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Lagrangian and Hamiltonian Mechanics on Fractals Subset of Real-Line

Alireza Khalili Golmankhaneh ·
Ali Khalili Golmankhaneh · Dumitru Baleanu

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Abstract A discontinuous media can be described by fractal dimensions. Fractal objects has special geometric properties, which are discrete and discontinuous structure. A fractal-time diffusion equation is a model for subdiffusive. In this work, we have generalized the Hamiltonian and Lagrangian dynamics on fractal using the fractional local derivative, so one can use as a new mathematical model for the motion in the fractal media. More, Poisson bracket on fractal subset of real line is suggested.

Keywords Fractal calculus · Lagrangian mechanics · Hamiltonian mechanics · Poisson bracket · Variational calculus

1 Introduction

It is well known that strange objects such as the Weierstrass continuous function, the van Koch curve, the Sierpiŝki gasket, coastlines, topographical surfaces, turbulence in fluid,

A.K. Golmankhaneh · A.K. Golmankhaneh
Department of Physics, Islamic Azad University, Urmia Branch, PO Box 969, Urmia, Iran

A.K. Golmankhaneh
e-mail: alirezakhalili2005@gmail.com

A.K. Golmankhaneh
e-mail: ali.khalili.gol@gmail.com

D. Baleanu (✉)
Department of Mathematics and Computer Science, Çankaya University, 06530 Ankara, Turkey
e-mail: dumitru@cankaya.edu.tr

D. Baleanu
Institute of Space Sciences, P.O. BOX, MG-23, R76900, Magurele-Bucharest, Romania

D. Baleanu
Department of Chemical and Materials Engineering, Faculty of Engineering, King Abdulaziz University, P.O. Box 80204, Jeddah, 21589, Saudi Arabia

etc. are called fractals [1, 2]. Initial motivation for analysis on fractals came from physicists working in disordered media. The heat and wave transfer in polymers, fractured and porous rocks and amorphous semiconductors can be modeled using fractals random walk on them see [3, 4]. Therefore, mathematicians is developing the analysis on fractals. For instance, probabilistic approach method namely they considered a sequence of random walks on fractals by taking a certain scaling factor, those random walks converge to diffusion on them. In the second approach, (analytic approach) instead of the sequence of random walks, a sequence of discrete Laplacians on a sequence of fractals. Here under the proper scaling these discrete Laplacians is converge to Laplacians on fractals. Also, using the measure-theoretical method and fractional calculous people has generalized a calculus on fractal since the ordinary calculus can't apply to them [1–34]. Riemann method for constructing a calculus has own place which has algorithmic method [34]. Local fractional derivative has suggested and applied in the science and engineering problems [10–14]. Recently, using F^α-calculus random walk on a fractal structure has studied. There are several researcher that they have studied anomalous diffusion but in most of these approaches simplest form central limit theorem is violated [15]. In this paper we have used F^α-calculus to generalized Lagrangian and Hamiltonian mechanics on fractal subset real line. So one can model the motion on fractal medium.

The organization of the paper is as follows:

We begin in Sect. 2 by reviewing the F^α-calculus. In Sect. 3 we introduced the Lagrangian and Hamiltonian mechanics on fractal subset of real line. Section 5 is dedicated to our conclusions.

2 Summery of Fractional F^α-Calculus

We begin by reviewing the F^α-calculus as follows [34].

2.1 The Mass Function and the Integral Staircase

Definition 1 F be a subset of real line ($\Re$). Let F be in the most cases a fractal. The flag function for a set F is denoted by $\theta(F,I)$ and define as [34].

$$\theta(F,I)=\begin{cases}1 & \text{if } F\cap I\neq\emptyset\\ 0 & \text{otherwise}\end{cases} \tag{1}$$

where $I=[a,b]$ is a interval in $\Re$.

Definition 2 For a set F and a subdivision $P_{[a,b]}$, $a<b$

$$\sigma^\alpha[F,I]=\sum_{i=1}^{n}\frac{(x_i-x_{i-1})^\alpha}{\Gamma(\alpha+1)}\theta\big(F,[x_{i-1},x_i]\big) \tag{2}$$

where $a<b$ and $0<\alpha\leq 1$.

Definition 3 Given $\delta>0$ and $a\leq b$ the coarse-grained mass $\gamma_\delta^\alpha(F,a,b)$ of $F\cap[a,b]$ is given by

$$\gamma_\delta^\alpha(F,a,b)=\inf_{P_{[a,b]}:|P|\leq\delta}\sigma^\alpha[F,I] \tag{3}$$

where $|P| = \max_{1\le i\le n}(x_i - x_{i-1})$. Taking infimum over all subdivisions P of $[a,b]$ satisfying $|P| \le \delta$.

