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ORGINAL ARTICLE

Cycle cost considerations in a continuous review inventory control model

Dinçer Konur[a] and Gonca Yıldırım[b]

[a]Department of Computer Information Systems and Quantitative Methods, McCoy College of Business Administration, Texas State University, San Marcos, TX, USA; [b]Department of Industrial Engineering, Cankaya University, Ankara, Turkey

ABSTRACT
In this study, the continuous review order-quantity–re-order point (*Q*, *R*) model is analysed with cycle cost considerations. First, we formulate the maximum cycle cost of a given (*Q*, *R*) policy using a distribution-free approach. Then, two approaches are introduced to minimize the maximum cycle cost: (i) adjusting *R* of a given (*Q*, *R*) policy and (ii) designing a new (*Q*, *R*) policy. Optimum inventory control decisions are characterized for each approach. A set of numerical studies is presented to compare the outcomes of both approaches to three long-term cost minimization approaches, namely the cost minimizing (*Q*, *R*) policy, the distribution-free minmax (*Q*, *R*) policy, and the distribution-free (*Q*, *R*) policy based on the maximum entropy principle. Our numerical results demonstrate the viability of the two approaches introduced and discuss implications of penalty costs and lead time demand's coefficient of variation. Later, we formulate a bi-objective model with the objectives of expected cost and maximum cycle cost minimizations and propose a bi-directional method to approximate the set of Pareto efficient solutions. Numerical examples are presented to illustrate the algorithm and demonstrate the Pareto front.

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Inventory; stochastic demand; cycle cost; multi-objective

1. Introduction and literature review

Inventory is present in almost any business, and efficient inventory control is crucial for the good financial standing of a company. In most practical settings, inventory management is typically challenged by the uncertain demand, and this demand uncertainty directly translates to the inventory related costs in the long run as well as inventory costs incurred recurrently in short intervals. This study revisits a well-known continuous review inventory control model under stochastic demand, namely order-quantity–re-order point model. A common approach for making inventory control decisions in continuous review systems is to minimize the long-term expected cost per unit time. This approach, nevertheless, ignores the short-term costs incurred recurrently. In this study, we incorporate short-term costs by taking cycle costs into account in designing an inventory control policy under a continuous review inventory control system with stochastic demand.

Particularly, consider a retailer who manages the inventory of a single product, for which the demand is stochastic. We assume that the inventory is continuously reviewed and the retailer adopts a (Q, R) policy, where $Q > 0$ is the order quantity in each replenishment and $R > 0$ is the re-order point to initiate an order. That is, whenever the on-hand inventory is R, an order of Q units is placed. The basic settings of continuous review inventory control systems are as follows (see, e.g., Hadley & Whitin, 1963). The demand rate obeys a continuous probability distribution with a constant mean $\lambda > 0$ and there is a fixed lead time for receiving an order, denoted by $\tau > 0$. The lead time demand is a random variable, denoted by D, and let $f(D)$ and $F(D)$ be the lead time demand's probability density function and cumulative density function, respectively, such that $\mu > 0$ and $\sigma > 0$ define the mean and the standard deviation of the lead time demand, respectively. Finally, there is at most one outstanding order at any time and we consider the case where all of the shortages are backordered.

(Q, R) policy for inventory systems with stochastic demand has been vastly investigated in the literature. We refer the reader to Hadley and Whitin (1963), Silver, Pyke, and Peterson (1998), and Nahmias (2009) for classical results and discussion. Under the basic settings, the retailer is subject to inventory holding, order setup, and shortage costs. In particular, let $h > 0$ denote the inventory holding cost per unit per unit time, $K > 0$ denote the setup cost per order, and $p > 0$ denote the penalty cost per unit short. Note that the shortages can only be observed during the lead time, and the number of shortages observed in one replenishment cycle

CONTACT Dinçer Konur d_k141@txstate.edu Department of Computer Information Systems and Quantitative Methods, McCoy College of Business Administration, Texas State University, 454 McCoy, 601 University Dr., San Marcos, TX 78666, USA.
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(simply referred to as *cycle* hereafter) depends on the re-order point R. Let $n(R)$ denote the expected number of shortages within a cycle. It is well known that under a (Q, R) model, the retailer's expected cost per unit time, denoted by $EC(Q, R)$, is (see also Hadley & Whitin, 1963)

$$EC(Q,R) = h\left(R - \mu + \frac{Q}{2}\right) + \frac{K\lambda}{Q} + \frac{p\lambda n(R)}{Q}. \quad (1)$$

The first, second, and third terms in Equation (1) define the expected inventory holding, order setup, and penalty costs per unit time, respectively. Main notation used throughout the paper is summarized in the online supplement (Section A) and additional notation is defined as needed.

Given the lead time demand distribution (i.e., $f(D)$ and $F(D)$), Hadley and Whitin (1963) proposed an iterative algorithm to approximate the minimizer of $EC(Q, R)$. This algorithm is presented in the online supplement (Section B) and we accept the output of this algorithm as the cost minimizing (Q, R) policy, denoted by (Q^C, R^C). It should be noted that there is a significant number of studies analyzing the characteristics of the optimal (Q, R) policy under various distributions (see, e.g., Braglia, Castellano, & Gallo, 2016; Burgin, 1975; C. Das, 1976; Halkos, Kevork, & Tziourtzioumis, 2018; Rossetti & Ünlü, 2011; Tyworth & Ganeshan, 2000; Tyworth, Guo, & Ganeshan, 1996; Tyworth & O'Neill, 1997; Vasconcelos & Marques, 2000). It is also worthwhile to note that there are many extensions of the basic (Q, R) model to settings with including but not limited to resource/service constraints (see, e.g., Aardal, Jonsson, & Jonsson, 1989; Hariga, 2010), quantity discounts (see, e.g., Tamjidzad & Mirmohammadi, 2015, 2017), multiple supply sources/transportation modes (see, e.g., Dullaert, Maes, Vernimmen, & Witlox, 2005; Moinzadeh & Nahmias, 1988; Sculli & Shum, 1990; Sculli & Wu, 1981), lead time crashing/order expediting (see, e.g., Bookbinder & Çakanyildirim, 1999; Chiang, 2010; Duran, Gutierrez, & Zequeira, 2004), additional objectives (see, e.g., Fattahi, Hajipour, & Nobari, 2015; Konur, Campbell, & Monfared, 2017; Schaefer & Konur, 2015), and deteriorating items with or without mixture of backorders and lost sales (see, e.g., Braglia, Castellano, Marrazzini, & Song, 2019; Braglia, Castellano, & Song, 2018; Chiu, 1995; Nahmias, 1981; Nahmias & Wang, 1979; Olsson, 2014; Uthayakumar & Parvathi, 2009). An alternative solution approach proposed, especially for settings with perishable items, is optimization via simulation (see, e.g., Braglia et al., 2019 for successful implementations; Chiu, 1995; Nahmias, 1981; Nahmias & Wang, 1979; Olsson, 2014). This study assumes the aforementioned basic settings (several extensions and modifications of some of our results are noted in Section 6); therefore, Hadley and Whitin's (1963) iterative algorithm is used to determine (Q^C, R^C) (in Section 5, we present the simulated outcomes of several (Q, R) policies with examples). Furthermore, we introduce new approaches to design a (Q, R) policy so as to avoid extremely high cycle costs that might be realized. To do so, we model the maximum cycle cost possible using a distribution-free approach.

One significant extension of the classical (Q, R) model addresses the case of unknown lead time demand distribution. Considering that the lead time demand distribution is not always known in practice, Gallego (1992) introduced a distribution-free minmax procedure for the (Q, R) model. This procedure, building on the approach of Scarf (1958), finds the (Q, R) policy that minimizes the maximum expected cost per unit time, which is realized under the worst possible lead time demand distribution fitting the given mean and standard deviation of the lead time demand. We refer the reader to Gallego (1992) for a description of this procedure and we denote the (Q, R) policy corresponding to the Gallego's (1992) distribution-free minmax approach by (Q^G, R^G). There is a significant number of studies building upon Gallego's (1992) approach for various (Q, R) models. Moon and Choi (1994), Agrawal and Seshadri (2000), and Tajbakhsh (2010) study the distribution-free (Q, R) model with a service level constraint. Moon and Gallego (1994), Chu (1999), and Achary and Geetha (2001) analyse the case with a mixture of backorders and lost sales. Moon, Shin, and Sarkar (2014) consider lead time crashing with a service level constraint and Shin, Guchhait, Sarkar, and Mittal (2016) consider lead time crashing, service level constraint, and transportation discounts. Several other extensions include multiple constraints with lead time crashing (Gholami-Qadikolaei & Mirzazadeh, 2013), two-echelon model with lead time crashing (Gutgutia & Jha, 2018), and production planning with fuzzy demand (Kumar & Goswami, 2015). We note that the distribution-free minmax approach has also been extensively studied for newsvendor models (see, e.g., Gallego & Moon 1993 for a review).

Another distribution-free approach utilizes the maximum entropy principle. Under the maximum entropy principle, the distribution with the maximum entropy is determined; and then, this distribution is used to find the corresponding optimum inventory decision. The maximum entropy principle has been typically employed for newsvendor models in the inventory management literature (see, e.g., Andersson, Jörnsten, Nonås, Sandal, & Ubøe, 2013; Fleischhacker & Fok, 2015a, 2015b; Guo, Chen, Wang, Yang, & Zhang, 2019; Han, Du, & Zuluaga, 2014; Ninh, Hu, & Allen, 2019; Perakis & Roels, 2008; Saghafian & Tomlin, 2016; C. X. Wang, Webster, & Suresh, 2009). To the best of our knowledge, the maximum entropy principle has been only recently adapted for a (Q, R) model by Castellano (2016). The reader is referred to Castellano (2016) for an overview of the maximum entropy principle applied to a (Q, R) model, where the author considers that the lead time demand's mean and standard deviation are known and its support is non-negative (i.e., $[0, \infty)$). In this study, we denote the (Q, R) policy minimizing the expected cost per unit time under the maximum entropy distribution by (Q^E, R^E) and explain the details of how (Q^E, R^E) is determined for our settings in Section 4 (and Section E of the online supplement).

The expected cost minimization approaches (i.e., cost minimizing (Q^C, R^C) policy and the distribution-free (Q^G, R^G) and (Q^E, R^E) policies) focus on minimizing long-term costs, i.e., the retailer's expected cost per unit time considering an infinite planning horizon. While this might be attractive for a retailer, an approach minimizing the expected cost per unit time ignores the costs incurred at the short-term operational level. For instance, Archibald, Thomas, Betts, and Johnston (2002) note that, rather than maximizing long-term economic performance, a start-up company might focus on short-term financial performance in order to maximize their survival probability. Specifically, because the demand is random, the cycle cost incurred is also random; and this uncertainty can burden a retailer with financial risks such as extremely high costs during some cycles.

Indeed, there is a significant number of studies investigating inventory control models with financial considerations. Typically, these studies focus on a capital-constrained retailer's financing her inventory related costs through her suppliers or other sources. For instance, single-echelon Economic Order Quantity models have been investigated with supplier/trade-credit financing and/or delayed payments (see, e.g., Aggarwal & Jaggi, 1995; Carlson, Miltenburg, & Rousseau, 1996; Chand & Ward, 1987; Chang, 2004; Feng, Li, & Zhao, 2013; Goyal, 1985; Haley & Higgins, 1973; Huang, 2004, 2007; Jamal, Sarker, & Wang, 1997; Kingsman, 1991; Luo & Huang, 2002; Mahata, 2012; Rachamadugu, 1989; Silver & Costa, 1998; Taleizadeh, Pentico, Jabalameli, & Aryanezhad, 2013; Taleizadeh, Pourmohammad-Zia, & Konstantaras, 2019; Teng, 2002; Teng & Chang, 2009; Teng, Min, & Pan, 2012). Similarly, multi-period (lot-sizing) or infinite horizon (periodic review) inventory control models have been integrated with various tools that a retailer can use to finance her inventory related costs (see, e.g., Chao, Chen, & Wang, 2008; Gao, Zhao, & Geng, 2014; Gong, Chao, & Simchi-Levi, 2014; Gupta & Wang, 2009; Hu & Sobel, 2007; Hu, Sobel, & Turcic, 2010; Katehakis, Melamed, & Shi, 2016; L. Li, Shubik, & Sobel, 2013; Maddah, Jaber, & Abboud, 2004; Protopappa-Sieke & Seifert, 2010; Xu & Birge, 2006). In addition, a diverse set of newsvendor models has been analysed for two-echelon supply chains with detailed financial implications of the decisions made by the parties at different echelons (see, e.g., Cai, Chen, & Xiao, 2014; Dada & Hu, 2008; Kouvelis & Zhao, 2012, 2016; Lai, Debo, & Sycara, 2009; Lee & Rhee, 2011; B. Li, An, & Song, 2018; Raghavan & Mishra, 2011; Reindorp, Tanrisever, & Lange, 2018; Xiao, Sethi, Liu, & Ma, 2017; Yan, He, & Liu, 2019; S. A. Yang & Birge, 2018). We refer the reader to L. Zhao and Huchzermeier (2015), Xu et al. (2018), and Chakuu, Masi, and Godsell (2019) for recent reviews of studies focusing on various aspects of supply chain finance.

The studies mentioned above emphasize the risks imposed by a retailer's inventory related decisions on her financial standing. Particularly, it has been discussed that unsold inventory or unmet demand due to the stochastic nature of the demand can result in serious financial risks (see, e.g., Bogataj & Bogataj, 2007; Christopher & Lee, 2004; Ghadge, Dani, & Kalawsky, 2012). For instance, Cisco had a dramatic financial loss in 2001 because of unsold inventory (see, e.g., Christopher & Lee, 2004; Lai, Debo, & Sycara, 2009; Supply Chain Digest, 2006) and Apple lost market share and faced financial problems because of unmet demand in 1995 (see, e.g., Supply Chain Digest, 2006). Some of the aforementioned studies even consider possibility of a retailer's bankruptcy due to inventory related decisions (see, e.g., Cai et al., 2014; Chao, Chen, & Wang, 2008; Gong et al., 2014; Hu & Sobel, 2007; Hu et al., 2010; Kouvelis & Zhao, 2012, 2016; Lai et al., 2009; Lee & Rhee, 2011; B. Li et al., 2018; L. Li et al., 2013; Raghavan & Mishra, 2011; Xu & Birge, 2006; Yan et al., 2019).

Inventory related financial risks arise because of the uncertainty in a retailer's ability to pay the credit repayments, interests on the credit, or potential penalties. For instance, capital-constrained retailers might use short-term loans to finance their inventory (Lai, Debo, & Sycara, 2009) and repayments rest on demand realization (see, e.g., S. A. Yang & Birge, 2018). These financial risks necessitate taking into account cycle costs in managing a continuous review inventory control model. Managing cycle costs can help the retailer accumulate her available capital, which then can be allocated for her payments. This, in return, will improve the retailer's short-term financial standing. While we do not explicitly include any financial considerations in our models, we address the need for cycle cost management with two approaches: (i) adjusting the re-order point R of a given (Q, R) policy and (ii) designing a new (Q, R) policy so as to minimize the maximum cycle cost possible.

