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Research Article

Behavior of the Solutions for Predator-Prey Dynamic Systems with Beddington-DeAngelis Type Functional Response on Periodic Time Scales in Shifts

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We consider two-dimensional predator-prey system with Beddington-DeAngelis type functional response on periodic time scales in shifts. For this special case we try to find under which conditions the system has $\delta_{\pm}$-periodic solution.

1. Introduction

This study is mainly about the predator-prey dynamic systems with Beddington-DeAngelis type functional response on periodic time scales in shifts. Therefore the main tools that we have used in this study are the time scale calculus, periodic time scales in shifts, predator-prey dynamic systems, and their functional response which shows the effect of predator and prey on each other.

First of all, the main tool we have used is time scales calculus, which first appeared in 1990 in the thesis of Hilger [1]. The main aim of this new topic is to unify the discrete and the continuous dynamic systems; in other words, the unification of dynamical systems obtained from differential equations and the difference equations is the principle target of this new area. After this study, many studies have been done on some properties of dynamical systems on time scale calculus such as [2] and our study can be seen as the continuation of those studies.

Secondly, the dynamic systems that we have considered in this study are the predator-prey ones which are very important in the mathematical ecology that is the branch of the mathematical biology. Many studies have been done on this type of dynamical systems, since these systems help us to understand the future of the considered species. For instance, in a determined territory, if there are two species (one of them is prey and the other is predator), using such a system which models their life gives us some clues about whether predator or prey goes to extinction or their life cycles are permanent or not.

Predator-prey equations are also known as the Lotka-Volterra equations. The Lotka-Volterra predator-prey model was initially proposed by Lotka in the theory of autocatalytic chemical reactions in 1910 [3, 4]. This was effectively the logistic equation [5], which was originally derived by Verhulst [6]. In 1920 Lotka extended, via Kolmogorov, the model to "organic systems" using a plant species and a herbivorous animal species as an example [7] and in 1925 he utilised the equations to analyse predator-prey interactions in his book on biomathematics [8] arriving at the equations that we know today. Vito Volterra, who made a statistical analysis of fish catches in the Adriatic Sea, independently investigated the equations in 1926.

This model was developed by several researchers in the following years. One of them is Holling who is the first to propose using the idea of functional response in [9, 10]. Both the Lotka Volterra model and Holling's extensions have been used to model the moose and wolf populations in Isle

Royale National Park [11], which with over 50 published papers is one of the best studied predator-prey relationships. In addition to these, there are many studies that use the predator-prey dynamic systems with Holling type functional responses that study the permanence, stability, periodicity, and such different aspects of these systems. The papers [12–14] can be some of its examples.

After the extention of Holling that is about the effect of predator and prey to each other, Arditi and Gizburg made some changes on the extention of Holling on the functional response and this new functional response is known as the ratio dependent functional response and as derivative of it there is also semi ratio dependent functional responses. Again there are many studies that are about the several structures of the predator-prey dynamic systems such as [12, 15–19].

After that Beddington and DeAngelis proposed another functional response separately, because of the some advantages of that new type of functional response. Nowadays, this is known as Beddington-DeAngelis type functional response. According to the studies [20, 21], the advantages are at low densities; this type of functional response can avoid some of the singular behaviors of ratio dependent models and predator feeding can be described much better over a range of predator-prey abundances by using this functional response. Therefore, we preferred to use this type of functional response in our model. The following studies are some of the examples that investigate the several aspects of this model: [12, 22–25].

In recent years, after the development of time scale calculus, the model of predator-prey dynamic systems models started to adapt to the time scales case because of some aspects of this new calculus. At the very beginning, as it is mentioned, time scale case is the unification of continuous and discrete systems. Because of the unordinary life cycle of some species like insects studying with the time scale model of these dynamical systems becomes important. When we consider the life cycle of an insect, most of them live in the summer and then die and their eggs become dormant in the winter. Thus, the life cycle of an insect contains both continuous and discrete time intervals. For such a system, using the model that is obtained by the time scale is more appropriate. The papers [12, 26, 27] are some examples for the studies that are done on the predator-prey dynamic systems on time scale calculus.

To investigate the periodic solutions on time scale case of the predator-prey dynamic systems, the notion of periodic time scale becomes important which is defined as follows: if the given time scale $\mathbb{T}$ is w-periodic, then for each $t \in \mathbb{T}$, also $t + w \in \mathbb{T}$. There are several papers such as [12, 24, 27] that study the w-periodic solutions of the predator-prey models. However, since there are many different kinds of species in the world this periodicity notion on an arbitrary time scale needs some development. This was first done by Adivar in his study [28] and we meet with the notion periodic time scales in shifts. According to the suggestion of Bohner in the conference PODE 2014, we started to study on the predator-prey dynamic systems with Beddington-DeAngelis type functional response with periodic time scales in shifts and we obtain the following results.

2. Preliminaries

Theorem 1 (continuation theorem, [12]). *Let L be a Fredholm mapping of index zero and let C be L-compact on Ω. Assume the following:*

(a) *For each $\lambda \in (0,1)$, any y satisfying $Ly = \lambda Cy$ is not on $\delta\Omega$,; that is, $y \notin \delta\Omega$.*

(b) *For each $y \in \delta\Omega \cap \operatorname{Ker} L, VCy \neq 0$ and the Brouwer degree* $\deg\{JVC, \delta\Omega \cap \operatorname{Ker} L, 0\} \neq 0$. *Then $Ly = Cy$ has at least one solution lying in* $\operatorname{Dom} L \cap \delta\Omega$.

