+1}(\tau)))d\tau\Big)\\ &\quad+\mathscr{G}\Big(s,\phi(\zeta_1(s)),\dots,\phi(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds\\ &\quad+\sum_{0<t_k<t} C(t,t_k)I_k(v(t_k)) + \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v(t_k)).\end{aligned} \tag{3.2}$$

Since $C(\cdot,0)(u_0+q(v))$ and $S(\cdot,0)(\widetilde{u}_0+\widetilde{q}(v))$ are belongs to $\mathcal{PC}(\mathscr{J},\mathbb{X})$, it follows from $(H1)-(H3)$ that $(\Upsilon_v\phi)(t) \in \mathcal{V}$ for all $\phi \in \mathcal{V}$. Let $\phi,\widetilde{\phi} \in \mathcal{V}$, we have

$$\begin{aligned}&e^{-\mathscr{L}_0 t}\|(\Upsilon_v\phi)(t)-(\Upsilon_v\widetilde{\phi})(t)\|\\ &\le e^{-\mathscr{L}_0 t}\int_0^t\Big\|S(t,s)\Big[\mathscr{F}\Big(s,\phi(\xi_1(s)),\dots,\phi(\xi_n(s)),\int_0^s k_1(s,\tau,\phi(\xi_{n+1}(\tau)))d\tau\Big)\\ &\quad-\mathscr{F}\Big(s,\widetilde{\phi}(\xi_1(s)),\dots,\widetilde{\phi}(\xi_n(s)),\int_0^s k_1(s,\tau,\widetilde{\phi}(\xi_{n+1}(\tau)))d\tau\Big)\Big]\Big\|ds\\ &\quad+e^{-\mathscr{L}_0 t}\int_0^t\Big\|S(t,s)\Big[\mathscr{G}\Big(s,\phi(\zeta_1(s)),\dots,\phi(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi(\zeta_{p+1}(\tau)))d\tau\Big)\\ &\quad-\mathscr{G}\Big(s,\widetilde{\phi}(\zeta_1(s)),\dots,\widetilde{\phi}(\zeta_p(s)),\int_0^s k_2(s,\tau,\widetilde{\phi}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]\Big\|ds\\ &\le \widehat{M}_2\mathscr{L}\int_0^t e^{-\mathscr{L}_0 t}\Big[\|\phi(\xi_1(s))-\widetilde{\phi}(\xi_1(s))\|+\cdots+\|\phi(\xi_n(s))-\widetilde{\phi}(\xi_n(s))\|\\ &\quad+\Big\|\int_0^s k_1(s,\tau,\phi(\xi_{n+1}(\tau)))d\tau-\int_0^s k_1(s,\tau,\widetilde{\phi}(\xi_{n+1}(\tau)))d\tau\Big\|\Big]ds\\ &\quad+\widehat{M}_2\widetilde{\mathscr{L}}\int_0^t e^{-\mathscr{L}_0 t}\Big[\|\phi(\zeta_1(s))-\widetilde{\phi}(\zeta_1(s))\|+\cdots+\|\phi(\zeta_p(s))-\widetilde{\phi}(\zeta_p(s))\|\\ &\quad+\Big\|\int_0^s k_2(s,\tau,\phi(\zeta_{p+1}(\tau)))d\tau-\int_0^s k_2(s,\tau,\widetilde{\phi}(\zeta_{p+1}(\tau)))d\tau\Big\|\Big]ds\\ &\le \widehat{M}_2\mathscr{L}\int_0^t e^{-\mathscr{L}_0 t}\Big[e^{\mathscr{L}_0\xi_1(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\dots\\ &\quad+e^{\mathscr{L}_0\xi_n(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\mathscr{N}\int_0^s\|\phi(\xi_{n+1}(\tau))-\widetilde{\phi}(\xi_{n+1}(\tau))\|d\tau\Big]ds\\ &\quad+\widehat{M}_2\widetilde{\mathscr{L}}\int_0^t e^{-\mathscr{L}_0 t}\Big[e^{\mathscr{L}_0\zeta_1(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\dots\\ &\quad+e^{\mathscr{L}_0\zeta_p(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\widetilde{\mathscr{N}}\int_0^s\|\phi(\zeta_{p+1}(\tau))-\widetilde{\phi}(\zeta_{p+1}(\tau))\|d\tau\Big]ds\\ &\le \widehat{M}_2\mathscr{L}\int_0^t e^{-\mathscr{L}_0 t}\Big[ne^{\mathscr{L}_0 s}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\mathscr{N}be^{\mathscr{L}_0\xi_{n+1}(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|\Big]ds\\ &\quad+\widehat{M}_2\widetilde{\mathscr{L}}\int_0^t e^{-\mathscr{L}_0 t}\Big[pe^{\mathscr{L}_0 s}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|+\widetilde{\mathscr{N}}be^{\mathscr{L}_0\zeta_{p+1}(s)}\sup_{s\in\mathscr{J}}e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|\Big]ds\end{aligned}$$

$$
\begin{aligned}
&\leq \widehat{M}_2\mathscr{L}\int_0^t e^{-\mathscr{L}_0(t-s)}\Big[n\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\| + \mathscr{N}b\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|\Big]ds\\
&\quad + \widehat{M}_2\widetilde{\mathscr{L}}\int_0^t e^{-\mathscr{L}_0(t-s)}\Big[p\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\| + \widetilde{\mathscr{N}}b\sup_{s\in\mathscr{J}} e^{-\mathscr{L}_0 s}\|\phi(s)-\widetilde{\phi}(s)\|\Big]ds\\
&\leq \widehat{M}_2\mathscr{L}(n+\mathscr{N}b)\int_0^t e^{-\mathscr{L}_0(t-s)}ds\|\phi-\widetilde{\phi}\|_{\mathcal{V}} + \widehat{M}_2\widetilde{\mathscr{L}}(p+\widetilde{\mathscr{N}}b)\int_0^t e^{-\mathscr{L}_0(t-s)}ds\|\phi-\widetilde{\phi}\|_{\mathcal{V}}\\
&\leq \widehat{M}_2\Big[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})b\Big]\int_0^t e^{-\mathscr{L}_0(t-s)}ds\|\phi-\widetilde{\phi}\|_{\mathcal{V}}\\
&\leq \frac{\widehat{M}_2\Big[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})b\Big]}{\mathscr{L}_0}\|\phi-\widetilde{\phi}\|_{\mathcal{V}},\ t\in\mathscr{J},
\end{aligned}
$$

which implies that

$$e^{-\mathscr{L}_0 t}\|(\Upsilon_v\phi)(t)-(\Upsilon_v\widetilde{\phi})(t)\| \leq \frac{1}{2}\|\phi-\widetilde{\phi}\|_{\mathcal{V}},\ t\in\mathscr{J}.$$

Thus

$$\|\Upsilon_v\phi-\Upsilon_v\widetilde{\phi}\|_{\mathcal{V}} \leq \frac{1}{2}\|\phi-\widetilde{\phi}\|_{\mathcal{V}},\ \phi,\widetilde{\phi}\in\mathcal{V}.$$

Therefore, Υ_v is a strict contraction. By the Banach contraction principle, we conclude that Υ_v has a unique fixed point $\phi_v\in\mathcal{V}$ and the equation (3.2) has a unique mild solution on $[0,b]$.

Set

$$\widetilde{v}(t) := \begin{cases} v(t) & \text{if } t\in(\delta,b],\\ v(\delta) & \text{if } t\in[0,\delta]. \end{cases}$$

From (3.2), we have

$$
\begin{aligned}
\phi_{\widetilde{v}}(t) &= C(t,0)[u_0+q(\widetilde{v})]+S(t,0)[\widetilde{u}_0+\widetilde{q}(\widetilde{v})]\\
&\quad+\int_0^t S(t,s)\Big[\mathscr{F}\Big(s,\phi_{\widetilde{v}}(\xi_1(s)),\dots,\phi_{\widetilde{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\\
&\quad+\mathscr{G}\Big(s,\phi_{\widetilde{v}}(\zeta_1(s)),\dots,\phi_{\widetilde{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds\\
&\quad+\sum_{0<t_k<t} C(t,t_k)I_k(v(t_k))+\sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v(t_k)).
\end{aligned}
\tag{3.3}
$$

Consider the map $\Gamma:\mathcal{PC}_\delta=\mathcal{PC}([\delta,b],\mathbb{X})\to\mathcal{PC}_\delta$ defined by

$$(\Gamma v)(t)=\phi_{\widetilde{v}}(t),\ t\in[\delta,b].$$

We should demonstrate that Γ fulfills every one of the states of Lemma 2.5. The proof will be given in a few stages.

Step 1. Γ maps bounded sets into bounded sets in $\mathcal{PC}_\delta$.

In fact, it is sufficient to demonstrate that there exists a positive constant Λ_2 such that for each $v\in B_r(\delta):=\Big\{\phi\in\mathcal{PC}_\delta;\sup_{\delta\leq t\leq b}\|\phi(t)\|\leq r\Big\}$ one has $\|\Gamma v\|_{\mathcal{PC}}\leq\Lambda_2$.