In particular, we first formulate the cycle cost and then characterize the maximum cycle cost function using a distribution-free approach in Section 2. After that, the two approaches taking the maximum cycle cost into account are discussed in detail. One of the reasons why a retailer might prefer adjusting R (i.e., approach (i)) rather than adjusting both Q and R (i.e., approach(ii)) is the potential increase in long-term average costs. Particularly, switching from a long-term cost minimizing (Q, R) policy (such as (Q^C, R^C), (Q^G, R^G), or (Q^E, R^E)) to a cycle cost minimizing (Q, R) policy might significantly increase the expected cost per unit time. It can be the case that such a drastic change is not preferred by a retailer. On the other hand, keeping Q fixed, for instance the Q of a (Q, R) policy minimizing long-term costs and then adjusting R accordingly (i.e., using approach (i)) does not completely ignore the long-term costs. These are illustrated in our numerical analyses in Section 4. Another reason why approach (i) can be preferred is that, in practice, Q might have binding restrictions considering the retailer's storage space, transportation capacity, capital availability, or relations with suppliers. Therefore, adjusting R can be considered as an easier modification than modifying both Q and R of a (Q, R) policy, which is currently being used. In Section 3, we characterize the optimal solutions under both approaches and discuss their implications.

For several lead time demand distributions, Section 4 compares various strategies through an extensive numerical study. Specifically, we investigate the effects of the penalty cost per unit short and the lead time demand's coefficient of variation on how the expected cost per unit time and the maximum cycle cost possible compare under different (Q, R) policies. Our results indicate that, in some settings, adjusting the R of the cost minimizing (Q, R) policy can significantly reduce the maximum cycle cost with a relatively small increase in the expected cost per unit time. It is also important to note that approach (ii) is a distribution-free procedure. Furthermore, when approach (i) is used on a (Q, R) policy determined via a distribution-free procedure, approach (i) is also a distribution-free procedure. Our numerical results indicate that, under some settings, approach (i), when applied on a distribution-free cost minimizing (Q, R) policy (i.e., (Q^G, R^G) or (Q^E, R^E)), might reduce both the expected cost per unit time and the maximum cycle cost after lead time demand distribution is revealed.

It can be noticed that a (Q, R) policy minimizing the maximum cycle cost possible based on approach (ii) ignores the long-term expected costs; and, as noted previously, a (Q, R) policy based on the minimization of the expected cost per unit time ignores the cycle costs. In this sense, using approach (i) with a (Q, R) policy based on minimizing the expected cost per unit time is a relatively moderate strategy. Indeed, in most cases, our numerical results presented in Section 4 agree with this. However, approach (i) gives the retailer only a single alternative (Q, R) policy to balance the expected cost per unit time and the maximum cycle cost. Furthermore, based on our numerical analysis, it is also possible that approach (ii) can be Pareto superior to approach (i), i.e., the (Q, R) policy generated by approach (ii) may result in lower expected cost per unit time in addition to lower maximum cycle cost. Therefore, to give a retailer more alternative policies that consider both the long-term expected costs and the short-term cycle costs, we introduce a bi-objective (Q, R) model in Section 5. We note that there are several multi-objective (Q, R) models in the literature (see, e.g., Fattahi et al., 2015; Konur et al., 2017; Schaefer & Konur, 2015). Section 5 presents the bi-objective (Q, R) model with the objectives of minimizing the expected cost per unit time and the maximum cycle cost possible, proposes a method to approximate its Pareto front, and demonstrates its application with several numerical examples.

To the best of our knowledge, this is the first study to incorporate cycle cost and its uncertainty in designing a (Q, R) policy. This study contributes to the literature by introducing approaches to account for maximum cycle costs in a continuous review inventory control system and characterizing the optimal policy decisions under these approaches. A set of numerical analyses documents the practical viability of the approaches introduced and discusses several managerial insights. The rest of the paper is organized as follows. Section 2 formulates the maximum cycle cost function and the optimal decisions under the two approaches are characterized in Section 3. The numerical studies are summarized in Section 4. Section 5 presents the bi-objective model, a solution method, and several examples. Concluding remarks and future research directions are noted in Section 6. An online supplement is provided and it includes the following: notation table (Section A); Hadley and Whitin's (1963) iterative algorithm (Section B); proofs of the propositions and theorems (Section C); details of problem instance generation for the problem instances used in Section 4 (Section D); details of the maximum entropy principle used to determine (Q^E, R^E) (Section E); description of a supplementary procedure and simulation process used in Section 5 (Section F); figures used for the analyses of cycle cost considerations on $(Q^C, R^C), (Q^G, R^G)$, and (Q^E, R^E) in Sections 4.1, 4.2, and 4.3, respectively (Sections G, H, and I, respectively); detailed tables of the numerical analyses in Sections 4.1, 4.2, and 4.3, respectively (Sections J, K, and L, respectively); and the sketch of proofs of analytical results in Sections 2 and 3 when p is defined per unit short per unit time (Section M).

2. Maximum cycle cost modelling

In this section, we formulate the maximum cycle cost function. In addition to the aforementioned basic settings in Section 1, the following assumptions are made:

Assumption 1. (i) $D \in \mathbf{d}$, where $\mathbf{d} = [d_\ell, d_u]$ such that $0 < d_\ell < d_u$. (ii) $Q \geq d_u$. (iii) $R \in \mathbf{d}$.

Assumption 1(i) suggests a specific range for the lead time demand between a lower limit, denoted by d_ℓ, and an upper limit, denoted by d_u. A similar assumption is also used in newsvendor settings by Vairaktarakis (2000), Andersson et al. (2013), and Guo et al. (2019). By definition, the demand should be non-negative; and in most practical scenarios, there will be a lower bound on the lead time demand that can be observed (we assume that $d_\ell > 0$ to avoid division by zero in our models). On the other hand, even if it is practically unlikely, when there is no upper bound on the lead time demand, it might be the case that the retailer targets a range of demand within specific probabilities in her inventory planning; hence, d_ℓ and d_u can be defined accordingly. Also, since $d_\ell = d_u$ implies deterministic lead time demand, which is not the scope of this study, we consider that $d_\ell < d_u$. We give further discussion on defining d_ℓ and d_u in our numerical analysis in Section 4. Recall that there is at most one outstanding order in the basic (Q, R) settings, which is guaranteed when $Q \geq d_u$; hence, we have Assumption 1(ii) so that, for any lead time demand realization, the on-hand inventory is guaranteed to be greater than or equal to R when an order is received (see, e.g., Bookbinder & Çakanyildirim, 1999; Chiang, 2010; Halkos et al., 2018; Konur et al., 2017). Note that assuming $R \leq d_u$ is reasonable because a rational retailer will not pay unnecessary inventory holding costs by having $R > d_u$; therefore, we have $R \leq d_u$ as stated in Assumption 1(iii). Furthermore, in most settings, it is considered that $R \geq \mu$ to guarantee non-

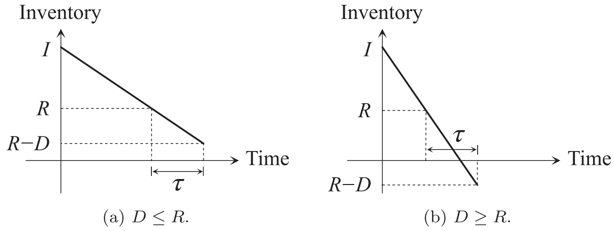


Figure 1. Illustration of two cases during lead time.

negative safety stock (see, e.g., Aardal et al., 1989; Hariga, 2010) because negative safety stock is not practically preferred as backordering is typically more costly than inventory holding in practice and/or low service levels are unlikely in practical settings (see, e.g., Tyworth & O'Neill, 1997). Assumption 1(iii) makes it possible to have a negative safety stock, if desired (our results can be easily modified for the case $R \geq \mu$).

Similar to the calculation of $EC(Q, R)$, when the inventory is considered to stationarily decrease from the beginning inventory to the ending inventory within a cycle, we can express the cycle cost in terms of beginning inventory, denoted by I, and lead time demand D. Note that the minimum (maximum) beginning inventory of a cycle is realized when the previous cycle's lead time demand is maximum (minimum); therefore, the beginning inventory $I \in \mathbf{i}$, where $\mathbf{i} = [i_\ell, i_u]$ such that $i_\ell = R - d_u + Q$ and $i_u = R - d_\ell + Q$. Furthermore, the ending inventory will be $R - D$ and two cases are possible: (a) $D - R \leq 0$ and (b) $D - R \geq 0$. These two cases are illustrated in Figure 1.

In case (a), the retailer carries unnecessary inventory and her cycle cost as a function of I and D for a given (Q, R) policy, denoted by $C_1(I, D|Q, R)$, amounts to

$$C_1(I, D|Q, R) = \frac{h\tau}{2D}(I^2 - R^2) + \frac{h\tau}{2}(2R - D) + K, \tag{2}$$

where the first term is the inventory holding cost from the beginning of the cycle until order initiation (i.e., until on-hand inventory reaches R), the second term is the inventory holding cost during the lead time, and the last term is the order setup cost. Note that, when $D - R \leq 0$ as in case (a), there is no penalty cost in the cycle. On the other hand, in case (b), the retailer pays inventory carrying and penalty costs due to the shortages and her cycle cost as a function of I and D for a given (Q, R) policy, denoted by $C_2(I, D|Q, R)$, amounts to

$$C_2(I, D|Q, R) = \frac{h\tau}{2D}(I^2 - R^2) + \frac{h\tau}{2D}R^2 + p(D - R) + K, \tag{3}$$

where the first term is the inventory holding cost from the beginning of the cycle until order initiation, the second term is the inventory holding cost during the lead time, the third term is the total penalty cost, and the last term is the order setup cost. It then follows from Equations (2) and (3) that the cycle cost as a function of I and D for a given (Q, R) policy, denoted by $CC(I,D|Q,R)$, reads as

$$CC(I,D|Q,R) = \begin{cases} C_1(I,D|Q,R) & \text{if } D \le R, \\ C_2(I,D|Q,R) & \text{if } D \ge R. \end{cases} \quad (4)$$

The maximum cycle cost will depend on the realizations of I and D. Let $MC(Q, R)$ denote the maximum cycle cost possible for a (Q, R) policy. $MC(Q, R)$ reads as

$$MC(Q,R) = \max_{I\in\mathbf{i},\,D\in\mathbf{d}} \{CC(I,D|Q,R)\}, \quad (5)$$

where $CC(I,D|Q,R)$ is defined in Equation (4). Next proposition gives an explicit characterization of $MC(Q, R)$.

Proposition 2.1. $MC(Q,R) = \max\{C_1(i_u,d_\ell|Q,R), C_2(i_u,d_u|Q,R)\}$.

Proposition 2.1 implies that, for a given (Q, R) policy, the maximum cycle cost is realized when the beginning inventory is at its maximum possible (i.e., when $I = i_u$) and the lead time demand is equal to either its lower or its upper limit (i.e., $D = d_\ell$ or $D = d_u$). Indeed, as aforementioned, from the moment the cycle starts until the inventory reaches R, the retailer pays only inventory holding costs, which is maximized when the beginning inventory is maximized. However, during the lead time, the retailer might incur penalty costs in addition to inventory holding costs. If it is more costly to hold inventory, then the cycle cost is maximized when the inventory held is maximized, which happens when the ending inventory is maximized (i.e., when the lead time demand is minimum). On the other hand, if it is more costly to pay penalties, then the cycle cost is maximized when the number of shortages is maximized, which happens when the lead time demand is maximum.

Considering Proposition 2.1, Equations (2)–(4), and recalling that $i_u = R - d_\ell + Q$, we can reformulate $MC(Q, R)$ as $MC(Q,R) = \max\{G_1(Q,R), G_2(Q,R)\}$, where

$$G_1(Q,R) = \left(R - d_\ell + \frac{Q}{2}\right)\frac{h\tau Q}{d_\ell} + K, \quad (6)$$

$$G_2(Q,R) = (R - d_\ell + Q)^2 \frac{h\tau}{2d_u} + p(d_u - R) + K. \quad (7)$$

Proposition 2.1 and properties of Equations (6) and (7) are utilized in the next section, where we discuss how to optimally adjust the re-order quantity R of a given (Q, R) policy and how to optimally design a new (Q, R) policy so as to minimize the maximum cycle cost possible.

3. Maximum cycle cost minimization

Recall that, in Section 1, two approaches are introduced to account for the maximum cycle cost possible. The first approach modifies the re-order quantity R of a given (Q, R) policy so as to minimize the maximum cycle cost possible. Let $R^m(Q)$ be the optimally adjusted R under the first approach. Note that $R^m(Q) = \text{argmin}_{R\in\mathbf{d}}\{MC(Q,R|Q)\}$ and it is assumed that $Q \ge d_u$. The second approach determines a new (Q, R) policy so as to minimize the maximum cycle cost possible. Let (Q^M, R^M) be the minimizer of the maximum cycle cost. Note that $(Q^M, R^M) = \text{argmin}_{Q\ge d_u,\,R\in\mathbf{d}}\{MC(Q,R)\}$. It is straightforward to notice that $R^M = R^m(Q^M)$. In what follows, we first characterize $R^m(Q)$ and then determine (Q^M, R^M). For notational brevity, we let $y(Q) = y(Q,R|R)$ and $y(R) = y(Q,R|Q)$ for a function $y(Q, R)$.

3.1. Re-order point adjustment

Suppose that a (Q, R) policy is given such that $Q \ge d_u$. It can be noticed from Equation (6) that $\frac{dG_1(R)}{dR} = \frac{h\tau Q}{d_\ell} > 0$, that is, $G_1(R)$ is an increasing linear function of R. Furthermore, it follows from Equation (7) that $\frac{dG_2^2(R)}{dR^2} = \frac{h\tau}{d_u} > 0$, that is, $G_2(R)$ is strictly convex with respect to R, and r_0 uniquely minimizes $G_2(R)$ (i.e., $G_2(R)$ is decreasing with R for $R \le r_0$ and increasing with R for $R \ge r_0$) such that

$$r_0 = \frac{pd_u}{h\tau} + d_\ell - Q. \quad (8)$$

These, along with $G_2(R)$ being a quadratic function of R, imply that $G_1(R) = G_2(R)$ only when $R = r_\ell$ and $R = r_u$ such that

$$r_\ell = Q\frac{(d_u - d_\ell)}{d_\ell} - d_\ell + \frac{pd_u}{h\tau} - d_u\sqrt{\left(\frac{p}{h\tau} + \frac{Q}{d_\ell}\right)^2 - \frac{2p}{h\tau}\left(1 + \frac{(Q-d_\ell)}{d_u}\right) - \frac{Q^2}{d_\ell d_u}}, \quad (9)$$

$$r_u = Q\frac{(d_u - d_\ell)}{d_\ell} - d_\ell + \frac{pd_u}{h\tau} + d_u\sqrt{\left(\frac{p}{h\tau} + \frac{Q}{d_\ell}\right)^2 - \frac{2p}{h\tau}\left(1 + \frac{(Q-d_\ell)}{d_u}\right) - \frac{Q^2}{d_\ell d_u}}. \quad (10)$$

Equations (9) and (10) follow from the quadratic formula. Given that r_ℓ and r_u exist, the aforementioned characteristics of $G_1(R)$ and $G_2(R)$ imply that $G_1(R) \ge G_2(R)$ for $R \in [r_\ell, r_u]$ and $G_1(R) < G_2(R)$ for $R \notin (r_\ell, r_u)$. These are illustrated in Figure 2 for two possible cases: (a) $r_0 < r_\ell$ and (b) $r_\ell \le r_0 \le r_u$

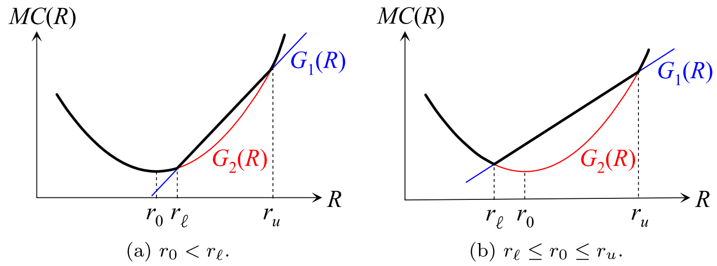


Figure 2. Maximum cycle cost realizations as a function of *R*.