We will also give the following lemma, which is essential for this paper.

Definition 2 (see [28]). Let the time scale $\mathbb{T}$ include a fixed number $t_0 \in \mathbb{T}^*$ where $\mathbb{T}^*$ is a nonempty subset of $\mathbb{T}$, such that there exist operators $\delta_{\pm} : [t_0;\infty)_{\mathbb{T}} \times \mathbb{T}^* \rightarrow \mathbb{T}^*$ which satisfy the following properties:

(P.1) With respect to their second arguments the functions $\delta_{\pm}$ are strictly increasing; that is, if

$$\begin{aligned}&(S_0, v), (S_0, s) \in D_{\pm}\\ &\quad := \{(u,v) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{T}^* : \delta_{\pm}(u,v) \in \mathbb{T}^*\},\end{aligned} \tag{1}$$

then

$$S_0 \le v < s \quad \text{implies } \delta_{\pm}(S_0, v) < \delta_{\pm}(S_0, s). \tag{2}$$

(P.2) If $(S_1, s), (S_2, s) \in D_-$ with $S_1 < S_2$, then $\delta_-(S_1, s) > \delta_-(S_2, s)$, and if $(S_1, s), (S_2, s) \in D_+$ with $S_1 < S_2$, then $\delta_+(S_1, s) < \delta_+(S_2, s)$.

(P.3) If $v \in [t_0;\infty)_{\mathbb{T}}$, then $(v, t_0) \in D_+$ and $\delta_+(v, t_0) = s$. Moreover, if $v \in \mathbb{T}^*$, then $(t_0, v) \in D_+$ and $\delta_+(t_0, v) = v$ holds.

(P.4) If $(u,v) \in D_{\pm}$, then $(u, \delta_{\pm}(u,v)) \in D_{\pm}$ and $\delta_{\mp}(u; \delta_{\pm}(u,v)) = v$, respectively.

(P.5) If $(u,v) \in D_{\pm}$ and $(s, \delta_{\pm}(u,v)) \in D_{\pm}$, then $(u, \delta_{\mp}(s, v)) \in D_{\pm}$ and $\delta_{\mp}(s, \delta_{\pm}(u,v)) = \delta_{\pm}(u, \delta_{\mp}(s,v))$, respectively.

Then the backward operator is δ_- and the forward operator is δ_+ which are associated with $t_0 \in \mathbb{T}^*$ (called the initial point). Shift size is the variable $u \in [t_0;\infty)_{\mathbb{T}}$ in $\delta_{\pm}(u,v)$. The values $\delta_+(u,v)$ and $\delta_-(u,v)$ in $\mathbb{T}^*$ indicate u units translation of the term $v \in \mathbb{T}^*$ to the right and left, respectively. The sets $D_{\pm}$ are the domains of the shift operators $\delta_{\pm}$, respectively.

Definition 3 (see [28]). Let $\mathbb{T}$ be a time scale with the shift operators $\delta_{\pm}$ associated with the initial point $t_0 \in \mathbb{T}^*$. The time scale $\mathbb{T}$ is said to be periodic in shifts $\delta_{\pm}$ if there exists $q \in (t_0,\infty)_{\mathbb{T}^*}$ such that $(q,t) \in D_{\pm}$ for all $t \in \mathbb{T}^*$. Furthermore, if

$$Q := \inf\{q \in (t_0,\infty)_{\mathbb{T}^*} : (q,t) \in D_{\pm}\ \forall t \in \mathbb{T}^*\} \neq t_0 \tag{3}$$

then P is called the period of the time scale $\mathbb{T}$.

Definition 4 (periodic function in shifts δ_+ and δ_-, [28]). Let $\mathbb{T}$ be a time scale that is periodic in shifts δ_+ and δ_- with the period Q. We say that a real valued function g defined on $\mathbb{T}^*$ is periodic in shifts if there exists $\widetilde{T} \in [Q, \infty)_{\mathbb{T}^*}$ such that

$$g\left(\delta_{\pm}\left(\widetilde{T}, t\right)\right) = g(t). \quad (4)$$

The smallest number $\widetilde{T} \in [Q, \infty)_{\mathbb{T}^*}$ such that it is called the period of f.

Definitions 2, 3, and 4 are from [28].

Notation 1. Consider

$$\begin{aligned}\delta_+^2(T,\kappa) &= \delta_+\left(T, \delta_+(T,\kappa)\right),\\ \delta_+^3(T,\kappa) &= \delta_+\left(T, \delta_+\left(T, \delta_+(T,\kappa)\right)\right),\\ &\vdots\\ \delta_+^n(T,\kappa) &= \delta_+\left(T, \delta_+\left(T, \delta_+\left(T, \delta_+(\cdots)\right)\right)\right).\end{aligned} \quad (5)$$

Lemma 5. *Let our time scale* $\mathbb{T}$ *be periodic in shifts and for each* $t \in \mathbb{T}^*$, $(\delta_+^n(T,t))^\Delta$ *is constant. Then* $\int_\kappa^{\delta_+(T,\kappa)} u(t)\Delta t / \mathrm{mes}(\delta_+(T,\kappa))$ *is also constant* $\forall \kappa \in \mathbb{T}$, *where* $\kappa = \delta_{\pm}^m(T, t_0)$ *for* $m \in \mathbb{N}$ *and* $\mathrm{mes}(\delta_+(T,\kappa)) = \int_\kappa^{\delta_+(T,\kappa)} 1\Delta t$. *Here* $u(t)$ *is a periodic function in shifts.*

