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An increasing variables singular system of fractional q-differential equations via numerical calculations

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Abstract

We investigate the existence of solutions for an increasing variables singular m-dimensional system of fractional q-differential equations on a time scale. In this singular system, the first equation has two variables and the number of variables increases permanently. By using some fixed point results, we study the singular system under some different conditions. Also, we provide two examples involving practical algorithms, numerical tables, and some figures to illustrate our main results.

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1 Introduction

The subject of q-difference equations was introduced by Jackson in the first decade of the last century [1]. The fractional calculus provides a meaningful generalization for the classical integration and differentiation to any order. It is known that working on quantum calculus is equivalent to traditional infinitesimal calculus without the notion of limits. In last decades, some researchers studied q-fractional difference equations [2–5]. Later, q-fractional boundary value problems have been considered by many researchers (see, for example, [6–13]). Nowadays many researchers focus on applications of fractional calculus [14–25] or analytical studies [26–36].

In 2013, Baleanu *et al.* investigated the coupled system of multi-term singular fractional integro-differential boundary value problem

$$\begin{cases} \mathcal{D}_{0^+}^{\sigma_1}[k](t) + w_1(t, k(t), l(t), \psi_{11}[k](t), \psi_{21}[l](t), \\ \quad \mathcal{D}_{0^+}^{\alpha_1}[k](t), \mathcal{D}_{0^+}^{\beta_{11}}[l](t), \mathcal{D}_{0^+}^{\beta_{12}}[l](t), \dots, \mathcal{D}_{0^+}^{\beta_{1m}}[l](t)) = 0, \\ \mathcal{D}_{0^+}^{\sigma_2}[l](t) + w_2(t, k(t), l(t), \psi_{12}[k](t), \psi_{22}[l](t), \\ \quad \mathcal{D}_{0^+}^{\alpha_2}[l](t), \mathcal{D}_{0^+}^{\beta_{21}}[k](t), \mathcal{D}_{0^+}^{\beta_{22}}[k](t), \dots, \mathcal{D}_{0^+}^{\beta_{2m}}[k](t)) = 0, \end{cases}$$

via boundary conditions $k^{(i)}(0) = l^{(i)}(0) = 0$ for $0 \leq i \leq n-2$, $\mathcal{D}_{0^+}^{\delta_1}[k](1) = 0$ for $2 < \delta_1 < n-1$, $\sigma_1 - \delta_1 \geq 1$, and $\mathcal{D}_{0^+}^{\delta_2}[l](1) = 0$ for $2 < \delta_2 < n-1$, $\sigma_2 - \delta_2 \geq 1$, where $n \geq 4$, $n-1 < \sigma_i < n$, $0 < \alpha_i < 1$, $1 < \beta_{ij} < 2$ for $i = 1,2$ and $j = 1,2,\dots,m$, γ_{ij} is positive-valued continuous functions on $[0,1] \times [0,1]$ $(i,j = 1,2)$, $\psi ij[k](t) = \int_0^t \gamma_{ij}(t,r)k(r)\,\mathrm{d}r$, w_1, w_2 satisfy the local Caratheodory condition on $[0,1] \times D(w_1,w_2 \in \mathrm{Car}([0,1] \times D))$, where $D \subset \mathbb{R}^{m+5}$ and w_i may be singular at the value zero of all its variables [37]. In 2016, Taieb *et al.* reviewed the fractional coupled system of nonlinear differential equations

$$\begin{cases} \mathcal{D}^{\sigma_1}[k](t) + \sum_{i=1}^m w_{1i}(t,k(t),l(t),\mathcal{D}^{\beta_1}[k](t),\mathcal{D}^{\beta_2}[l](t)) = 0, \\ \mathcal{D}^{\sigma_2}[l](t) + \sum_{i=1}^m w_{2i}(t,k(t),l(t),\mathcal{D}^{\beta_1}[k](t),\mathcal{D}^{\beta_2}[l](t)) = 0, \end{cases}$$

with boundary conditions $k(0) = k_0^*$, $l(0) = l_0^*$, $k'(0) = k''(0) = l'(0) = l''(0) = 0$, $k'''(0) = \mathcal{J}^{\alpha_1}[k](a_1)$, and $l'''(0) = \mathcal{J}^{\alpha_2}[k](a_2)$, where $t \in [0,1]$, $m \in \mathbb{N}^*$, $\alpha_j > 0$, $\sigma_j \in (3,4)$, $a_j \in (0,1)$, $\mathcal{D}^{\sigma_j}$, $\mathcal{D}^{\beta_j}$ are the Caputo derivatives and $\mathcal{J}^{\alpha_j}$ are the Riemann–Liouville fractional integrals [38]. In 2017, El Abidine studied the coupled system of nonlinear fractional equations

$$\begin{cases} \mathcal{D}^{\sigma_1}[k](t) = w_{1i}(t,l(t),\mathcal{D}^{\beta_1}[l](t)), \\ \mathcal{D}^{\sigma_2}[l](t) = w_{2i}(t,k(t),\mathcal{D}^{\beta_2}[k](t)), \end{cases}$$

with boundary conditions $k(0) = k^{(j)}(0) = 0$ and $l(0) = l^{(j)}(0) = 0$ for $1 \leq j \leq m-2$ with $m \geq 2$, where $t \in \mathbb{R}^+ = (0,\infty)$, $m-1 < \sigma_i \leq m$, $\beta_i \in (0,3)$ for $i = 1,2$, $0 < \beta_1 \leq \sigma_2 - 1$, $0 < \beta_2 \leq \sigma_1 - 1$, the differential operator is in the Riemann–Liouville sense and w_i are Borel measurable functions in $\mathbb{R}^{+3}$ satisfying some conditions [39].