Let $v \in B_r(\delta)$, then for $t \in (0, b]$, we have

$$
\begin{aligned}
\|\phi_{\widetilde{v}}(t)\| &\leq \|C(t,0)[u_0 + q(\widetilde{v})]\| + \|S(t,0)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\| \\
&\quad + \int_0^t \left\| S(t,s)\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\right\| ds \\
&\quad + \int_0^t \left\| S(t,s)\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\right\| ds \\
&\quad + \left\| \sum_{0<t_k<t} C(t,t_k) I_k(v(t_k))\right\| + \left\| \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v(t_k))\right\| \\
&\leq \widehat{M}_1[\|u_0 + q(\widetilde{v})\|] + \widehat{M}_2[\|\widetilde{u}_0 + \widetilde{q}(\widetilde{v})\|] + \widehat{M}_1 \sum_{k=1}^m \|I_k(v(t_k))\| + \widehat{M}_2 \sum_{k=1}^m \|\bar{I}_k(v(t_k))\| \\
&\quad + \widehat{M}_2 \int_0^t \Big[\Big\| \mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \ldots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big) \\
&\quad - \mathscr{F}(s,0,\ldots,0)\Big\| + \|\mathscr{F}(s,0,\ldots,0)\|\Big] ds \\
&\quad + \widehat{M}_2 \int_0^t \Big[\Big\| \mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \ldots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big) \\
&\quad - \mathscr{G}(s,0,\ldots,0)\Big\| + \|\mathscr{G}(s,0,\ldots,0)\|\Big] ds \\
&\leq \widehat{M}_1[\|u_0\| + \|q(\widetilde{v})\|] + \widehat{M}_2[\|\widetilde{u}_0\| + \|\widetilde{q}(\widetilde{v})\|] + \widehat{M}_1 \sum_{k=1}^m \Psi_k(\|v(t_k)\|) + \widehat{M}_2 \sum_{k=1}^m \widetilde{\Psi}_k(\|v(t_k)\|) \\
&\quad + \widehat{M}_2 \int_0^t \Big\{ \mathscr{L}\Big[\sup_{s\in(0,b]} \|\phi_{\widetilde{v}}(s)\| + \cdots + \sup_{s\in(0,b]} \|\phi_{\widetilde{v}}(s)\| \\
&\quad + \int_0^s [\|k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau))) - k_1(s,\tau,0)\| + \|k_1(s,\tau,0)\|]d\tau\Big] + \mathscr{L}_1\Big\} ds \\
&\quad + \widehat{M}_2 \int_0^t \Big\{ \widetilde{\mathscr{L}}\Big[\sup_{s\in(0,b]} \|\phi_{\widetilde{v}}(s)\| + \cdots + \sup_{s\in(0,b]} \|\phi_{\widetilde{v}}(s)\| \\
&\quad + \int_0^s [\|k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau))) - k_2(s,\tau,0)\| + \|k_2(s,\tau,0)\|]d\tau\Big] + \widetilde{\mathscr{L}_1}\Big\} ds \\
&\leq \widehat{M}_1[\|u_0\| + \Lambda(\|\widetilde{v}\|_{\mathcal{PC}})] + \widehat{M}_2[\|\widetilde{u}_0\| + \widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}})] + \widehat{M}_1 \sum_{k=1}^m \Psi_k(\|v(t_k)\|) + \widehat{M}_2 \sum_{k=1}^m \widetilde{\Psi}_k(\|v(t_k)\|) \\
&\quad + \widehat{M}_2 \int_0^t \left\{ \mathscr{L}\left[n \sup_{s\in(0,b]} \|\phi_{\widetilde{v}}(s)\| + b\left(\mathscr{N} \sup_{s\in(0,b]} \|\phi_{\widetilde{v}}(s)\| + \mathscr{N}_1\right)\right] + \mathscr{L}_1\right\} ds \\
&\quad + \widehat{M}_2 \int_0^t \left\{ \widetilde{\mathscr{L}}\left[p \sup_{s\in(0,b]} \|\phi_{\widetilde{v}}(s)\| + b\left(\widetilde{\mathscr{N}} \sup_{s\in(0,b]} \|\phi_{\widetilde{v}}(s)\| + \widetilde{\mathscr{N}_1}\right)\right] + \widetilde{\mathscr{L}_1}\right\} ds \\
&\leq \widehat{M}_* + \widehat{M}_1\Lambda(r) + \widehat{M}_2\widetilde{\Lambda}(r) + \widehat{M}_1 \sum_{k=1}^m \Psi_k(r) + \widehat{M}_2 \sum_{k=1}^m \widetilde{\Psi}_k(r) \\
&\quad + \widehat{M}_2\Big[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big] \int_0^t \sup_{s\in(0,b]} \|\phi_{\widetilde{v}}(s)\| ds,
\end{aligned}
$$

where $\widehat{M}_* = \widehat{M}_1\|u_0\| + \widehat{M}_2\|\widetilde{u}_0\| + \widehat{M}_2 b\left[b(\mathscr{L}\mathscr{N}_1 + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1}) + (\mathscr{L}_1 + \widetilde{\mathscr{L}_1})\right]$.

Utilizing the Gronwall's inequality, we obtain

$$\sup_{s\in(0,b]}\|\phi_{\widetilde{v}}(s)\| \leq e^{\widehat{M}_2 b\left[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+b(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\right]}\left[\widehat{M}_* + \widehat{M}_1\left[\Lambda(r)+\sum_{k=1}^{m}\Psi_k(r)\right]+\widehat{M}_2\left[\widetilde{\Lambda}(r)+\sum_{k=1}^{m}\widetilde{\Psi}_k(r)\right]\right].$$

Thus

$$\|\Gamma v\|_{\mathcal{PC}} \leq e^{\eta}\left[\widehat{M}_* + \widehat{M}_1\left[\Lambda(r)+\sum_{k=1}^{m}\Psi_k(r)\right]+\widehat{M}_2\left[\widetilde{\Lambda}(r)+\sum_{k=1}^{m}\widetilde{\Psi}_k(r)\right]\right]. := \Lambda_2.$$

Step 2. Γ is continuous on $B_r(\delta)$.

From (3.2) and $(H1)-(H5)$, we deduce that for $v_1, v_2 \in B_r(\delta)$, $t \in (0,b]$,

$$\begin{aligned}
&\|\phi_{\widetilde{v}_1}(t)-\phi_{\widetilde{v}_2}(t)\|\\
&\leq \|C(t,0)[q(\widetilde{v_1})-q(\widetilde{v_2})]\| + \|S(t,0)[\widetilde{q}(\widetilde{v_1})-\widetilde{q}(\widetilde{v_2})]\| + \Big\|\sum_{0<t_k<t} C(t,t_k)I_k(v_1(t_k))\\
&\quad - \sum_{0<t_k<t} C(t,t_k)I_k(v_2(t_k))\Big\| + \Big\|\sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v_1(t_k)) - \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v_2(t_k))\Big\|\\
&\quad + \int_0^t \Big\|S(t,s)\Big[\mathscr{F}\Big(s,\phi_{\widetilde{v}_1}(\xi_1(s)),\dots,\phi_{\widetilde{v}_1}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}_1}(\xi_{n+1}(\tau)))d\tau\Big)\\
&\quad - \mathscr{F}\Big(s,\phi_{\widetilde{v}_2}(\xi_1(s)),\dots,\phi_{\widetilde{v}_2}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\widetilde{v}_2}(\xi_{n+1}(\tau)))d\tau\Big)\Big]\Big\|ds\\
&\quad + \int_0^t \Big\|S(t,s)\Big[\mathscr{G}\Big(s,\phi_{\widetilde{v}_1}(\zeta_1(s)),\dots,\phi_{\widetilde{v}_1}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}_1}(\zeta_{p+1}(\tau)))d\tau\Big)\\
&\quad - \mathscr{G}\Big(s,\phi_{\widetilde{v}_2}(\zeta_1(s)),\dots,\phi_{\widetilde{v}_2}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\widetilde{v}_2}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]\Big\|ds\\
&\leq \widehat{M}_1\|q(\widetilde{v_1})-q(\widetilde{v_2})\| + \widehat{M}_2\|\widetilde{q}(\widetilde{v_1})-\widetilde{q}(\widetilde{v_2})\| + \widehat{M}_1\sum_{k=1}^{m}\|I_k(v_1(t_k))-I_k(v_2(t_k))\|\\
&\quad + \widehat{M}_2\sum_{k=1}^{m}\|\bar{I}_k(v_1(t_k))-\bar{I}_k(v_2(t_k))\| + \widehat{M}_2\mathscr{L}\int_0^t\Big[\|\phi_{\widetilde{v}_1}(\xi_1(s))-\phi_{\widetilde{v}_2}(\xi_1(s))\|+\dots\\
&\quad + \|\phi_{\widetilde{v}_1}(\xi_n(s))-\phi_{\widetilde{v}_2}(\xi_n(s))\| + \int_0^s[\|k_1(s,\tau,\phi_{\widetilde{v}_1}(\xi_{n+1}(\tau)))-k_1(s,\tau,\phi_{\widetilde{v}_2}(\xi_{n+1}(\tau)))\|]d\tau\Big]ds\\
&\quad + \widehat{M}_2\widetilde{\mathscr{L}}\int_0^t\Big[\|\phi_{\widetilde{v}_1}(\zeta_1(s))-\phi_{\widetilde{v}_2}(\zeta_1(s))\|+\dots+\|\phi_{\widetilde{v}_1}(\zeta_p(s))-\phi_{\widetilde{v}_2}(\zeta_p(s))\|\\
&\quad + \int_0^s[\|k_2(s,\tau,\phi_{\widetilde{v}_1}(\zeta_{p+1}(\tau)))-k_2(s,\tau,\phi_{\widetilde{v}_2}(\zeta_{p+1}(\tau)))\|]d\tau\Big]ds\\
&\leq \widehat{M}_1\|q(\widetilde{v_1})-q(\widetilde{v_2})\| + \widehat{M}_2\|\widetilde{q}(\widetilde{v_1})-\widetilde{q}(\widetilde{v_2})\| + \widehat{M}_1\sum_{k=1}^{m}\|I_k(v_1(t_k))-I_k(v_2(t_k))\|\\
&\quad + \widehat{M}_2\sum_{k=1}^{m}\|\bar{I}_k(v_1(t_k))-\bar{I}_k(v_2(t_k))\| + \widehat{M}_2\mathscr{L}\int_0^t\Big[\sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s)-\phi_{\widetilde{v}_2}(s)\|+\dots\\
&\quad + \sup_{s\in[0,b]}\|\phi_{\widetilde{v}_1}(s)-\phi_{\widetilde{v}_2}(s)\| + \mathscr{N}\int_0^s\|\phi_{\widetilde{v}_1}(\xi_{n+1}(\tau))-\phi_{\widetilde{v}_2}(\xi_{n+1}(\tau))\|d\tau\Big]ds
\end{aligned}$$

