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Bounds of Generalized Proportional Fractional Integrals in General Form via Convex Functions and Their Applications

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Abstract: In this paper, our objective is to apply a new approach to establish bounds of sums of left and right proportional fractional integrals of a general type and obtain some related inequalities. From the obtained results, we deduce some new inequalities for classical generalized proportional fractional integrals as corollaries. These inequalities have a connection with some known and existing inequalities which are mentioned in the literature. In addition, some applications of the main results are presented.

Keywords: fractional integrals; generalized proportional fractional integrals; inequalities; convex functions; bounds

MSC: 26A33; 26D10; 26D53; 05A30

1. Introduction

Fractional calculus is an area of mathematics that studies the differentiation and integration of arbitrary order. This calculus has been attracting many researchers because of the astoishing results obtained when fractional operators were used in modeling a variety of real world problems. Thus, these operators have been conisdred as one the most powerful tools in the area of mathematical modeling. Many engineering, physical, chemical, and biological phenomena can be modeled by employing differential equations containing fractional derivatives. The applications of fractional integrals and derivatives can be found in [1–13].

It can be observed from the works in the literature that one of most important pecularities of the fractional operators is the fact that they are non-local. However, for the last few years, there has been an interest in the local derivatives with non-integer order. Although these types of derivatives can be used in modeling too, they are usually not considered as fractional operators. Nevertheless, these local derivatives succeeded to allure many scientists. There are various definitions of the local derivatives. One of the most well known local derivatives is the conformable integrals and derivatives, which were introduced for the first time by Khalil et al. [14]. In [15], Abdeljawad introduced certain properties of the fractional conformable derivative operators. Also, he gave the idea of how to employ the conformable derivative operators to define further more general fractional integral and derivative operators. The disadvantage of the conformable derivative is that the function is not obtained when the order of the conformable derivative is zero. In [16], Anderson and Unless introduced the idea of local proportional derivatives that produce the function when the order of the derivative is zero. Later on, Jarad et al. [17] introduced non-local fractional derivatives and integrals benefiting from the iteration of the proportional integrals. Abdeljawad and Baleanu [18] studied certain monotonicity results for fractional difference operators with discrete exponential kernels. In [19], Abdeljawad and Baleanu introduced fractional derivative with exponential kernel and their discrete version. Atangana and Baleanu [20] established certain new fractional derivatives with non-local and non-singular kernels. In [21], Caputo and Fabrizio defined fractional derivatives without a singular kernel. Losada and Nieto [22] studied certain properties of fractional derivatives without a singular kernel. A verity of such type of new definitions of fractional integrals and derivatives promotes future research to establish more new ideas and fractional integral inequalities by utilizing new fractional derivative and integral operators.

In [23,24], the authors established certain weighted Grüss type inequalities and some other inequalities containing Riemann–Liouville fractional integrals. Nisar et al. [25] studied several inequalities for extended gamma and confluent hypergeometric k-functions. Nisar et al. [26] presented Gronwall inequalities involving the generalized Riemann–Liouville and Hadamard k-fractional derivatives with applications. In [27], Rahman et al. proved certain inequalities involving the generalized fractional integral operators. In [28,29], Grüss type inequalities in the setting of generalized fractional integrals were found and some applictions were introduced. Liu et al. [30] presented several interesting integral inequalities. Sarikaya and Budak [31] have presented the generalization of Riemann–Liouville fractional integrals and their applications. In [32], using some fractional integral operators, Set et al. established Hermite–Hadamard type inequalities. Meanwhile, Agarwal et al. [33] employed generalized k-fractional integral operators for the sake of establishing Hermite–Hadamard type inequalities. Dahmani [34] presented a variety of integral inequalities by using some families of n positive functions. In [35], Aldhaifallah et al. introduced some integral inequalities for a certain family of $n (n \in \mathbb{N})$ positive continuous and decreasing functions on some intervals employing what is called generalized (k, s)-fractional integral operators. Recently, some researchers introduced a verity of certain interesting inequalities, applications, and properties for the conformable integrals [36–40].

2. Preliminaries

In this section, we present some well known results.

Definition 1. *Let $f : I \to \mathbb{R}$ be a real valued function. We say that f is convex on interval I, if for all $\lambda \in [0,1]$ and $x, y \in I$, the following inequality is satisfied*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y). \tag{1}$$

Moreover, we say f is concave if the inequality (1) is reversed.

In [41], Jarad et al. defined the following left and right sided generalized proportional fractional integrals.

Definition 2. *The left and right fractional proportional integrals in their general forms are defined by*

$$\begin{aligned}&\left({}_r\mathcal{I}^{\eta,\delta} f\right)(\rho)\\ &= \frac{1}{\delta^\eta \Gamma(\eta)} \int_r^\rho \exp[\frac{\delta-1}{\delta}(\rho-\theta)](\rho-\theta)^{\eta-1} f(\theta) d\theta, r < \rho\end{aligned} \tag{2}$$

and

$$\begin{aligned}&\left(\mathcal{I}_s^{\eta,\delta}f\right)(\rho)\\&=\frac{1}{\delta^{\eta}\Gamma(\eta)}\int_{\rho}^{s}\exp[\frac{\delta-1}{\delta}(\theta-\rho)](\theta-\rho)^{\eta-1}f(\theta)d\theta,\rho<s,\end{aligned}\tag{3}$$

where the proportional index $\delta\in(0,1]$ *and* $\eta\in\mathbb{C}$ *and* $\Re(\eta)>0$ *and* Γ *is the complete gamma function.*

Remark 1. *Setting* $\delta=1$ *in* (2) *and* (3)*, then the following left and right Riemann–Liouville integrals are respectively obtained as:*

$$\left({}_r\mathcal{I}^{\eta}f\right)(\rho)=\frac{1}{\Gamma(\eta)}\int_{r}^{\rho}(\rho-\theta)^{\eta-1}f(\theta)d\theta,r<\rho\tag{4}$$

and

$$\left(\mathcal{I}_s^{\eta}f\right)(\rho)=\frac{1}{\Gamma(\eta)}\int_{\rho}^{s}(\theta-\rho)^{\eta-1}f(\theta)d\theta,\rho<s\tag{5}$$

where $\eta\in\mathbb{C}$ *and* $\Re(\eta)>0$.