(note that, $r_0 > r_u$ is not possible when r_ℓ and r_u exist since $G_1(R)$ is increasing).

In the next proposition, we show that r_ℓ and r_u do exist (i.e., they are real-valued) and compare them with d_ℓ and d_u.

Proposition 3.1. *(i)* $\exists r_\ell \in \mathbb{R}$ *and* $\exists r_u \in \mathbb{R}$ *such that* $G_1(r_\ell) = G_2(r_\ell)$ *and* $G_1(r_u) = G_2(r_u)$*; and (ii)* $r_\ell < d_u < r_u$*. Furthermore, (iii) if* $h\tau Q^2 \geq 2pd_u d_\ell$*, then* $r_\ell \leq d_\ell$*; and (iv) if* $h\tau Q^2 < 2pd_u d_\ell$*, then* $d_\ell < r_\ell$*.*

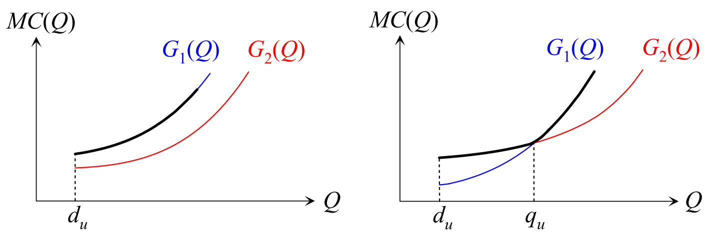
Proposition 3.1(i) then implies that $MC(R) = G_1(R)$ for $R \in [r_\ell, r_u]$ and $MC(R) = G_2(R)$ for $R \notin (r_\ell, r_u)$. Furthermore, Proposition 3.1(ii)-(iv) indicate that there are two possible cases: $r_\ell \leq d_\ell < d_u < r_u$ when $h\tau Q^2 \geq 2pd_u d_\ell$ and $d_\ell < r_\ell < d_u < r_u$ when $h\tau Q^2 < 2pd_u d_\ell$. Proposition 3.1 suggests evaluating $R^m(Q)$ in these two cases. Next theorem characterizes $R^m(Q)$ as well as $MC(R^m(Q))$ for each case.

Theorem 3.2. *(i) If* $h\tau Q^2 \geq 2pd_u d_\ell$*, then* $R^m(Q) = d_\ell$ *and* $MC(R^m(Q)) = G_1(d_\ell)$*. (ii) If* $h\tau Q^2 < 2pd_u d_\ell$*, then* $R^m(Q) = r_\ell$ *and* $MC(R^m(Q)) = G_2(r_\ell)$ *when* $r_\ell \leq r_0$*, and* $R^m(Q) = \max\{r_0, d_\ell\}$ *and* $MC(R^m(Q)) = G_2(\max\{r_0, d_\ell\})$ *when* $r_\ell > r_0$*.*

It can be noted from Theorem 3.2 that $R^m(Q) < d_u$ because $R^m(Q) = d_\ell < d_u$ when $h\tau Q^2 \geq 2pd_u d_\ell$ and $R^m(Q) \leq r_\ell < d_u$ when $h\tau Q^2 < 2pd_u d_\ell$. Note that setting $R = d_u$ guarantees that there will be no shortages in any cycle. Therefore, Theorem 3.2 implies that the retailer will not consider to have a *no-shortages-at-all* policy (i.e., a policy indicating no shortages ever) in order to minimize the maximum cycle cost given $Q \geq d_u$. Also, $R^m(Q) \geq d_\ell$ by definition, i.e., the retailer does not adopt a policy indicating *guaranteed-shortages-at-every-cycle*, which is not practical as backorders are typically more costly and/or less preferred than holding inventories in practice. Nevertheless, these results do not specify that, under the optimum *R* adjustment for the given (*Q*, *R*) policy, the maximum cycle cost is observed in a cycle with or without shortages. Indeed, it is possible that the minimized maximum cycle cost is observed in either cycle (with or without shortages). In what follows, we discuss several key insights from Theorem 3.2 on adjusting *R* of a given (*Q*, *R*) policy in order to minimize the maximum cycle cost possible.

In particular, let us consider the penalty cost per unit short (p) vs. inventory holding cost per unit per unit time (h). Suppose that $p \ll h$, i.e., the penalty cost per unit short is significantly small compared to the inventory holding cost per unit per unit time, such that $h\tau Q^2 \geq 2pd_u d_\ell$. In such a situation, one would expect the maximum cycle cost to be observed in a cycle with no shortages. Indeed, if we consider that $h\tau Q^2 \geq 2pd_u d_\ell$ when $p \ll h$, it can be shown that $MC(R) = G_1(R)$ (see, e.g., the proof of Theorem 3.2) and recall that $G_1(R)$ defines the cycle cost for a cycle with no shortages. Knowing this, the retailer would then aim to minimize the inventory holding cost during a cycle so as to minimize the maximum cycle cost possible; therefore, the retailer is expected to reduce *R* as much as possible and the minimum *R* value is d_ℓ. Theorem 3.2 validates this discussion as it states that $R^m(Q) = d_\ell$ when $h\tau Q^2 \geq 2pd_u d_\ell$.

Next, suppose that $p \gg h$, i.e., the penalty cost per unit short is significantly large compared to the inventory holding cost per unit per unit time, such that $h\tau Q^2 < 2pd_u d_\ell$. In such a situation, however, we cannot directly say that the maximum cycle cost is expected to be observed in a cycle with or without shortages because the possible number of shortages can be relatively small. Indeed, since $d_\ell < r_\ell$ when $h\tau Q^2 < 2pd_u d_\ell$, regardless of how large p is, it is possible that the maximum cycle cost is observed in a cycle with or without shortages. Particularly, for relatively large *R* values (for $R \geq r_\ell$ to be specific), it can be noticed that $MC(R) = G_1(R)$, i.e., when $R \geq r_\ell$, the maximum cycle cost is observed in a cycle without a shortage. Knowing that, if the retailer is restricted to set $R \geq r_\ell$, she would set $R = r_\ell$ to minimize the inventory holding costs, which would minimize the maximum cycle cost given the restriction $R \geq r_\ell$. On the other hand, for relatively small *R* values (for $R \leq r_\ell$ to be specific), it can be noticed that $MC(R) = G_2(R)$, which defines the cycle cost for a cycle with shortages. Knowing that, the retailer's optimum *R* adjustment would be to minimize $G_2(R)$ over $d_\ell \leq R \leq r_\ell$ given that she is restricted to have $R \leq r_\ell$. Nonetheless, what Theorem 3.2(ii) indicates that, when $p \gg h$ such that $h\tau Q^2 < 2pd_u d_\ell$, the maximum cycle cost is observed in a cycle with shortages under the



(a) $G_2(q_0) \geq G_1(q_0)$ and $q_u \geq d_u$. (b) $G_2(q_0) \geq G_1(q_0)$ and $q_u < d_u$, or $G_2(q_0) < G_1(q_0)$.

Figure 3. Maximum cycle cost realizations as a function of Q.

optimum R adjustment minimizing the maximum cycle cost. That is, $MC(R^m(Q)) = G_2(R^m(Q))$ when $h\tau Q^2 < 2pd_u d_\ell$. Similar to the preceding discussion, we can evaluate the effect of Q in the retailer's optimum R adjustment. In particular, indications of Theorem 3.2 for a relatively large (small) Q such that $h\tau Q^2 \geq 2pd_u d_\ell$ ($h\tau Q^2 < 2pd_u d_\ell$) are analogous to the relatively low (high) penalty cost per unit short.

3.2. (Q, R) policy design

Here, we characterize (Q^M, R^M). Similar to the definition of $R^m(Q)$, let us first define $Q^m(R) = \text{argmin}_{Q \geq d_u}\{MC(Q)\}$ given that $R \in \mathbf{d}$. Note that $MC(Q) = \max\{G_1(Q), G_2(Q)\}$. It can be noticed from Equations (6) and (7) that $\frac{dG_1^2(Q)}{dQ^2} = \frac{h\tau}{d_\ell} > 0$ and $\frac{dG_2^2(Q)}{dQ^2} = \frac{h\tau}{d_u} > 0$, that is, both $G_1(Q)$ and $G_2(Q)$ are strictly convex with respect to Q. Furthermore, $\frac{dG_1(Q)}{dQ} = \frac{h\tau}{d_\ell}(R - d_\ell + Q)$ and $\frac{dG_2(Q)}{dQ} = \frac{h\tau}{d_u}(R - d_\ell + Q)$ imply that q_0 uniquely minimizes both $G_1(Q)$ and $G_2(Q)$, where

$$q_0 = d_\ell - R. \tag{11}$$

In addition, $\frac{dG_1(Q)}{dQ} > \frac{dG_2(Q)}{dQ} > 0$ for $Q > q_0$ and $\frac{dG_1(Q)}{dQ} < \frac{dG_2(Q)}{dQ} < 0$ for $Q < q_0$. Therefore, if $G_2(q_0) \geq G_1(q_0)$, there exist $q_\ell \in \mathbb{R}$ and $q_u \in \mathbb{R}$ such that $G_2(Q) \geq G_1(Q)$ for $Q \in [q_\ell, q_u]$ and $G_2(Q) < G_1(Q)$ for $Q \notin (q_\ell, q_u)$, where

$$q_\ell = -(R - d_\ell) - \sqrt{\frac{2pd_\ell d_u(d_u - R)}{h\tau(d_u - d_\ell)} + \frac{d_u(R - d_\ell)^2}{(d_u - d_\ell)}}, \tag{12}$$

$$q_u = -(R - d_\ell) + \sqrt{\frac{2pd_\ell d_u(d_u - R)}{h\tau(d_u - d_\ell)} + \frac{d_u(R - d_\ell)^2}{(d_u - d_\ell)}}. \tag{13}$$

Equations (12) and (13) follow from the quadratic formula. Furthermore, by convexity of $G_1(Q)$ and $G_2(Q)$, and the definition of q_0, we have $q_\ell \leq q_0$. On the other hand, if $G_2(q_0) < G_1(q_0)$, then $G_2(Q) < G_1(Q)$ for any Q (i.e., there do not exist real-valued q_ℓ and q_u). The preceding discussion indicates that, for $Q \geq d_u$, there are two possible cases as illustrated in Figure 3: (a) $G_2(q_0) \geq G_1(q_0)$ and $q_u \geq d_u$ and (b) $G_2(q_0) \geq G_1(q_0)$ and $q_u < d_u$, or $G_2(q_0) < G_1(q_0)$.

In the next proposition, we define $Q^m(R)$.

Proposition 3.3. $Q^m(R) = d_u$.

As expected, Proposition 3.3 implies that the retailer should minimize her order quantity if she wants to minimize the maximum cycle cost. This is because, from the beginning of a cycle until order initiation, the retailer incurs only inventory holding cost, which is maximized when the beginning inventory is at its maximum level possible. Furthermore, noting that the order quantity does not affect the portion of the cycle cost incurred during the lead time, the retailer should minimize the maximum beginning inventory possible so as to minimize the maximum cycle cost possible. To do so, for any reorder point, the retailer sets the order quantity to the minimum possible, i.e., $Q^m(R) = d_u$.

Next theorem, which follows from Proposition 3.3 and Theorem 3.2, defines (Q^M, R^M).

Theorem 3.4. $(Q^M, R^M) = (d_u, R^m(d_u))$.

Following section discusses the results of a set of numerical studies investigating the effects of the approaches introduced in this section. Later, we present a bi-objective optimization problem where both the expected cost per unit time and the maximum cycle cost possible are minimized.

4. Numerical experimentation

In this section, we present a set of numerical studies demonstrating the effects of accounting for the maximum cycle cost possible, either by just adjusting R of a given (Q, R) policy or by designing a new (Q, R) policy, on the retailer's expected cost per unit time. To do so, we consider the three aforementioned (Q, R) policies based on long-term expected cost minimization, namely, the cost minimizing policy (Q^C, R^C), Gallego's (1992) distribution-free minmax policy (Q^G, R^G), and the cost minimizing (Q, R) policy under the maximum entropy distribution (Q^E, R^E). For each $j \in \{C, G, E\}$, we compare the given (Q^j, R^j) policy to $(Q^j, R^m(Q^j))$ and (Q^M, R^M).

The pertinent figures and tables are provided in the online supplement (Sections G–L).

In our analyses, we use six different lead time demand distributions: normal, truncated normal, symmetric triangular, gamma, right-skewed triangular, and left-skewed triangular. Note that normal, truncated normal, and symmetric triangular distributions are symmetric distributions; gamma and right-skewed triangular are right-skewed distributions; and left-skewed triangular is a left-skewed distribution. Even though left-skewness is noted to be an uncommon demand pattern in practice (see, e.g., Bagchi, Hayya, & Chu, 1986; Burgin, 1975; H. Yang & Schrage, 2009), we believe that these six distributions capture three basic demand patterns. It is worthwhile to note that triangular distribution is also commonly used in case of lack of data due to its simplicity and ability to capture different patterns (Konur & Golias, 2013; Smith, Sturrock, & Kelton, 2018). Furthermore, recall that one of the assumptions of our models is that the lead time demand has lower and upper limits, i.e., its distribution should have a compact support. Among the distributions we use, normal and gamma have unbounded supports. These distributions are commonly used in (Q, R) analyses (see, e.g., Tyworth & Ganeshan, 2000; Tyworth et al., 1996; Tyworth & O'Neill, 1997); and it is possible that one of these distributions is the best fitted one to a retailer's observed data. In such a case, we suggest that the retailer defines lower and upper limits for the lead time demand such that a specific percentage of the possible lead time demand realizations are captured. In our numerical analyses, for a lead time demand distributed according to either normal or gamma distribution, we define $d_\ell = F^{-1}(0.005)$ and $d_u = F^{-1}(0.995)$, where $F^{-1}(\cdot)$ denotes the inverse cumulative distribution function, so that the probability of $D \in \mathbf{d}$ is 0.99 (i.e., the defined range $\mathbf{d}$ covers 99% of the possible lead time demand realizations from the corresponding distribution). This notion of coverage also allows flexibility for the retailer in terms of how robust she prefers to be while minimizing the maximum cycle cost possible. The other four distributions have bounded supports, and thus, d_ℓ and d_u are naturally defined for them.

As discussed in Section 3, penalty cost per unit short (p) and inventory holding cost per unit per unit time (h) play an important role in minimizing the maximum cycle cost. Another key parameter, which is commonly considered in (Q, R) analyses is the lead time demand's coefficient of variance, denoted by CV, where $CV = \sigma/\mu$. To demonstrate the effects of p vs. h, we make comparisons for increasing p values while keeping h fixed over four different CV ranges. To do so, 100 problem instances within a specific CV range are randomly generated, and each problem instance is solved for 10 different p values while keeping $h = 1$. To demonstrate the effects of CV, we make comparisons for increasing CV values over four different p ranges (keeping $h = 1$ fixed). To do so, 100 problem instances within a specific p range and with $h = 1$ are randomly generated, and each problem instance is solved for 10 different CV values. The details of problem instance generation are explained in the online supplement (Section D). This way, we are also able to see the potential cross-effects of p vs. h and CV on how $(Q^j, R^j), (Q^j, R^m(Q^j))$, and (Q^M, R^M) compare for $j \in \{C, G, E\}$. When a problem instance is solved, we determine: (Q^C, R^C) using Hadley and Whitin's (1963) iterative algorithm, (Q^G, R^G) using Gallego's (1992) minmax approach, and (Q^E, R^E) using Hadley and Whitin's (1963) iterative algorithm after the maximum entropy distribution is calculated. The details of determining (Q^E, R^E) are explained in the online supplement (Section E). Also, $(Q^j, R^m(Q^j))\ \forall j \in \{C, G, E\}$ are determined using Theorem 3.2, and (Q^M, R^M) is calculated using Theorem 3.4. Hadley and Whitin's (1963) iterative algorithm requires calculating $n(R)$, which depends on the distribution. Online supplement (Section B) further discusses how we calculate $n(R)$ values for the distributions considered.