Definition 4 The mass function $\gamma^\alpha(F,a,b)$ is given by [34]

$$\gamma^\alpha(F,a,b) = \lim_{\delta\to 0} \gamma_\delta^\alpha(F,a,b) \tag{4}$$

Definition 5 Let a_0 be an arbitrary but fixed real number. The integral staircase function $S_F^\alpha(x)$ of order α for a set F is given by [34]

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F,a_0,x) & \text{if } x \ge a_0 \\ -\gamma^\alpha(F,a_0,x) & \text{otherwise} \end{cases} \tag{5}$$

Definition 6 We say that a point x is a point of change of a function f if f is not constant over any open interval (a,d) containing x. The set of all points of change of f is called the set of change of f and is denoted by $\mathbf{S}\mathrm{ch}\, f$ [34].

Definition 7 The γ-dimension of $F \cap [a,b]$ denoted by $\dim_\gamma(F\cap[a,b])$ is

$$\begin{aligned}\dim_\gamma\big(F\cap[a,b]\big) &= \inf\big\{\alpha : \gamma^\alpha(F,a,b) = 0\big\} \\ &= \sup\big\{\alpha : \gamma^\alpha(F,a,b) = \infty\big\}\end{aligned} \tag{6}$$

Definition 8 Let $F \subset R$ be such that $S_F^\alpha(x)$ is finite for all $x \in R$ for $\alpha = \dim_\gamma F$. Then the $\mathbf{S}\mathrm{ch}(S_F^\alpha)$ is said to be α-perfect (closed and every point of $\mathbf{S}\mathrm{ch}(S_F^\alpha)$ is its limit point).

2.2 F-Limit and F-Continuity

Definition 9 Let $F \subset R$, $f : R \to R$ and $x \in F$. A number l is said to be the limit of f through the points of F or simply F-limit of f as $y \to x$ if given any $\epsilon > 0$ there exists $\delta > 0$ such that [34]

$$y \in F \quad \text{and} \quad |y-x| < \delta \quad \Rightarrow \quad \big|f(y) - l\big| < \epsilon \tag{7}$$

If such a number exists then it is denoted by

$$l = F - \lim_{y\to x} f(y) \tag{8}$$

Definition 10 A function $f : R \Rightarrow R$ is said to be F-continues at $x \in F$ if

$$f(x) = F - \lim_{y\to x} f(x) \tag{9}$$

2.3 F^α-Integration

Definition 11 Let f be bounded function on F and I be a closed interval [34]. Then

$$M[f,F,I] = \begin{cases} \sup_{x\in F\cap I} f(x) & \text{if } F\cap I \ne 0 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

and similarly

$$m[f,F,I]=\begin{cases}\inf_{x\in F\cap I} f(x) & \text{if } F\cap I\neq 0\\ 0 & \text{otherwise}\end{cases} \tag{11}$$

Definition 12 Let $S_F^\alpha(x)$ be finite for $x\in[a,b]$. Let P be a subdivision of $[a,b]$ with points $x_0,\dots,x_n$ The upper F^α-sum and the lower F^α-sum for function f over the subdivision P are given respectively by [34]

$$U^\alpha[f,F,P]=\sum_{i=1}^{n} M\big[f,F,[x_i,x_{i-1}]\big]\big(S_F^\alpha(x_i,x_{i-1})\big) \tag{12}$$

and

$$L^\alpha[f,F,P]=\sum_{i=1}^{n} m\big[f,F,[x_i,x_{i-1}]\big]\big(S_F^\alpha(x_i,x_{i-1})\big) \tag{13}$$

Definition 13 If f be a bounded function on F. we say that f is F^α-integrable on $[a,b]$ if [34]

$$\underline{\int_a^b} f(x)d_F^\alpha x=\sup_{P[a,b]} L^\alpha[f,F,P]=\overline{\int_a^b} f(x)d_F^\alpha x=\inf_{P[a,b]} L^\alpha[f,F,P] \tag{14}$$

In that case the F^α-integral of f on $[a,b]$ denoted by $\int_a^b f(x)d_F^\alpha x$ is given by the common value [34].