$$
\begin{aligned}
&+ \widehat{M}_2\widetilde{\mathscr{L}}\int_0^t \Big[\sup_{s\in[0,b]} \|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\| + \cdots + \sup_{s\in[0,b]} \|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\| \\
&+ \widetilde{\mathscr{N}}\int_0^s \|\phi_{\widetilde{v}_1}(\zeta_{p+1}(\tau)) - \phi_{\widetilde{v}_2}(\zeta_{p+1}(\tau))\| d\tau \Big] ds \\
&\leq \widehat{M}_1\|q(\widetilde{v}_1) - q(\widetilde{v}_2)\| + \widehat{M}_2\|\widetilde{q}(\widetilde{v}_1) - \widetilde{q}(\widetilde{v}_2)\| + \widehat{M}_1\sum_{k=1}^m \|I_k(v_1(t_k)) - I_k(v_2(t_k))\| \\
&+ \widehat{M}_2\sum_{k=1}^m \|\bar{I}_k(v_1(t_k)) - \bar{I}_k(v_2(t_k))\| + \widehat{M}_2\mathscr{L}\int_0^t \Big[n \sup_{s\in[0,b]} \|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\| \\
&+ \mathscr{N} b \sup_{s\in[0,b]} \|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\| \Big] ds \\
&+ \widehat{M}_2\widetilde{\mathscr{L}}\int_0^t \Big[p \sup_{s\in[0,b]} \|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\| + \widetilde{\mathscr{N}} b \sup_{s\in[0,b]} \|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\| \Big] ds \\
&\leq \widehat{M}_1\Big[\|q(\widetilde{v}_1) - q(\widetilde{v}_2)\| + \sum_{k=1}^m \|I_k(v_1(t_k)) - I_k(v_2(t_k))\| \Big] + \widehat{M}_2\Big[\|\widetilde{q}(\widetilde{v}_1) - \widetilde{q}(\widetilde{v}_2)\| \\
&+ \sum_{k=1}^m \|\bar{I}_k(v_1(t_k)) - \bar{I}_k(v_2(t_k))\| \Big] \\
&+ \widehat{M}_2\Big[(\mathscr{L} n + \widetilde{\mathscr{L}} p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}}) \Big] \int_0^t \sup_{s\in[0,b]} \|\phi_{\widetilde{v}_1}(s) - \phi_{\widetilde{v}_2}(s)\| ds.
\end{aligned}
$$

Using Gronwall's inequality again, for t, v_1, v_2 as above

$$
\begin{aligned}
&\sup_{s\in[0,b]} \|\phi_{\widetilde{v}_1}(t) - \phi_{\widetilde{v}_2}(t)\| \\
&\leq \Big[\widehat{M}_1\Big[\|q(\widetilde{v}_1) - q(\widetilde{v}_2)\| + \sum_{k=1}^m \|I_k(v_1(t_k)) - I_k(v_2(t_k))\| \Big] + \widehat{M}_2\Big[\|\widetilde{q}(\widetilde{v}_1) - \widetilde{q}(\widetilde{v}_2)\| \\
&+ \sum_{k=1}^m \|\bar{I}_k(v_1(t_k)) - \bar{I}_k(v_2(t_k))\| \Big] \Big] e^{\eta},
\end{aligned}
$$

for all $t \in [0, b]$, which implies that

$$
\begin{aligned}
\|\Gamma v_1 - \Gamma v_2\|_{\mathcal{PC}} \leq \Big[&\widehat{M}_1\Big[\|q(\widetilde{v}_1) - q(\widetilde{v}_2)\| + \sum_{k=1}^m \|I_k(v_1(t_k)) - I_k(v_2(t_k))\| \Big] + \widehat{M}_2\Big[\|\widetilde{q}(\widetilde{v}_1) - \widetilde{q}(\widetilde{v}_2)\| \\
&+ \sum_{k=1}^m \|\bar{I}_k(v_1(t_k)) - \bar{I}_k(v_2(t_k))\| \Big] \Big] e^{\eta},
\end{aligned}
$$

for all $t \in [\delta, b]$, $v_1, v_2 \in B_r(\delta)$. Therefore, Γ is continuous.

Step 3. Γ is a compact operator.

To this end, we consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 and Γ_2 are the operators on $B_r(\delta)$ defined respectively by

$$
(\Gamma_1 v)(t) = C(t, 0)[u_0 + q(\widetilde{v})] + S(t, 0)[\widetilde{u_0} + \widetilde{q}(\widetilde{v})]
$$

$$+\int_0^t S(t,s)\mathscr{F}\Big(s,\phi_{\overline{v}}(\xi_1(s)),\dots,\phi_{\overline{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big)ds$$
$$+\int_0^t S(t,s)\mathscr{G}\Big(s,\phi_{\overline{v}}(\zeta_1(s)),\dots,\phi_{\overline{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)ds,\ t\in[\delta,b],$$
$$(\Gamma_2 v)(t)=\sum_{0<t_k<t}C(t,t_k)I_k(v(t_k))+\sum_{0<t_k<t}S(t,t_k)\bar{I}_k(v(t_k)),\ t\in[\delta,b].$$

We first show that Γ_1 is a compact operator.
(i) $\Gamma_1(B_r(\delta))$ is equicontinuous.

Let $\delta\le\tau_1<\tau_2\le b$, and $\epsilon>0$ be small, note that

$$\begin{aligned}
&\Big\|\mathscr{F}\Big(s,\phi_{\overline{v}}(\xi_1(s)),\dots,\phi_{\overline{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\|\\
&\le\Big\|\mathscr{F}\Big(s,\phi_{\overline{v}}(\xi_1(s)),\dots,\phi_{\overline{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big)-\mathscr{F}(s,0,\dots,0)\Big\|+\|\mathscr{F}(s,0,\dots,0)\|\\
&\le\mathscr{L}\Big[\|\phi_{\overline{v}}(\xi_1(s))\|\cdots+\|\phi_{\overline{v}}(\xi_n(s))\|+\Big\|\int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big\|\Big]+\mathscr{L}_1\\
&\le\mathscr{L}\Big[\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+\cdots+\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+\int_0^s[\|k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))-k_1(s,\tau,0)\|\\
&\quad+\|k_1(s,\tau,0)\|]d\tau\Big]+\mathscr{L}_1\\
&\le\mathscr{L}\Big\{n\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+b\Big[\mathscr{N}\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+\mathscr{N}_1\Big]\Big\}+\mathscr{L}_1\\
&\le\mathscr{L}\Big[(n+\mathscr{N}b)\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+b\mathscr{N}_1\Big]+\mathscr{L}_1\\
&\le\mathscr{L}[(n+\mathscr{N}b)r+b\mathscr{N}_1]+\mathscr{L}_1:=\widehat{M}^{**}
\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\mathscr{G}\Big(s,\phi_{\overline{v}}(\zeta_1(s)),\dots,\phi_{\overline{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\|\\
&\le\Big\|\mathscr{G}\Big(s,\phi_{\overline{v}}(\zeta_1(s)),\dots,\phi_{\overline{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)-\mathscr{G}(s,0,\dots,0)\Big\|+\|\mathscr{G}(s,0,\dots,0)\|\\
&\le\widetilde{\mathscr{L}}\Big[\|\phi_{\overline{v}}(\zeta_1(s))\|+\cdots+\|\phi_{\overline{v}}(\zeta_p(s))\|+\Big\|\int_0^s k_2(s,\tau,\phi_{\overline{v}}(\xi_{p+1}(\tau)))d\tau\Big\|\Big]+\widetilde{\mathscr{L}_1}\\
&\le\widetilde{\mathscr{L}}\Big[\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+\cdots+\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+\int_0^s[\|k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))-k_2(s,\tau,0)\|\\
&\quad+\|k_2(s,\tau,0)\|]d\tau\Big]+\widetilde{\mathscr{L}_1}\\
&\le\widetilde{\mathscr{L}}\Big\{p\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+b\Big[\widetilde{\mathscr{N}}\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+\widetilde{\mathscr{N}_1}\Big]\Big\}+\widetilde{\mathscr{L}_1}\\
&\le\widetilde{\mathscr{L}}\Big[(p+\widetilde{\mathscr{N}}b)\sup_{s\in[\delta,b]}\|\phi_{\overline{v}}(s)\|+b\widetilde{\mathscr{N}_1}\Big]+\widetilde{\mathscr{L}_1}\\
&\le\widetilde{\mathscr{L}}[(p+\widetilde{\mathscr{N}}b)r+b\widetilde{\mathscr{N}_1}]+\widetilde{\mathscr{L}_1}:=\widehat{M}_1^{**}.
\end{aligned}$$