Proof. We get the desired result, if we are able to show that for any $\kappa_1 \neq \kappa_2$ ($\kappa_1, \kappa_2 \in \mathbb{T}$):

$$\frac{\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{\mathrm{mes}\left(\delta_+(T,\kappa_1)\right)} = \frac{\int_{\kappa_2}^{\delta_+(T,\kappa_2)} u(t)\Delta t}{\mathrm{mes}\left(\delta_+(T,\kappa_2)\right)}. \quad (6)$$

Since $\mathbb{T}$ is a periodic time scale in shifts (WLOG $\kappa_2 > \kappa_1$) there exits $n \in \mathbb{N}$ such that $\kappa_2 = \delta_+^n(T,\kappa_1)$. Hence it is also enough to show that

$$\frac{\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{\mathrm{mes}\left(\delta_+(T,\kappa_1)\right)} = \frac{\int_{\delta_+^n(T,\kappa_1)}^{\delta_+(T,\delta_+^n(T,\kappa_1))} u(t)\Delta t}{\mathrm{mes}\left(\delta_+\left(T,\delta_+^n(T,\kappa_1)\right)\right)}. \quad (7)$$

Because of the definition of the time scale and u, $u(\kappa_1) = u(\delta_+^n(T,\kappa_1))$, $u(\delta_+(T,\kappa_1)) = u(\delta_+^{n+1}(T,\kappa_1))$ and for each $t \in [\kappa_1, \delta_+(T,\kappa_1)]$, $u(t) = u(\delta_+^n(T,t))$. By using change of variables we get the result. If $s = \delta_+^n(T,t)$, then by the assumption of the lemma $\Delta s = \widetilde{c}\Delta t$. When $s = \delta_+^n(T,\kappa_1)$, then $t = \delta_-^n(T,s) = \kappa_1$ and when $s = \delta_+^{n+1}(T,\kappa_1)$, then $t = \delta_-^n(T,s) = \delta_+(T,\kappa_1)$:

$$\begin{gathered}\int_{\delta_+^n(T,\kappa_1)}^{\delta_+^{n+1}(T,\kappa_1)} u(s)\Delta s = \widetilde{c}\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t,\\ \int_{\delta_+^n(T,\kappa_1)}^{\delta_+^{n+1}(T,\kappa_1)} 1\Delta t = \widetilde{c}\int_{\kappa_1}^{\delta_+(T,\kappa_1)} 1\Delta t,\\ \frac{\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{\mathrm{mes}\left(\delta_+(T,\kappa_1)\right)} = \frac{\widetilde{c}\int_{\kappa_1}^{\delta_+(T,\kappa_1)} u(t)\Delta t}{\widetilde{c}\,\mathrm{mes}\left(\delta_+(T,\kappa_1)\right)}.\end{gathered} \quad (8)$$

Hence proof follows. □

3. Main Result

The equation that we investigate is

$$\begin{aligned}x^\Delta(t) &= a(t) - b(t)\exp(x(t))\\ &\quad - \frac{c(t)\exp(y(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))},\\ y^\Delta(t) &= -d(t)\\ &\quad + \frac{f(t)\exp(x(t))}{\alpha(t) + \beta(t)\exp(x(t)) + m(t)\exp(y(t))}.\end{aligned} \quad (9)$$

In (9), let $a(t) = a(\delta_{\pm}(T,t))$, $b(\delta_{\pm}(T,t)) = b(t)$, $c(\delta_{\pm}(T,t)) = c(t)$, $d(\delta_{\pm}(T,t)) = d(t)$, $f(\delta_{\pm}(T,t)) = f(t)$, $\alpha(\delta_{\pm}(T,t)) = \alpha(t)$, $\beta(\delta_{\pm}(T,t)) = \beta(t)$, $m(\delta_{\pm}(T,t)) = m(t)$, and $\int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t$, $\int_\kappa^{\delta_+(T,\kappa)} b(t)\Delta t$, $\int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t > 0$. $\beta^l = \min_{t\in[\kappa,\delta_+(T,\kappa)]}\beta(t)$, $m^l = \min_{t\in[\kappa,\delta_+(T,\kappa)]}m(t)$, $\beta^u = \max_{t\in[\kappa,\delta_+(T,\kappa)]}\beta(t)$, and $m^u = \max_{t\in[\kappa,\delta_+(T,\kappa)]}m(t)$, such that $\kappa = \delta_{\pm}^m(T,t_0)$ for $m \in \mathbb{N}$ $m(t) > 0$ and $c(t), f(t), b(t) > 0$ $\alpha(t) \geq 0$, $\beta(t) > 0$. Each function is from $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$.

Lemma 6. *Let* $t_1, t_2 \in [\kappa, \delta_+(T,\kappa)]$ *and* $t \in \mathbb{T}$. κ *is defined as in Lemma 5. If* $g : \mathbb{T} \to R$ *is periodic function in shifts, then*

$$\begin{aligned}g(t) &\leq g(t_1) + \int_\kappa^{\delta_+(T,\kappa)} \left|g^\Delta(s)\right|\Delta s,\\ g(t) &\geq g(t_2) - \int_\kappa^{\delta_+(T,\kappa)} \left|g^\Delta(s)\right|\Delta s.\end{aligned} \quad (10)$$

Proof. We only show the first inequality as the proof of the second inequality is similar to the proof of the other one. Since g is periodic function in shifts, without loss of generality, it suffices to show that the inequality is valid for $t \in [\kappa, \delta_+(T,\kappa)]$. If $t = t_1$ then the first inequality is obviously true. If $t > t_1$

$$\begin{aligned}g(t) - g(t_1) &\leq \left|g(t) - g(t_1)\right| = \left|\int_{t_1}^t g^\Delta(s)\Delta s\right|\\ &\leq \int_{t_1}^t \left|g^\Delta(s)\right|\Delta s \leq \int_\kappa^{\delta_+(T,\kappa)} \left|g^\Delta(s)\right|\Delta s.\end{aligned} \quad (11)$$

Therefore $g(t) \leq g(t_1) + \int_\kappa^{\delta_+(T,\kappa)} |g^\Delta(s)|\Delta s$.