By using the main idea of the above works, we investigate the increasing variables m-dimensional singular system of fractional q-differential equations

$$\begin{cases} {}^c\mathcal{D}_q^{\sigma_1}[k_1](t) = w_1(t,k_1(t)), \\ {}^c\mathcal{D}_q^{\sigma_2}[k_2](t) = w_2(t,k_1(t),k_2(t)), \\ \quad\vdots \\ {}^c\mathcal{D}_q^{\sigma_m}[k_m](t) = w_m(t,k_1(t),k_2(t),\dots,k_m(t)), \end{cases} \tag{1}$$

with boundary conditions $k_1(0) = {}_1b_0$, $k_i^{(j)}(0) = {}_ib_j$ for $j = 0,1,\dots i-2$ and $2 \leq i \leq m$, ${}^c\mathcal{D}_q^{\zeta_{i-1}}k_i(1) = 0$ for $\zeta_{i-1} \in [i-2,i-1]$ and $2 \leq i \leq m$, where $t \in J := (0,1]$, $m \geq 2$, $\sigma_i \in (i-1,i)$ for $1 \leq i \leq m$, ${}^c\mathcal{D}_q^{\sigma_i}$ denotes the Caputo fractional q-derivative of order σ_i, $w_i : J \times \mathbb{R}^i \to \mathbb{R}$ are continuous, $w_i(t,k_1,k_2,\dots,k_i)$ may be singular at $t = 0$ of its space variables, $\lim_{t\to 0^+} w_i(t,k_1,k_2,\dots,k_i) = \infty$, and there exists $0 < \alpha_1,\dots,\alpha_m < 1$ such that $t^{\alpha_1}w_1,\dots,t^{\alpha_m}w_m$ are continuous on $\overline{J} := [0,1]$.

2 Essential preliminaries

Throughout this article, we apply the time scales calculus notation [40]. In fact, we consider the fractional q-calculus on the time scale $\mathbb{T}_{t_0} = \{0\} \cup \{t : t = t_0q^n\}$, where $n \geq 0$, $t_0 \in \mathbb{R}$, and $q \in (0,1)$. Let $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$ [1]. The power function $(x-y)_q^{(n)}$ with $n \in \mathbb{N}_0$ is defined by $(x-y)_q^{(n)} = \prod_{k=0}^{n-1}(x-yq^k)$ for $n \geq 1$ and $(x-y)_q^{(0)} = 1$, where x and y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ [1, 2]. Also, $(x-y)_q^{(\alpha)} = x^\alpha \prod_{k=0}^{\infty}(x-yq^k)/(x-yq^{\alpha+k})$

Algorithm 1 The proposed method for calculated $(a-b)_q^{(\alpha)}$

```
function p = powerfunction(a, b, n, q)
%Power Gamma (a-b)^(n)
s=1;
if n==0
p=1
else
for k=1:n-1
s=s*(a-b*q^k)/(a- b*q^(alpha+k));
end
p=a^alpha * s;
end
end
```

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

```
function g = qGamma(q, x, n)
%q-Gamma Function
p=1;
for k=0:n
p=p*(1-q^(k+1))/(1- q^(x+k));
end;
g=p/(1-q)^(x-1);
end
```

Algorithm 3 The proposed method for calculated $(D_q f)(x)$

```
function g = Dq(q, x, n, fun)
if x==0
g=limit ((fun(x)-fun(q*x))/((1-q)*x),x,0);
else
g=(fun(x)-fun(q*x))/((1-q)*x);
end;
end
```

for $\alpha \in \mathbb{R}$ and $q \neq 0$. If $y = 0$, then it is clear that $x^{(\alpha)} = x^\alpha$ [6] (see Algorithm 1). The q-gamma function is given by $\Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1}$, where $z \in \mathbb{R}\backslash\{0,-1,-2,\dots\}$ [1]. Note that $\Gamma_q(z+1) = [z]_q\Gamma_q(z)$. Algorithm 2 shows a pseudo-code description of the technique for estimating q-gamma function of order n. The q-derivative of function f is defined by $(\mathcal{D}_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(\mathcal{D}_q f)(0) = \lim_{x\to 0}(\mathcal{D}_q f)(x)$, which is shown in Algorithm 3 [2, 3]. Furthermore, the higher order q-derivative of a function f is defined by $(D_q^n f)(x) = D_q(D_q^{n-1} f)(x)$ for $n \geq 1$, where $(D_q^0 f)(x) = f(x)$ [2, 3]. The q-integral of a function f is defined on $[0,b]$ by $I_q f(x) = \int_0^x f(s)\,\mathrm{d}_q s = x(1-q)\sum_{k=0}^\infty q^k f(xq^k)$ for $0 \leq x \leq b$, provided the series absolutely converges [2, 3]. If x in $[0,T]$, then