From the above estimations, we have

$$
\begin{aligned}
&\|\Gamma_1 v(\tau_2) - \Gamma_1 v(\tau_1)\| \\
&\le \|[C(\tau_2,0) - C(\tau_1,0)][u_0 + q(\widetilde{v})]\| + \|[S(\tau_2,0) - S(\tau_1,0)][\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\| \\
&\quad + \Big\| \int_0^{\tau_2} S(\tau_2,s)\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \dots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)ds \\
&\quad - \int_0^{\tau_1} S(\tau_1,s)\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \dots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)ds\Big\| \\
&\quad + \Big\| \int_0^{\tau_2} S(\tau_2,s)\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \dots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)ds \\
&\quad - \int_0^{\tau_1} S(\tau_1,s)\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \dots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)ds\Big\| \\
&\le \|[C(\tau_2,0) - C(\tau_1,0)][u_0 + q(\widetilde{v})]\| + \|[S(\tau_2,0) - S(\tau_1,0)][\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\| \\
&\quad + \int_0^{\tau_1-\epsilon} \Big\|[S(\tau_2,s) - S(\tau_1,s)]\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \dots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_{\tau_1-\epsilon}^{\tau_1} \Big\|[S(\tau_2,s) - S(\tau_1,s)]\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \dots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_{\tau_1}^{\tau_2} \Big\|S(\tau_2,s)\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \dots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_0^{\tau_1-\epsilon} \Big\|[S(\tau_2,s) - S(\tau_1,s)]\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \dots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_{\tau_1-\epsilon}^{\tau_1} \Big\|[S(\tau_2,s) - S(\tau_1,s)]\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \dots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\|ds \\
&\quad + \int_{\tau_1}^{\tau_2} \Big\|S(\tau_2,s)\mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \dots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\|ds \\
&\le \|[C(\tau_2,0) - C(\tau_1,0)][u_0 + q(\widetilde{v})]\| + \|[S(\tau_2,0) - S(\tau_1,0)][\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\| \\
&\quad + \widehat{M}^{**} \int_0^{\tau_1-\epsilon} \|S(\tau_2,s) - S(\tau_1,s)\|ds + \widehat{M}^{**} \int_{\tau_1-\epsilon}^{\tau_1} \|S(\tau_2,s) - S(\tau_1,s)\|ds \\
&\quad + \widehat{M}^{**} \int_{\tau_1}^{\tau_2} \|S(\tau_2,s)\|ds + \widehat{M}_1^{**} \int_0^{\tau_1-\epsilon} \|S(\tau_2,s) - S(\tau_1,s)\|ds \\
&\quad + \widehat{M}_1^{**} \int_{\tau_1-\epsilon}^{\tau_1} \|S(\tau_2,s) - S(\tau_1,s)\|ds + \widehat{M}_1^{**} \int_{\tau_1}^{\tau_2} \|S(\tau_2,s)\|ds.
\end{aligned}
$$

We see that $\|\Gamma_1 v(\tau_2) - \Gamma v(\tau_1)\|$ tends to zero independently of $v \in B_r(\delta)$ as $\tau_2 - \tau_1 \to 0$, since the compactness of the operator $S(t,s)$ for $t - s > 0$, implies the continuity in the uniform operator topology. Thus, Γ_1 maps $B_r(\delta)$ into an equicontinuous family of the functions.

(ii) The set $\Gamma_1(B_r(\delta))(t)$ is precompact in $\mathbb{X}$.

Let $\delta < t \le s \le b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $v \in B_r(\delta)$, we define

$$
\begin{aligned}
(\Gamma_{1,\epsilon} v)(t) &= C(t,0)[u_0 + q(\widetilde{v})] + S(t,0)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v})] \\
&\quad + \int_0^{t-\epsilon} S(t,s)\Big[\mathscr{F}\Big(s, \phi_{\widetilde{v}}(\xi_1(s)), \dots, \phi_{\widetilde{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\widetilde{v}}(\xi_{n+1}(\tau)))d\tau\Big) \\
&\quad + \mathscr{G}\Big(s, \phi_{\widetilde{v}}(\zeta_1(s)), \dots, \phi_{\widetilde{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\widetilde{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds.
\end{aligned}
$$

Using the compactness of $S(t,s)$ for $t - s > 0$, we deduce that the set $\{(\Gamma_{1,\epsilon} v)(t) : v \in B_r(\delta)\}$ is precompact

$v \in B_r(\delta)$ for $\epsilon, 0 < \epsilon < t$. Moreover, for every $v \in B_r(\delta)$ we have

$$\begin{aligned}
&\|(\Gamma_1 v)(t) - (\Gamma_{1,\epsilon} v)(t)\| \\
&\leq \int_{t-\epsilon}^{t} \Big\| S(t,s)\Big[\mathscr{F}\Big(s, \phi_{\overline{v}}(\xi_1(s)), \ldots, \phi_{\overline{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big) \\
&\quad + \mathscr{G}\Big(s, \phi_{\overline{v}}(\zeta_1(s)), \ldots, \phi_{\overline{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]\Big\| ds \\
&\leq \widehat{M}_2 \int_{t-\epsilon}^{t} \Big(\widehat{M}^{**} + \widehat{M}_1^{**}\Big) ds \\
&\leq \widehat{M}_2\Big(\widehat{M}^{**} + \widehat{M}_1^{**}\Big)\epsilon.
\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $\{(\Gamma_1 v) : v \in B_r(\delta)\}$. Hence the set $\{(\Gamma_1 v) : v \in B_r(\delta)\}$ is a precompact in $\mathbb{X}$. It is easy to see that $\Gamma_1(B_r(\delta))$ is uniformly bounded. Since we have shown that $\Gamma_1(B_r(\delta))$ is an equicontinuous collection, by the Arzela-Ascoli theorem, we conclude that Γ_1 is compact operator.

Next, it stays to check that Γ_2 is also a compact operator. From [11, Theorem 3.2], we observe that Γ_2 is compact operator and hence Γ is a compact operator.

Step 4. We now show that there exists an open set $U \subseteq \mathcal{PC}_\delta$ with $v \notin \lambda\Gamma v$ for $\lambda \in (0,1)$ and $v \in \partial U$. Let $\lambda \in (0,1)$ and let $v \in \mathcal{PC}_\delta$ be a possible solution of $v = \lambda\Gamma(v)$ for some $0 < \lambda < 1$. Thus, for each $t \in (0,b]$,

$$\begin{aligned}
v(t) = \lambda\phi_{\overline{v}}(t) &= \lambda C(t,0)[u_0 + q(\widetilde{v})] + \lambda S(t,0)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v})] \\
&+ \lambda \int_0^t S(t,s)\Big[\mathscr{F}\Big(s, \phi_{\overline{v}}(\xi_1(s)), \ldots, \phi_{\overline{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big) \\
&+ \mathscr{G}\Big(s, \phi_{\overline{v}}(\zeta_1(s)), \ldots, \phi_{\overline{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big] ds \\
&+ \lambda \sum_{0<t_k<t} C(t,t_k) I_k(v(t_k)) + \lambda \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v(t_k)).
\end{aligned}$$

This suggests by $(H1) - (H5)$ and for each $t \in (0,b]$, we have $\|v(t)\| \leq \|\phi_{\overline{v}}(t)\|$ and

$$\begin{aligned}
\|\phi_{\overline{v}}(t)\| &\leq \|C(t,0)[u_0 + q(\widetilde{v})]\| + \|S(t,0)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v})]\| \\
&\quad + \int_0^t \Big\| S(t,s)\mathscr{F}\Big(s, \phi_{\overline{v}}(\xi_1(s)), \ldots, \phi_{\overline{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big)\Big\| ds \\
&\quad + \int_0^t \Big\| S(t,s)\mathscr{G}\Big(s, \phi_{\overline{v}}(\zeta_1(s)), \ldots, \phi_{\overline{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big\| ds \\
&\quad + \Big\| \sum_{0<t_k<t} C(t,t_k) I_k(v(t_k))\Big\| + \Big\| \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v(t_k))\Big\| \\
&\leq \widehat{M}_* + \widehat{M}_1\Big[\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \Psi_k(\|v(t_k)\|)\Big] + \widehat{M}_2\Big[\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \widetilde{\Psi}_k(\|v(t_k)\|)\Big] \\
&\quad + \widehat{M}_2\Big[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big]\int_0^t \sup_{s\in(0,b]} \|\phi_{\overline{v}}(s)\| ds,
\end{aligned}$$

where $\widehat{M}_* = \widehat{M}_1\|u_0\| + \widehat{M}_2\|\widetilde{u}_0\| + \widehat{M}_2 b\Big[b(\mathscr{L}\mathscr{N}_1 + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1}) + (\mathscr{L}_1 + \widetilde{\mathscr{L}_1})\Big]$.

Utilizing the Gronwall's inequality, we obtain

$$\sup_{s\in(0,b]} \|\phi_{\overline{v}}(s)\| \leq e^{\eta}\Big[\widehat{M}_* + \widehat{M}_1\Big[\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \Psi_k(\|v\|_{\mathcal{PC}})\Big] + \widehat{M}_2\Big[\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\Big]\Big].$$

and the previous inequality holds. Consequently,

$$\|v\|_{\mathcal{PC}} \leq e^{\eta}\left[\widehat{M}_* + \widehat{M}_1\left[\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\Psi_k(\|v\|_{\mathcal{PC}})\right] + \widehat{M}_2\left[\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^{m}\widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right]\right],$$

and therefore

$$\frac{\|v\|_{\mathcal{PC}}}{e^{\eta}\left[\widehat{M}_* + \widehat{M}_1\left[\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum\limits_{k=1}^{m}\Psi_k(\|v\|_{\mathcal{PC}})\right] + \widehat{M}_2\left[\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum\limits_{k=1}^{m}\widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right]\right]} \leq 1.$$

Then, by $(H6)$, there exists $\widehat{M}^*$ such that $\|v\|_{\mathcal{PC}} \neq \widehat{M}^*$. Set

$$U = \left\{v \in \mathcal{PC}([\delta, b], \mathbb{X}); \sup_{\delta\leq t\leq b}\|v(t)\| < \widehat{M}^*\right\}.$$

As an outcome of Steps 1-3 in Theorem 3.1, it suffices to demonstrate that $\Gamma : \overline{U} \to \mathcal{PC}_\delta$ is a compact map.