If $t < t_1$

$$\begin{aligned}g(t_1) - g(t) &\geq -\left|g(t_1) - g(t)\right| = -\left|\int_t^{t_1} g^\Delta(s)\Delta s\right|\\ &\geq -\int_t^{t_1} \left|g^\Delta(s)\right|\Delta s\\ &\leq -\int_\kappa^{\delta_+(T,\kappa)} \left|g^\Delta(s)\right|\Delta s,\end{aligned} \quad (12)$$

which gives $g(t) \leq g(t_1) + \int_\kappa^{\delta_+(T,\kappa)} |g^\Delta(s)|\Delta s$.

The proof is complete. □

Remark 7 (see [12]). Consider the following equation:

$$\begin{aligned}\tilde{x}'(t) &= a(t)\tilde{x}(t) - b(t)\tilde{x}^2(t) \\ &\quad - \frac{c(t)\tilde{y}(t)\tilde{x}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}, \\ \tilde{y}'(t) &= -d(t)\tilde{y}(t) + \frac{f(t)\tilde{x}(t)\tilde{y}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}.\end{aligned} \tag{13}$$

This is the predator-prey dynamic system that is obtained from ordinary differential equations. Let $\mathbb{T} = \mathbb{R}$. In (9), by taking $\exp(x(t)) = \tilde{x}(t)$ and $\exp(y(t)) = \tilde{y}(t)$, we obtain equality (13), which is the standard predator-prey system with Beddington-DeAngelis functional response. Many studies have been done on this system and [22, 26, 29] are some of their examples.

Let $\mathbb{T} = \mathbb{Z}$. By using equality (9), we obtain

$$\begin{aligned}x(t+1) &- x(t) \\ &= a(t) - b(t)\exp(x(t)) \\ &\quad - \frac{c(t)\exp(y(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}, \\ y(t+1) &- y(t) \\ &= -d(t) \\ &\quad + \frac{f(t)\exp(x(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}.\end{aligned} \tag{14}$$

Here again by taking $\exp(x(t)) = \tilde{x}(t)$ and $\exp(y(t)) = \tilde{y}(t)$, we obtain

$$\begin{aligned}\tilde{x}(t+1) &= \tilde{x}(t)\exp\Big[a(t) - b(t)\tilde{x}(t) \\ &\quad - \frac{c(t)\tilde{y}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}\Big], \\ \tilde{y}(t+1) &= \tilde{y}(t)\exp\Big[-d(t) \\ &\quad + \frac{f(t)\tilde{x}(t)}{\alpha(t)+\beta(t)\tilde{x}(t)+m(t)\tilde{y}(t)}\Big],\end{aligned} \tag{15}$$

which is the discrete time predator-prey system with Beddington De-Angelis type functional response and also the discrete analogue of (13). This system was studied in [23, 30, 31]. Since (9) incorporates (13) and (15) as special cases, we call (9) the predator-prey dynamic system with Beddington-DeAngelis functional response on time scales.

For (9), $\exp(x(t))$ and $\exp(y(t))$ denote the density of prey and the predator. Therefore $x(t)$ and $y(t)$ could be negative. By taking the exponentials of $x(t)$ and $y(t)$, we obtain the amount of prey and predators that are living per unit of an area. In other words, for the general time scale case, our equation is based on the natural logarithm of the density of the predator and prey. Hence $x(t)$ and $y(t)$ could be negative.

For (13) and (15), since $\exp(x(t)) = \tilde{x}(t)$ and $\exp(y(t)) = \tilde{y}(t)$, the given dynamic systems directly depend on the density of the prey and predator.

Theorem 8. *Assume that for the given time scale $\mathbb{T}$ while T is constant,* $\operatorname{mes}(\delta_+(T,t))$ *is equal for each $t \in \mathbb{T}$. In addition to conditions on coefficient functions and Lemma 5 if $\int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t - \int_\kappa^{\delta_+(T,\kappa)} (c(t)/m(t))\Delta t > 0$ and*

$$\begin{aligned}&\left(\frac{\int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t - \int_\kappa^{\delta_+(T,\kappa)} (c(t)/m(t))\Delta t}{\int_\kappa^{\delta_+(T,\kappa)} b(t)\Delta t}\right) \\ &\cdot \exp\left[-\left(\int_\kappa^{\delta_+(T,\kappa)} |a(t)|\Delta t + \int_\kappa^{\delta_+(T,\kappa)} a(t)\Delta t\right)\right] \\ &\cdot \left(\int_\kappa^{\delta_+(T,\kappa)} f(t)\Delta t - \beta^u\left(\int_\kappa^{\delta_+(T,\kappa)} d(t)\Delta t\right)\right) \\ &- \alpha^u\left(\int_\kappa^{\delta_+(T,\kappa)} d(t)\right)\Delta t > 0\end{aligned} \tag{16}$$

are satisfied then there exist at least one $\delta_\pm$-periodic solution.