4.1. Analysis of cycle cost considerations on (Q^C, R^C)

Below, we first summarize the effects of p and CV on $EC(Q, R)$ and $MC(Q, R)$ under three alternative (Q, R) policies, namely $(Q^C, R^C), (Q^C, R^m(Q^C))$, and (Q^M, R^M). Then, we state two key observations. The reader is referred to Section G of the online supplement for Figures G.1, G.2, and G.3 and Section J for the corresponding tables.

4.1.1. Effects of p and CV

The following discussion is based on Figures G.1 and G.2, which illustrate the averages of the $EC(Q, R)$ and $MC(Q, R)$ values over the 100 problem instances solved from each CV and p interval with varying p and CV values, respectively, for each of the six lead time demand distributions considered.

- $EC(Q^C, R^C)$ increases with p and CV, which is an expected result. Particularly, for any problem instance solved under different CV intervals and lead time demand distributions, we observe that $EC(Q^C, R^C)$ increases with p; and for any problem instance solved under different p intervals and lead time demand distributions, we observe that $EC(Q^C, R^C)$ increases with CV. Indeed, it can be argued that $EC(Q^C, R^C)$ is non-decreasing with p and CV. On the other hand, while $MC(Q^C, R^C)$ tends to increase with p and CV, it is not guaranteed that $MC(Q^C, R^C)$ will always increase with an increase in p or CV. This is because (Q^C, R^C) minimizes $EC(Q, R)$ (in several problem instances, it is observed that $MC(Q^C, R^C)$ decreased with an increase in p, however, we did not observe a problem instance where $MC(Q^C, R^C)$ decreased with an increase in CV).

- $EC(Q^C, R^m(Q^C))$ might increase or decrease with an increase in p or CV. Indeed, one can note from Figure G.1, especially under high CV intervals, that $EC(Q^C, R^m(Q^C))$ might increase or decrease with an increase in p. Similarly, even though $EC(Q^C, R^m(Q^C))$ tends to increase with CV as depicted in Figure G.2, we observed several problem instances where $EC(Q^C, R^m(Q^C))$ decreased with an increase in CV. Furthermore, $MC(Q^C, R^m(Q^C))$ might as well increase or decrease with an increase in p or CV. One can note from Figure G.1, especially under high CV intervals, that $MC(Q^C, R^m(Q^C))$ might increase or decrease with an increase in p. Also, while we have not observed a problem instance where $MC(Q^C, R^m(Q^C))$ decreased with an increase in CV, it is not guaranteed that $MC(Q^C, R^m(Q^C))$ increases with CV. These results follow because $(Q^C, R^m(Q^C))$ is a mixed policy; Q^C is leaned towards minimizing $EC(Q, R)$, whereas $R^m(Q^C)$ aims at minimizing $MC(Q^C, R)$.
- $MC(Q^M, R^M)$ increases with p and CV, which is an expected result. Particularly, for any problem instance solved under different CV intervals and lead time demand distributions, we observe that $MC(Q^M, R^M)$ increases with p; and for any problem instance solved under different p intervals and lead time demand distributions, we observe that $MC(Q^M, R^M)$ increases with CV. Indeed, it can be argued that $MC(Q^M, R^M)$ is non-decreasing with p and CV. Also, while we have not observed any problem instance where $EC(Q^M, R^M)$ decreased with an increase in CV, it can be seen in Figure G.1, especially for larger CV intervals, that $EC(Q^M, R^M)$ might increase or decrease with an increase in p. As (Q^M, R^M) minimizes $MC(Q, R)$, it is not guaranteed that $EC(Q^M, R^M)$ will increase with an increase in p or CV.

4.1.2. Observations

One can notice that $EC(Q^C, R^C) \leq EC(Q^C, R^m(Q^C))$ and $MC(Q^C, R^m(Q^C)) \leq MC(Q^C, R^C)$ by definition. That is, adjusting the R of the cost minimizing (Q, R) policy, i.e., (Q^C, R^C), so as to reduce the maximum cycle cost possible comes at a cost of increased expected cost per unit time. Nevertheless, the magnitude of the changes in $EC(Q, R)$ and $MC(Q, R)$ are considerable factors when preferring one policy over the other. For instance, increasing the expected cost per unit time by less than 5% while reducing the maximum cycle cost possible by more than 20% might be preferable for the retailer. Figure G.3 demonstrates the average percent changes in $EC(Q, R)$ and $MC(Q, R)$ due to switching from (Q^C, R^C) policy to $(Q^C, R^m(Q^C))$ as well as to (Q^M, R^M). Percent changes for a problem instance are defined as follows: $\Delta EC = 100\% \times \frac{EC(Q', R') - EC(Q^C, R^C)}{EC(Q^C, R^C)}$ and $\Delta MC = 100\% \times \frac{MC(Q', R') - MC(Q^C, R^C)}{MC(Q^C, R^C)}$ for (Q^C, R^C) vs. (Q', R'), where (Q', R') is either $(Q^C, R^m(Q^C))$ or (Q^M, R^M). Observation O.1, stated below, emphasizes that, in some settings, $(Q^C, R^m(Q^C))$ can be an attractive policy to adopt based on the magnitude of these changes.

> O.1 By adjusting the R of the cost minimizing (Q, R) policy, the maximum cycle cost possible can be significantly reduced with a relatively small increase in the expected cost per unit time under settings with: (i) relatively high p values under right-skewed lead time demand distributions considered, and (ii) relatively high p and low CV values under symmetric lead time demand distributions considered.

We note that similar observation can be, but weakly, stated for settings with relatively high p and low CV values under the left-skewed lead time distribution we considered. Nevertheless, as noted previously, left-skewness is not common in practice. For switching from one extreme policy (Q^C, R^C) to the other extreme policy (Q^M, R^M), Observation O.1 does not necessarily hold, especially considering the significant amount of changes in both $EC(Q, R)$ and $MC(Q, R)$.

It can also be noticed that $MC(Q^M, R^M) \leq MC(Q^C, R^m(Q^C))$ by definition. Furthermore, one would expect that $EC(Q^M, R^M) \geq EC(Q^C, R^m(Q^C))$ because $(Q^C, R^m(Q^C))$ includes the cost minimizing order quantity Q^C. For low CV values, we do observe that $EC(Q^M, R^M) \geq EC(Q^C, R^m(Q^C))$ (see, e.g., Figures G.1 and G.2). On the contrary, that is not always the case as noted in the next observation.

> O.2 It is possible that $EC(Q^M, R^M) \leq EC(Q^C, R^m(Q^C))$; therefore, (Q^M, R^M) might be Pareto superior to $(Q^C, R^m(Q^C))$.

Observation O.2 implies that, the retailer should not completely disregard modifying both Q and R in order to minimize the maximum cycle cost possible. This is because doing so might be a better alternative than just adjusting R, especially under high CV settings. The tools presented in this study can be used towards such policy comparisons. Furthermore, in Section 5, we present a bi-objective model that minimizes both $EC(Q, R)$ and $MC(Q, R)$.

4.2. Analysis of cycle cost considerations on (Q^G, R^G)

Below, we analyse the effects of p and CV on $EC(Q, R)$ and $MC(Q, R)$ under three alternative (Q, R) policies, namely $(Q^G, R^G), (Q^G, R^m(Q^G))$, and (Q^M, R^M). Note that all of these three policies are based on distribution-free approaches. In the following analyses, we solve the same problem instances used in Section 4.1, assuming that the mean (μ), standard deviation (σ), and upper and lower limits (d_ℓ and d_u) of the lead time demand distribution are known before determining the policy, but the exact lead time demand distribution is observed after the policy is implemented. Recall that, for normal and gamma distributions, d_ℓ and d_u are considered to cover 99% of possible demand realizations. The reader is referred to Section H of the online supplement for Figures H.1, H.2, and H.3 and Section K for the corresponding tables.

Note that (Q^G, R^G) does not minimize $EC(Q, R)$; it minimizes the worst (i.e., the maximum possible) value of $EC(Q, R)$ against the lead time demand distributions with mean μ and standard deviation σ. Therefore, $(Q^G, R^m(Q^G))$ does not explicitly regard $EC(Q, R)$ while minimizing $MC(Q, R)$. Furthermore, (Q^M, R^M) does not target minimizing $EC(Q, R)$ by definition. The purpose of our comparison here is twofold. First, recalling that it might be easier to modify R for a retailer, we naively discuss that it is possible to decrease both $EC(Q, R)$ and $MC(Q, R)$ by adjusting the R of (Q^G, R^G) after the lead time demand distribution is learned. That is, if the retailer will gradually transition her (Q, R) policy from (Q^G, R^G) as the lead time demand is being learned, it is a possibility that adjusting R to $R^m(Q^G)$ is a beneficial intermediate policy. Second, given that (Q^G, R^G) is one extreme approach to deal with the uncertainty of the lead time demand distribution (i.e., it focuses on the worst case possible), we would like to test whether $(Q^G, R^m(Q^G))$ and/or (Q^M, R^M) can offer benefits in addition to reducing the maximum cycle cost possible. Moreover, we are able to demonstrate approach (i) with another (Q, R) policy, compare (Q^G, R^G) and (Q^M, R^M) (two policies that use different types of limited information about the lead time demand distribution, i.e., μ and σ vs. d_ℓ and d_u), and observe the changes in $MC(Q^G, R^G)$ and the changes in $EC(Q^G, R^G)$ when the lead time demand distribution becomes known.

Next, the effects of p and CV on the three policies are discussed. After that, we state several observations.

4.2.1. Effects of p and CV

The following discussion is based on Figures H.1 and H.2, which illustrate the averages of the $EC(Q, R)$ and $MC(Q, R)$ over the 100 problem instances solved under each CV and p interval with varying p and CV values, respectively, for each of the six lead time demand distributions considered. That is, we discuss how $EC(Q, R)$ and $MC(Q, R)$ change with p and CV given that the lead time demand distribution turns out to be the distribution evaluated.

- Similarly to $EC(Q^C, R^C), EC(Q^G, R^G)$ increases with p and CV for any problem instance solved, which is an expected result. On the other hand, while $MC(Q^G, R^G)$ tends to increase with p and CV as depicted in Figures H.1 and H.2, respectively, it cannot be guaranteed that $MC(Q^G, R^G)$ increases or decreases with p or CV. This is because (Q^G, R^G) minimizes the maximum $EC(Q, R)$ value over all possible distributions (while it can be seen in Figure H.1, especially for lower CV values, that $MC(Q^G, R^G)$ can increase or decrease with p, we did not observe a problem instance where $MC(Q^G, R^G)$ decreased with an increase in CV).
- $EC(Q^G, R^m(Q^G))$ might increase or decrease with an increase in p or CV. Indeed, one can note from Figure H.1, especially under high CV values, that $EC(Q^G, R^m(Q^G))$ might increase or decrease with an increase in p. Similarly, even though $EC(Q^G, R^m(Q^G))$ tends to increase with CV as depicted in Figure H.2, we observed several problem instances where $EC(Q^G, R^m(Q^G))$ decreased with an increase in CV. It can be further seen in Figures H.1 and H.2 that $MC(Q^G, R^m(Q^G))$ tends to increase with p and CV, and we have not observed a problem instance where $MC(Q^G, R^m(Q^G))$ decreased with p or CV. However, it is not guaranteed that $MC(Q^G, R^m(Q^G))$ will always increase with p or CV because Q^G is based on minimizing the maximum $EC(Q, R)$ possible, whereas $R^m(Q^G)$ minimizes $MC(Q^G, R)$.
- $EC(Q^M, R^M)$ and $MC(Q^M, R^M)$ behave the same way as in Section 4.2.1 because (Q^M, R^M) is the same.

4.2.2. Observations

One can notice that $MC(Q^M, R^M) \le MC(Q^G, R^m(Q^G)) \le MC(Q^G, R^G)$ by definition. Nonetheless, unlike (Q^C, R^C) vs. $(Q^C, R^m(Q^C))$ or

(Q^M, R^M), it cannot be guaranteed that $EC(Q^G, R^G) \leq EC(Q^G, R^m(Q^G))$ or $EC(Q^G, R^G) \leq EC(Q^M, R^M)$ unless the lead time demand distribution is indeed the worst possible distribution maximizing $EC(Q, R)$. Therefore, as noted in the next observation, it is possible that $EC(Q^G, R^G) \geq EC(Q^G, R^m(Q^G))$ or $EC(Q^G, R^G) \geq EC(Q^M, R^M)$ if the lead time demand distribution is not the worst possible one.

O.3 If the lead time demand distribution is in fact not the worst one for $EC(Q, R)$, it is possible that $EC(Q^G, R^m(Q^G)) \leq EC(Q^G, R^G)$ and/or $EC(Q^M, R^M) \leq EC(Q^G, R^G)$; therefore, $(Q^G, R^m(Q^G))$ and/or (Q^M, R^M) might be Pareto superior to (Q^G, R^G)

Especially, the noted Pareto superiority of $(Q^G, R^m(Q^G))$ can be recognized in Figures H.1 and H.2 for the lead time demand distributions we tested under settings with high p and low CV values. This suggests that, in such settings, a retailer, who is not completely risk-averse, might prefer $EC(Q^G, R^m(Q^G))$ over (Q^G, R^G) and reduce both the expected costs per unit time and the maximum cycle cost possible. Also, even though it is not very apparent in Figures H.1 and H.2, we observed problem instances where (Q^M, R^M) was Pareto superior to (Q^G, R^G) under each of the distributions considered. On the other hand, it should be highlighted that, (Q^G, R^G) may perform significantly better than both $(Q^G, R^m(Q^G))$ and (Q^M, R^M) in terms of the expected costs per unit time, especially under settings with high CV and low p values. This suggests that, in such settings, Gallego's (1992) minmax approach is preferable for long-term expected cost minimization. Nevertheless, by definition, (Q^G, R^G) cannot outperform $(Q^G, R^m(Q^G))$ or $(Q^G, R^m(Q^G))$ in terms of the maximum cycle cost possible; therefore, (Q^G, R^G) cannot be Pareto superior to either policy. Finally, we note that Observation O.2 of Section 4.1.2 can also be noticed in Figures H.1 and H.2, and Figure H.3 (defined similarly as Figure G.3). This suggests that a discussion similar to Observation O.1 can be noted for (Q^G, R^G) vs. $(Q^G, R^m(Q^G))$.