From the choice of U, there is no $u \in \partial U$ such that $v \in \lambda\Gamma v$ for $\lambda \in (0, 1)$. As a consequence of Lemma 2.5, we deduce that Γ has a fixed point $\widetilde{v}_* \in \overline{U}$. Then, we have

$$\begin{aligned}
u(t) &= C(t,0)[u_0 + q(\widetilde{v}_*)] + S(t,0)[\widetilde{u}_0 + \widetilde{q}(\widetilde{v}_*)] \\
&+ \int_0^t S(t,s)\Big[\mathscr{F}\Big(s, u(\xi_1(s)), \ldots, u(\xi_n(s)), \int_0^s k_1(s,\tau,u(\xi_{n+1}(\tau)))d\tau\Big) \\
&+ \mathscr{G}\Big(s, u(\zeta_1(s)), \ldots, u(\zeta_p(s)), \int_0^s k_2(s,\tau,u(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds \\
&+ \sum_{0<t_k<t} C(t,t_k)I_k(v(t_k)) + \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v(t_k)).
\end{aligned} \tag{3.4}$$

Noting that $u = \phi_{\widetilde{v}_*} = (\Gamma\widetilde{v}_*)(t) = \widetilde{v}_*$, $t \in [\delta, b]$. By $(H5)(i)(ii)$, we obtain $q(u) = q(\widetilde{v}_*)$ and $\widetilde{q}(u) = \widetilde{q}(\widetilde{v}_*)$. This suggests, joined with (3.4), that $u(t)$ is a mild solution of problem (1.1)-(1.3). This completes the proof of this theorem. □

4. Approximate Results

As an application of Theorem 3.1, we shall consider the system (1.1) with control parameters such as:

$$\begin{aligned}
u''(t) &= \mathscr{A}(t)u(t) + \mathscr{F}\left(t, u(\xi_1(t)), \ldots, u(\xi_n(t)), \int_0^t k_1(t,s,u(\xi_{n+1}(s)))ds\right) \\
&+ \mathscr{G}\left(t, u(\zeta_1(t)), \ldots, u(\zeta_p(t)), \int_0^t k_2(t,s,u(\zeta_{p+1}(s)))ds\right) + B\widehat{u}(t), \\
&t \in \mathscr{J} = [0,b],\ t \neq t_k,\ k = 1,2,\ldots,m,
\end{aligned} \tag{4.1}$$

with the conditions (1.2) and (1.3). The functions $\mathscr{F}(\cdot), \mathscr{G}(\cdot), k_1(\cdot), k_2(\cdot), q(\cdot), \widetilde{q}(\cdot), I_k(\cdot), \bar{I}_k(\cdot), \xi_i$, $i = 1, 2, \ldots, n+1$ and ζ_l, $l = 1, \ldots, p+1$, are same as defined in (1.1)-(1.3). The control function $\widehat{u}(\cdot) \in \mathscr{L}^2(\mathscr{J}, U)$, a Banach space of admissible control function with U as a Banach space and B is a bounded linear operator from U to $\mathbb{X}$.

Definition 4.1. *A function $u(\cdot) \in \mathcal{PC}(\mathscr{J}, \mathbb{X})$ is said to be a mild solution of problem* (4.1) *with the conditions (1.2) and (1.3) if it satisfies the following integral equation*

$$\begin{aligned} u(t) &= C(t,0)[u_0 + q(u)] + S(t,0)[\widetilde{u_0} + \widetilde{q}(u)] \\ &+ \int_0^t S(t,s)\Big[\mathscr{F}\Big(s, u(\xi_1(s)), \ldots, u(\xi_n(s)), \int_0^s k_1(s,\tau,u(\xi_{n+1}(\tau)))d\tau\Big) \\ &+ \mathscr{G}\Big(s, u(\zeta_1(s)), \ldots, u(\zeta_p(s)), \int_0^s k_2(s,\tau,u(\zeta_{p+1}(\tau)))d\tau\Big) + B\widehat{u}(s)\Big]ds \\ &+ \sum_{0<t_k<t} C(t,t_k)I_k(u(t_k)) + \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(u(t_k)), \quad t \in \mathscr{J}. \end{aligned}$$

Definition 4.2. *The control system (4.1) with the conditions (1.2) and (1.3) is said to be approximately controllable on $\mathscr{J}$ if for all $u_0 \in \mathbb{X}_\alpha$, there is some control $\widehat{u} \in \mathscr{L}^2(\mathscr{J}, U)$, the closure of the reachable set, $\overline{R(b,u_0)}$ is dense in $\mathbb{X}$, i.e., $\overline{R(b,u_0)} = \mathbb{X}$, where $R(b,u_0) = \{u(b,\widehat{u}) : \widehat{u} \in \mathscr{L}^2(\mathscr{J}, U), u(0,\widehat{u}) = u_0\}$ is a mild solution of the system (4.1) with the conditions (1.2) and (1.3).*

In order to address the problem, it is helpful now to present two significant operators and essential hypotheses on these operators:

$$\begin{aligned} \overline{\Upsilon}_0^b &= \int_0^b S(b,s)BB^*S^*(b,s)ds : \mathbb{X} \to \mathbb{X}, \\ R(\gamma, \overline{\Upsilon}_0^b) &= (\gamma I + \overline{\Upsilon}_0^b)^{-1} : \mathbb{X} \to \mathbb{X}, \quad 0 < \gamma < 1, \end{aligned}$$

where B^* denotes the adjoint of B and $S^*(t)$ is the adjoint of $S(t)$. It is straightforward that the operator $\overline{\Upsilon}_0^b$ is a linear bounded operator.

To investigate the approximate controllability of system (4.1) with the conditions (1.2) and (1.3), we impose the following condition:

(H0) $\gamma R(\gamma, \overline{\Upsilon}_0^b) \to 0$ as $\gamma \to 0^+$ in the strong operator topology.

In view of [27], hypothesis (H0) holds if and only if the linear system

$$u'(t) = \mathscr{A}(t)u(t) + B\widehat{u}(t), \quad t \in [0,b], \tag{4.2}$$

$$u(0) = u_0 \tag{4.3}$$

is approximate controllability on $\mathscr{J}$.

It will be shown that the system (1.4) with the conditions (1.2) and (1.3) is approximately controllable, if for all $\gamma > 0$, there exists a function $u(\cdot) \in \mathcal{PC}$ and $u_b \in \mathbb{X}$ such that

$$\begin{aligned} u(t) &= C(t,0)[u_0 + q(u)] + S(t,0)[\widetilde{u_0} + \widetilde{q}(u)] \\ &+ \int_0^t S(t,s)\Big[\mathscr{F}\Big(s, u(\xi_1(s)), \ldots, u(\xi_n(s)), \int_0^s k_1(s,\tau,u(\xi_{n+1}(\tau)))d\tau\Big) \\ &+ \mathscr{G}\Big(s, u(\zeta_1(s)), \ldots, u(\zeta_p(s)), \int_0^s k_2(s,\tau,u(\zeta_{p+1}(\tau)))d\tau\Big) + B\widehat{u}(s,u)\Big]ds \\ &+ \sum_{0<t_k<t} C(t,t_k)I_k(u(t_k)) + \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(u(t_k)), \\ \widehat{u}(t,u) &= B^*S^*(b,t)R(\gamma, \overline{\Upsilon}_0^b)\widehat{p}(u(\cdot)), \end{aligned}$$

where

$$\begin{aligned}\widehat{p}(u(\cdot)) &= u_b - C(b,0)[u_0 + q(u)] - S(b,0)[\widetilde{u}_0 + \widetilde{q}(u)]\\ &\quad - \int_0^b S(b,s)\Big[\mathscr{F}\Big(s, u(\xi_1(s)), \dots, u(\xi_n(s)), \int_0^s k_1(s,\tau,u(\xi_{n+1}(\tau)))d\tau\Big)\\ &\quad + \mathscr{G}\Big(s, u(\zeta_1(s)), \dots, u(\zeta_p(s)), \int_0^s k_2(s,\tau,u(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds\\ &\quad - \sum_{k=1}^m C(b,t_k)I_k(u(t_k)) - \sum_{k=1}^m S(b,t_k)\bar{I}_k(u(t_k)).\end{aligned}$$