Proof. $X := \{\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right] \in C(\mathbb{T},\mathbb{R}^2) : u(\delta_\pm(T,t)) = u(t), v(\delta_\pm(T,t)) = v(t)\}$ with the norm

$$\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\| = \max_{t\in[t_0,\delta_+(T,t_0)]_{\mathbb{T}}} (|u(t)|, |v(t)|). \tag{17}$$

$Y := \{\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right] \in C(\mathbb{T},\mathbb{R}^2) : u(\delta_\pm(T,t)) = u(t), v(\delta_\pm(T,t)) = v(t)\}$ with the norm

$$\left\|\begin{bmatrix}u\\v\end{bmatrix}\right\| = \max_{t\in[t_0,\delta_+(T,t_0)]_{\mathbb{T}}} (|u(t)|, |v(t)|). \tag{18}$$

Let us define the mappings L and C by $L : \operatorname{Dom} L \subset X \to Y$ such that

$$L\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}u^\Delta\\v^\Delta\end{bmatrix} \tag{19}$$

and $C : X \to Y$ such that

$$C\left(\begin{bmatrix}u\\v\end{bmatrix}\right) = \begin{bmatrix}a(t) - b(t)\exp(u(t)) - \dfrac{c(t)\exp(v(t))}{\alpha(t)+\beta(t)\exp(u(t))+m(t)\exp(v(t))} \\ -d(t) + \dfrac{f(t)\exp(u(t))}{\alpha(t)+\beta(t)\exp(u(t))+m(t)\exp(v(t))}\end{bmatrix}. \tag{20}$$

Then $\operatorname{Ker} L = \{[\begin{smallmatrix} u \\ v \end{smallmatrix}] : [\begin{smallmatrix} u \\ v \end{smallmatrix}] = [\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}]\}$, where c_1 and c_2 are constants:

$$\operatorname{Im} L = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} : \begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} u(t)\,\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} v(t)\,\Delta t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}. \tag{21}$$

$\operatorname{Im} L$ is closed in Y. It is obvious that $\dim \operatorname{Ker} L = 2$. To show $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 2$, we have to prove that $\operatorname{Ker} L \oplus \operatorname{Im} L = Y$. It is obvious that when we take an element from $\operatorname{Ker} L$ and an element from $\operatorname{Im} L$, we find an element of Y by summing these two elements. If we take an element $[\begin{smallmatrix} u \\ v \end{smallmatrix}] \in Y$, and WLOG taking $u(t)$ we have $\int_{\kappa}^{\delta_+(T,\kappa)} u(t)\Delta t = I$, where I is a constant. Let us define a new function $g = u - I/\operatorname{mes}(\delta_+(T,\kappa))$. Since $I/\operatorname{mes}(\delta_+(T,\kappa))$ is constant by Lemma 5 if we take the integral of g from κ to $\delta_+(T,\kappa)$, we get

$$\int_{\kappa}^{\delta_+(T,\kappa)} g(t)\,\Delta t = \int_{\kappa}^{\delta_+(T,\kappa)} u(t)\,\Delta t - I = 0. \tag{22}$$

Similar steps are used for v. $[\begin{smallmatrix} u \\ v \end{smallmatrix}] \in Y$ can be written as the summation of an element from $\operatorname{Im} L$ and an element from $\operatorname{Ker} L$. Also it is easy to show that any element in Y is uniquely expressed as the summation of an element $\operatorname{Ker} L$ and an element from $\operatorname{Im} L$. We get the desired result, since $\operatorname{codim} \operatorname{Im} L$ is 2. Hence L is a Fredholm mapping of index zero. There exist continuous projectors $U : X \to X$ and $V : Y \to Y$ such that

$$\begin{aligned} U\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) &= \frac{1}{\operatorname{mes}(\delta_+(T,\kappa))} \begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} u(t)\,\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} v(t)\,\Delta t \end{bmatrix}, \\ V\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) &= \frac{1}{\operatorname{mes}(\delta_+(T,\kappa))} \left(\begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} u(t)\,\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} v(t)\,\Delta t \end{bmatrix} \right). \end{aligned} \tag{23}$$

The generalized inverse $K_U = \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} U$ is given,

$$\begin{aligned} K_U\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) &= \begin{bmatrix} \int_{\kappa}^{t} u(s)\,\Delta s - \frac{1}{\operatorname{mes}(\delta_+(T,\kappa))} \int_{\kappa}^{\delta_+(T,\kappa)} \int_{\kappa}^{t} u(s)\,\Delta s \\ \int_{\kappa}^{t} v(s)\,\Delta s - \frac{1}{\operatorname{mes}(\delta_+(T,\kappa))} \int_{\kappa}^{\delta_+(T,\kappa)} \int_{\kappa}^{t} v(s)\,\Delta s \end{bmatrix}, \\ VC\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) &= \frac{1}{\operatorname{mes}(\delta_+(T,\kappa))} \left(\begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} a(s) - b(s)\exp(u(s)) - \frac{c(s)\exp(v(s))}{\alpha(s) + \beta(s)\exp(u(s)) + m(s)\exp(v(s))} \Delta s \\ \int_{\kappa}^{\delta_+(T,\kappa)} -d(s) + \frac{f(s)\exp(u(s))}{\alpha(s) + \beta(s)\exp(u(s)) + m(s)\exp(v(s))} \Delta s \end{bmatrix} \right). \end{aligned} \tag{24}$$