$$\int_x^T f(r)\,\mathrm{d}_q r = I_q f(T) - I_q f(x) = (1-q)\sum_{k=0}^\infty q^k\bigl[Tf\bigl(Tq^k\bigr) - xf\bigl(xq^k\bigr)\bigr],$$

whenever the series exists. In addition, we can interchange the order of double q-integral by $\int_0^t \int_0^s h(r)\,\mathrm{d}_q r\,\mathrm{d}_q s = \int_0^t \int_{qr}^t h(r)\,\mathrm{d}_q s\,\mathrm{d}_q r$ [41]. Actually, the interchange of order is true since

$$\int_0^t \int_{qr}^t \mathrm{d}_q s\,\mathrm{d}_q r = \int_0^t (t-qr)^{(\sigma-1)} h(r)\,\mathrm{d}_q r$$
$$= t(1-q)\sum_{i=0}^{\infty} q^i h(q^i t)(t - q^{i+1} t)$$
$$= t^2(1-q)^2 \sum_{i=0}^{\infty} q^i h(q^i t)\left(\sum_{i=0}^{\infty} q^i\right).$$

In addition the left-hand side can be written as follows:

$$\int_0^t \int_0^r h(s)\,\mathrm{d}_q s\,\mathrm{d}_q r = t(1-q)\sum_{i=0}^{\infty} q^i \int_0^{tq^i} h(r)\,\mathrm{d}_q r$$
$$= t^2(1-q)^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q^{i+2j} h(q^{i+j} t). \tag{2}$$

The operator I_q^n is given by $(I_q^0 h)(x) = h(x)$ and $(I_q^n h)(x) = (I_q(I_q^{n-1} h))(x)$ for all $n \geq 1$ and $h \in C([0,T])$ [2, 3]. It has been proved that $(D_q(I_q h))(x) = h(x)$ and $(I_q(D_q h))(x) = h(x) - h(0)$ whenever h is continuous at $x = 0$ [2, 3]. The fractional Riemann–Liouville type q-integral of the function h on $J = (0,1)$ for $\sigma \geq 0$ is defined by $\mathcal{I}_q^0[h](t) = h(t)$ and

$$\mathcal{I}_q^\sigma[h](t) = \frac{1}{\Gamma_q(\sigma)}\int_0^t (t-qr)^{(\sigma-1)} h(r)\,\mathrm{d}_q r$$
$$= t^\sigma (1-q)^\sigma \sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1}(1-q^{\sigma+i})}{\prod_{i=1}^{k-1}(1-q^{i+1})} h(tq^k)$$

for $t \in J$ [42]. Also, the Caputo fractional q-derivative of a function h is defined by

$${}^c\mathcal{D}_q^\sigma[h](t) = \mathcal{I}_q^{[\sigma]-\sigma}\left[\mathcal{D}_q^{[\sigma]}[h]\right](t)$$
$$= \frac{1}{\Gamma_q([\sigma]-\sigma)}\int_0^t (t-qr)^{([\sigma]-\sigma-1)} \mathcal{D}_q^{[\sigma]}[h](r)\,\mathrm{d}_q r,$$

where $t \in J$ and $\sigma > 0$ [42]. It has been proved that $\mathcal{I}_q^\beta(\mathcal{I}_q^\alpha[h])(x) = \mathcal{I}_q^{\alpha+\beta}[h](x)$ and $\mathcal{D}_q^\alpha[\mathcal{I}_q^\alpha[h]](x) = h(x)$, where $\alpha, \beta \geq 0$ [42]. Algorithm 5 shows MATLAB lines for $\mathcal{I}_q^\alpha[h](x)$.

Let $(\mathcal{E}, \rho)$ be a metric space. Denote by $\mathcal{P}(\mathcal{E})$ and $2^{\mathcal{E}}$ the class of all subsets and the class of all nonempty subsets of $\mathcal{E}$, respectively. Thus, $\mathcal{P}_{cl}(\mathcal{E})$, $\mathcal{P}_{bd}(\mathcal{E})$, $\mathcal{P}_{cv}(\mathcal{E})$, and $\mathcal{P}_{cp}(\mathcal{E})$ denote the class of all closed, bounded, convex, and compact subsets of $\mathcal{E}$, respectively. For each i, consider the space $E_i = \{k_i(t) : k_i(t) \in \mathcal{A}\}$ endowed with the norm $\|k_i\|_\infty = \max_{t \in \bar{J}} |k_i(t)|$, where $\mathcal{A} = C(\bar{J}, \mathbb{R})$. Also, define the product space $\mathcal{E} = E_1 \times \cdots \times E_m$ endowed with the norm $\|(k_1, \dots, k_m)\| = \max_{1 \leq i \leq m} \|k_i\|_\infty$. Then $(\mathcal{E}, \|.\|)$ is a Banach space [43]. Similar to the idea of the works [44, 45], define the set of the selections of $\mathcal{S}$ at k

by

$$S = \{k = (k_1, k_2, \dots, k_m) : k_i \in \mathcal{A}, i = 1, 2, \dots, m\}$$

for all $t \in \overline{J}$ and $k = (k_1, \dots, k_m) \in \mathcal{E}$. One can check that $S \neq \emptyset$ for all $k \in \mathcal{E}$ whenever $\dim \mathcal{E} < \infty$ [46]. We need next results.