Remark 4.3. *In view of equations (3.2), (3.3) and Step 1 of Theorem 3.1, if $v \in \mathcal{PC}_\delta$, we calculate the following estimate:*

$$\begin{aligned}&\Big\|\int_0^t S(t,s)B\widehat{u}(s,v)ds\Big\|\\ &\le \int_0^t S(t,s)BB^*S^*(b,t)R(\gamma,\overline{\Upsilon}_0^b)\Big[u_b - C(b,0)[u_0 + q(\overline{v})] - S(b,0)[\widetilde{u}_0 + \widetilde{q}(\overline{v})]\\ &\quad - \int_0^b S(b,s)\Big[\mathscr{F}\Big(s, \phi_{\overline{v}}(\xi_1(s)), \dots, \phi_{\overline{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big)\\ &\quad + \mathscr{G}\Big(s, \phi_{\overline{v}}(\zeta_1(s)), \dots, \phi_{\overline{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds - \sum_{k=1}^m C(b,t_k)I_k(v(t_k))\\ &\quad - \sum_{k=1}^m S(b,t_k)\bar{I}_k(v(t_k))\Big](s)ds\\ &\le \left(\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Big[\|u_b\| + \widehat{M}_1\Big[\|u_0\| + \Lambda(r) + \sum_{k=1}^m \Psi_k(r)\Big] + \widehat{M}_2\Big[\|\widetilde{u}_0\| + \widetilde{\Lambda}(r) + \sum_{k=1}^m \widetilde{\Psi}_k(r)\Big]\\ &\quad + \widehat{M}_2 b\Big[b(\mathscr{L}\mathscr{N}_1 + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1}) + (\mathscr{L}_1 + \widetilde{\mathscr{L}_1})\Big]\Big]\\ &\quad + \left(\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\widehat{M}_2\Big[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big]\int_0^b \sup_{s\in(0,b]}\|\phi_{\overline{v}}(s)\|ds.\end{aligned}$$

Theorem 4.4. *Suppose that the hypotheses (H0)-(H5) are satisfied. Then the system (4.1) with the conditions (1.2) and (1.3) has at least one mild solution on $\mathscr{J}$ provided*

$$\frac{\widehat{M}_{***}}{e^{\widetilde{\eta}}\Big[\widehat{M}_{**} + \Big(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\Big)\Big[\widehat{M}_1\Big(\Lambda(\widehat{M}_{***}) + \sum_{k=1}^m \Psi_k(\widehat{M}_{***})\Big) + \widehat{M}_2\Big(\widetilde{\Lambda}(\widehat{M}_{***}) + \sum_{k=1}^m \widetilde{\Psi}_k(\widehat{M}_{***})\Big)\Big]\Big]} > 1,$$

where $\widehat{M}_B = \|B\|$, $\widehat{M}_{**} = \Big(\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\Big)\|u_b\| + \Big(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\Big)\Big[\widehat{M}_1\|u_0\| + \widehat{M}_2\|\widetilde{u}_0\| + \widehat{M}_2 b\Big[b(\mathscr{L}\mathscr{N}_1 + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1})$ $+ (\mathscr{L}_1 + \widetilde{\mathscr{L}_1})\Big]\Big]$ *and* $\widetilde{\eta} = \Big(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\Big)\widehat{M}_2[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})]b.$

Proof. By thinking of Theorem 3.1, we define

$$\begin{aligned}
\phi_{\overline{v}}(t) &= C(t,0)[u_0 + q(\overline{v})] + S(t,0)[\widetilde{u_0} + \widetilde{q}(\overline{v})] \\
&+ \int_0^t S(t,s)\Big[\mathscr{F}\Big(s, \phi_{\overline{v}}(\xi_1(s)), \ldots, \phi_{\overline{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big) \\
&+ \mathscr{G}\Big(s, \phi_{\overline{v}}(\zeta_1(s)), \ldots, \phi_{\overline{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds \\
&+ \int_0^t S(t,s)BB^*S^*(b-t)R(\gamma,\overline{\Upsilon}_0^b)\Big[u_b - C(b,0)[u_0 + q(\overline{v})] - S(b,0)[\widetilde{u_0} + \widetilde{q}(\overline{v})] \\
&- \int_0^b S(b,s)\Big[\mathscr{F}\Big(s, \phi_{\overline{v}}(\xi_1(s)), \ldots, \phi_{\overline{v}}(\xi_n(s)), \int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big) \\
&+ \mathscr{G}\Big(s, \phi_{\overline{v}}(\zeta_1(s)), \ldots, \phi_{\overline{v}}(\zeta_p(s)), \int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds \\
&- \sum_{k=1}^m C(b,t_k)I_k(v(t_k)) - \sum_{k=1}^m S(b,t_k)\bar{I}_k(v(t_k))\Big](s)ds \\
&+ \sum_{0<t_k<t} C(t,t_k)I_k(v(t_k)) + \sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v(t_k)).
\end{aligned}$$

Consider the map $\widetilde{\Gamma} : \mathcal{PC}_\delta = \mathcal{PC}([\delta,b],\mathbb{X}) \to \mathcal{PC}_\delta$ defined by

$$(\widetilde{\Gamma}v)(t) = \phi_{\overline{v}}(t),\ t \in [\delta,b].$$

We might demonstrate that $\widetilde{\Gamma}$ fulfills every one of the states of Lemma 2.5. The proof will be given in two stages.

Step 1*. $\widetilde{\Gamma}$ maps bounded sets into bounded sets in $\mathcal{PC}_\delta$.

Indeed, it is enough to show that there exists a positive constant $\widetilde{\Lambda}_2$ such that for each $v \in B_r(\delta) := \left\{\phi \in \mathcal{PC}_\delta; \sup_{\delta\le t\le b}\|\phi(t)\| \le r\right\}$ one has $\|\widetilde{\Gamma}v\|_{\mathcal{PC}} \le \widetilde{\Lambda}_2$.

Let $v \in B_r(\delta)$, then for $t \in (0,b]$, we have

$$\begin{aligned}
\|\phi_{\overline{v}}(t)\| &\le \widehat{M}_{**} + \left(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\left[\widehat{M}_1\left(\Lambda(r) + \sum_{k=1}^m \Psi_k(r)\right) + \widehat{M}_2\left(\widetilde{\Lambda}(r) + \sum_{k=1}^m \widetilde{\Psi}_k(r)\right)\right] \\
&+ \left(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\widehat{M}_2\Big[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big]\int_0^b \sup_{s\in(0,b]}\|\phi_{\overline{v}}(s)\|ds,
\end{aligned}$$

where $\widehat{M}_{**} = \left(\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\|u_b\| + \left(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Big[\widehat{M}_1\|u_0\| + \widehat{M}_2\|\widetilde{u_0}\| + \widehat{M}_2 b\Big[b(\mathscr{L}\mathscr{N}_1 + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}_1}) + (\mathscr{L}_1 + \widetilde{\mathscr{L}_1})\Big]\Big]$.

Utilizing the Gronwall's inequality, we receive

$$\begin{aligned}
\sup_{s\in(0,b]}\|\phi_{\overline{v}}(t)\| &\le e^{\left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\widehat{M}_2[(\mathscr{L}n+\widetilde{\mathscr{L}}p)+b(\mathscr{L}\mathscr{N}+\widetilde{\mathscr{L}}\widetilde{\mathscr{N}})]b}\Bigg[\widehat{M}_{**} \\
&+ \left(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\left[\widehat{M}_1\left(\Lambda(r) + \sum_{k=1}^m \Psi_k(r)\right) + \widehat{M}_2\left(\widetilde{\Lambda}(r) + \sum_{k=1}^m \widetilde{\Psi}_k(r)\right)\right].
\end{aligned}$$

Thus

$$\|\widetilde{\Gamma}v\|_{\mathcal{PC}} \le e^{\overline{\eta}}\left[\widehat{M}_{**} + \left(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\left[\widehat{M}_1\left(\Lambda(r) + \sum_{k=1}^m \Psi_k(r)\right) + \widehat{M}_2\left(\widetilde{\Lambda}(r) + \sum_{k=1}^m \widetilde{\Psi}_k(r)\right)\right] := \widetilde{\Lambda}_2.$$

By implementing the techniques applied in Theorem 3.1 (Step 2 & Step 3), we deduce that the operator $\widetilde{\Gamma}$ is continuous and compact with simple modifications.

Step 4*. We now show that there exists an open set $U \subseteq \mathcal{PC}_\delta$ with $v \notin \lambda\widetilde{\Gamma}v$ for $\lambda \in (0,1)$ and $v \in \partial U$. Let $\lambda \in (0,1)$ and let $v \in \mathcal{PC}_\delta$ be a possible solution of $v = \lambda\widetilde{\Gamma}(v)$ for some $0 < \lambda < 1$. Thus, for each $t \in (0,b]$,

$$\begin{aligned}
v(t) = \lambda\phi_{\overline{v}}(t) &= \lambda C(t,0)[u_0 + q(\widetilde{v})] + \lambda S(t,0)[\widetilde{u_0} + \widetilde{q}(\widetilde{v})] \\
&+ \lambda\int_0^t S(t,s)\Big[\mathscr{F}\Big(s,\phi_{\overline{v}}(\xi_1(s)),\ldots,\phi_{\overline{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big) \\
&+ \mathscr{G}\Big(s,\phi_{\overline{v}}(\zeta_1(s)),\ldots,\phi_{\overline{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds \\
&+ \lambda\int_0^t S(t,s)BB^*S^*(b,t)R(\gamma,\overline{\Upsilon}_0^b)\Big[u_b - C(b,0)[u_0 + q(\widetilde{v})] - S(b,0)[\widetilde{u_0} + \widetilde{q}(\widetilde{v})] \\
&- \int_0^b S(b,s)\Big[\mathscr{F}\Big(s,\phi_{\overline{v}}(\xi_1(s)),\ldots,\phi_{\overline{v}}(\xi_n(s)),\int_0^s k_1(s,\tau,\phi_{\overline{v}}(\xi_{n+1}(\tau)))d\tau\Big) \\
&+ \mathscr{G}\Big(s,\phi_{\overline{v}}(\zeta_1(s)),\ldots,\phi_{\overline{v}}(\zeta_p(s)),\int_0^s k_2(s,\tau,\phi_{\overline{v}}(\zeta_{p+1}(\tau)))d\tau\Big)\Big]ds \\
&- \sum_{k=1}^m C(b,t_k)I_k(v(t_k)) - \sum_{k=1}^m S(b,t_k)\bar{I}_k(v(t_k))\Big](s)ds \\
&+ \lambda\sum_{0<t_k<t} C(t,t_k)I_k(v(t_k)) + \lambda\sum_{0<t_k<t} S(t,t_k)\bar{I}_k(v(t_k)).
\end{aligned}$$