Let

$$\begin{aligned} &a(t) - b(t)\exp(u(t)) - \frac{c(t)\exp(v(t))}{\alpha(t) + \beta(t)\exp(u(t)) + m(t)\exp(v(t))} = C_1, \\ &-d(t) + \frac{f(t)\exp(u(t))}{\alpha(t) + \beta(t)\exp(u(t)) + m(t)\exp(v(t))} = C_2, \\ &\frac{1}{\operatorname{mes}(\delta_+(T,\kappa))} \int_{\kappa}^{\delta_+(T,\kappa)} a(s) - b(s)\exp(u(s)) - \frac{c(s)\exp(v(s))}{\alpha(s) + \beta(s)\exp(u(s)) + m(s)\exp(v(s))} \Delta s = \overline{C_1}, \\ &\frac{1}{\operatorname{mes}(\delta_+(T,\kappa))} \int_{\kappa}^{\delta_+(T,\kappa)} -d(s) + \frac{f(s)\exp(u(s))}{\alpha(s) + \beta(s)\exp(u(s)) + m(s)\exp(v(s))} \Delta s = \overline{C_2}, \\ &K_U(I-V)C\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = K_U\left(\begin{bmatrix} C_1 - \overline{C_1} \\ C_2 - \overline{C_2} \end{bmatrix}\right) = \begin{bmatrix} \int_{\kappa}^{t} C_1(s) - \overline{C_1}(s)\,\Delta s - \frac{1}{\operatorname{mes}(\delta_+(T,\kappa))} \int_{\kappa}^{\delta_+(T,\kappa)} \int_{\kappa}^{t} C_1(s) - \overline{C_1}(s)\,\Delta s \\ \int_{\kappa}^{t} C_2(s) - \overline{C_2}(s)\,\Delta s - \frac{1}{\operatorname{mes}(\delta_+(T,\kappa))} \int_{\kappa}^{\delta_+(T,\kappa)} \int_{\kappa}^{t} C_2(s) - \overline{C_2}(s)\,\Delta s \end{bmatrix}. \end{aligned} \tag{25}$$

Clearly, VC and $K_U(I-V)C$ are continuous. Here X and Y are Banach spaces. Since for the given time scale $\mathbb{T}$ while T is constant, $\operatorname{mes}(\delta_+(T,t))$ is equal for each $t \in \mathbb{T}$ then we can apply Arzela-Ascoli theorem and by using Arzela-Ascoli theorem we can find $\overline{K_U(I-V)C(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Additionally, $VC(\overline{\Omega})$ is bounded. Thus, C is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

To apply the continuation theorem we investigate the following operator equation:

$$\begin{aligned} x^{\Delta}(t) &= \lambda\left[a(t) - b(t)\exp(x(t)) - \frac{c(t)\exp(y(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}\right], \\ y^{\Delta}(t) &= \lambda\left[-d(t) + \frac{f(t)\exp(x(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}\right]. \end{aligned} \tag{26}$$

Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in X$ be any solution of system (26). Integrating both sides of system (26) over the interval $[0,w]$ we obtain

$$\begin{aligned} &\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\,\Delta t \\ &= \int_{\kappa}^{\delta_+(T,\kappa)} b(t)\exp(x(t)) + \frac{c(t)\exp(y(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}\Delta t, \\ &\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t \\ &= \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)\exp(x(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}\Delta t. \end{aligned} \tag{27}$$

From (26) and (27) we get

$$\begin{aligned} &\int_{\kappa}^{\delta_+(T,\kappa)} \left|x^{\Delta}(t)\right|\Delta t \le \lambda\left[\int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} b(t)\exp(x(t)) + \frac{c(t)\exp(y(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}\Delta t\right] \\ &\le \lambda\left[\int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\,\Delta t\right] \\ &\le \int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\,\Delta t := M_1, \\ &\int_{\kappa}^{\delta_+(T,\kappa)} \left|y^{\Delta}(t)\right|\Delta t \le \lambda\left[\int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)\exp(x(t))}{\alpha(t)+\beta(t)\exp(x(t))+m(t)\exp(y(t))}\Delta t\right] \\ &\le \lambda\left[\int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t\right] \\ &\le \int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t := M_2. \end{aligned} \tag{28}$$

Since $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in X$, then there exist $\eta_i, \xi_i,\ i = 1,2$, such that

$$\begin{aligned} x(\xi_1) &= \min_{t\in[\kappa,\delta_+(T,\kappa)]} x(t), \\ x(\eta_1) &= \max_{t\in[\kappa,\delta_+(T,\kappa)]} x(t), \\ y(\xi_2) &= \min_{t\in[\kappa,\delta_+(T,\kappa)]} y(t), \\ y(\eta_2) &= \max_{t\in[\kappa,\delta_+(T,\kappa)]} y(t). \end{aligned} \tag{29}$$

If ξ_1 is the minimum point of $x(t)$ on the interval $[\kappa, \delta_+(T,\kappa)]$ because $x(t)$ is a function that is periodic in shifts for any $n \in \mathbb{N}$ on the interval $[\delta_+^n(T,\kappa_1), \delta_+^{n+1}(T,\kappa_1)]$ the minimum point of $x(t)$ is $\delta_+^n(T,\xi_1)$ and $x(\xi_1) = x(\delta_+^n(T,\xi_1))$. We have similar results for the other points for ξ_2, η_1, η_2.