The Gronwall inequalities which involve the proportional fractional integral operator can be found in work of Alzabut et al. [42]. Rahman et al. [43] established the Minkowski inequality and other types of inequalities in the frame of the proportional fractional integrals. In [44], Rahman et al. discussed some specific new types of integral inequalities for a class of n ($n\in\mathbb{N}$) positive continuous and decreasing functions on $[r,b]$. Rahman et al. [45] defined the generalized proportional Hadamard fractional integrals and established certain new integral inequalities for convex functions. In [46–50], certain remarkable inequalities, properties, and applications can be found.

In [51], general forms of the proportional fractional integrals (4) and (5) were given as

Definition 3. *Let* $f:[r,s]\to\mathbb{R}$ *be an integrable function and let* $g\in C^1[r,s]$ *such that* $g'>0$ *on* $[r,s]$*. Then, the left (forword) and right (backward) proportional fractional integrals of the function* f *with respect to the function* g *are respectively defined by*

$$\begin{aligned}&\left({}_r^g\mathcal{I}^{\eta,\delta}f\right)(\rho)\\&=\frac{1}{\delta^{\eta}\Gamma(\eta)}\int_{r}^{\rho}\exp[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))]\\&\times(g(\rho)-g(\theta))^{\eta-1}g'(\theta)f(\theta)d\theta,r<\rho\end{aligned}\tag{6}$$

and

$$\begin{aligned}&\left({}^g\mathcal{I}_s^{\eta,\delta}f\right)(\rho)\\&=\frac{1}{\delta^{\eta}\Gamma(\eta)}\int_{\rho}^{s}\exp[\frac{\delta-1}{\delta}(g(\theta)-g(\rho))]\\&\times(g(\theta)-g(\rho))^{\eta-1}g'(\theta)f(\theta)d\theta,\rho<s,\end{aligned}\tag{7}$$

where the proportional index $\delta\in(0,1]$ *and* $\eta\in\mathbb{C}$ *and* $\Re(\eta)>0$ *and* Γ *is the well-known gamma function.*

Remark 2. *The generalized proportional fractional integrals defined in* (6) *and* (7) *are the generalization of the following fractional integrals respectively:*

i. *if we take $g(\tau)=\tau$, one obtains the left and right sided generalized proportional fractional integral defined in* (2) *and* (3)*,*
ii. *if we take $g(\rho)=\frac{\rho^{\eta}}{\eta}$, $\eta>0$ and $\delta=1$, we get the left and right sided Katugampola fractional integral operators,*
iii. *if we take $\delta=1$, then it reduces to the general form of Riemann–Liouville fractional integral given in* [52]*,*
iv. *if we take $g(\rho)=\rho$ and $\delta=1$, then it reduces to the left and right Riemann–Liouville fractional integrals* (4) *and* (5)*,*
v. *if we take $g(\rho)=\frac{\rho^{\delta+\lambda}}{\delta+\lambda}$ and $\delta=1$ (where $\delta\in(0,1]$, $\lambda\in\mathbb{R}^{+}$ and $\delta+\lambda\neq 0$), then it reduces to the generalized fractional conformable integrals given in* [53]*.*

3. Main Results

In this section, we first obtain a bound for the sum of the left and right-sided generalized proportional fractional integrals in their general forms. For this sake, we use convexity and monotonicity of the functions.

Theorem 1. *Let $f,g:[r,s]\to\mathbb{R}$ be the functions such that f is convex and positive and g is increasing and differentiable with $g'\in L[r,s]$. Then, for $\eta,\xi\geq 1$ and $\rho\in[r,s]$ and $\delta\in(0,1]$, we have*

$$\begin{aligned}
&\Gamma(\eta)\delta^{\eta}\,{}_{r}^{g}\mathcal{I}^{\eta,\delta}f(\rho)+\Gamma(\xi)\delta^{\xi}\,{}^{g}\mathcal{I}_{s}^{\xi,\delta}f(\rho)\\
&\leq\frac{\exp[\frac{\delta-1}{\delta}(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta-1}}{\rho-r}\\
&\times\Big[(\rho-r)f(\rho)g(\rho)-(\rho-r)f(r)g(r)\\
&-(f(\rho)-f(r))\int_{r}^{\rho}g(\theta)d\theta\Big]\\
&+\frac{\exp[\frac{\delta-1}{\delta}(g(s)-g(\rho))]\,(g(s)-g(\rho))^{\xi-1}}{s-\rho}\\
&\times\Big[(s-\rho)f(s)g(s)-(s-\rho)f(\rho)g(\rho)\\
&-(f(s)-f(\rho))\int_{\rho}^{s}g(\theta)d\theta\Big].
\end{aligned}\tag{8}$$

Proof. Since g is differentiable and increasing, we obtain

$$\begin{aligned}
&\exp\Big[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))\Big](g(\rho)-g(\theta))^{\eta-1}\\
&\leq\exp\Big[\frac{\delta-1}{\delta}(g(\rho)-g(r))\Big](g(\rho)-g(r))^{\eta-1},
\end{aligned}$$

where $\rho\in[r,s]$, $\theta\in[r,\rho]$, $\eta\geq 1$, $\delta\in(0,1]$ and $g'(\theta)>0$. Hence, the following inequality holds true

$$\begin{aligned}
&g'(\theta)\exp\Big[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))\Big](g(\rho)-g(\theta))^{\eta-1}\\
&\leq g'(\theta)\exp\Big[\frac{\delta-1}{\delta}(g(\rho)-g(r))\Big](g(\rho)-g(r))^{\eta-1}.
\end{aligned}\tag{9}$$

From the convexity of f, we have

$$f(\theta)\leq\frac{\rho-\theta}{\rho-r}f(r)+\frac{\theta-r}{\rho-r}f(\rho).\tag{10}$$

From (9) and (10), we can write

$$\int_r^\rho \exp[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))]\,(g(\rho)-g(\theta))^{\eta-1}g'(\theta)f(\theta)d\theta$$
$$\leq \frac{\exp[\frac{\delta-1}{\delta}(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta-1}}{\rho-r}$$
$$\times\left[f(r)\int_r^\rho(\rho-\theta)g'(\theta)d\theta+f(\rho)\int_r^\rho(\theta-r)g'(\theta)d\theta\right].$$

By using (6), we get

$$\begin{aligned}&\Gamma(\eta)\delta^\eta\,{}^g_r\mathcal{I}^{\eta,\delta}f(\rho)\\ &\leq \frac{\exp[\frac{\delta-1}{\delta}(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta-1}}{\rho-r}\\ &\times\Big[(\rho-r)f(\rho)g(\rho)-(\rho-r)f(r)g(r)\\ &-(f(\rho)-f(r))\int_r^\rho g(\theta)d\theta\Big].\end{aligned}\tag{11}$$