Lemma 1 ([47, 48]) *The general solution of the q-fractional equation* ${}^{c}\mathcal{D}_q^{\sigma}[k](t) = 0$ *is given by* $k(t) = d_0 + d_1 t + d_2 t^2 + \cdots + d_{m-1} t^{m-1}$ *for* $\sigma > 0$, *where* $d_i \in \mathbb{R}$ *for* $i = 0, 1, \dots, m-1$ *and* $m = [\sigma] + 1$.

Theorem 2 ([43], Schauder's fixed point) *Assume that* $(\mathcal{E}, \rho)$ *is a complete metric space, S is a closed convex subset of* $\mathcal{E}$, *and* $\mathcal{N} : \mathcal{E} \to \mathcal{E}$ *is a map such that the set* $K = \{\mathcal{N}(k) : k \in S\}$ *is relatively compact in* $\mathcal{E}$. *Then* $\mathcal{N}$ *has at least one fixed point.*

3 Main results

Now, we are ready to provide our results about the m-dimensional system of singular fractional q-differential equations. First, we prove next basic result to give the integral representation of problem (1).

Lemma 3 *Let* $m \geq 2$ *for* $i \in \{1, 2, \dots, m\}$, $\sigma_i \in (i-1, i)$, $\varrho_1, \dots, \varrho_m \in \mathcal{A}$, *and* $t \in J$. *Then the m-dimensional system*

$$\begin{cases} {}^{c}\mathcal{D}_q^{\sigma_1}[k_1](t) = \varrho_1(t), \\ {}^{c}\mathcal{D}_q^{\sigma_2}[k_2](t) = \varrho_2(t), \\ \quad \vdots \\ {}^{c}\mathcal{D}_q^{\sigma_m}[k_m](t) = \varrho_m(t), \end{cases} \tag{3}$$

under the conditions

$$\begin{cases} k_1(0) = {}_1b_0, \\ k_i^{(j)}(0) = {}_ib_j, \qquad j = 0, 1, \dots i-2, \\ {}^{c}\mathcal{D}_q^{\zeta_{i-1}} k_i(1) = 0, \quad i-2 < \zeta_{i-1} < i-1 (2 \leq i \leq m), \end{cases} \tag{4}$$

has a unique solution $k = (k_1, k_2, \dots, k_m)$, *where*

$$k_i(t) = \begin{cases} \mathcal{I}_q^{\sigma_i}[\varrho_i](t) + {}_1b_0, & i = 1, \\ \mathcal{I}_q^{\sigma_i}[\varrho_i](t) + \sum_{j=0}^{i-2} \frac{{}_ib_j}{j!} t^j \\ \quad - \frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!} t^{i-1} \mathcal{I}_q^{\sigma_i - \zeta_{i-1}}[\varrho_i](1), & 2 \leq i \leq m. \end{cases} \tag{5}$$

Proof By using Lemma 1, we obtain the fractional q-integral equation

$$k_i(t) = \mathcal{I}_q^{\sigma_i}[\varrho_i](t) - \sum_{j=0}^{i-1} {}_id_j t^j \tag{6}$$

for $1 \leq i \leq m$. Let

$$D = \begin{pmatrix} {}_1d_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ {}_2d_0 & {}_2d_1 & 0 & 0 & \cdots & 0 & 0 \\ {}_3d_0 & {}_3d_1 & {}_3d_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ {}_{m-1}d_0 & {}_{m-1}d_1 & {}_{m-1}d_2 & {}_{m-1}d_3 & \cdots & {}_{m-1}d_{m-2} & 0 \\ {}_md_0 & {}_md_1 & {}_md_2 & {}_md_3 & \cdots & {}_md_{m-2} & {}_md_{m-1} \end{pmatrix} \in M_m(\mathbb{R}).$$

By using the assumptions, we find $k_1(0) = -{}_1d_0 = {}_1b_0$, $k_i^{(j)}(0) = -j!{}_id_j = {}_ib_j$ for $j = 0, 1, 2, \ldots, i-2$ and

$${}^c\mathcal{D}_q^{\zeta_{i-1}}[k_i](1) = \mathcal{I}_q^{\sigma_i-\zeta_{i-1}}[\varrho_i](1) - \frac{\Gamma_q(i)}{\Gamma_q(i-\zeta_{i-1})}{}_id_{i-1} = 0$$

for $i-2 < \zeta_{i-1} < i-1$, where $2 \leq i \leq m$. Thus, ${}_1d_0 = -{}_1b_0$ and

$${}_id_j = \begin{cases} -\frac{{}_ib_j}{j!}, & j = 0, 1, \ldots, i-2, \\ \frac{\Gamma_q(i-\zeta_{i-1})}{\Gamma_q(i)}\mathcal{I}_q^{\sigma_i-\zeta_{i-1}}[\varrho_i](1), & j = i-1, \end{cases} \tag{7}$$

for $2 \leq i \leq m$. By substituting these constants and (7) in (6), we find (5). □