By the first equation of (27) and (29)

$$\begin{aligned} &\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\,\Delta t \\ &\le \int_{\kappa}^{\delta_+(T,\kappa)} \left[b(t)\exp(x(\eta_1)) + \frac{c(t)}{m(t)}\Delta t\right] \\ &= \exp(x(\eta_1))\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} \frac{c(t)}{m(t)}\Delta t. \end{aligned} \tag{30}$$

Since $\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\Delta t > 0$ we get

$$\begin{aligned} &x(\eta_1) \\ &\ge \ln\left(\frac{\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\,\Delta t - \int_{\kappa}^{\delta_+(T,\kappa)} (c(t)/m(t))\,\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\,\Delta t}\right) \\ &:= l_1. \end{aligned} \tag{31}$$

Using the second inequality in Lemma 6 we have

$$\begin{aligned} x(t) &\ge x(\eta_1) - \int_{\kappa}^{\delta_+(T,\kappa)} \left|x^{\Delta}(t)\right|\Delta t \\ &\ge x(\eta_1) - \left(\int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\,\Delta t\right) \\ &= l_1 - M_1 := H_1. \end{aligned} \tag{32}$$

By the first equation of (27) and (29)

$$\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\,\Delta t \geq \int_{\kappa}^{\delta_+(T,\kappa)} b(t)\exp(x(\xi_1))\,\Delta t = \exp(x(\xi_1))\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\,\Delta t. \tag{33}$$

Then we get

$$x(\xi_1) \leq \ln\left(\frac{\int_{\kappa}^{\delta_+(T,\kappa)} a(t)\,\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} b(t)\,\Delta t}\right) := l_2; \tag{34}$$

using the first inequality in Lemma 6 we have

$$\begin{aligned} x(t) &\leq x(\xi_1) + \int_{\kappa}^{\delta_+(T,\kappa)} \left|x^{\Delta}(t)\right|\Delta t \\ &\leq x(\xi_1) + \left(\int_{\kappa}^{\delta_+(T,\kappa)} |a(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} a(t)\,\Delta t\right) \\ &= l_2 + M_1 := H_2. \end{aligned} \tag{35}$$

By (32) and (35) $\max_{t\in[\kappa,\delta_+(T,\kappa)]}|x(t)| \leq \max\{|H_1|,|H_2|\} := B_1$. From the second equation of (27) and the second equation of (32), we can derive that

$$\begin{aligned} &\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t \\ &\quad\leq \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)\exp(x(t))}{\beta^l\exp(x(t)) + m^l\exp(y(t))}\Delta t \\ &\quad\leq \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)\,e^{H_2}}{\beta^l e^{H_2} + m^l\exp(y(\xi_2))}\Delta t \\ &\quad= \frac{e^{H_2}}{\beta^l e^{H_2} + m^l\exp(y(\xi_2))}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\,\Delta t. \end{aligned} \tag{36}$$

Therefore

$$\exp(y(\xi_2)) \leq \frac{1}{m^l}\left(\frac{e^{H_2}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\,\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t} - \beta^l e^{H_2}\right). \tag{37}$$

By the assumption of Theorem 8 we get

$$\begin{aligned} &\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\,\Delta t - \beta^l\left(\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\right)\Delta t > 0, \\ &y(\xi_2) \leq \ln\left(\frac{1}{m^l}\left(\frac{e^{H_2}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\,\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t} - \beta^l e^{H_2}\right)\right) \\ &\quad := L_1. \end{aligned} \tag{38}$$

Hence, by using the first inequality in Lemma 6 and the second equation of (27),

$$\begin{aligned} y(t) &\leq y(\xi_2) + \int_{\kappa}^{\delta_+(T,\kappa)} \left|y^{\Delta}(t)\right|\Delta t \\ &\leq y(\xi_2) \\ &\quad + \left(\int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t\right) \\ &\leq L_1 + M_2 := H_3. \end{aligned} \tag{39}$$

Again using the second equation of (27) we obtain

$$\begin{aligned} &\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t \\ &\quad\geq \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)\exp(x(t))}{\alpha^u + \beta^u\exp(x(t)) + m^u\exp(y(t))}\Delta t \\ &\quad\geq \int_{\kappa}^{\delta_+(T,\kappa)} \frac{f(t)\,e^{H_1}}{\alpha^u + \beta^u e^{H_1} + m^u\exp(y(\eta_2))}\Delta t \\ &\quad= \frac{e^{H_1}}{\alpha^u + \beta^u e^{H_1} + m^u\exp(y(\eta_2))}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\,\Delta t, \\ &\exp(y(\eta_2)) \\ &\quad\geq \frac{1}{m^u}\left(\frac{e^{H_1}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\,\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t} - \beta^u e^{H_1} - \alpha^u\right). \end{aligned} \tag{40}$$

Using the assumption of Theorem 8 we obtain

$$\begin{aligned} &e^{H_1}\left(\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\,\Delta t - \beta^u\left(\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t\right)\right) \\ &\quad - \alpha^u\left(\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t\right) > 0, \\ &y(\eta_2) \\ &\quad\geq \ln\left(\frac{1}{m^u}\left(\frac{e^{H_1}\int_{\kappa}^{\delta_+(T,\kappa)} f(t)\,\Delta t}{\int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t} - \beta^u e^{H_1} - \alpha^u\right)\right) \\ &\quad := L_2. \end{aligned} \tag{41}$$

By using the second inequality in Lemma 6,

$$\begin{aligned} y(t) &\geq y(\eta_2) - \int_{\kappa}^{\delta_+(T,\kappa)} \left|y^{\Delta}(t)\right|\Delta t \\ &\geq y(\eta_2) \\ &\quad - \left(\int_{\kappa}^{\delta_+(T,\kappa)} |d(t)|\,\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} d(t)\,\Delta t\right) \\ &= L_2 - M_2 := H_4. \end{aligned} \tag{42}$$