Properties (See [34])

$$\int_a^b f(x)d_F^\alpha x=\int_a^c f(x)d_F^\alpha x+\int_c^b f(x)d_F^\alpha x \tag{15}$$

$$\int_a^b \lambda f(x)d_F^\alpha x=\lambda\int_a^b f(x)d_F^\alpha x \quad \text{where } \lambda \text{ is constant.} \tag{16}$$

$$\int_a^b \big(f(x)+g(x)\big)d_F^\alpha x=\int_a^b f(x)d_F^\alpha x+\int_a^b g(x)d_F^\alpha x \tag{17}$$

$$\int_a^b f(x)\,d_F^\alpha x\geq\int_a^b g(x)d_F^\alpha x \quad \text{if } f(x)\geq g(x) \tag{18}$$

$$\int_b^a f(x)\,d_F^\alpha x=-\int_a^b g(x)\,d_F^\alpha x \quad \text{if } f(x)\geq g(x) \tag{19}$$

2.4 F^α-Differentiation

Definition 14 If F is an α-perfect set then the F^α-derivative of f at x is [34]

$$D_F^\alpha f(x)=\begin{cases}F\text{-}\lim_{y\to x}\frac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)} & \text{if } x\in F\\ 0 & \text{otherwise}\end{cases} \tag{20}$$

if the limit exists.

Properties

$$D_F^\alpha \lambda f(x) = \lambda D_F^\alpha f(x)$$

$$D_F^\alpha \big(f(x) + g(x)\big) = D_F^\alpha f(x) + D_F^\alpha g(x)$$

$$D_F^\alpha c = 0 \quad \text{if } c \text{ is constant.}$$

$$D_F^\alpha \big(u(x)v(x)\big) = \big(D_F^\alpha u(x)\big)v(x) + u(x)\big(D_F^\alpha v(x)\big)$$

2.5 Fundamental Theorem of F^α-Calculus

Theorem 1 *Let $F \subset R$ be an α-perfect set. If f is bounded on F is an F-continuous function on $F \cap [a,b]$ and* [34]

$$g(x) = \int_a^x f(y) d_F^\alpha y \tag{21}$$

for all $x \in [a,b]$ then

$$D_F^\alpha g(x) = f(x)\chi_F(x) \tag{22}$$

Theorem 2 *Let $f : R \to R$ be a continuous F^α-differentiable function such that $\mathbf{S}ch(f)$ is contained in an α-perfect set F and $h : R \to R$ be F-continuous such that*

$$D_F^\alpha f(x) = h(x)\chi_F(x) \tag{23}$$

then

$$\int_a^x h(x) d_F^\alpha x = f(b) - f(a) \tag{24}$$

Theorem 3 *Let the functions $u : R \to R$ $v : R \to R$ be such that*

1. *$u(x)$ is continuous on $[a,b]$ and $\mathbf{S}ch(u) \subset F$*
2. *$D_F^\alpha u(x)$ exists and is F-continuous on $[a,b]$*
3. *$v(x)$ is F-continuous on $[a,b]$* [34].

Then

$$\int_a^b u(x)v(x) d_F^\alpha x = \left[u(x)\int_a^x v(x') d_F^\alpha x'\right]_a^b - \int_a^b D_F^\alpha u(x) \int_a^x v(x') d_F^\alpha x' d_F^\alpha x. \tag{25}$$

Remark (See [34])

$$\int_0^y \big(S_F^\alpha(x)\big)^n d_F^\alpha x = \frac{1}{n+1}\big(S_F^\alpha(y)\big)^{n+1}$$

$$D_F^\alpha \big(S_F^\alpha(x)\big)^n = n\big(S_F^\alpha(x)\big)^{n-1}\chi_F(x)$$

3 Lagrangian and Hamiltonian Mechanics on Fractals

Our goal in this section is to derive the equations of motion for a particle the position is $x(t): F \to R$ that varies on fractal subset of real F. We aim to find a function L_F^α such that the paths of the particles between times $t_1 \in F$ and $t_2 \in F$ extremize the integral

$$A_F^\alpha = \int_{t_1}^{t_2} L_F^\alpha\big(t, x(t), {}^tD_F^\alpha x(t)\big) d_F^\alpha t \quad L_F^\alpha : F \times R \times R \to R,$$

where integral A_F^α is the action of the particle and the function L_F^α its Lagrangian. So we have

$$\begin{aligned}\delta A_F^\alpha &= \int_{t_1}^{t_2} \big[{}^xD_F^\alpha\, L_F^\alpha\, \delta x + {}^{{}^tD_F^\alpha x} D_F^\alpha\, L_F^\alpha\, \delta\big({}^tD_F^\alpha x(t)\big)\big] d_F^\alpha t = 0 \\ &= \int_{t_1}^{t_2} \big[{}^xD_F^\alpha\, L_F^\alpha\, \delta x + {}^{{}^tD_F^\alpha x} D_F^\alpha\, L_F^\alpha\, {}^tD_F^\alpha \delta\big(x(t)\big)\big] d_F^\alpha t = 0. \end{aligned} \tag{26}$$