Now, define the nonlinear operator $\mathcal{N} : S \to S$ by

$$\mathcal{N}[k_1, k_2, \ldots, k_m](t) = \begin{pmatrix} N_1(k_1)(t) \\ N_2(k_1, k_2)(t) \\ N_3(k_1, k_2, k_3)(t) \\ \vdots \\ N_m(k_1, k_2, \ldots, k_m)(t) \end{pmatrix}, \tag{8}$$

where

$$N_i(k_1, k_2, \ldots, k_i)(t) = \begin{cases} \mathcal{I}_q^{\sigma_i}[w_i](t, k_1(t)) + {}_1b_0, & i = 1, \\ \mathcal{I}_q^{\sigma_i}[w_i](t, k_1(t), \ldots, k_i(t)) & \\ \quad + [\sum_{j=0}^{i-2} \frac{{}_ib_j}{j!} t^j] - \frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!} t^{i-1} & \\ \quad \times \mathcal{I}_q^{\sigma_i-\zeta_{i-1}}[w_i](1, k_1(1), \ldots, k_i(1)), & 2 \leq i \leq m, \end{cases}$$

for $t \in \overline{J}$.

Lemma 4 *Let $m \geq 2$, $\sigma_1 \in (0,1)$, $\sigma_1 > \alpha_1$, $\sigma_i \in (i-1, i)$ for $i = 2, \ldots, m$, $\alpha_i \in (0,1)$ for $i = 1, 2, \ldots, m$, $f_i : J \to \mathbb{R}$ be a function with $\lim_{t\to 0^+} f_i(t) = \infty$, and the maps $t^{\alpha_i} f_i(t)$ be continuous on $\overline{J}$. Then the maps*

$$k_i(t) = \begin{cases} \mathcal{I}_q^{\sigma_i}[f_i](t) + {}_ib_0, & i = 1, \\ \mathcal{I}_q^{\sigma_i}[f_i](t) + \sum_{j=0}^{i-2} \frac{{}_ib_j}{j!} t^j & \\ \quad - \frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!} t^{i-1} \mathcal{I}_q^{\sigma_i-\zeta_{i-1}}[f_i](1) & 2 \leq i \leq m, \end{cases}$$

are continuous on $\overline{J}$.

Proof By using the definition of the maps $k_i(t)$, we have

$$k_i(t) = \begin{cases} \frac{1}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} f_i(r)\,\mathrm{d}_q r + {}_ib_0, & i=1, \\ \frac{1}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} f_i(r)\,\mathrm{d}_q r + \sum_{j=0}^{i-2} \frac{{}_ib_j}{j!} t^j \\ \quad - \frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} t^{i-1} \\ \quad \times \int_0^1 (1-qr)^{(\sigma_i-\zeta_{i-1}-1)} f_i(r)\,\mathrm{d}_q r, & 2 \le i \le m, \end{cases}$$

$$= \begin{cases} \frac{1}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r + {}_ib_0, & i=1, \\ \frac{1}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r \\ \quad + \sum_{j=1}^{i-2} \frac{{}_ib_j}{j!} t^j - \frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} t^{i-1} \\ \quad \times \int_0^1 (1-qr)^{(\sigma_i-\zeta_{i-1}-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r, & 2 \le i \le m, \end{cases}$$

and by the continuity of the maps $t^{\alpha_i} f_i(t)$, we get $k_i(0) = {}_ib_0$ for $i = 1, 2, \ldots, m$. Now, we consider some cases.

(I) Let $t_0 = 0$ and $t \in J$. Since $t^{\alpha_i} f_i(t)$ is continuous, there exist $M_1, \ldots, M_n > 0$ such that $|t^{\alpha_i} f_i(t)| \le M_i$ for all $t \in \overline{J}$. Thus,

$$\begin{aligned} &\left|k_i(t) - k_i(0)\right| \\ &= \begin{cases} \left|\frac{1}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r\right|, & i=1, \\ \left|\frac{1}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r \right. \\ \quad + \sum_{j=0}^{i-2} \frac{{}_ib_j}{j!} t^j - \frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} t^{i-1} \\ \quad \left. \times \int_0^1 (t-qr)^{(\sigma_i-\zeta_{i-1}-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r\right|, & 2 \le i \le m, \end{cases} \\ &\le \begin{cases} \frac{M_i}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r, & i=1, \\ \frac{M_i}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r \\ \quad + \sum_{j=1}^{i-2} \frac{|{}_ib_j|}{j!} t^j + \frac{\Gamma_q(i-\zeta_{i-1})M_i}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} t^{i-1} \\ \quad \times \int_0^1 (1-qr)^{(\sigma_i-\zeta_{i-1}-1)} r^{-\alpha_i}\,\mathrm{d}_q r. & 2 \le i \le m. \end{cases} \end{aligned}$$