4.3. Analysis of cycle cost considerations on (Q^E, R^E)

Finally, we analyse the effects of considering cycle costs under the cost minimizing (Q, R) policy corresponding to the maximum entropy distribution of the lead time demand. Here, we consider that the mean (μ), standard deviation (σ), and upper and lower limits (d_ℓ and d_u) of the lead time demand distribution are known and these are used in determining the maximum entropy distribution. Therefore, we consider the problem instances used in Sections 4.1 and 4.2 with bounded lead time demand distributions. That is, we consider the problem instances, where the actual lead time demand distribution is truncated normal, symmetric, right-skewed, and left-skewed triangular. We solve the same problem instances from these distributions with three alternative (Q, R) policies, namely $(Q^E, R^E), (Q^E, R^m(Q^E))$, and (Q^M, R^M). Note that all of these three policies are based on distribution-free approaches. The purpose of this section is the same as that of Section 4.2: after the lead time demand distribution becomes known, how $EC(Q, R)$ and $MC(Q, R)$ of different approaches, which use varying information about the lead time demand distribution, compare for various p and CV values. Note that (Q^E, R^E) uses μ, σ, d_ℓ, and d_u and (Q^M, R^M) uses d_ℓ and d_u. Also, we further demonstrate and compare approaches (i) and (ii) with another (Q, R) policy.

4.3.1. Effects of p and CV

Figures I.1–I.3 are defined similar to Figures G.1–G.3 (and H.1–H.3), respectively. It can be noticed that the expected costs per unit time and the maximum cycle costs of (Q^E, R^E) and, consequentially, $(Q^E, R^m(Q^E))$ behave very similarly to the expected costs per unit time and the maximum cycle costs of (Q^C, R^C) and $(Q^C, R^m(Q^C))$, respectively. The discussions of Section 4.1.1 hold for comparing $EC(Q, R)$ and $MC(Q, R)$ under $(Q^E, R^E), (Q^E, R^m(Q^E))$, and (Q^M, R^M) for various p and CV values. The only difference would be that we cannot argue that $EC(Q^E, R^E)$ would always increase with CV as $EC(Q^C, R^C)$ does for a given problem instance. This is because the underlying maximum entropy distribution changes as CV changes. Nevertheless, we did not observe a problem instance where $EC(Q^E, R^E)$ decreased with an increase in CV for a problem instance. On the other hand, it can be argued that $EC(Q^E, R^E)$ would increase with p as $EC(Q^C, R^C)$ does for a given problem instance, because the maximum entropy distribution does not change as p changes for the same problem instance.

4.3.2. Observations

Recall from Observation O.3 that $(Q^G, R^m(Q^G))$ and/or (Q^M, R^M) might be Pareto superior to (Q^G, R^G). Comparing $(Q^E, R^E), (Q^E, R^m(Q^E))$, and (Q^M, R^M), we have a similar observation as noted below.

O.4 If the lead time demand distribution is in fact not the maximum entropy distribution, it is possible that $EC(Q^E, R^m(Q^E)) \leq EC(Q^E, R^E)$ and/or $EC(Q^M, R^M) \leq EC(Q^E, R^E)$; therefore, $(Q^E, R^m(Q^E))$ and/or (Q^M, R^M) might be Pareto superior to (Q^E, R^E).

Observation O.4 follows from the fact that, even though $MC(Q^M, R^M) \leq MC(Q^E, R^m(Q^E)) \leq MC(Q^E, R^E)$ by definition, it is not guaranteed that $EC(Q^E, R^E) \leq MC(Q^E, R^m(Q^E))$ or $EC(Q^E, R^E) \leq MC(Q^M, R^M)$ when the actual lead time demand distribution is not the maximum entropy distribution. Unlike Section 4.2.2, we did not notice a specific pattern related to p and CV values for Pareto superiority of $(Q^E, R^m(Q^E))$ and/or (Q^M, R^M) over (Q^E, R^E). However, in most cases, (Q^E, R^E) is not Pareto dominated or Pareto dominated with small margins. Furthermore, it is worthwhile to note that Observation O.1 can be accepted for (Q^E, R^E) vs. $(Q^E, R^m(Q^E))$ and Observation O.2 is valid for comparing $(Q^E, R^m(Q^E))$ and (Q^M, R^M).

Even though it is not our purpose to compare the two distribution-free policies minimizing the expected costs per unit time, i.e., (Q^G, R^G) based on Gallego's (1992) minmax approach and (Q^E, R^E) based on the maximum entropy distribution for the lead time demand, we observe that (Q^E, R^E) tends to outperform (Q^G, R^G) both in terms of $EC(Q, R)$ and $MC(Q, R)$. This is expected because the maximum entropy approach uses more information about the lead time demand distribution than Gallego's (1992) minmax approach does: (Q^E, R^E) is determined by utilizing μ, σ, d_ℓ, and d_u; whereas, (Q^G, R^G) is determined by utilizing μ and σ. Furthermore, Gallego's (1992) minmax approach focuses on the worst-case lead time demand distribution and minimizes the expected cost per unit time against this worst-case lead time demand distribution; whereas, the maximum entropy approach focuses on determining the best representation for the lead time demand and uses this best representation for minimizing the expected cost per unit time. In this sense, we can cautiously say that Gallego's (1992) minmax approach is risk-averse while the maximum entropy approach is risk-neutral. We refer to the reader to Castellano (2016) for a detailed comparison of the long-term expected costs under these two policies.

5. Extension to bi-objective modelling

As observed in the previous section, solely minimizing the maximum cycle cost and using (Q^M, R^M) policy can significantly increase the expected cost per unit time. On the other hand, when the expected cost per unit time is minimized and (Q^C, R^C) is used, the maximum cycle cost significantly increases. Adjusting only the R of the cost minimizing policy, i.e., $(Q^C, R^m(Q^C))$ policy, can be accepted as a moderate approach, and we noted that it can be a preferable policy in several settings (i.e., see Observation O.1). However, $(Q^C, R^m(Q^C))$ is a single policy aimed to balance $EC(Q, R)$ and $MC(Q, R)$, and (Q^M, R^M) might be Pareto superior to $(Q^C, R^m(Q^C))$ (i.e., see Observation O.2). These results necessitate a systematic approach to simultaneously take $EC(Q, R)$ and $MC(Q, R)$ into account and offer a retailer multiple alternative (Q, R) policies.

In order to balance the expected cost per unit time and the maximum cycle cost, we formulate a bi-objective optimization model such that both $EC(Q, R)$ and $MC(Q, R)$ are minimized given the lead time demand distribution. This bi-objective optimization model (**P**) is:

$$\begin{aligned} \mathbf{P}\colon \min\ & EC(Q, R) \\ \min\ & MC(Q, R) \\ \text{s.t. }\ & Q \geq d_u, d_\ell \leq R \leq d_u. \end{aligned}$$

Note that since Q and R are continuous decision variables, the set of Pareto efficient solutions of **P** is infinite unless $(Q^C, R^C) = (Q^M, R^M)$; therefore, we focus on approximating the set of Pareto efficient solutions of **P**, denoted by $\mathcal{PF}$. Let $F = \{(Q, R) : Q \geq d_u, d_\ell \leq R \leq d_u\}$, i.e., F is the set of feasible (Q, R) policies. Note that a policy $(Q^e, R^e) \in F$ is a Pareto efficient solution of **P** if and only if $\nexists (Q', R') \in F$ such that $[EC(Q', R'), MC(Q', R')] \neq [EC(Q^e, R^e), MC(Q^e, R^e)], EC(Q', R') \leq EC(Q^e, R^e)$, and $MC(Q', R') \leq MC(Q^e, R^e)$. We let (Q^e, R^e) to denote an arbitrary solution in $\mathcal{PF}$.

One common method to approximate $\mathcal{PF}$ is the ϵ-constraint method (Lin, 1976). In this method, one of the objective functions is included as a constraint with an upper limit while minimizing the other objective function (see, e.g., Schaefer & Konur, 2015). For problem **P**, when the expected cost per unit time is selected for minimization, ϵ-constraint would require solving $\min_{Q,R}\{EC(Q, R) : MC(Q, R) \leq \Delta, (Q, R) \in F\}$ for decreasing Δ values, i.e., decreasing limits on the maximum cycle costs. In this sense, ϵ-constraint method implies iteratively solving single-objective cost minimization (Q, R) models with budget restrictions, where budget refers to the maximum cycle cost in our study. We note that various (Q, R) models with budget restrictions have been analysed in the literature (see, e.g., Bera, Rong, Mahapatra, & Maiti, 2009; K. Das, Roy, & Maiti, 2004; Fattahi et al., 2015; Ghalebsaz-Jeddi, Shultes, & Haji, 2004; Kundu & Chakrabarti, 2012; Tamjidzad & Mirmohammadi, 2015, 2017, 2018; T.-Y. Wang & Hu, 2008; X. Zhao, Qiu, Xie, & He, 2012). The resulting solution will be a Pareto efficient solution for problem **P**. This suggests that a capital restriction on a retailer's short-term spending will increase her long-term expected cost, which is an anticipated result. Indeed, the studies cited in Section 1 on the intersection of inventory management and supply chain finance emphasize such tradeoffs between a retailer's capital availability and long-term profitability.

As noted above, ϵ-constraint method requires solving nonlinear optimization models with a non-continuously differentiable function ($MC(Q, R)$ being either the objective or the constraint function) and it is noted to be computationally burdensome (Schaefer & Konur, 2015). In what follows, a simple search produce is proposed to determine such Pareto efficient solutions by utilizing several characteristics of (Q^e, R^e).

First, let us define $Q^c(R) = \text{argmin}\{EC(Q, R|R) : Q \geq d_u\}$ and $R^c(Q) = \text{argmin}\{EC(Q, R|Q) : R \in \mathbf{d}\}$. It can be discussed that $EC(Q, R|R)$ is strictly convex with respect to Q and one can show that $Q^c(R) = \max\{d_u, \sqrt{\frac{2\lambda(K+pn(R))}{h}}\}$. Similarly, it can be discussed that $EC(Q, R|Q)$ is strictly convex with respect to R and one can show that $R^c(Q) = d_\ell$ if $F^{-1}\left(1 - \frac{Qh}{p\lambda}\right) < d_\ell, R^c(Q) = F^{-1}\left(1 - \frac{Qh}{p\lambda}\right)$ if $d_\ell \leq F^{-1}\left(1 - \frac{Qh}{p\lambda}\right) < d_u$, and $R^c(Q) = d_u$ if $d_u < F^{-1}\left(1 - \frac{Qh}{p\lambda}\right)$. Notice that $Q^C = Q^c(R^C)$ and $R^C = R^c(Q^C)$ by definition. Also recall that $Q^m(R) = \text{argmin}\{MC(Q, R|R) : Q \geq d_u\} = d_u \quad \forall R \in \mathbf{d}$ as defined in Proposition 3.3 and $R^m(Q) = \text{argmin}\{EC(Q, R|Q) : R \in \mathbf{d}\}$, which is defined in Theorem 3.2.

Proposition 5.1. *If $(Q^e, R^e) \in \mathcal{PF}$, then (i) $Q^e \in [d_u, Q^c(R^e)]$ and (ii) $R^e \in [\min\{R^c(Q^e), R^m(Q^e)\}, \max\{R^c(Q^e), R^m(Q^e)\}]$.*

We utilize Proposition 5.1 to devise a bi-directional search method to approximate $\mathcal{PF}$ as follows. Let $PF(R)$ and $PF(Q)$ denote the Pareto efficient solutions of bi-objective models **P-R** and **P-Q**, respectively, such that

$$\begin{array}{llll} \textbf{P-R}: & \min\ EC(Q,R|R) & \textbf{P-Q}: & \min\ EC(Q,R|Q) \\ & \min\ MC(Q,R|R) & & \min\ MC(Q,R|Q) \\ & \text{s.t. } (Q,R) \in F, & & \text{s.t. } (Q,R) \in F. \end{array}$$

That is, $PF(R)$ is the set of Pareto efficient Q values of **P-R** corresponding to the given R and $PF(Q)$ is the set of Pareto efficient R values of **P-Q** corresponding to the given Q. Note that $PF \subseteq \{\cup_{Q \geq d_u} PF(Q)\} \cup \{\cup_{R \in \mathbf{d}} PF(R)\}$. Prior to giving details of the search method, a routine is presented in the online supplement (Section F), Routine 0, that returns the set of Pareto efficient solutions, $PE(S)$, out of a given set of solutions S (for similar routines, see also, Konur & Golias, 2013; Konur & Dagli, 2015; Schaefer & Konur, 2015; Konur & Schaefer, 2016).

Proposition 5.1(i) implies that $PF(R) \subseteq [d_u, Q^c(R)]$, and it can be further discussed that $PF(R) \equiv [d_u, Q^c(R)]$ due the convexity of $EC(Q, R|R)$ and $MC(Q, R|R)$ being an increasing linear function of Q. Therefore, $PF(R)$ can be approximated by generating an arbitrary number of Q values from $[d_u, Q^c(R)]$. This approximation of $PF(R)$ can then be used to approximate $\cup_{R \in \mathbf{d}} PF(R)$ by considering an arbitrary number of R values from within $\mathbf{d}$. Routine 1 states a one-directional search method, based on R, that approximates $\cup_{R \in \mathbf{d}} PF(R)$.

Routine 1: One-directional search to approximate $\bigcup_{R \in \mathbf{d}} PF(R)$

```
Let δ_r^1 = (d_u − d_ℓ)/φ^1. Set R = d_ℓ and PF^1 = ∅.
while R ≤ d_u do
    Let δ_q^1 = (Q^c(R) − d_u)/ψ^1. Set Q = d_u and PF_R^1 = ∅.
    while Q ≤ Q^c(R) do
        Set PF_R^1 := PF_R^1 ∪ {Q} and Q := Q + δ_q^1.
    Set PF^1 := PF^1 ∪ PF_R^1 and R := R + δ_r^1.
Return PF^1 := PE(PF^1) using Routine 0.
```

Routine 1 returns at most $\phi^1 \times \psi^1$ solutions, where ϕ^1 and ψ^1 are user-defined parameters. Note that neither (Q^C, R^C) nor (Q^M, R^M), the two extreme Pareto efficient solutions, are in PF^1. These will be included in the next routine. If only Routine 1 is to be used to approximate $\mathcal{PF}$, then both (Q^C, R^C) and (Q^M, R^M) should be added to PF^1 before executing Routine 0.

Proposition 5.1(ii) implies that $PF(Q) \subseteq [\min\{R^c(Q), R^m(Q)\}, \max\{R^c(Q), R^m(Q)\}]$, and it can be further discussed that $PF(Q) \equiv [\min\{R^c(Q), R^m(Q)\}, \max\{R^c(Q), R^m(Q)\}]$ due to the convexity of both $EC(Q, R|Q)$ and $MC(Q, R|Q)$. Therefore, $PF(Q)$ can be approximated by generating an arbitrary number of R values from $[\min\{R^c(Q), R^m(Q)\}, \max\{R^c(Q), R^m(Q)\}]$. This approximation of $PF(Q)$ can then be used to approximate $\cup_{Q \geq d_u} PF(Q)$ by considering an arbitrary number of Q values. We know that $Q \geq d_u$; however, unlike approximating $\cup_{R \in \mathbf{d}} PF(R)$, Q does not have a direct upper bound and it is therefore necessary to define an upper bound on Q in order to approximate $\cup_{Q \geq d_u} PF(Q)$. We know from Proposition 5.1(i) that $Q^e \in [d_u, Q^c(R^e)]$ for a Pareto efficient solution (Q^e, R^e). Also, recall that $Q^c(R) = \max\{d_u, \sqrt{\frac{2\lambda(K+pn(R))}{h}}\} \leq \sqrt{\frac{2\lambda(K+pn(R))}{h}}$. Noting that $n(R)$ is maximized when R is minimized, it can be concluded that $Q^c(R) \leq \sqrt{\frac{2\lambda(K+pn(d_\ell))}{h}}\ \forall R \in \mathbf{d}$. Based on the preceding discussion, Routine 2, stated below, is also a one-directional search method, based on Q, that approximates $\cup_{Q \geq d_u} PF(Q)$.