Now, for $\rho\in[r,s]$, $\theta\in[r,\rho]$, $\xi\geq1$, $\delta\in(0,1]$ and $g'(\theta)>0$, the following inequality holds true

$$\begin{aligned}&g'(\theta)\exp\left[\frac{\delta-1}{\delta}(g(\theta)-g(\rho))\right](g(\theta)-g(\rho))^{\xi-1}\\ &\leq g'(\theta)\exp\left[\frac{\delta-1}{\delta}(g(s)-g(\rho))\right](g(s)-g(\rho))^{\xi-1}.\end{aligned}\tag{12}$$

Again, from the convexity of f, we have

$$f(\theta)\leq\frac{\theta-\rho}{s-\rho}f(s)+\frac{s-\theta}{s-\rho}f(\rho).\tag{13}$$

From (12) and (13), the following can be written

$$\begin{aligned}&g'(\theta)\exp\left[\frac{\delta-1}{\delta}(g(\theta)-g(\rho))\right](g(\theta)-g(\rho))^{\xi-1}f(\theta)\\ &\leq\frac{\exp\left[\frac{\delta-1}{\delta}(g(s)-g(\rho))\right](g(s)-g(\rho))^{\xi-1}}{s-\rho}\\ &\times\left[(\theta-\rho)f(s)g'(\theta)+(s-\theta)g'(\theta)f(\rho)\right].\end{aligned}\tag{14}$$

Integrating (14) with respect to θ over $[\rho,s]$ and then applying (7), we get

$$\begin{aligned}&\Gamma(\xi)\delta^\xi\,{}^g\mathcal{I}_s^{\eta,\delta}f(\rho)\\ &\leq\frac{\exp\left[\frac{\delta-1}{\delta}(g(s)-g(\rho))\right](g(s)-g(\rho))^{\xi-1}}{s-\rho}\\ &\times\Big[(s-\rho)f(s)g(s)-(s-\rho)f(\rho)g(\rho)-(f(s)\\ &-f(\rho))\int_\rho^s g(\theta)d\theta\Big].\end{aligned}\tag{15}$$

Hence, from (11) and (15), we get the desired result. □

Corollary 1. *Let $f, g : [r,s] \to \mathbb{R}$ be functions such that f is convex and positive and let g be increasing and differentiable with $g' \in L[r,s]$. Then, for $\eta, \xi \geq 1$ and $\rho \in [r,s]$ and $\delta \in (0,1]$, we have*

$$\begin{aligned}
&\Gamma(\eta)\delta^{\eta}\, {}_{r}^{g}\mathcal{I}^{\eta,\delta} f(\rho) + \Gamma(\eta)\delta^{\eta}\, {}^{g}\mathcal{I}_{s}^{\eta,\delta} f(\rho) \\
\leq &\frac{\exp[\frac{\delta-1}{\delta}\,(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta-1}}{\rho-r} \\
&\times \Big[(\rho-r)f(\rho)g(\rho) - (\rho-r)f(r)g(r) - (f(\rho)-f(r)) \\
&\int_r^{\rho} g(\theta)d\theta\Big] \\
+&\frac{\exp[\frac{\delta-1}{\delta}\,(g(s)-g(\rho))]\,(g(s)-g(\rho))^{\eta-1}}{s-\rho} \\
&\times \Big[(s-\rho)f(s)g(s) - (s-\rho)f(\rho)g(\rho) - (f(s)-f(\rho)) \\
&\int_{\rho}^{s} g(\theta)d\theta\Big].
\end{aligned} \tag{16}$$

Proof. By setting $\eta = \xi$ in Theorem 1, we get the desired Corollary 1. □

Corollary 2. *Setting $g(\rho) = \rho$ in Theorem 1, then we get the following inequality for generalized proportional fractional integrals (2) and (3)*

$$\begin{aligned}
&\Gamma(\eta)\delta^{\eta}\, {}_{r}\mathcal{I}^{\eta,\delta} f(\rho) + \Gamma(\xi)\delta^{\xi}\mathcal{I}_{s}^{\xi,\delta} f(\rho) \\
&\leq \frac{1}{2}\Big[\exp[\frac{\delta-1}{\delta}\,(\rho-r)]\,(\rho-r)^{\eta} f(r) \\
&+\exp[\frac{\delta-1}{\delta}\,(s-\rho)]\,(s-\rho)^{\xi} f(s)\Big] \\
&+f(\rho)\frac{1}{2}\Big[\exp[\frac{\delta-1}{\delta}\,(\rho-r)]\,(\rho-r)^{\eta} \\
&+\exp[\frac{\delta-1}{\delta}\,(s-\rho)]\,(s-\rho)^{\xi}\Big].
\end{aligned}$$

Remark 3. *Setting $\delta = 1$ in Theorem 1, we get the following inequality for generalized Riemann–Liouville fractional integral ([52], Theorem 1).*

$$\begin{aligned}
&\Gamma(\eta)\, {}_{r}^{g}\mathcal{I}^{\eta} f(\rho) + \Gamma(\xi)\, {}^{g}\mathcal{I}_{s}^{\xi} f(\rho) \\
\leq &\frac{(g(\rho)-g(r))^{\eta-1}}{\rho-r} \\
&\times \Big[(\rho-r)f(\rho)g(\rho) - (\rho-r)f(r)g(r) - (f(\rho)-f(r)) \\
&\int_r^{\rho} g(\theta)d\theta\Big] \\
+&\frac{(g(s)-g(\rho))^{\xi-1}}{s-\rho} \\
&\Big[(s-\rho)f(s)g(s) - (s-\rho)f(\rho)g(\rho) - (f(s)-f(\rho)) \\
&\int_{\rho}^{s} g(\theta)d\theta\Big].
\end{aligned} \tag{17}$$

Remark 4. *Setting $g(\rho) = \rho$ and $\delta = 1$ in Theorem 1, we get integral inequality involving Riemann–Liouville fractional integrals ([54], Theorem 2).*