Hence, by using the q-beta function, we get

$$\begin{aligned} &\left|k_i(t) - k_i(0)\right| \\ &\le \begin{cases} \frac{M_i t^{\sigma_i-\alpha_i}}{\Gamma_q(\sigma_i)} \int_0^1 (1-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r, & i=1, \\ \frac{M_i t^{\sigma_i-\alpha_i}}{\Gamma_q(\sigma_i)} \int_0^1 (1-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r \\ \quad + \sum_{j=1}^{i-2} \frac{|{}_ib_j|}{j!} t^j \\ \quad + \frac{\Gamma_q(i-\zeta_{i-1}) M_i B_q(\sigma_i-\zeta_{i-1}, 1-\alpha_i)}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} t^{i-1}, & 2 \le i \le m, \end{cases} \\ &\le \begin{cases} \frac{M_i B_q(\sigma_i, 1-\alpha_i) t^{\sigma_i-\alpha_i}}{\Gamma_q(\sigma_i)}, & i=1, \\ \frac{M_i B_q(\sigma_i, 1-\alpha_i) t^{\sigma_i-\alpha_i}}{\Gamma_q(\sigma_i)} + \sum_{j=1}^{i-2} \frac{|{}_ib_j|}{j!} t^j \\ \quad + \frac{\Gamma_q(i-\zeta_{i-1}) M_i B_q(\sigma_i-\zeta_{i-1}, 1-\alpha_i)}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} t^{i-1}, & 2 \le i \le m, \end{cases} \end{aligned}$$

which, by assumption $\sigma_1 > \alpha_1$ and the fact $\sigma_i > \alpha_i$, tend to zero as $t \to 0$ for $i = 1, 2, \ldots, m$.

(II) Let $t_0 \in (0,1)$ and $t \in (t_0, 1]$. Then we have

$$\begin{aligned}
&\bigl|k_i(t) - k_i(t_0)\bigr| \\
&= \begin{cases}
|\frac{1}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r \\
\quad - \frac{1}{\Gamma_q(\sigma_i)} \int_0^{t_0} (t_0-qr)^{(\sigma_i-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r|, \quad i=1, \\
|\frac{1}{\Gamma_q(\sigma_i)} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r \\
\quad - \frac{1}{\Gamma_q(\sigma_i)} \int_0^{t_0} (t_0-qr)^{(\sigma_i-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r| \\
\quad + \sum_{j=0}^{i-2} \frac{{}_ib_j}{j!} (t^j - t_0^j) \\
\quad + \frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} (t^{i-1} - t_0^{i-1}) \\
\quad \times |\int_0^1 (1-qr)^{(\sigma_i-\zeta_{i-1}-1)} r^{-\alpha_i} r^{\alpha_i} f_i(r)\,\mathrm{d}_q r|, \quad 2 \le i \le m,
\end{cases} \\
&\le \begin{cases}
\frac{M_i}{\Gamma_q(\sigma_i)} [\int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r \\
\quad - \int_0^{t_0} (t_0-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r], \quad i=1, \\
\frac{M_i}{\Gamma_q(\sigma_i)} [\int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r \\
\quad - \int_0^{t_0} (t_0-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r] \\
\quad + \sum_{j=0}^{i-2} \frac{|{}_ib_j|}{j!} (t^j - t_0^j) \\
\quad + \frac{\Gamma_q(i-\zeta_{i-1})M_i}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} (t^{i-1} - t_0^{i-1}) \\
\quad \times \int_0^1 (1-qr)^{(\sigma_i-\zeta_{i-1}-1)} r^{-\alpha_i}\,\mathrm{d}_q r. \quad 2 \le i \le m.
\end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigl|k_i(t) - k_i(t_0)\bigr| \\
&\le \begin{cases}
\frac{M_i B_q(\sigma_i, 1-\alpha_i)}{\Gamma_q(\sigma_i)} (t^{\sigma_i-\alpha_i} - t_0^{\sigma_i-\alpha_i}), \quad i=1, \\
\frac{M_i B_q(\sigma_i, 1-\alpha_i)}{\Gamma_q(\sigma_i)} (t^{\sigma_i-\alpha_i} - t_0^{\sigma_i-\alpha_i}) \\
\quad + \sum_{j=0}^{i-2} \frac{|{}_ib_j|}{j!} (t^j - t_0^j) \\
\quad + \frac{\Gamma_q(i-\zeta_{i-1}) M_i B_q(\sigma_i-\zeta_{i-1}, 1-\alpha_i)}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} \\
\quad \times (t^{i-1} - t_0^{i-1}), \quad 2 \le i \le m,
\end{cases}
\end{aligned}$$

which similar to case I tends to zero as $t \to 0$ for $i = 1, 2, \ldots, m$.

(III) Let $t_0 = 1$ and $t \in [0, t_0)$. By using similar arguments as in the previous case, one can obtain

$$\begin{aligned}
&\bigl|k_i(t) - k_i(t_0)\bigr| \\
&\le \begin{cases}
\frac{M_i B_q(\sigma_i, 1-\alpha_i)}{\Gamma_q(\sigma_i)} (t_0^{\sigma_i-\alpha_i} - t^{\sigma_i-\alpha_i}), \quad i=1, \\
\frac{M_i B_q(\sigma_i, 1-\alpha_i)}{\Gamma_q(\sigma_i)} (t_0^{\sigma_i-\alpha_i} - t^{\sigma_i-\alpha_i}) \\
\quad + \sum_{j=0}^{i-2} \frac{|{}_ib_j|}{j!} (t_0^j - t^j) \\
\quad + \frac{\Gamma_q(i-\zeta_{i-1}) M_i B_q(\sigma_i-\zeta_{i-1}, 1-\alpha_i)}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} \\
\quad \times (t_0^{i-1} - t^{i-1}), \quad 2 \le i \le m,
\end{cases}
\end{aligned}$$

which similar to the previous case tends to zero as $t \to 1$ for $i = 1, 2, \ldots, m$. This completes the proof. □