Using Eq. (25) and $\delta(x(t_2)) = \delta(x(t_1)) = 0$ and let ${}^tD_F^\alpha x = \varrho$ we have

$$= \int_{t_1}^{t_2} \big[{}^xD_F^\alpha\, L_F^\alpha - {}^tD_F^\alpha\big({}^\varrho D_F^\alpha\, L_F^\alpha\big)\big]\delta(x) d_F^\alpha t = 0. \tag{27}$$

We arrive at Euler-Lagrange equation on fractal time set as

$${}^xD_F^\alpha\, L_F^\alpha - {}^tD_F^\alpha\big({}^\varrho D_F^\alpha\, L_F^\alpha\big) = 0. \tag{28}$$

Now if we define the generalized momentum $p_F^\alpha = {}^\varrho D_F^\alpha\, L_F^\alpha$ so then we have Hamiltonian as

$$H_F^\alpha = {}^tD_F^\alpha x\ p_F^\alpha - L_F^\alpha. \tag{29}$$

Applying d_F^α to both side of Eq. (29) we obtain

$$d_F^\alpha H_F^\alpha = {}^tD_F^\alpha x\ d_F^\alpha p_F^\alpha - {}^tD_F^\alpha\, L_F^\alpha\, d_F^\alpha t - {}^xD_F^\alpha\, L_F^\alpha\, d_F^\alpha x, \tag{30}$$

Eq. (30) shows that H_F^α is function of t, p_F^α, x so we have Hamilton equation on fractal subset of real line F as follows:

$${}^tD_F^\alpha x = {}^\zeta D_F^\alpha H_F^\alpha \quad \zeta = p_F^\alpha\ {}^tD_F^\alpha\, L_F^\alpha = {}^tD_F^\alpha H_F^\alpha\ {}^tD_F^\alpha\, p_F^\alpha = -{}^xD_F^\alpha H_F^\alpha. \tag{31}$$

Example 1 Suppose the Lagrangian of a particle is

$$L_F^\alpha\big(t, x(t), {}^tD_F^\alpha x\big) = a\big({}^tD_F^\alpha x\big)^2 - b\big(x(t)\big)^2, \quad c, e \text{ are constant.} \tag{32}$$

Therefore, using Eq. (28) the Lagrange equation will be as following

$$-2b\big(x(t)\big) = 2a\big({}^tD_F^\alpha\big)^2 x(t) \tag{33}$$

Example 2 Let the Hamiltonian of particle is

$$H_F^\alpha = c\big(p_F^\alpha\big)^2 + e\big(x(t)\big)^2, \quad c, e \text{ are constant.}$$

So the Hamilton's equation is

$$ {}^{t}D_F^{\alpha} p_F^{\alpha} = -2ex(t) \qquad {}^{t}D_F^{\alpha} x = 2cp_F^{\alpha}. \tag{34}$$

4 Poisson Bracket on Fractals

If $F(t, x(t), p_F^{\alpha})$ and $G(t, x(t), p_F^{\alpha})$ be any two function of dynamical variables the generalized Poisson bracket is define

$$ [F, G]_F^{\alpha} = {}^{x}D_F^{\alpha} F \, {}^{p_F^{\alpha}}D_F^{\alpha} G - {}^{p_F^{\alpha}}D_F^{\alpha} F \, {}^{x}D_F^{\alpha} G. \tag{35}$$

Therefore the Hamiltonian equation using Poisson brackets will be

$$ \left[p_F^{\alpha}, H_F^{\alpha}\right]_F^{\alpha} = {}^{t}D_F^{\alpha} p_F^{\alpha} \qquad \left[x, H_F^{\alpha}\right]_F^{\alpha} = {}^{t}D_F^{\alpha} x. \tag{36}$$

Poisson brackets are invariant under canonical transformation.

5 Conclusion

In this work F^{α}-action functional on the Sobolev Space is introduced. Then, we use least action to get the Hamiltonian and Lagrangian mechanics on fractal subset of real line. F^{α}-calculus is involving local and fractional derivative so that it's properties is similar to the ordinary calculus. Therefor, the time evolution of dynamical system has group properties unlike fractional nonlocal derivative that has semigroup property. Using this property we define generalized Poisson Bracket.

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