Theorem 2. *Let $f, g : [r,s] \to \mathbb{R}$ be functions such that f is differentiable, $|f'|$ is convex, and g is also differentiable and increasing with $g' \in L[r,s]$. Then, for $\eta, \xi \geq 0$ and $\rho \in [r,s]$ and $\delta \in (0,1]$, we have*

$$\begin{aligned}
&\Big|\Gamma(\eta+1)\delta^{\eta}\,{}^{g}_{r}\mathcal{I}^{\eta,\delta}f(\rho) + \frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}^{g}_{r}\mathcal{I}^{\eta+1,\delta}f(\rho) \\
&+\Gamma(\xi+1)\delta^{\xi}\,{}^{g}\mathcal{I}^{\xi,\delta}_{s}f(\rho) + \frac{\delta-1}{\delta}\Gamma(\xi+1)\delta^{\xi+1} \\
&\times{}^{g}\,\mathcal{I}^{\xi+1,\delta}_{s}f(\rho) - \exp[\frac{\delta-1}{\delta}(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta} \\
&f(r) + \exp[\frac{\delta-1}{\delta}(g(s)-g(\rho))]\,(g(s)-g(\rho))^{\xi}\,f(s)\Big| \\
\leq&\frac{1}{2}\Big[\exp[\frac{\delta-1}{\delta}\,(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta}\,(\rho-r)|f'(r)| \\
&+\exp[\frac{\delta-1}{\delta}\,(g(s)-g(\rho))]\,(g(s)-g(\rho))^{\xi}\,(s-\rho)|f'(s)|\Big] \\
&+|f'(\rho)|\frac{1}{2}\Big[\exp[\frac{\delta-1}{\delta}\,(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta}\,(\rho-r) \\
&+\exp[\frac{\delta-1}{\delta}\,(g(s)-g(\rho))]\,(g(s)-g(\rho))^{\xi}\,(s-\rho)\Big].
\end{aligned} \tag{18}$$

Proof. From the convexity of $|f'|$, we have

$$|f'(t)| \leq \frac{\rho-\theta}{\rho-r}|f'(r)| + \frac{\theta-r}{\rho-r}|f'(\rho)|. \tag{19}$$

It follows that

$$f'(t) \leq \frac{\rho-\theta}{\rho-r}|f'(r)| + \frac{\theta-r}{\rho-r}|f'(\rho)|. \tag{20}$$

Since g is differentiable and increasing, we have

$$\begin{aligned}
&\exp[\frac{\delta-1}{\delta}\,(g(\rho)-g(\theta))]\,(g(\rho)-g(\theta))^{\eta} \\
&\leq \exp[\frac{\delta-1}{\delta}\,(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta}\,,
\end{aligned} \tag{21}$$

where $\rho \in [r,s]$, $\theta \in [r,\rho]$, and $\eta > 0$.

From (20) and (21), we can write

$$\begin{aligned}
&\exp[\frac{\delta-1}{\delta}\,(g(\rho)-g(\theta))]\,(g(\rho)-g(\theta))^{\eta}\,f'(\theta) \\
\leq&\frac{\exp[\frac{\delta-1}{\delta}\,(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta}}{\rho-r}\Big((\rho-\theta)|f'(r)| \\
&+(\theta-r)|f'(\rho)|\Big).
\end{aligned} \tag{22}$$

Integrating (22) with respect to θ over $[r,\rho]$, we get

$$\begin{aligned}
&\int_r^\rho \exp[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))]\,(g(\rho)-g(\theta))^\eta f'(\theta)d\theta \\
\leq&\frac{\exp[\frac{\delta-1}{\delta}(g(\rho)-g(r))]\,(g(\rho)-g(r))^\eta}{\rho-r} \\
&\times\left[|f'(r)|\int_r^\rho(\rho-\theta)d\theta+|f'(\rho)|\int_r^\rho(\theta-r)d\theta\right] \\
=&\exp[\frac{\delta-1}{\delta}(g(\rho)-g(r))]\,(g(\rho)-g(r))^\eta\,(\rho-r) \\
&\times\left[\frac{|f'(r)|+|f'(\rho)|}{2}\right],
\end{aligned}\tag{23}$$

and

$$\begin{aligned}
&\int_r^\rho \exp[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))]\,(g(\rho)-g(\theta))^\eta f'(\theta)d\theta \\
=&\exp[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))]\,(g(\rho)-g(\theta))^\eta f(\theta)\Big|_r^\rho \\
&-\int_r^\rho\frac{d}{d\theta}\left[\exp[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))]\,(g(\rho)-g(\theta))^\eta\right]f(\theta)d\theta \\
=&-\exp\left[\frac{\delta-1}{\delta}(g(\rho)-g(r))\right](g(\rho)-g(r))^\eta f(r) \\
&+\eta\int_r^\rho\exp[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))]\,(g(\rho)-g(\theta))^{\eta-1}g'(\theta)f(\theta)d\theta \\
+&\frac{\delta-1}{\delta}\int_r^\rho\exp[\frac{\delta-1}{\delta}(g(\rho)-g(\theta))]\,(g(\rho)-g(\theta))^\eta g'(\theta)f(\theta)d\theta \\
=&-\exp\left[\frac{\delta-1}{\delta}(g(\rho)-g(r))\right](g(\rho)-g(r))^\eta f(r)+\Gamma(\eta+1)\delta^\eta \\
&\times{}_r^g\mathcal{I}^{\eta,\delta}f(\rho) \\
&+\frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}_r^g\mathcal{I}^{\eta+1,\delta}f(\rho).
\end{aligned}\tag{24}$$

Therefore, (23) becomes

$$\begin{aligned}
&\Gamma(\eta+1)\delta^\eta\,{}_r^g\mathcal{I}^{\eta,\delta}f(\rho)+\frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}_r^g\mathcal{I}^{\eta+1,\delta}f(\rho) \\
&-\exp\left[\frac{\delta-1}{\delta}(g(\rho)-g(r))\right](g(\rho)-g(r))^\eta f(r) \\
\leq&\exp[\frac{\delta-1}{\delta}(g(\rho)-g(r))] \\
&\times(g(\rho)-g(r))^\eta\,(\rho-r)\left[\frac{|f'(r)|+|f'(\rho)|}{2}\right].
\end{aligned}\tag{25}$$

Also, from (19), we can write

$$f'(t)\geq-\left(\frac{\rho-\theta}{\rho-r}|f'(r)|+\frac{\theta-r}{\rho-r}|f'(\rho)|\right).\tag{26}$$

Applying a similar procedure as we applied for (20), we have

$$\begin{aligned}
&\exp\left[\frac{\delta-1}{\delta}\left(g(\rho)-g(r)\right)\right]\left(g(\rho)-g(r)\right)^{\eta} f(r) \\
&-\Gamma(\eta+1)\delta^{\eta}\,{}^{g}_{r}\mathcal{I}^{\eta,\delta} f(\rho)-\frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}^{g}_{r}\mathcal{I}^{\eta+1,\delta} f(\rho) \\
\leq &\exp\left[\frac{\delta-1}{\delta}\left(g(\rho)-g(r)\right)\right]\left(g(\rho)-g(r)\right)^{\eta}(\rho-r) \\
&\times\left[\frac{|f'(r)|+|f'(\rho)|}{2}\right].
\end{aligned} \tag{27}$$