Lemma 5 *Let $m \geq 2$, $\sigma_1 \in (0,1)$, $\sigma_1 > \alpha_1$, $\sigma_i \in (i-1, i)$ for $i = 2, \ldots, m$, $\alpha_i \in (0,1)$ for $i = 1, 2, \ldots, m$, $w_i : J \times \mathbb{R}^i \to \mathbb{R}$ be a function with $\lim_{t \to 0^+} w_i(t, \ldots) = \infty$, and $t^{\alpha_i} w_i(t)$ be continuous on $\overline{J} \times \mathbb{R}^i$. Then the operator $\mathcal{N} : S \to S$ defined by Eq. (8) is completely continuous.*

Proof Let $({}_0k_1, {}_0k_2, \ldots, {}_0k_m) \in S$ with

$$\left\| (k_1, k_2, \ldots, k_m) - ({}_0k_1, {}_0k_2, \ldots, {}_0k_m) \right\| < 1,$$

and $\|({}_0k_1, {}_0k_2, \ldots, {}_0k_m)\| = l_0$ for all $(k_1, k_2, \ldots, k_m) \in S$. Hence,

$$\left\| (k_1, k_2, \ldots, k_m) \right\| < 1 + l_0 := l.$$

By using the continuity of the map $t^{\alpha_i} \varrho_i(t, k_1, k_2, \ldots, k_m)$, we get the map

$$t^{\alpha_i} \varrho_i(t, k_1, k_2, \ldots, k_m)$$

is uniformly continuous on $\overline{J} \times [-l, l]^i$. For each $\varepsilon > 0$, choose $\lambda \in (0,1)$ such that

$$\left| t^{\alpha_i} w_i\big(t, k_1(t), k_2(t), \ldots, k_i(t)\big) - t^{\alpha_i} w_i\big(t, {}_0k_1(t), {}_0k_2(t), \ldots, {}_0k_i(t)\big) \right| < \varepsilon \tag{9}$$

for all $t \in \overline{J}$ whenever $\|(k_1, k_2, \ldots, k_m) - ({}_0k_1, {}_0k_2, \ldots, {}_0k_m)\| < \lambda$. Thus,

$$\begin{aligned} &\left\| \mathcal{N}[k_1, k_2, \ldots, k_m](t) - \mathcal{N}[{}_0k_1, {}_0k_2, \ldots, {}_0k_m](t) \right\| \\ &\quad = \max_{1 \leq i \leq m} \|N_i(k_1, k_2, \ldots, k_i)(t) - N_i({}_0k_1, {}_0k_2, \ldots, {}_0k_i)(t)\|_\infty \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\left\| N_i(k_1, k_2, \ldots, k_i)(t) - N_i({}_0k_1, {}_0k_2, \ldots, {}_0k_i)(t) \right\|_\infty \\ &\quad \leq \begin{cases} \max_{t \in \overline{J}} \int_0^t \frac{(t-qr)^{(\sigma_i - 1)} r^{-\alpha_i}}{\Gamma_q(\sigma_i)} \\ \quad \times |r^{\alpha_i} w_i(r, k_i(r)) - r^{\alpha_i} w_i(r, {}_0k_i(r))| \, \mathrm{d}_q r, \quad i = 1, \\ \max_{t \in \overline{J}} \int_0^t \frac{(t-qr)^{(\sigma_i - 1)} r^{-\alpha_i}}{\Gamma_q(\sigma_i)} \\ \quad \times |r^{\alpha_i} w_i(r, k_1(r), \ldots, k_i(r)) \\ \quad - r^{\alpha_i} w_i(r, {}_0k_1(r), \ldots, {}_0k_i(r))| \, \mathrm{d}_q r \\ \quad + \max_{t \in \overline{J}} \frac{\Gamma_q(i - \zeta_{i-1})}{(i-1)!} t^{i-1} \\ \quad \times \int_0^1 \frac{(1-qr)^{(\sigma_i - \zeta_{i-1} - 1)} r^{-\alpha_i}}{\Gamma_q(\sigma_i - \zeta_{i-1})} \\ \quad \times |r^{\alpha_i} w_i(r, k_1(r), \ldots, k_i(r)) \\ \quad - r^{\alpha_i} w_i(r, {}_0k_1(r), \ldots, {}_0k_i(r))| \, \mathrm{d}_q r, \qquad 2 \leq i \leq m. \end{cases} \end{aligned}$$