Routine 2: One-directional search to approximate $\bigcup_{Q \geq d_u} PF(Q)$

Let $\delta_q^2 = \left(\sqrt{2\lambda(K + pn(d_\ell))/h} - d_u\right)/\phi^2$. Set $Q = d_u$ and $PF^2 = \emptyset$.
while $Q \leq \sqrt{2\lambda(K + pn(d_\ell))/h}$ **do**
  Let $\delta_r^2 = (\max\{R^c(Q), R^m(Q)\} - \min\{R^c(Q), R^m(Q)\})/\psi^2$. Set $R = \min\{R^c(Q), R^m(Q)\}$ and $PF_Q^2 = \emptyset$.
  while $R \leq \max\{R^c(Q), R^m(Q)\}$ **do**
    Set $PF_Q^2 := PF_Q^2 \cup \{R\}$ and $R := R + \delta_r^2$.
  Set $PF^2 := PF^2 \cup PF_Q^2$ and $Q := Q + \delta_q^2$.
Return $PF^2 := PE(PF^2 \cup \{(Q^C, R^C)\})$ using Routine 0.

Routine 2 returns at most $\phi^2 \times \psi^2 + 1$ solutions, where ϕ^2 and ψ^2 are user-defined parameters. Notice that (Q^M, R^M) is guaranteed to be in PF^2. Also, since Q^C might be missed during the search over Q, we add it to PF^2 before executing Routine 0.

The bi-directional search method to approximate $\mathcal{PF}$, Algorithm 1, is stated below. Algorithm 1 simply combines PF^1 and PF^2, and returns the approximated $\mathcal{PF}$, denoted by $\widehat{PF}$, as the Pareto efficient solutions with the union of PF^1 and PF^2. Note that both PF^1 and PF^2 will have a piecewise representation on the objective space, where each piece is an approximated convex curve corresponding to either $PF(Q)$ or $PF(R)$. Therefore, one can expect that $\widehat{PF}$ has a piecewise representation as well. Indeed, that is the case as we demonstrate Algorithm 1 with several examples next.

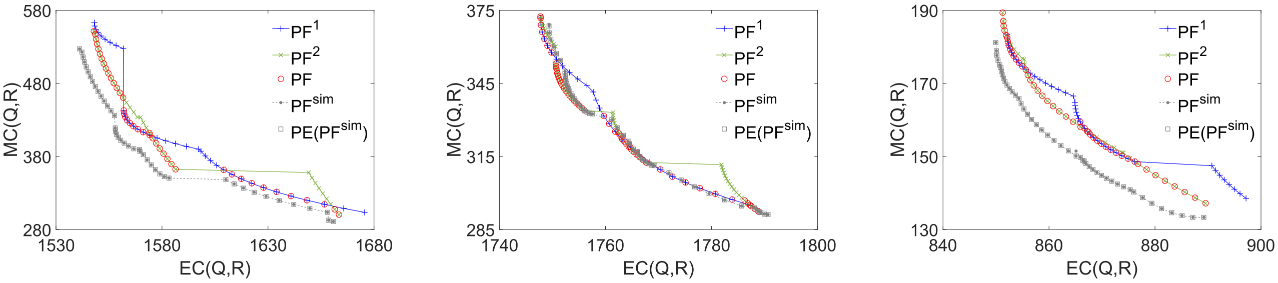
Algorithm 1: A bi-directional search method to generate an approximated $\mathcal{PF}$, denoted by $\widehat{PF}$.

Generate PF^1 using Routine 1.
Generate PF^2 using Routine 2.
Return $\widehat{PF} = PE(PF^1 \cup PF^2)$ using Routine 0.

Prior to these examples, we note that $EC(Q, R)$ and $MC(Q, R)$ values for any given (Q, R) in Routines 1 and 2, and Algorithm 1, are calculated using Equations (1) and (5) right before Routine 0 is executed. One can also use simulation to evaluate $EC(Q, R)$ and $MC(Q, R)$ values for any given (Q, R) before executing Routine 0. In this case, these methods would return the Pareto efficient solutions by comparing individual policies based on their simulated average cost per unit time and maximum cycle cost. We demonstrate such a simulation approach with some of the examples discussed below. To do so, we let SPF^1, SPF^2, and $\widehat{SPF}$ be defined similar to PF^1, PF^2, and $\widehat{PF}$, respectively. SPF^1, SPF^2, and $\widehat{SPF}$ correspond to the outcomes of Routines 1 and 2, and Algorithm 1, respectively, when simulation is used to evaluate $EC(Q, R)$ and $MC(Q, R)$ values for any given (Q, R) before executing Routine 0. PF^1, PF^2, and $\widehat{PF}$ correspond to the outcomes of Routines 1 and 2, and Algorithm 1, respectively, when Equations (1) and (5) are used to calculate $EC(Q, R)$ and $MC(Q, R)$ values for any given (Q, R) before executing Routine 0. We explain the details of our simulation setup in the online supplement (Section F) and we refer the reader to Nahmias and Wang (1979), Nahmias (1981), Chiu (1995), Olsson (2014), and Braglia et al. (2019) for using detailed simulation approaches in determining near-optimal policies for several challenging single-objective (Q, R) models.

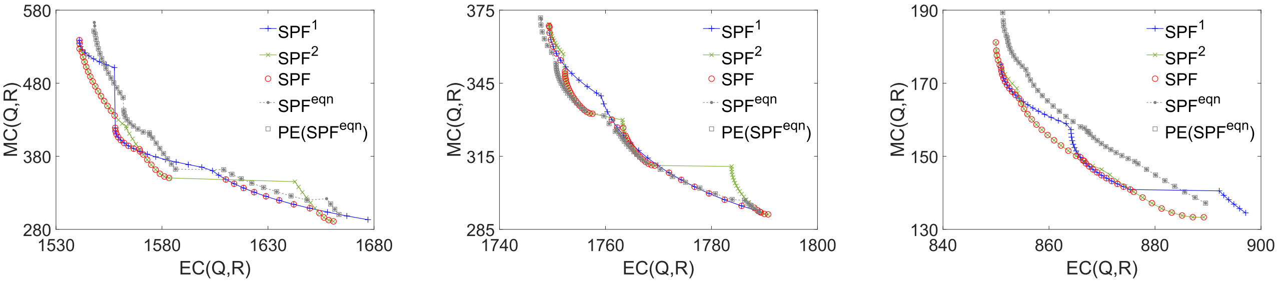
Figure 4(a–c) illustrates PF^1, PF^2, and $\widehat{PF}$ and Figure 5(a–c) illustrate SPF^1, SPF^2, and $\widehat{SPF}$ for a problem instance, where the lead time demand distribution is normal, gamma, and truncated normal, respectively. In both Figures 4 and 5, Routines 1 and 2 parameters are $\phi^1 = \phi^2 = 10$ and $\psi^1 = \psi^2 = 20$ for illustration purposes. Furthermore, Figure 4 includes the simulated $EC(Q, R)$ and $MC(Q, R)$ values for the (Q, R) policies in $\widehat{PF}$, denoted by PF^{sim}, as well as $PE(PF^{sim})$, i.e., the Pareto efficient solutions based on these simulated $EC(Q, R)$ and $MC(Q, R)$ values of the solutions within $\widehat{PF}$. Similarly, Figure 5 includes the $EC(Q, R)$ and $MC(Q, R)$ values calculated using Equations (1) and (5) for the (Q, R) policies in $\widehat{SPF}$, denoted by SPF^{eqn}, as well as $PE(SPF^{eqn})$, i.e., the Pareto efficient solutions based on these calculated $EC(Q, R)$ and $MC(Q, R)$ values of the solutions within $\widehat{SPF}$.

As it can be observed, both $\widehat{PF}$ and $\widehat{SPF}$ have a piecewise structure, where each piece is a convex curve defined partially or fully by one of the convex pieces returned by either Routine 1 or 2. Similarly, both $PE(PF^{sim})$ and $PE(SPF^{eqn})$ have piecewise structures. One can also note that $\widehat{PF}$ and $\widehat{SPF}$ are very close to each other, indicating that the methods proposed return similar Pareto points when they are executed with calculated or simulated $EC(Q, R)$ and $MC(Q, R)$ values. Furthermore, when $\widehat{PF}$ and $PE(PF^{sim})$ are compared in Figure 4, we can see that the range of $EC(Q, R)$ values are very close for all three problem instances. Indeed, it can be observed that the calculated $EC(Q, R)$ values can be lower or higher than the simulated $EC(Q, R)$ values. On the other hand, the simulated $MC(Q, R)$ values are lower than the calculated $MC(Q, R)$ values, which is an expected result because the calculated $MC(Q, R)$



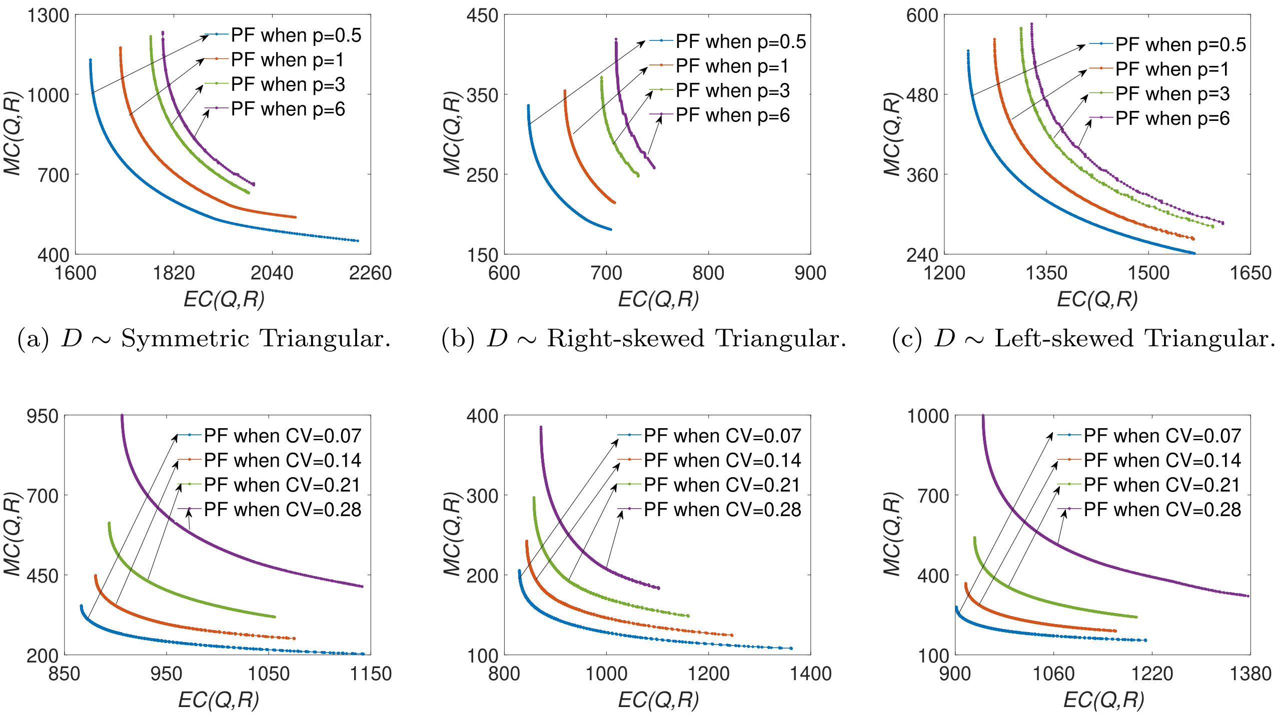
(a) $D \sim$ Normal. (b) $D \sim$ Truncated Normal. (c) $D \sim$ Gamma.

Figure 4. Illustration of Algorithm 1 with calculated $EC(Q, R)$ and $MC(Q, R)$ values.



(a) $D \sim$ Normal. (b) $D \sim$ Truncated Normal. (c) $D \sim$ Gamma.

Figure 5. Illustration of Algorithm 1 with simulated $EC(Q, R)$ and $MC(Q, R)$ values.



(a) $D \sim$ Symmetric Triangular. (b) $D \sim$ Right-skewed Triangular. (c) $D \sim$ Left-skewed Triangular.

(d) $D \sim$ Symmetric Triangular. (e) $D \sim$ Right-skewed Triangular. (f) $D \sim$ Left-skewed Triangular.

Figure 6. Illustrations of $\widehat{PF}$ over varying p and CV values.

value is the maximum cycle cost possible, whereas the simulated $MC(Q, R)$ value is based on the simulated cycles. Similar observations can be noted when $\widehat{SPF}$ and $PE(SPF^{eqn})$ are compared in Figure 5. Finally, it can be noticed from Figure 4 that PF^{sim} and $PE(PF^{sim})$ differ only by several points (0, 3, and 1 point(s) in Figure 4(a–c), respectively). These points within PF^{sim} that are not in $PE(PF^{sim})$ are Pareto dominated by small margins. Similarly, it can be noticed from Figure 5 that SPF^{eqn} and $PE(PF^{eqn})$ differ only by several points (3, 4, and 0 point(s) in Figure 5(a–c), respectively). Again, these points are Pareto dominated by small margins. These observations further suggest that the methods proposed approximate the Pareto front similarly when they are executed with calculated or simulated $EC(Q, R)$ and $MC(Q, R)$ values.

Increasing ϕ^1, ϕ^2, ψ^1, and ψ^2 will result in a better approximation. In Figure 6, Algorithm 1 is executed with $\phi^1 = \phi^2 = 100$ and $\psi^1 = \psi^2 = 100$ using calculated $EC(Q, R)$ and $MC(Q, R)$ values. Figure 6(a–c) demonstrates $\widehat{PF}$ for increasing p values for a problem instance such that D is distributed according to symmetric, right-skewed, and left-skewed triangular distributions, respectively. Similarly, Figure 6(d–f) demonstrates $\widehat{PF}$ for increasing CV values for a problem instance such that D is distributed according to symmetric, right-skewed, and left-skewed triangular distributions, respectively. It can be noticed that $\widehat{PF}$ tends to move up-right direction as p or CV increases, which is expected (but not guaranteed) based on the numerical results discussed in Section 4. Specifically, as p or CV increases, the left extreme of $\widehat{PF}$ (i.e., the point corresponding to (Q^C, R^C)) is guaranteed to move right but not up and the right extreme of $\widehat{PF}$ (i.e., the point corresponding to (Q^M, R^M)) is guaranteed to move up but not right. We finally note that, even though it is not very apparent in Figure 6 due to the number of Pareto efficient solutions, $\widehat{PF}$ s have the piecewise structure similar to those in Figures 4 and 5.

6. Conclusions

Considering that demand uncertainty can pose short-term financial risks due to recurring inventory related costs at the operational level, this study analyzes approaches to minimize such risks by minimizing the maximum cycle cost possible in a continuous review inventory control system, i.e., a (Q, R) model. After formulating the maximum cycle cost possible, two approaches to account for maximum cycle cost minimization are proposed. The first approach, a relatively moderate approach, suggests adjusting the re-order point R while keeping the order quantity Q fixed at a specific value, such as the order quantity minimizing the long-term expected costs. The second approach focuses on designing a (Q, R) policy that directly minimizes the maximum cycle cost possible. We characterize the optimum policy parameters under each approach. A set of numerical analyses suggests that, especially in settings with high shortage penalty and/or low lead time demand variation, using the first approach to account for cycle costs can be a viable practical strategy. This is because it might help reduce the maximum cycle cost significantly while increasing the long-term expected cost by a relatively small amount. Another set of numerical analyses suggests that, considering maximum cycle costs during the process of learning the lead time demand distribution might be preferred. This is because doing so can help reduce the long-term expected cost in addition to reducing the maximum cycle cost. Our numerical results further indicate that long-term expected cost minimization and short-term maximum cycle cost minimization can be conflicting objectives. Therefore, we presented a bi-objective (Q, R) model with these two objectives. A method to approximate the set of Pareto efficient solutions is discussed and the bi-objective model and the approximation method are demonstrated with several examples.