By (39) and (42) we have $\max_{t\in[t_0,\delta_+(T,t_0)]}|y(t)| \leq \max\{|H_3|,|H_4|\} := B_2$. Obviously, B_1 and B_2 are both independent of λ. Let $M = B_1 + B_2 + 1$. Then $\max_{t\in[t_0,\delta_+(T,t_0)]} \|[\begin{smallmatrix} x \\ y \end{smallmatrix}]\| < M$. Let $\Omega = \{\|[\begin{smallmatrix} x \\ y \end{smallmatrix}]\| \in X : \|[\begin{smallmatrix} x \\ y \end{smallmatrix}]\| < M\}$; then Ω verifies requirement (a) in Theorem 1. When $[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in \operatorname{Ker} L \cap \partial\Omega$, $[\begin{smallmatrix} x \\ y \end{smallmatrix}]$ is a constant with $\|[\begin{smallmatrix} x \\ y \end{smallmatrix}]\| = M$, then

$$VC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(\begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} a(s) - b(s)\exp(x) - \dfrac{c(s)\exp(y)}{\alpha(s)+\beta(s)\exp(x)+m(s)\exp(y)}\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} -d(s) + \dfrac{f(s)\exp(x)}{\alpha(s)+\beta(s)\exp(x)+m(s)\exp(y)}\Delta t \end{bmatrix}\right) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$JVC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = VC\left(\begin{bmatrix} x \\ y \end{bmatrix}\right), \tag{43}$$

where $J : \operatorname{Im} V \to \operatorname{Ker} L$ is the identity operator.

Let us define the homotopy such that $H_\nu = \nu(JVC) + (1-\nu)G$, where

$$G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \int_{\kappa}^{\delta_+(T,\kappa)} a(s) - b(s)\exp(x)\Delta t \\ \int_{\kappa}^{\delta_+(T,\kappa)} d(s) - \dfrac{f(s)\exp(x)}{\alpha(s)+\beta(s)\exp(x)+m(s)\exp(y)}\Delta t \end{bmatrix}. \tag{44}$$

Take DJ_G as the determinant of the Jacobian of G. Since $[\begin{smallmatrix} x \\ y \end{smallmatrix}] \in \operatorname{Ker} L$, then Jacobian of G is

$$\begin{bmatrix} -e^x \int_{\kappa}^{\delta_+(T,\kappa)} b(s)\Delta t & 0 \\ \int_{\kappa}^{\delta_+(T,\kappa)} \dfrac{-e^x f(s)}{\alpha(s)+\beta(s)e^x+m(s)e^y}\Delta t + \int_{\kappa}^{\delta_+(T,\kappa)} \dfrac{(e^x)^2 f(s)\beta(s)}{(\alpha(s)+\beta(s)e^x+m(s)e^y)^2}\Delta t & -\int_{\kappa}^{\delta_+(T,\kappa)} \dfrac{e^x e^y f(s) m(s)}{(\alpha(s)+\beta(s)e^x+m(s)e^y)^2}\Delta t \end{bmatrix}. \tag{45}$$

All the functions in Jacobian of G are positive; then sign DJ_G is always positive. Hence

$$\deg(JVC, \Omega\cap\operatorname{Ker} L, 0) = \deg(G, \Omega\cap\operatorname{Ker} L, 0)$$
$$= \sum_{[\begin{smallmatrix} x \\ y \end{smallmatrix}]\in G^{-1}([\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}])} \operatorname{sign} DJ_G\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \neq 0. \tag{46}$$

Thus all the conditions of Theorem 1 are satisfied. Therefore system (9) has at least a positive $\delta_\pm$-periodic solution. □

Example 9. Let $\mathbb{T} = \{0\} \cup q^{\mathbb{Z}}$. $\delta_\pm(q,t)$ is the shift operator and $t_0 = 1$:

$$x^\Delta(t) = \left((-1)^{\ln|t|/\ln(q)} + 4\right) - \left((-1)^{\ln|t|/\ln(q)} + 0.5\right)\exp(x(t)) - \frac{\exp(y(t))}{\exp(x(t)) + 2\exp(y(t))},$$
$$y^\Delta(t) = -0.3 + \frac{\left((-1)^{\ln|t|/\ln(q)} + 7\right)\exp(x(t))}{\exp(x(t)) + 2\exp(y(t))}. \tag{47}$$

Each function in system (47) is $\delta_\pm(q^2,t)$ periodic and satisfies Theorem 1; then the system has at least one $\delta_\pm(q^2,t)$ periodic solution. Here $\operatorname{mes}(\delta_+(q^2,t)) = 2$.

4. Discussion

There are many studies about the predator-prey dynamic systems on time scale calculus such as [12, 17, 27, 32]. All of these cited studies are about the periodic solutions of the considered system on a periodic time scale. However, in the world, there are many different species. While investigating the periodicity notion of the different life cycle of the species, the w-periodic time scales could be a little bit restricted. Therefore, if the life cycle of this kind of species is appropriate to the functional response Beddington De-Angelis, then the results that we have found in that study become more useful and important.

In addition to these, the $\delta_\pm$-periodic solutions for predator-prey dynamic systems with Holling type functional response, semi ratio dependent functional response, and monotype functional response can be also taken into account for future studies. In that dynamic system, delay conditions and impulsive conditions can also be added for new investigations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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