Therefore, from (25) and (27), we get

$$\begin{aligned}
&\Big|\Gamma(\eta+1)\delta^{\eta}\,{}^{g}_{r}\mathcal{I}^{\eta,\delta} f(\rho)+\frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}^{g}_{r}\mathcal{I}^{\eta+1,\delta} f(\rho)- \\
&\quad\exp\left[\frac{\delta-1}{\delta}\left(g(\rho)-g(r)\right)\right]\left(g(\rho)-g(r)\right)^{\eta} f(r)\Big| \\
\leq &\exp\left[\frac{\delta-1}{\delta}\left(g(\rho)-g(r)\right)\right]\left(g(\rho)-g(r)\right)^{\eta}(\rho-r) \\
&\times\left[\frac{|f'(r)|+|f'(\rho)|}{2}\right].
\end{aligned} \tag{28}$$

Again, from convexity of $|f'|$, we have

$$|f'(t)| \leq \frac{\theta-\rho}{s-\rho}|f'(s)|+\frac{s-\theta}{s-\rho}|f'(\rho)|. \tag{29}$$

Now, for $\rho \in [r,s]$, $\theta \in [\rho,s]$ and $\xi > 0$, we have

$$\begin{aligned}
&\exp\left[\frac{\delta-1}{\delta}\left(g(\theta)-g(\rho)\right)\right]\left(g(\theta)-g(\rho)\right)^{\xi} \\
&\leq \exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(\rho)\right)\right]\left(g(s)-g(\rho)\right)^{\xi}.
\end{aligned} \tag{30}$$

Following the similar procedure as we did for (20), (21) and (26), one can obtain the following result from (29) and (30)

$$\begin{aligned}
&\Big|\Gamma(\xi+1)\delta^{\xi}\,{}^{g}\mathcal{I}^{\eta,\delta}_{s} f(\rho)+\frac{\delta-1}{\delta}\Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}^{\xi+1,\delta}_{s} f(\rho)- \\
&\quad\exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(\rho)\right)\right]\left(g(s)-g(\rho)\right)^{\xi} f(s)\Big| \\
\leq &\exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(\rho)\right)\right]\left(g(s)-g(\rho)\right)^{\xi}(s-\rho) \\
&\times\left[\frac{|f'(s)|+|f'(\rho)|}{2}\right].
\end{aligned} \tag{31}$$

Thus, from (28), (31), and together with triangular inequality, we get the desired inequality (18). □

Corollary 3. *Setting $\eta = \xi$ in Theorem 2, we deduce the following for the generalized proportional fractional integral in general form*

$$\begin{aligned}
&\Big|\Gamma(\eta+1)\delta^{\eta}\, {}^{g}_{r}\mathcal{I}^{\eta,\delta} f(\rho) + \frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\, {}^{g}_{r}\mathcal{I}^{\eta+1,\delta} f(\rho) \\
&\quad + \Gamma(\eta+1)\delta^{\eta}\, {}^{g}\mathcal{I}^{\eta,\delta}_{s} f(\rho) + \frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1} \\
&\times {}^{g}\mathcal{I}^{\eta+1,\delta}_{s} f(\rho) - \exp[\frac{\delta-1}{\delta}(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta} \\
&\quad + \exp[\frac{\delta-1}{\delta}(g(s)-g(\rho))]\,(g(s)-g(\rho))^{\eta}\, f(s)\Big| \\
&\leq \frac{1}{2}\exp[\frac{\delta-1}{\delta}\,(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta}\,(\rho-r)|f'(r)| \\
&\quad + \exp[\frac{\delta-1}{\delta}\,(g(s)-g(\rho))]\,(g(s)-g(\rho))^{\eta}\,(s-\rho)|f'(s)| \\
&+ |f'(\rho)|\frac{1}{2}\exp[\frac{\delta-1}{\delta}\,(g(\rho)-g(r))]\,(g(\rho)-g(r))^{\eta}\,(\rho-r) \\
&\quad + \exp[\frac{\delta-1}{\delta}\,(g(s)-g(\rho))]\,(g(s)-g(\rho))^{\eta}\,(s-\rho).
\end{aligned}$$

Corollary 4. *By taking $g(\rho) = \rho$ in Theorem 2, we get the following generalized proportional fractional integral inequality for the integrals (2) and (3)*

$$\begin{aligned}
&\Big|\Gamma(\eta+1)\delta^{\eta}\, {}_{r}\mathcal{I}^{\eta,\delta} f(\rho) + \frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\, {}_{r}\mathcal{I}^{\eta+1,\delta} f(\rho) \\
&+ \Gamma(\xi+1)\delta^{\xi}\, \mathcal{I}^{\xi,\delta}_{s} f(\rho) \\
&+ \frac{\delta-1}{\delta}\Gamma(\xi+1)\delta^{\xi+1}\mathcal{I}^{\xi+1,\delta}_{s} f(\rho) \\
&- \exp\,[\frac{\delta-1}{\delta}(\rho-r)]\,(\rho-r)^{\eta} \\
&+ \exp\,[\frac{\delta-1}{\delta}(s-\rho)]\,(s-\rho)^{\xi}\, f(s)\Big| \\
&\leq \frac{1}{2}\exp[\frac{\delta-1}{\delta}\,(\rho-r)]\,(\rho-r)^{\eta+1}\,|f'(r)| \\
&\quad + \exp[\frac{\delta-1}{\delta}\,(s-\rho)]\,(s-\rho)^{\xi+1}\,|f'(s)| \\
&+ |f'(\rho)|\frac{1}{2}\exp[\frac{\delta-1}{\delta}\,(\rho-r)]\,(\rho-r)^{\eta+1} \\
&\quad + \exp[\frac{\delta-1}{\delta}\,(s-\rho)]\,(s-\rho)^{\xi+1}.
\end{aligned}$$

Remark 5. *By setting $\delta = 1$ in Theorem 2, we get the integral inequality for Riemann–Liouville fractional integrals in general form ([52], Theorem 2).*

Remark 6. *By taking $g(\rho) = \rho$ and $\delta = 1$ in 2, we get the integral inequality for classical Riemann–Liouville fractional integrals ([54], Theorem, 1).*

Now, recalling the following Lemma from [54] which will be helpful in the proof of next result.