We made several assumptions in this study. First, it is assumed that the lead time is fixed. We note that, in the case of lead time uncertainty, our analytical results will still hold because they are based on the definition of the bounds on the demand during lead time (i.e., d_ℓ and d_u). Therefore, considering that the lead time uncertainty is translated to the lead time demand uncertainty and d_ℓ and d_u can be defined accordingly, our analytical results will not change. In addition, we did not enforce safety stock to be non-negative, i.e., it is assumed that $R \geq d_\ell$ rather than $R \geq \mu$. We note that Theorem 3.2, and thereby Theorem 3.4, can be easily modified for $R \geq \mu$. One further remark is that the results of Sections 2 and 3 will hold true if the retailer is presented with quantity discounts as long as the total procurement cost in a cycle increases with the order quantity. That is, if the procurement cost in a replenishment cycle is $C(Q) = c(Q)Q$, where $c(Q)$ is a discount schedule, and $C(Q)$ is increasing with Q (which is practically true as a retailer would not be paying less by buying more even if the unit purchase cost decreases with the order quantity), it can be easily discussed that the definitions of $R^m(Q)$ and (Q^M, R^M) remain the same. However, it should be noted that the aforementioned changes in the model might change the insights generated in Section 4.

On the other hand, changing several settings might require modifications. For instance, one of the critical assumptions is that the penalty cost is defined per unit short. Alternatively, in other practical settings, penalty cost can be defined per unit short per unit time, similar to inventory holding cost. In such a case, some of the results presented in Sections 2 and 3 hold; nevertheless, definitions of several functions, specific R values used in the analyses, and different cases investigated will need to be modified. The online supplement (Section M) provides a sketch for the proofs of these results when the penalty cost is defined per unit short per unit time. Furthermore, the discussions pertaining to the numerical analyses might change. Another practical setting would be the case of lost sales and/or partial backordering for the shortages (see, e.g., Achary & Geetha, 2001; Braglia et al., 2018, 2019; Chu, 1999; Moon & Gallego, 1994; Olsson, 2014; Uthayakumar & Parvathi, 2009). When a mixture of lost sales and backorders is allowed, our results will change. We also note that detailed simulation approaches can be developed for analysing such settings and other possible extensions (see, e.g., Braglia et al., 2019; Chiu, 1995; Nahmias, 1981; Nahmias & Wang, 1979; Olsson, 2014). We believe that the results presented in this study can help with such as well as other future research problems noted below.

The contribution of this study is in developing methodologies to incorporate maximum cycle cost in designing a (Q, R) policy. Several future research directions are as follows. We considered continuous review inventory system, one possible research question is to investigate cycle costs in periodic review inventory systems. Another potential and practical setting that can be explored is multi-item and/or multi-source inventory systems with shared resources. In such settings, analysing the interactions among multiple items and/or multiple sources and their contributions to the cycle costs can provide practical insights for a retailer. Also, as noted above, (Q, R) models extended to other practical settings can be studied with cycle cost considerations.

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ORCID

Dinçer Konur http://orcid.org/0000-0002-5728-584X
Gonca Yıldırım http://orcid.org/0000-0003-1308-5122

References

Aardal, K., Jonsson, O., & Jonsson, H. (1989). Optimal inventory policies with service-level constraints. *Journal of the Operational Research Society*, *40*(1), 65–73. doi:10.1057/jors.1989.6

Achary, K. K., & Geetha, K. K. (2001). A note on the analysis of (*Q*, *r*) policy: GLD approximation vs distribution free approach. *Opsearch*, *38*(2), 223–234. doi:10.1007/BF03399227

Aggarwal, S. P., & Jaggi, C. K. (1995). Ordering policies of deteriorating items under permissible delay in payments. *Journal of the Operational Research Society*, *46*(5), 658–662. doi:10.1057/jors.1995.90

Agrawal, V., & Seshadri, S. (2000). Distribution free bounds for service constrained (*Q*, *r*) inventory systems. *Naval Research Logistics*, *47*(8), 635–656. doi:10.1002/1520-6750(200012)47:8 < 635::AID-NAV3 > 3.3.CO;2-3

Andersson, J., Jörnsten, K., Nonås, S. L., Sandal, L., & Ubøe, J. (2013). A maximum entropy approach to the newsvendor problem with partial information. *European Journal of Operational Research*, *228*(1), 190–200. doi:10.1016/j.ejor.2013.01.031

Archibald, T. W., Thomas, L. C., Betts, J. M., & Johnston, R. B. (2002). Should start-up companies be cautious? Inventory policies which maximise survival probabilities. *Management Science*, *48*(9), 1161–1174. doi:10.1287/mnsc.48.9.1161.176

Bagchi, U., Hayya, J. C., & Chu, C. (1986). The effect of lead-time variability: The case of independent demand. *Journal of Operations Management*, *6*(2), 159–177. doi:10.1016/0272-6963(86)90023-9

Bera, U., Rong, M., Mahapatra, N., & Maiti, M. (2009). A multi-item mixture inventory model involving random lead time and demand with budget constraint and surprise function. *Applied Mathematical Modelling*, *33*(12), 4337–4344. doi:10.1016/j.apm.2009.03.025

Bogataj, D., & Bogataj, M. (2007). Measuring the supply chain risk and vulnerability in frequency space. *International Journal of Production Economics*, *108*(1–2), 291–301. doi:10.1016/j.ijpe.2006.12.017

Bookbinder, J. H., & Çakanyildirim, M. (1999). Random lead times and expedited orders in (*Q*, *r*) inventory systems. *European Journal of Operational Research*, *115*(2), 300–313. doi:10.1016/S0377-2217(98)00304-X

Braglia, M., Castellano, D., & Gallo, M. (2016). Approximated closed-form minimum-cost solution to the (*r*, *q*) policy with complete backordering and further developments. *Applied Mathematical Modelling*, *40*(19–20), 8406–8423. doi:10.1016/j.apm.2016.05.001

Braglia, M., Castellano, D., Marrazzini, L., & Song, D. (2019). A continuous review, (*Q*, *r*) inventory model for a deteriorating item with random demand and positive lead time. *Computers & Operations Research*, *109*, 102–121. doi:10.1016/j.cor.2019.04.019

Braglia, M., Castellano, D., & Song, D. (2018). Efficient near-optimal procedures for some inventory models with backorders-lost sales mixture and controllable lead time, under continuous or periodic review. *International Journal of Mathematics in Operational Research*, *13*(2), 141–177. doi:10.1504/IJMOR.2018.10014162

Burgin, T. A. (1975). The gamma distribution and inventory control. *Journal of the Operational Research Society*, *26*(3), 507–525. doi:10.1057/jors.1975.110

Cai, G. G., Chen, X., & Xiao, Z. (2014). The roles of bank and trade credits: Theoretical analysis and empirical evidence. *Production and Operations Management*, *23*(4), 583–598. doi:10.1111/poms.12035

Carlson, M. L., Miltenburg, G. J., & Rousseau, J. J. (1996). Economic order quantity and quantity discounts under date-terms supplier credit: A discounted cash flow approach. *Journal of the Operational Research Society*, *47*(3), 384–394. doi:10.1057/jors.1996.38

Castellano, D. (2016). Stochastic reorder point-lot size (*r*, *Q*) inventory model under maximum entropy principle. *Entropy*, *18*(1), 1–18. doi:10.3390/e18010016
Chakuu, S., Masi, D., & Godsell, J. (2019). Exploring the relationship between mechanisms, actors and instruments in supply chain finance: A systematic literature review. *International Journal of Production Economics*, *216*, 35–53. doi:10.1016/j.ijpe.2019.04.013
Chand, S., & Ward, J. (1987). A note on "Economic order quantity under conditions of permissible delay in payments". *Journal of the Operational Research Society*, *38*(1), 83–84. doi:10.1057/jors.1987.10
Chang, C.-T. (2004). An EOQ model with deteriorating items under inflation when supplier credits linked to order quantity. *International Journal of Production Economics*, *88*(3), 307–316. doi:10.1016/S0925-5273(03)00192-0
Chao, X., Chen, J., & Wang, S. (2008). Dynamic inventory management with cash flow constraints. *Naval Research Logistics*, *55*(8), 758–768. doi:10.1002/nav.20322
Chiang, C. (2010). An order expediting policy for continuous review systems with manufacturing lead-time. *European Journal of Operational Research*, *203*(2), 526–531. doi:10.1016/j.ejor.2009.08.012
Chiu, H. N. (1995). An approximation to the continuous review inventory model with perishable items and lead times. *European Journal of Operational Research*, *87*(1), 93–108. doi:10.1016/0377-2217(94)00060-P
Christopher, M., & Lee, H. (2004). Mitigating supply chain risk through improved confidence. *International Journal of Physical Distribution & Logistics Management*, *34*(5), 388–396. doi:10.1108/09600030410545436
Chu, P. (1999). A note on the distribution free procedures for some inventory models. *Journal of Information and Optimization Sciences*, *20*(3), 365–372. doi:10.1080/02522667.1999.10699424
Dada, M., & Hu, Q. (2008). Financing newsvendor inventory. *Operations Research Letters*, *36*(5), 569–573. doi:10.1016/j.orl.2008.06.004
Das, C. (1976). Approximate solution to the (*Q*, *r*) inventory model for gamma lead time demand. *Management Science*, *22*(9), 1043–1047. doi:10.1287/mnsc.22.9.1043
Das, K., Roy, T., & Maiti, M. (2004). Multi-item stochastic and fuzzy-stochastic inventory models under two restrictions. *Computers & Operations Research*, *31*(11), 1793–1806. doi:10.1016/S0305-0548(03)00120-5
Dullaert, W., Maes, B., Vernimmen, B., & Witlox, F. (2005). An evolutionary algorithm for order splitting with multiple transport alternatives. *Expert Systems with Applications*, *28*(2), 201–208. doi:10.1016/j.eswa.2004.10.002
Duran, A., Gutierrez, G., & Zequeira, R. I. (2004). A continuous review inventory model with order expediting. *International Journal of Production Economics*, *87*(2), 157–169. doi:10.1016/S0925-5273(03)00100-2
Fattahi, P., Hajipour, V., & Nobari, A. (2015). A bi-objective continuous review inventory control model: Pareto-based meta-heuristic algorithms. *Applied Soft Computing*, *32*, 211–223. doi:10.1016/j.asoc.2015.02.044
Feng, H., Li, J., & Zhao, D. (2013). Retailer's optimal replenishment and payment policies in the EPQ model under cash discount and two-level trade credit policy. *Applied Mathematical Modelling*, *37*(5), 3322–3339. doi:10.1016/j.apm.2012.07.012
Fleischhacker, A. J., & Fok, P.-W. (2015a). An entropy-based methodology for valuation of demand uncertainty reduction. *Decision Sciences*, *46*(6), 1165–1198. doi:10.1111/deci.12170
Fleischhacker, A. J., & Fok, P.-W. (2015b). On the relationship between entropy, demand uncertainty, and expected loss. *European Journal of Operational Research*, *245*(2), 623–628. doi:10.1016/j.ejor.2015.03.014
Gallego, G. (1992). A minmax distribution free procedure for the (*Q*, *R*) inventory model. *Operations Research Letters*, *11*(1), 55–60. doi:10.1016/0167-6377(92)90063-9
Gallego, G., & Moon, I. (1993). The distribution free newsboy problem: Review and extensions. *Journal of the Operational Research Society*, *44*(8), 825–834. doi:10.1057/jors.1993.141
Gao, D., Zhao, X., & Geng, W. (2014). A delay-in-payment contract for pareto improvement of a supply chain with stochastic demand. *Omega*, *49*, 60–68. doi:10.1016/j.omega.2014.05.008
Ghadge, A., Dani, S., & Kalawsky, R. (2012). Supply chain risk management: Present and future scope. *The International Journal of Logistics Management*, *23*(3), 313–339. doi:10.1108/09574091211289200
Ghalebsaz-Jeddi, B., Shultes, B. C., & Haji, R. (2004). A multi-product continuous review inventory system with stochastic demand, backorders, and a budget constraint. *European Journal of Operational Research*, *158*(2), 456–469. doi:10.1016/S0377-2217(03)00363-1
Gholami-Qadikolaei, A., & Mirzazadeh, A. (2013). Optimizing a convex multi-constraint distribution-free inventory model with fixed and variable crash costs. *International Journal of Management Science and Engineering Management*, *8*(3), 219–228. doi:10.1080/17509653.2013.812338
Gong, X., Chao, X., & Simchi-Levi, D. (2014). Dynamic inventory control with limited capital and short-term financing. *Naval Research Logistics*, *61*(3), 184–201. doi:10.1002/nav.21576
Goyal, S. K. (1985). Economic order quantity under conditions of permissible delay in payments. *Journal of the Operational Research Society*, *36*(4), 335–338. doi:10.1057/jors.1985.56
Guo, M., Chen, Y.-W., Wang, H., Yang, J.-B., & Zhang, K. (2019). The single-period (newsvendor) problem under interval grade uncertainties. *European Journal of Operational Research*, *273*(1), 198–216. doi:10.1016/j.ejor.2018.07.048
Gupta, D., & Wang, L. (2009). A stochastic inventory model with trade credit. *Manufacturing & Service Operations Management*, *11*(1), 4–18. doi:10.1287/msom.1070.0191
Gutgutia, A., & Jha, J. K. (2018). A closed-form solution for the distribution free continuous review integrated inventory model. *Operational Research*, *18*(1), 159–186. doi:10.1007/s12351-016-0258-5
Hadley, G., & Whitin, T. M. (1963). *Analysis of inventory systems*. Englewood Cliffs, NJ: Prentice-Hall.
Haley, C. W., & Higgins, R. C. (1973). Inventory policy and trade credit financing. *Management Science*, *20*(4-part-i), 464–471. doi:10.1287/mnsc.20.4.464
Halkos, G. E., Kevork, I. S., & Tziourtzioumis, C. N. (2018). On the existence of unique minimum for the cost function in a (*Q*, *R*) inventory model with backorders. *IMA Journal of Management Mathematics*, *29*(1), 1–21. doi:10.1093/imaman/dpw010

Han, Q., Du, D., & Zuluaga, L. F. (2014). Technical note—A risk- and ambiguity-averse extension of the max-min newsvendor order formula. *Operations Research*, *62*(3), 535–542. doi:10.1287/opre.2014.1262

Hariga, M. A. (2010). A single-item continuous review inventory problem with space restriction. *International Journal of Production Economics*, *128*(1), 153–158. doi: 10.1016/j.ijpe.2010.06.008

Hu, Q., & Sobel, M. J. (2007). Echelon base-stock policies are financially sub-optimal. *Operations Research Letters*, *35*(5), 561–566. doi:10.1016/j.orl.2006.11.001

Hu, Q., Sobel, M. J., & Turcic, D. (2010). Optimization of inventory and dividends with risky debt. In *Proceedings of the 2010 MSOM Society Conference*.