This implies by $(H1)-(H5)$ and for each $t \in (0,b]$ we have $\|v(t)\| \le \|\phi_{\overline{v}}(t)\|$ and

$$\begin{aligned}
\|\phi_{\overline{v}}(t)\| &\le \widehat{M}_{**} + \left(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Big[\widehat{M}_1\left(\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \Psi_k(\|v\|_{\mathcal{PC}})\right) \\
&+ \widehat{M}_2\left(\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right)\Big] \\
&+ \left(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\widehat{M}_2\Big[(\mathscr{L}n + \widetilde{\mathscr{L}}p) + b(\mathscr{L}\mathscr{N} + \widetilde{\mathscr{L}}\widetilde{\mathscr{N}})\Big]\int_0^b \sup_{s\in(0,b]}\|\phi_{\overline{v}}(s)\|ds.
\end{aligned}$$

Utilizing the Gronwall's inequality, we receive

$$\begin{aligned}
\sup_{s\in(0,b]}\|\phi_{\overline{v}}(t)\| &\le e^{\overline{\eta}}\Big[\widehat{M}_{**} + \left(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Big[\widehat{M}_1\left(\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \Psi_k(\|v\|_{\mathcal{PC}})\right) \\
&+ \widehat{M}_2\left(\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right)\Big]\Big],
\end{aligned}$$

and the previous inequality holds. Consequently,

$$\begin{aligned}
\|v\|_{\mathcal{PC}} &\le e^{\overline{\eta}}\Big[\widehat{M}_{**} + \left(1 + \frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\Big[\widehat{M}_1\left(\Lambda(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \Psi_k(\|v\|_{\mathcal{PC}})\right) \\
&+ \widehat{M}_2\left(\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}}) + \sum_{k=1}^m \widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right)\Big]\Big],
\end{aligned}$$

and therefore

$$\frac{\|v\|_{\mathcal{PC}}}{e^{\widetilde{\eta}}\left[\widehat{M}_{**}+\left(1+\frac{1}{\gamma}\widehat{M}_2^2\widehat{M}_B^2 b\right)\left[\widehat{M}_1\left(\Lambda(\|\widetilde{v}\|_{\mathcal{PC}})+\sum_{k=1}^{m}\Psi_k(\|v\|_{\mathcal{PC}})\right)+\widehat{M}_2\left(\widetilde{\Lambda}(\|\widetilde{v}\|_{\mathcal{PC}})+\sum_{k=1}^{m}\widetilde{\Psi}_k(\|v\|_{\mathcal{PC}})\right)\right]\right]}\leq 1.$$

There exists a constant $\widehat{M}_{***}>0$ such that $\|v\|_{\mathcal{PC}}\neq\widehat{M}_{***}$. Set

$$U=\left\{v\in\mathcal{PC}([\delta,b],\mathbb{X});\sup_{\delta\leq t\leq b}\|v(t)\|<\widehat{M}_{***}\right\}.$$

As a consequence of Step 1* and Step 4* in Theorem 4.4, it suffices to show that $\widetilde{\Gamma}:\overline{U}\to\mathcal{PC}_\delta$ is a compact map.

From the choice of U, there is no $u\in\partial U$ such that $v=\lambda\widetilde{\Gamma}v$ for $\lambda\in(0,1)$. As a consequence of Lemma 2.5, we deduce that $\widetilde{\Gamma}$ has a fixed point $\widetilde{v}_*\in\overline{U}$. From the equation (3.4), we infer that $u(t)$ is a mild solution of the system (4.1) with the conditions (1.2). The proof is now completed. □

Theorem 4.5. *Assume that the conditions (H0)-(H5) hold and linear system (4.2)-(4.3) is approximately controllable on $\mathcal{J}$. The functions $\mathcal{F}:J\times\mathbb{X}^{n+1}\to\mathbb{X}$ and $\mathcal{G}:J\times\mathbb{X}^{p+1}\to\mathbb{X}$ are continuous and uniformly bounded and there exist constants $\mathcal{N}^*>0,\mathcal{N}^{**}>0$ such that $\|\mathcal{F}(t,u_1,u_2,\ldots,u_{n+1})\|_\alpha\leq\mathcal{N}^*$ and $\|\mathcal{G}(t,u_1,u_2,\ldots,u_{p+1})\|_\alpha\leq\mathcal{N}^{**}$, then the system (4.1) with the conditions (1.2) and (1.3) is approximately controllable on $\mathcal{J}$.*

Proof. Let $u^\gamma(\cdot)$ be a fixed point of $\widetilde{\Gamma}$. By Theorem 4.1, any fixed point of $\widetilde{\Gamma}$ is a mild solution of (4.1) with the conditions (1.2) and (1.3) under the control

$$\hat{u}^\gamma(t)=B^*S^*(b,t)R(\gamma,\overline{\Upsilon}_0^b)\widehat{p}(u^\gamma)$$

and satisfies the inequality

$$u^\gamma(b)=u_b+\gamma R(\gamma,\overline{\Upsilon}_0^b)\widehat{p}(u^\gamma). \tag{4.2}$$

Moreover by assumptions on $\mathcal{F}$ and $\mathcal{G}$ with Dunford-Pettis theorem, we have that $\{f^\gamma(s)\}$ and $\{g^\gamma(s)\}$ are weakly compact in $\mathscr{L}^1(\mathcal{J},\mathbb{X})$, so there is a subsequence, still denoted by $\{f^\gamma(s)\}$ and $\{g^\gamma(s)\}$, that converges weakly to say $f(s)$ and $g(s)$ in $\mathscr{L}^1(\mathcal{J},\mathbb{X})$ respectively.

Define

$$w=u_b-C(b,0)[u_0+q(u)]-S(b,0)[\widetilde{u}_0+\widetilde{q}(u)]-\int_0^b S(b,s)[f(s)+g(s)]ds-\sum_{k=1}^{m}C(b,t_k)I_k(u(t_k))$$
$$-\sum_{k=1}^{m}S(b,t_k)\bar{I}_k(u(t_k)).$$

Now, we have

$$\begin{aligned}\|\widehat{p}(u^\gamma)-w\|&=\left\|\int_0^b S(b,s)[f(s,u_1^\gamma(s),u_2^\gamma(s),\ldots,u_{n+1}^\gamma(s))-f(s)]ds\right\|\\&\quad+\left\|\int_0^b S(b,s)[g(s,u_1^\gamma(s),u_2^\gamma(s),\ldots,u_{p+1}^\gamma(s))-g(s)]ds\right\|\\&\leq\sup_{t\in[0,b]}\left[\left\|\int_0^t S(t,s)[f(s,u_1^\gamma(s),u_2^\gamma(s),\ldots,u_{n+1}^\gamma(s))-f(s)]ds\right\|\right.\\&\quad\left.+\left\|\int_0^t S(t,s)[g(s,u_1^\gamma(s),u_2^\gamma(s),\ldots,u_{p+1}^\gamma(s))-g(s)]ds\right\|\right].\end{aligned}\tag{4.3}$$

By using infinite-dimensional version of the Ascoli-Arzela theorem, one can show that an operator $l(\cdot) \to \int_0^{\cdot} S(\cdot, s)l(s)ds : \mathscr{L}^1(\mathscr{J}, \mathbb{X}) \to \mathscr{C}(\mathscr{J}, \mathbb{X})$ is compact. Consequently, we obtain that $\|\widehat{p}(u^\gamma) - w\| \to 0$ as $\gamma \to 0^+$. Moreover, from (4.2), we obtain

$$\begin{aligned}
\|u^\gamma(b) - u_b\| &\leq \|\gamma R(\gamma, \overline{\Upsilon}_0^b)\widehat{p}(u^\gamma)\| \\
&\leq \|\gamma R(\gamma, \overline{\Upsilon}_0^b)(\widehat{p}(u^\gamma) - w + w)\| \\
&\leq \|\gamma R(\gamma, \overline{\Upsilon}_0^b)(w)\| + \|\gamma R(\gamma, \overline{\Upsilon}_0^b)\|\|\widehat{p}(u^\gamma) - w\| \\
&\leq \|\gamma R(\gamma, \overline{\Upsilon}_0^b)w\| + \|\widehat{p}(u^\gamma) - w\|.
\end{aligned}$$

It follows from assumption (H0) and the estimation (4.3) that $\|u^\gamma(b) - u_b\| \to 0$ as $\gamma \to 0^+$. This proves the approximate controllability of (4.1) with the conditions (1.2) and (1.3). □

5. Example

In this section, we apply our abstract results on a concrete impulsive partial differential equation. In order to establish our results, we need to introduce the required technical tools. From the equations (2.7)-(2.8), here we consider $\mathscr{A}(t) = \mathscr{A} + \widetilde{B}(t)$ where $\mathscr{A}$ is the infinitesimal generator of a cosine function $C(t)$ with associated sine function $S(t)$, and $\widetilde{B}(t) : \mathscr{D}(\widetilde{B}(t)) \to \mathbb{X}$ is a closed linear operator with $\mathscr{D} \subseteq \mathscr{D}(\widetilde{B}(t))$ for all $t \in \mathscr{J}$.