Lemma 1. *Let $f : [r,s] \to \mathbb{R}$ be a symmetric function which is symmetric about $\frac{r+s}{2}$, then we have*

$$f\left(\frac{r+s}{2}\right) \leq f(\rho), \rho \in [r,s]. \tag{32}$$

Theorem 3. *Let f, $g : [r,s] \to \mathbb{R}$ be the functions such that f is convex and positive and g is increasing and differentiable with $g' \in L[r,s]$. Then for $\eta, \xi \geq 0$ and $\rho \in [r,s]$ and $\delta \in (0,1]$, we have*

$$\begin{aligned}
& f\left(\frac{r+s}{2}\right)\Big[\Gamma(\eta+1)\delta^{\eta}\,{}_{r}^{g}\mathcal{I}^{\eta,\delta}g(s) \\
& + \frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}_{r}^{g}\mathcal{I}^{\eta+1,\delta}g(s) \\
& -\Gamma(\xi+1)\delta^{\xi}\,{}^{g}\mathcal{I}_{s}^{\xi,\delta}g(r) - \frac{\delta-1}{\delta}\Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}_{s}^{\xi+1,\delta}g(r) \\
& - \exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\right]\left(g(s)-g(r)\right)^{\eta}g(r) \\
& + \exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\right]\left(g(s)-g(r)\right)^{\eta}g(s)\Big] \\
& \leq \Gamma(\eta+1)\delta^{\eta+1}\,{}_{r}^{g}\mathcal{I}^{\eta+1,\delta}f(s) + \Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}_{s}^{\xi+1,\delta}f(r) \\
& \leq \exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\right]\frac{\left(g(s)-g(r)\right)^{\eta}+\left(g(s)-g(r)\right)^{\xi}}{s-r} \\
& \Big[(s-r)f(s)g(s)-(s-r)f(r)g(r)-\left(f(s)-f(r)\right) \\
& \times \int_{r}^{s} g(\rho)d\rho\Big]. \qquad (33)
\end{aligned}$$

Proof. Since g is differentiable and increasing, therefore

$$\begin{aligned}
& \exp\left[\frac{\delta-1}{\delta}\left(g(\rho)-g(r)\right)\right]\left(g(\rho)-g(r)\right)^{\eta} \\
& \leq \exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\right]\left(g(s)-g(r)\right)^{\eta},
\end{aligned}$$

where $\rho \in [r,s]$, $\theta \in [r,\rho]$, $\eta > 0$, $\delta \in (0,1]$, and $g'(\theta) > 0$. Hence, the following inequality holds true

$$\begin{aligned}
& g'(\rho)\exp\left[\frac{\delta-1}{\delta}\left(g(\rho)-g(r)\right)\right]\left(g(\rho)-g(r)\right)^{\eta} \\
& \leq g'(\rho)\exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\right]\left(g(s)-g(r)\right)^{\eta}. \qquad (34)
\end{aligned}$$

From the convexity of f, we have

$$f(\theta) \leq \frac{\rho-r}{s-r}f(s) + \frac{s-\rho}{s-r}f(r). \qquad (35)$$

From (34) and (35), we can write

$$\begin{aligned}
& \int_{r}^{s}\exp\left[\frac{\delta-1}{\delta}\left(g(\rho)-g(r)\right)\right]\left(g(\rho)-g(r)\right)^{\eta}g'(\rho)f(\rho)d\rho \\
& \leq \frac{\exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\right]\left(g(s)-g(r)\right)^{\eta}}{s-r} \\
& \times\left[f(s)\int_{r}^{s}(\rho-r)g'(\rho)d\rho + f(r)\int_{r}^{s}(s-\rho)g'(\rho)d\rho\right].
\end{aligned}$$

By using (6), we get

$$\begin{aligned}
&\Gamma(\eta+1)\delta^{\eta+1}\,{}_{r}^{g}\mathcal{I}^{\eta+1,\delta}f(s)\\
\leq&\frac{\exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\right]\left(g(s)-g(r)\right)^{\eta}}{s-r}\\
&\times\Big[(s-r)f(\rho)g(s)-(s-r)f(r)g(r)-(f(s)-f(r))\\
&\times\int_r^s g(\rho)d\rho\Big].
\end{aligned}\tag{36}$$

Now, for $\rho\in[r,s]$, $\xi\geq 0$, $\delta\in(0,1]$, and $g'(\rho)>0$, the following inequality holds true

$$\begin{aligned}
&g'(\rho)\exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(\rho)\right)\right]\left(g(s)-g(\rho)\right)^{\xi}\\
&\leq g'(\rho)\exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\right]\left(g(s)-g(r)\right)^{\xi}.
\end{aligned}\tag{37}$$

Applying a similar procedure to (35) and (37) as we did for (34) and (35), we have

$$\begin{aligned}
&\Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}_{s}^{\xi+1,\delta}f(r)\\
\leq&\frac{\exp[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)]\left(g(s)-g(r)\right)^{\xi}}{s-r}\\
&\times\Big[(s-r)f(s)g(s)-(s-r)f(r)g(r)-(f(s)-f(r))\\
&\times\int_r^s g(\rho)d\rho\Big].
\end{aligned}\tag{38}$$

Hence, from (36) and (38), we get the following result

$$\begin{aligned}
&\Gamma(\eta+1)\delta^{\eta+1}\,{}_{r}^{g}\mathcal{I}^{\eta+1,\delta}f(s)+\Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}_{s}^{\xi+1,\delta}f(r)\\
\leq&\exp\left[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\right]\frac{\left(g(s)-g(r)\right)^{\eta}+\left(g(s)-g(r)\right)^{\xi}}{s-r}\\
&\times\Big[(s-r)f(s)g(s)-(s-r)f(r)g(r)-(f(s)-f(r))\\
&\times\int_r^s g(\theta)d\theta\Big].
\end{aligned}\tag{39}$$

Now, multiplying (32) by

$$\exp[\frac{\delta-1}{\delta}\left(g(s)-g(\rho)\right)\left(g(s)-g(\rho)\right)^{\eta}g'(\rho)]$$

and applying Lemma 1 and then integrating with respect to ρ over $[r,s]$, we have

$$\begin{aligned}
&f\left(\frac{r+s}{2}\right)\int_r^s\exp[\frac{\delta-1}{\delta}\left(g(s)-g(\rho)\right)\left(g(s)-g(\rho)\right)^{\eta}g'(\rho)d\rho\\
\leq&\int_r^s\exp[\frac{\delta-1}{\delta}\left(g(s)-g(\rho)\right)\left(g(s)-g(\rho)\right)^{\eta}g'(\rho)f(\rho)d\rho.
\end{aligned}\tag{40}$$