Huang, Y.-F. (2004). Optimal retailer's replenishment policy for the EPQ model under the supplier's trade credit policy. *Production Planning & Control*, *15*(1), 27–33. doi:10.1080/09537280310001649372

Huang, Y.-F. (2007). Optimal retailer's replenishment decisions in the EPQ model under two levels of trade credit policy. *European Journal of Operational Research*, *176*(3), 1577–1591. doi:10.1016/j.ejor.2005.10.035

Jamal, A. M. M., Sarker, B. R., & Wang, S. (1997). An ordering policy for deteriorating items with allowable shortage and permissible delay in payment. *Journal of the Operational Research Society*, *48*(8), 826–833. doi: 10.1038/sj.jors.2600428

Katehakis, M. N., Melamed, B., & Shi, J. J. (2016). Cash-flow based dynamic inventory management. *Production and Operations Management*, *25*(9), 1558–1575. doi:10.1111/poms.12571

Kingsman, B. G. (1991). EOQ under date-terms supplier credit: A near-optimal solution. *Journal of the Operational Research Society*, *42*(9), 803–809. doi:10.1057/jors.1991.151

Konur, D., Campbell, J. F., & Monfared, S. A. (2017). Economic and environmental considerations in a stochastic inventory control model with order splitting under different delivery schedules among suppliers. *Omega*, *71*, 46–65. doi:10.1016/j.omega.2016.08.013

Konur, D., & Dagli, C. (2015). Military system of systems architecting with individual system contracts. *Optimization Letters*, *9*(8), 1749–1767. doi:10.1007/s11590-014-0821-z

Konur, D., & Golias, M. M. (2013). Cost-stable truck scheduling at a cross-dock facility with unknown truck arrivals: A meta-heuristic approach. *Transportation Research Part E: Logistics and Transportation Review*, *49*(1), 71–91. doi:10.1016/j.tre.2012.06.007

Konur, D., & Schaefer, B. (2016). Economic and environmental comparison of grouping strategies in coordinated multi-item inventory systems. *Journal of the Operational Research Society*, *67*(3), 421–436. doi:10.1057/jors.2015.67

Kouvelis, P., & Zhao, W. (2012). Financing the newsvendor: Supplier vs. bank, and the structure of optimal trade credit contracts. *Operations Research*, *60*(3), 566–580. doi:10.1287/opre.1120.1040

Kouvelis, P., & Zhao, W. (2016). Supply chain contract design under financial constraints and bankruptcy costs. *Management Science*, *62*(8), 2341–2357. doi:10.1287/mnsc.2015.2248

Kumar, R. S., & Goswami, A. (2015). A continuous review production-inventory system in fuzzy random environment: Minmax distribution free procedure. *Computers & Industrial Engineering*, *79*, 65–75. doi:10.1016/j.cie.2014.10.022

Kundu, A., & Chakrabarti, T. (2012). A multi-product continuous review inventory system in stochastic environment with budget constraint. *Optimization Letters*, *6*(2), 299–313. doi:10.1007/s11590-010-0245-3

Lai, G., Debo, L. G., & Sycara, K. (2009). Sharing inventory risk in supply chain: The implication of financial constraint. *Omega*, *37*(4), 811–825. doi:10.1016/j.omega.2008.06.003

Lee, C. H., & Rhee, B.-D. (2011). Trade credit for supply chain coordination. *European Journal of Operational Research*, *214*(1), 136–146. doi:10.1016/j.ejor.2011.04.004

Li, B., An, S-M., & Song, D-P. (2018). Selection of financing strategies with a risk-averse supplier in a capital-constrained supply chain. *Transportation Research Part E: Logistics and Transportation Review*, *118*, 163–183. doi:10.1016/j.tre.2018.06.007

Li, L., Shubik, M., & Sobel, M. J. (2013). Control of dividends, capital subscriptions, and physical inventories. *Management Science*, *59*(5), 1107–1124. doi:10.1287/mnsc.1120.1629

Lin, J. (1976). Multiple-objective problems: Pareto-optimal solutions by method of proper equality constraints. *IEEE Transactions on Automatic Control*, *21*(5), 641–650. 10.1109/TAC.1976.1101338

Luo, J., & Huang, P. (2002). EOQ under date-term supplier credit: A comment on a paper by Carlson and Rousseau. *Journal of the Operational Research Society*, *53*(10), 1150–1154. doi:10.1057/palgrave.jors.2601382

Maddah, B. S., Jaber, M. Y., & Abboud, N. E. (2004). Periodic review (*s*, *S*) inventory model with permissible delay in payments. *Journal of the Operational Research Society*, *55*(2), 147–159. doi:10.1057/palgrave.jors.2601675

Mahata, G. C. (2012). An EPQ-based inventory model for exponentially deteriorating items under retailer partial trade credit policy in supply chain. *Expert Systems with Applications*, *39*(3), 3537–3550. doi:10.1016/j.eswa.2011.09.044

Moinzadeh, K., & Nahmias, S. (1988). A continuous review model for an inventory system with two supply modes. *Management Science*, *34*(6), 761–773. doi:10.1287/mnsc.34.6.761

Moon, I., & Choi, S. (1994). The distribution free continuous review inventory system with a service level constraint. *Computers & Industrial Engineering*, *27*(1-4), 209–212. doi:10.1016/0360-8352(94)90272-0

Moon, I., & Gallego, G. (1994). Distribution free procedures for some inventory models. *Journal of the Operational Research Society*, *45*(6), 651–658. doi:10.1057/jors.1994.103

Moon, I., Shin, E., & Sarkar, B. (2014). Min-max distribution free continuous-review model with a service level constraint and variable lead time. *Applied Mathematics and Computation*, *229*, 310–315. doi:10.1016/j.amc.2013.12.034

Nahmias, S. (1981). Approximation techniques for several stochastic inventory models. *Computers & Operations Research*, *8*(3), 141–158. doi:10.1016/0305-0548(81)90004-6

Nahmias, S. (2009). *Production and operations analysis* (6th ed.). New York, NY: McGraw-Hill Irwin.

Nahmias, S., & Wang, S. S. (1979). A heuristic lot size reorder point model for decaying inventories. *Management Science*, *25*(1), 90–97. doi:10.1287/mnsc.25.1.90

Ninh, A., Hu, H., & Allen, D. (2019). Robust newsvendor problems: Effect of discrete demands. *Annals of Operations Research*, *275*(2), 607–621. doi:10.1007/s10479-018-3016-7

Olsson, F. (2014). Analysis of inventory policies for perishable items with fixed leadtimes and lifetimes. *Annals of Operations Research*, *217*(1), 399–423. doi:10.1007/s10479-014-1590-x

Perakis, G., & Roels, G. (2008). Regret in the newsvendor model with partial information. *Operations Research*, *56*(1), 188–203. doi:10.1287/opre.1070.0486

Protopappa-Sieke, M., & Seifert, R. W. (2010). Interrelating operational and financial performance measurements in inventory control. *European Journal of Operational Research*, *204*(3), 439–448. doi:10.1016/j.ejor.2009.11.001

Rachamadugu, R. (1989). Effect of delayed payments (trade credit) on order quantities. *Journal of the Operational Research Society*, *40*(9), 805–813. doi:10.1057/jors.1989.143

Raghavan, N. S., & Mishra, V. K. (2011). Short-term financing in a cash-constrained supply chain. *International Journal of Production Economics*, *134*(2), 407–412. doi:10.1016/j.ijpe.2009.11.014

Reindorp, M., Tanrisever, F., & Lange, A. (2018). Purchase order financing: Credit, commitment, and supply chain consequences. *Operations Research*, *66*(5), 1287–1303. doi:10.1287/opre.2018.1727

Rossetti, M. D., & Ünlü, Y. (2011). Evaluating the robustness of lead time demand models. *International Journal of Production Economics*, *134*(1), 159–176. doi:10.1016/j.ijpe.2011.06.010

Saghafian, S., & Tomlin, B. (2016). The newsvendor under demand ambiguity: Combining data with moment and tail information. *Operations Research*, *64*(1), 167–185. doi:10.1287/opre.2015.1454

Scarf, H. (1958). *A min-max solution of an inventory problem* (pp. 201–209). Stanford, CA: Stanford University Press.

Schaefer, B., & Konur, D. (2015). Economic and environmental considerations in a continuous review inventory control system with integrated transportation decisions. *Transportation Research Part E: Logistics and Transportation Review*, *80*, 142–165. doi:10.1016/j.tre.2015.05.006

Sculli, D., & Shum, Y. W. (1990). Analysis of a continuous review stock-control model with multiple suppliers. *Journal of the Operational Research Society*, *41*(9), 873–877. doi:10.1057/jors.1990.125

Sculli, D., & Wu, S. Y. (1981). Stock control with two suppliers and normal lead times. *The Journal of the Operational Research Society*, *32*(11), 1003–1009. doi:10.2307/2581854

Shin, D., Guchhait, R., Sarkar, B., & Mittal, M. (2016). Controllable lead time, service level constraint, and transportation discounts in a continuous review inventory model. *RAIRO – Operations Research*, *50*(4–5), 921–934. doi:10.1051/ro/2015055

Silver, E. A., & Costa, D. (1998). Some replenishment rules considering payment periods and risk of outdating. *Journal of the Operational Research Society*, *49*(8), 861–869. doi:10.1057/palgrave.jors.2600574

Silver, E. A., Pyke, D. P., & Peterson, R. (1998). *Inventory management and production planning and scheduling*. New York, NY: Wiley.

Smith, J. S., Sturrock, D. T., & Kelton, W. D. (2018). *Simio and simulation: Modeling, analysis, applications*. Pittsburgh, PA: Simio LLC.

Supply Chain Digest. (2006). *The 11 greatest supply chain disasters* (Tech. Rep.). Supply Chain Digest.

Tajbakhsh, M. M. (2010). On the distribution free continuous-review inventory model with a service level constraint. *Computers & Industrial Engineering*, *59*(4), 1022–1024. doi:10.1016/j.cie.2010.08.022

Taleizadeh, A. A., Pentico, D. W., Jabalameli, M. S., & Aryanezhad, M. (2013). An EOQ model with partial delayed payment and partial backordering. *Omega*, *41*(2), 354–368. doi:10.1016/j.omega.2012.03.008

Taleizadeh, A. A., Pourmohammad-Zia, N., & Konstantaras, I. (2019). Partial linked-to-order delayed payment and life time effects on decaying items ordering. *Operational Research An International Journal*, 1–23. doi:10.1007/s12351-019-00501-4

Tamjidzad, S., & Mirmohammadi, S. H. (2015). An optimal (*r*, *Q*) policy in a stochastic inventory system with all-units quantity discount and limited sharable resource. *European Journal of Operational Research*, *247*(1), 93–100. doi:10.1016/j.ejor.2015.05.073

Tamjidzad, S., & Mirmohammadi, S. H. (2017). Optimal (*r*, *Q*) policy in a stochastic inventory system with limited resource under incremental quantity discount. *Computers & Industrial Engineering*, *103*, 59–69. doi:10.1016/j.cie.2016.11.012

Tamjidzad, S., & Mirmohammadi, S. H. (2018). A two-stage heuristic approach for a multi-item inventory system with limited budgetary resource and all-units discount. *Computers & Industrial Engineering*, *124*, 293–303. doi:10.1016/j.cie.2018.07.032

Teng, J.-T. (2002). On the economic order quantity under conditions of permissible delay in payments. *Journal of the Operational Research Society*, *53*(8), 915–918. doi:10.1057/palgrave.jors.2601410

Teng, J.-T., & Chang, C.-T. (2009). Optimal manufacturer's replenishment policies in the EPQ model under two levels of trade credit policy. *European Journal of Operational Research*, *195*(2), 358–363. doi:10.1016/j.ejor.2008.02.001

Teng, J.-T., Min, J., & Pan, Q. (2012). Economic order quantity model with trade credit financing for non-decreasing demand. *Omega*, *40*(3), 328–335. doi:10.1016/j.omega.2011.08.001

Tyworth, J. E., & Ganeshan, R. (2000). A note on solutions to the (*Q*, *r*) inventory model for gamma lead-time demand. *International Journal of Physical Distribution & Logistics Management*, *30*(6), 534–539.

Tyworth, J. E., Guo, Y., & Ganeshan, R. (1996). Inventory control under gamma demand and random lead time. *Journal of Business Logistics*, *17*(1), 291–304.

Tyworth, J. E., & O'Neill, L. (1997). Robustness of the normal approximation of lead-time demand in a distribution setting. *Naval Research Logistics*, *44*(2), 165–186. doi:10.1002/(SICI)1520-6750(199703)44:2 < 165::AID-NAV2 > 3.0.CO;2-7

Uthayakumar, R., & Parvathi, P. (2009). A continuous review inventory model with controllable backorder rate and investments. *International Journal of Systems Science*, *40*(3), 245–254. doi:10.1080/00207720802299028

Vairaktarakis, G. L. (2000). Robust multi-item newsboy models with a budget constraint. *International Journal of Production Economics*, *66*(3), 213–226. doi:10.1016/S0925-5273(99)00129-2

Vasconcelos, B. C., & Marques, M. P. (2000). Reorder quantities for (*Q*, *R*) inventory models. *Journal of the Operational Research Society*, *51*(5), 635–638. doi:10.1057/palgrave.jors.2600933

Wang, C. X., Webster, S., & Suresh, N. C. (2009). Would a risk-averse newsvendor order less at a higher selling price? *European Journal of Operational Research*, *196*(2), 544–553. doi:10.1016/j.ejor.2008.04.002

Wang, T.-Y., & Hu, J.-M. (2008). An inventory control system for products with optional components under service level and budget constraints. *European Journal of Operational Research*, *189*(1), 41–58. doi:10.1016/j.ejor.2007.05.025

Xiao, S., Sethi, S. P., Liu, M., & Ma, S. (2017). Coordinating contracts for a financially constrained supply chain. *Omega*, *72*, 71–86. doi:10.1016/j.omega.2016.11.005

Xu, X., & Birge, J. R. (2006). Equity valuation, production, and financial planning: A stochastic programming approach. *Naval Research Logistics*, *53*(7), 641–655. doi:10.1002/nav.20182

Xu, X., Chen, X., Jia, F., Brown, S., Gong, Y., & Xu, Y. (2018). Supply chain finance: A systematic literature review and bibliometric analysis. *International Journal of Production Economics*, *204*, 160–173. doi:10.1016/j.ijpe.2018.08.003

Yan, N., He, X., & Liu, Y. (2019). Financing the capital-constrained supply chain with loss aversion: Supplier finance vs. supplier investment. *Omega*, *88*, 162–178. doi:10.1016/j.omega.2018.08.003

Yang, H., & Schrage, L. (2009). Conditions that cause risk pooling to increase inventory. *European Journal of Operational Research*, *192*(3), 837–851. doi:10.1016/j.ejor.2007.10.064

Yang, S. A., & Birge, J. R. (2018). Trade credit, risk sharing, and inventory financing portfolios. *Management Science*, *64*(8), 3667–3689. doi:10.1287/mnsc.2017.2799

Zhao, L., & Huchzermeier, A. (2015). Operations–finance interface models: A literature review and framework. *European Journal of Operational Research*, *244*(3), 905–917. doi:10.1016/j.ejor.2015.02.015

Zhao, X., Qiu, M., Xie, J., & He, Q. (2012). Computing (*r*, *Q*) policy for an inventory system with limited sharable resource. *Computers & Operations Research*, *39*(10), 2368–2379. doi:10.1016/j.cor.2011.12.012