We model this problem in the space $\mathbb{X} = L^2(\mathbb{T}, \mathbb{C})$, where the group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi Z$. We will use the identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. Specifically, in what follows we denote by $L^2(\mathbb{T}, \mathbb{C})$ the space of 2π-periodic 2-integrable functions from $\mathbb{R}$ into $\mathbb{C}$. Similarly, $H^2(\mathbb{T}, \mathbb{C})$ denotes the Sobolev space of 2π-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that $x'' \in L^2(\mathbb{T}, \mathbb{C})$.

We consider the operator $\mathscr{A}x(\xi) = x''(\xi)$ with domain $\mathscr{D}(\mathscr{A}) = H^2(\mathbb{T}, \mathbb{C})$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function $C(t)$ on $\mathbb{X}$. Moreover, $\mathscr{A}$ has discrete spectrum, the spectrum of $\mathscr{A}$ consists of eigenvalues $-n^2$ for $n \in \mathbb{Z}$, with associated eigenvectors

$$w_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi}, \quad n \in \mathbb{Z},$$

the set $\{w_n : n \in Z\}$ is an orthonormal basis of $\mathbb{X}$. In particular,

$$\mathscr{A}x = -\sum_{n=1}^{\infty} n^2 \langle x, w_n \rangle w_n$$

for $x \in \mathscr{D}(\mathscr{A})$. The cosine function $C(t)$ is given by

$$C(t)x = \sum_{n=1}^{\infty} \cos(nt)\langle x, w_n \rangle w_n, \quad t \in \mathbb{R},$$

with associated sine function

$$S(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle x, w_n \rangle w_n, \quad t \in \mathbb{R}.$$

It is clear that $\|C(t)\| \leq 1$ for all $t \in \mathbb{R}$. Thus, $C(\cdot)$ is uniformly bounded on $\mathbb{R}$.

Consider the following impulsive partial functional integro-differential equation of the form:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}z(t,x) &= \frac{\partial^2}{\partial x^2}z(t,x) + b(t)\frac{\partial}{\partial t}z(t,x) + a_1(t)z(\sin t,x) + a_2(t)\sin z(t,x) + \widehat{\mu}(t,x)\\ &\quad + \frac{1}{1+t^2}\int_0^t a_3(s)z(\sin s,x)ds + \widetilde{a}_1(t)z(\sin t,x) + \widetilde{a}_2(t)\sin z(t,x)\\ &\quad + \frac{1}{1+t^2}\int_0^t \widetilde{a}_3(s)z(\sin s,x)ds, \qquad (5.1)\end{aligned}$$

$$\Delta z(t_k,x) = \int_0^\pi p_k(x,y)z(t_k,y)dy \quad \text{and} \quad \Delta z'(t_k,x) = \int_0^\pi \widetilde{p}_k(x,y)z(t_k,y)dy, k = 1,\cdots,m, \qquad (5.2)$$

$$z(t,0) = z(t,\pi) = 0; \quad z(0,x) = z_0(x); \quad z_t(0,x) = z_1(x), \quad t \in \mathscr{J} = [0,1], 0 \le x \le \pi, \qquad (5.3)$$

$$z(0,x) = z_0(x) + \sum_{k=1}^m \phi_k z(t_k,x), \text{ and } \quad z_t(0,x) = z_1(x) + \sum_{k=1}^m \widetilde{\phi}_k z(t_k,x),\ 0 \le x \le \pi, \qquad (5.4)$$

where we assume that $b : \mathbb{R} \to \mathbb{R}, \widehat{\mu} : \mathscr{J} \times [0,\pi] \to [0,\pi]$ are continuous functions, and

(a) the functions $a_i(\cdot)$ and $\widetilde{a}_i(\cdot), i = 1,2,3$, are continuous on $[0,1]$, $n_i = \sup\limits_{0\le s\le 1}|a_i(s)| < 1, i = 1,2,3$; and $\widetilde{n}_i = \sup\limits_{0\le s\le 1}|\widetilde{a}_i(s)| < 1, i = 1,2,3$.

(b) the functions $p_k, \widetilde{p}_k : [0,\pi]\times[0,\pi] \to \mathbb{R}, k = 1,2,\cdots,m$, are continuously differentiable and

$$\gamma_k = \Big(\int_0^\pi\int_0^\pi \Big(\frac{\partial}{\partial x}p_k(x,y)\Big)^2 dxdy\Big)^{\frac{1}{2}} < \infty, \quad \text{and} \quad \widetilde{\gamma}_k = \Big(\int_0^\pi\int_0^\pi \Big(\frac{\partial}{\partial x}\widetilde{p}_k(x,y)\Big)^2 dxdy\Big)^{\frac{1}{2}} < \infty$$

for every $k = 1,2,\cdots,m$.

(c) The functions $\phi_k, \widetilde{\phi}_k \in \mathbb{R}$, $k = 1,2,\ldots,m$.

(d) Denote $\beta = \sup_{t\in\mathscr{J}}|b(t)|$.

We take $\widetilde{B}(t)u(x) = b(t)u'(x)$ defined on $H^1(\mathbb{T},\mathbb{C})$. It is easy to see that $\mathscr{A}(t) = \mathscr{A} + \widetilde{B}(t)$ is a closed linear operator. Initially we will show that $\mathscr{A} + \widetilde{B}(t)$ generates an evolution operator. It is well known that the solution of the scalar initial value problem

$$\begin{aligned}q''(t) &= -n^2q(t) + p(t),\\ q(s) &= 0, \quad q'(s) = q_1,\end{aligned}$$

is given by

$$q(t) = \frac{q_1}{n}\sin n(t-s) + \frac{1}{n}\int_s^t \sin n(t-\tau)p(\tau)d\tau.$$

Therefore, the solution of the scalar initial value problem

$$q''(t) = -n^2q(t) + inb(t)q(t), \qquad (5.5)$$

$$q(s) = 0, \quad q'(s) = q_1, \qquad (5.6)$$

satisfies the integral equation

$$q(t) = \frac{q_1}{n}\sin n(t-s) + i\int_s^t \sin n(t-\tau)b(\tau)q(\tau)d\tau.$$

Applying the Gronwall-Bellman lemma, we can affirm that

$$|q(t)| \leq \frac{|q_1|}{n} e^{\beta(t-s)} \tag{5.7}$$

for $s \leq t$. We denote by $q_n(t,s)$ the solution of (5.5)-(5.6). We define

$$S(t,s)x = \sum_{n=1}^{\infty} q_n(t,s)\langle x, w_n\rangle w_n.$$

It follows from the estimate (5.7) that $S(t,s) : \mathbb{X} \to \mathbb{X}$ is well defined and satisfies the conditions of Definition 2.1.

To treat this system, we define the operators respectively $\mathscr{F} : \mathscr{J} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}, \mathscr{G} : \mathscr{J} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}, k_1 : \mathscr{J} \times \mathscr{J} \times \mathbb{X} \to \mathbb{X}, k_2 : \mathscr{J} \times \mathscr{J} \times \mathbb{X} \to \mathbb{X}$, $I_k, \bar{I}_k : \mathbb{X} \to \mathbb{X}$, $k = 1, 2, \ldots, m; q, \widetilde{q} : \mathcal{PC}(\mathscr{J}, \mathbb{X}) \to \mathbb{X}$ by

$$\begin{aligned}
\mathscr{F}\Big(t, z(\xi(t)), \int_0^t k_1(t,s,z(\xi(s)))ds\Big)(x) &= a_1(t)z(\sin t, x) + a_2(t)\sin z(t,x) + \frac{1}{1+t^2}\int_0^t a_3(s)z(\sin s, x)ds,\\
\mathscr{G}\Big(t, z(\zeta(t)), \int_0^t k_2(t,s,z(\zeta(s)))ds\Big)(x) &= \widetilde{a_1}(t)z(\sin t, x) + \widetilde{a_2}(t)\sin z(t,x) + \frac{1}{1+t^2}\int_0^t \widetilde{a_3}(s)z(\sin s, x)ds,\\
\int_0^t k_1(t,s,\xi(z(s)))(x)ds &= \frac{1}{1+t^2}\int_0^t a_3(s)z(\sin s, x)ds,\\
\int_0^t k_2(t,s,\zeta(z(s)))(x)ds &= \frac{1}{1+t^2}\int_0^t \widetilde{a_3}(s)z(\sin s, x)ds,\\
I_k(z)(x) &= \int_0^\pi p_k(x,y)z(t_k,y)dy,\ k = 1, \cdots, m,\\
\bar{I}_k(z)(x) &= \int_0^\pi \widetilde{p_k}(x,y)z(t_k,y)dy,\ k = 1, \cdots, m,\\
q(z)(x) &= \sum_{k=1}^m \phi_k z(t_k, x),\\
\widetilde{q}(z)(x) &= \sum_{k=1}^m \widetilde{\phi_k} z(t_k, x).
\end{aligned}$$

Further $\widehat{u} : \mathscr{J} \to U$ be defined as

$$B(\widehat{u}(t))(x) = \widehat{\mu}(t,x), \quad x \in [0,\pi],$$

where $\widehat{\mu} : \mathscr{J} \times [0,\pi] \to [0,\pi]$ is continuous.

Then equations (5.1) – (5.4) takes the abstract form (1.1) – (1.3). It is easy to see that with the choices of the above functions, assumptions $(H0)$ – $(H5)$ of Theorem 4.4 are satisfied. Hence by Theorem 4.4, we deduce that nonlocal impulsive Cauchy problem (5.1) – (5.4) is approximately controllable on $\mathscr{J}$.

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