Using (6), we get

$$f\left(\frac{r+s}{2}\right)\Big[\Gamma(\eta+1)\delta^{\eta}\,{}^{g}_{r}\mathcal{I}^{\eta,\delta}g(s) \tag{41}$$

$$+\frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}^{g}_{r}\mathcal{I}^{\eta+1,\delta}g(s)$$

$$-\exp[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\left(g(s)-g(r)\right)^{\eta}g(r)\Big]$$

$$\leq\Gamma(\eta+1)\delta^{\eta+1}\,{}^{g}_{r}\mathcal{I}^{\eta+1,\delta}f(s). \tag{42}$$

Similarly, multiplying (32) by

$$\exp[\frac{\delta-1}{\delta}\left(g(\rho)-g(r)\right)\left(g(\rho)-g(r)\right)^{\xi}g'(\rho)]$$

and applying Lemma 1 and then integrating with respect to ρ over $[r,s]$, we have

$$f\left(\frac{r+s}{2}\right)\Big[\exp[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)\left(g(s)-g(r)\right)^{\xi}g(s)$$

$$-\Gamma(\xi+1)\delta^{\xi}\,{}^{g}\mathcal{I}^{\xi,\delta}_{s}g(r)$$

$$-\frac{\delta-1}{\delta}\Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}^{\xi+1,\delta}g(r)\Big]$$

$$\leq\Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}^{\xi+1,\delta}_{s}f(r). \tag{43}$$

From (41) and (43), we get

$$f\left(\frac{r+s}{2}\right)\Big[\Gamma(\eta+1)\delta^{\eta}\,{}^{g}_{r}\mathcal{I}^{\eta,\delta}g(s)$$

$$+\frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}^{g}_{r}\mathcal{I}^{\eta+1,\delta}g(s)$$

$$-\Gamma(\xi+1)\delta^{\xi}\,{}^{g}\mathcal{I}^{\xi,\delta}_{s}g(r)-\frac{\delta-1}{\delta}\Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}^{\xi+1,\delta}_{s}g(r)$$

$$-\exp[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)]\left(g(s)-g(r)\right)^{\eta}g(r)$$

$$+\exp[\frac{\delta-1}{\delta}\left(g(s)-g(r)\right)]\left(g(s)-g(r)\right)^{\eta}g(s)\Big]$$

$$\leq\Gamma(\eta+1)\delta^{\eta+1}\,{}^{g}_{r}\mathcal{I}^{\eta+1,\delta}f(s)+\Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}^{\xi+1,\delta}_{s}f(r) \tag{44}$$

Hence, from (39) and (44), we get the desired inequality (33). □

Corollary 5. *By taking $\eta = \xi$ in (33), we get the following generalized proportional fractional integral inequality in general form*

$$
\begin{aligned}
& f\left(\frac{r+s}{2}\right)\Big[\Gamma(\eta+1)\delta^{\eta}\, {}^{g}_{r}\mathcal{I}^{\eta,\delta} g(s) \\
& + \frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\, {}^{g}_{r}\mathcal{I}^{\eta+1,\delta} g(s) \\
& -\Gamma(\eta+1)\delta^{\eta}\, {}^{g}\mathcal{I}^{\eta,\delta}_{s} g(r) - \frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\, {}^{g}\mathcal{I}^{\eta+1,\delta}_{s} g(r) \\
& - \exp[\frac{\delta-1}{\delta}\,(g(s)-g(r))]\,(g(s)-g(r))^{\eta+1}\Big] \\
& \leq \Gamma(\eta+1)\delta^{\eta+1}\, {}^{g}_{r}\mathcal{I}^{\eta+1,\delta} f(s) + \Gamma(\eta+1)\delta^{\eta+1}\, {}^{g}\mathcal{I}^{\eta+1,\delta}_{s} f(r) \\
& \leq \frac{2\exp[\frac{\delta-1}{\delta}\,(g(s)-g(r))]\,(g(s)-g(r))^{\eta}}{s-r} \\
& \quad \times \Big[(s-r)f(s)g(s) - (s-r)f(r)g(r) - (f(s)-f(r)) \\
& \quad \times \int_r^s g(\rho)d\rho\Big].
\end{aligned}
$$

Remark 7. *If we set $\delta = 1$ in Theorem 3, we get integral inequality for Riemann–Liouville fractional integrals proved by ([52], Theorem 3).*

Remark 8. *By setting $g(\rho) = \rho$ and $\delta = 1$ in Theorem 3, we get the integral inequality for classical Riemann–Liouville fractional integrals ([54], Theorem 3).*

4. Applications

In the following, we study some applications of the results obtained in Section 3. In particular, we establish bounds of generalized proportional fractional integrals which contain bounds of all fractional integrals which are given in Remark 2. By applying Theorem 1, we get the following result.

Theorem 4. *Assume that the conditions of Theorem 1 are satisfied, then we have*

$$
\begin{aligned}
& \Gamma(\eta)\delta^{\eta}\, {}^{g}_{r}\mathcal{I}^{\eta,\delta} f(\rho) + \Gamma(\xi)\delta^{\xi}\, {}^{g}\mathcal{I}^{\xi,\delta}_{s} f(\rho) \\
& \leq \exp[\frac{\delta-1}{\delta}\,(g(s)-g(r))] \\
& \times \frac{(g(s)-g(r))^{\eta-1} + (g(s)-g(r))^{\xi-1}}{s-r} \\
& \times \Big[(s-r)f(s)g(s) - (s-r)f(r)g(r) - (f(s)-f(r)) \\
& \times \int_r^s g(\theta)d\theta\Big].
\end{aligned} \tag{45}
$$

Proof. If we set $\rho = r$ and $\rho = s$ in (8), we get the desired result (45). □

Corollary 6. *If we set $\eta = \xi$ in (45), then we get the following generalized proportional fractional integral inequality in general form*

$$
\begin{aligned}
&\Gamma(\eta)\delta^{\eta}\,{}_{r}^{g}\mathcal{I}^{\eta,\delta}f(\rho) + \Gamma(\eta)\delta^{\eta}\,{}^{g}\mathcal{I}_{s}^{\eta,\delta}f(\rho)\\
\leq& 2\frac{\exp[\frac{\delta-1}{\delta}(g(s)-g(r))]\,(g(s)-g(r))^{\eta-1}}{s-r}\\
&\times\Big[(s-r)f(s)g(s)-(s-r)f(r)g(r)-(f(s)-f(r))\\
&\times\int_{r}^{s} g(\theta)d\theta\Big]. \qquad (46)
\end{aligned}
$$

Corollary 7. *If we set $\eta = \delta = 1$ and $g(\rho) = \rho$ in (46), then we get the right Hadamard inequality*

$$
\frac{1}{s-r}\int_{r}^{s} f(\rho)d\rho \leq \frac{f(r)+f(s)}{2}. \qquad (47)
$$

Next, we present the applications of Theorem 2.

Theorem 5. *Assume that the conditions of Theorem 2 are satisfied, then we have*

$$
\begin{aligned}
&\Big|\Gamma(\eta+1)\delta^{\eta}\,{}_{r}^{g}\mathcal{I}^{\eta,\delta}f(\frac{r+s}{2}) + \frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\\
&\times {}_{r}^{g}\mathcal{I}^{\eta+1,\delta}f(\frac{r+s}{2}) + \Gamma(\xi+1)\delta^{\xi}\,{}^{g}\mathcal{I}_{s}^{\xi,\delta}f(\frac{r+s}{2})\\
&+\frac{\delta-1}{\delta}\Gamma(\xi+1)\delta^{\xi+1}\,{}^{g}\mathcal{I}_{s}^{\xi+1,\delta}f(\frac{r+s}{2})\\
&-\exp[\frac{\delta-1}{\delta}(g(\frac{r+s}{2})-g(r))]\left(g(\frac{r+s}{2})-g(r)\right)^{\eta}f(r)\\
&+\exp[\frac{\delta-1}{\delta}(g(s)-g(\frac{r+s}{2}))]\left(g(s)-g(\frac{r+s}{2})\right)^{\xi}f(s)\Big|\\
&\leq\left(\frac{s-r}{2}\right)\frac{1}{2}\Big[\exp[\frac{\delta-1}{\delta}(g(\frac{r+s}{2})-g(r))]\\
&\times\left(g(\frac{r+s}{2})-g(r)\right)^{\eta}|f'(r)|\\
&+\exp[\frac{\delta-1}{\delta}\left(g(s)-g(\frac{r+s}{2})\right)]\\
&\times\left(g(s)-g(\frac{r+s}{2})\right)^{\xi}|f'(s)|\Big]\\
&+|f'(\frac{r+s}{2})|\left(\frac{s-r}{2}\right)\\
&\times\frac{1}{2}\Big[\exp[\frac{\delta-1}{\delta}(g(\frac{r+s}{2})-g(r))]\left(g(\frac{r+s}{2})-g(r)\right)^{\eta}\\
&+\exp[\frac{\delta-1}{\delta}(g(s)-g(\frac{r+s}{2}))]\left(g(s)-g(\frac{r+s}{2})\right)^{\xi}\Big]. \qquad (48)
\end{aligned}
$$

Proof. If we set $\rho = \frac{r+s}{2}$ in (18), we get the desired inequality (48). □

Corollary 8. *If we set $\eta = \xi$ in (48), we get the following inequality*

$$
\begin{aligned}
&\Big|\Gamma(\eta+1)\delta^{\eta}\,{}^{g}_{r}\mathcal{I}^{\eta,\delta}f(\frac{r+s}{2}) \\
&+\frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}^{g}_{r}\mathcal{I}^{\eta+1,\delta}f(\frac{r+s}{2}) \\
&+\Gamma(\eta+1)\delta^{\xi}\,{}^{g}\mathcal{I}^{\eta,\delta}_{s}f(\frac{r+s}{2}) \\
&+\frac{\delta-1}{\delta}\Gamma(\eta+1)\delta^{\eta+1}\,{}^{g}\mathcal{I}^{\eta+1,\delta}_{s}f(\frac{r+s}{2}) \\
&-\exp[\frac{\delta-1}{\delta}(g(\frac{r+s}{2})-g(r))]\left(g(\frac{r+s}{2})-g(r)\right)^{\eta}f(r) \\
&+\exp[\frac{\delta-1}{\delta}(g(s)-g(\frac{r+s}{2}))]\left(g(s)-g(\frac{r+s}{2})\right)^{\eta}f(s)\Big| \\
&\le\left(\frac{s-r}{2}\right)\frac{1}{2}\Big[\exp[\frac{\delta-1}{\delta}(g(\frac{r+s}{2})-g(r))] \\
&\times\left(g(\frac{r+s}{2})-g(r)\right)^{\eta}|f'(r)| \\
&+\exp[\frac{\delta-1}{\delta}(g(s)-g(\frac{r+s}{2}))] \\
&\times\left(g(s)-g(\frac{r+s}{2})\right)^{\eta}|f'(s)|\Big] \\
&+|f'(\frac{r+s}{2})|\left(\frac{s-r}{2}\right)\frac{1}{2}\Big[\exp[\frac{\delta-1}{\delta} \\
&\times(g(\frac{r+s}{2})-g(r))]\left(g(\frac{r+s}{2})-g(r)\right)^{\eta} \\
&+\exp[\frac{\delta-1}{\delta}(g(s)-g(\frac{r+s}{2}))]\left(g(s)-g(\frac{r+s}{2})\right)^{\eta}\Big].
\end{aligned}
\tag{49}
$$

Corollary 9. *If we set $\eta = \delta = 1$ and $g(\rho) = \rho$, then we get the following inequality*

$$
\begin{aligned}
&\Big|\frac{1}{s-r}\int_r^s f(\rho)d\rho-\frac{f(r)+f(s)}{2}\Big| \\
&\le\frac{s-r}{8}\left[|f'(r)|+|f'(s)|+2\Big|f'\left(\frac{r+s}{2}\right)\Big|\right].
\end{aligned}
\tag{50}
$$

5. Concluding Remarks

The generalized proportional fractional integral inequalities for the generalized proportional fractional integrals in general form via convex functions are established in this paper. The obtained results contain a bound for the sum of left and right generalized proportional fractional integrals with dependence on a kernel function and some other inequalities for functions, the absolute values of the derivatives of which are convex. In addition, generalized Hadamard type inequalities for symmetric and convex functions are presented. In particular, these inequalities hold for all the fractional integrals comprises in Remark 2. The inequalities proved in this paper are the generalization of inequalities established earlier by Farid et al. [52] and Farid [54]. In conclusion, one can follow these inequalities to establish further inequalities for other classes of functions related to convex functions by employing generalized proportional fractional integrals.

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