Now, by using (9), we obtain

$$\begin{aligned}
&\bigl\|N_i(k_1,k_2,\ldots,k_i)(t) - N_i({}_0k_1,{}_0k_2,\ldots,{}_0k_i)(t)\bigr\|_\infty \\
&\quad\le \begin{cases} \frac{\varepsilon}{\Gamma_q(\sigma_i)} \max_{t\in\overline{J}} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r, & i=1, \\ \frac{\varepsilon}{\Gamma_q(\sigma_i)} \max_{t\in\overline{J}} \int_0^t (t-qr)^{(\sigma_i-1)} r^{-\alpha_i}\,\mathrm{d}_q r \\ \quad + \frac{\varepsilon\Gamma_q(i-\zeta_{i-1})}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})} \\ \max_{t\in\overline{J}} \int_0^t (1-qr)^{(\sigma_i-\zeta_{i-1}-1)} r^{-\alpha_i}\,\mathrm{d}_q r, & 2\le i\le m, \end{cases} \\
&\quad\le \begin{cases} \frac{\varepsilon B_q(\sigma_i,1-\alpha_i)}{\Gamma_q(\sigma_i)} \max_{t\in\overline{J}} t^{\sigma_i-\alpha_i}, & i=1, \\ \varepsilon\bigl[\frac{B_q(\sigma_i,1-\alpha_i)}{\Gamma_q(\sigma_i)} \max_{t\in\overline{J}} t^{\sigma_i-\alpha_i} \\ \quad + \frac{\Gamma_q(i-\zeta_{i-1})B_q(\sigma_i-\zeta_{i-1},1-\alpha_i)}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})}\bigr], & 2\le i\le m, \end{cases} \\
&\quad\le \varepsilon\Lambda_i,
\end{aligned}$$

where $\Lambda_i = \frac{\Gamma_q(1-\alpha_i)}{\Gamma_q(\sigma_i+1-\alpha_i)}$ whenever $i=1$ and

$$\Lambda_i = \frac{\Gamma_q(1-\alpha_i)}{\Gamma_q(\sigma_i+1-\alpha_i)} + \frac{\Gamma_q(i-\zeta_{i-1})\Gamma_q(1-\alpha_i)}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1}+1-\alpha_i)}, \tag{11}$$

whenever $2\le i\le m$. Now, by applying last result and (11), we get

$$\bigl\|N_i(k_1,k_2,\ldots,k_i)(t) - N_i({}_0k_1,{}_0k_2,\ldots,{}_0k_i)(t)\bigr\|_\infty \le \begin{cases} \varepsilon\Lambda_1, & i=1, \\ \varepsilon\Lambda_i, & 2\le i\le m. \end{cases} \tag{12}$$

Also, (10) and (11) imply that

$$\bigl\|\mathcal{N}[k_1,k_2,\ldots,k_m](t) - \mathcal{N}[{}_0k_1,{}_0k_2,\ldots,{}_0k_m](t)\bigr\| \le \varepsilon \max_{1\le i\le m} \Lambda_i$$

for all $t\in\overline{J}$. Hence, $\|\mathcal{N}[k_1,k_2,\ldots,k_m](t) - \mathcal{N}[{}_0k_1,{}_0k_2,\ldots,{}_0k_m](t)\| \to 0$ as

$$\bigl\|(k_1,k_2,\ldots,k_m) - ({}_0k_1,{}_0k_2,\ldots,{}_0k_m)\bigr\| \to 0.$$

Thus, the operator $\mathcal{N}$ is continuous. Now consider a bounded subset $K\subset S$. Then there exists a positive constant δ such that $\|(k_1,k_2,\ldots,k_m)\|\le\delta$ for all $(k_1,k_2,\ldots,k_m)\in K$. Since the maps $t^{\alpha_i}w_i(t,k_1,k_2,\ldots,k_i)$ are continuous on $\overline{J}\times[-\delta,\delta]^i$ for $i=1,2,\ldots,m$, there exist positive constants L_i such that

$$\bigl|t^{\alpha_i}w_i\bigl(t,k_1(t),k_2(t),\ldots,k_i(t)\bigr)\bigr| \le L_i \tag{13}$$

for all $t\in\overline{J}$ and $(k_1,k_2,\ldots,k_m)\in K$. Consider the norm

$$\bigl\|\mathcal{N}[k_1,k_2,\ldots,k_m](t)\bigr\| = \max_{1\le i\le m} \bigl\|N_i(k_1,k_2,\ldots,k_i)(t)\bigr\|_\infty. \tag{14}$$

Note that

$$\begin{aligned}
&\bigl\|N_i(k_1,k_2,\dots,k_i)(t)\bigr\|_\infty \\
&\quad\le \begin{cases} \max_{t\in\overline{J}}\int_0^t \frac{(t-qr)^{(\sigma_i-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i)} \\ \quad\times |r^{\alpha_i}w_i(r,k_i(r))|\,\mathrm{d}_qr + |{}_1b_0|, & i=1, \\ \max_{t\in\overline{J}}\int_0^t \frac{(t-qr)^{(\sigma_i-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i)} \\ \quad\times |r^{\alpha_i}w_i(r,k_1(r),\dots,k_i(r))|\,\mathrm{d}_qr \\ \quad+\sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!}\max_{t\in\overline{J}}t^j + \frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!}\max_{t\in\overline{J}}t^{i-1} \\ \quad\times\int_0^1\frac{(1-qr)^{(\sigma_i-\zeta_{i-1}-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i-\zeta_{i-1})} \\ \quad\times |r^{\alpha_i}w_i(r,k_1(r),\dots,k_i(r))|\,\mathrm{d}_qr, & 2\le i\le m. \end{cases}
\end{aligned}$$

Now, by using (13), we get

$$\begin{aligned}
&\bigl\|N_i(k_1,k_2,\dots,k_i)(t)\bigr\|_\infty \\
&\quad\le \begin{cases} \frac{L_i}{\Gamma_q(\sigma_i)}\max_{t\in\overline{J}}\int_0^t(t-qr)^{(\sigma_i-1)}r^{-\alpha_i}\,\mathrm{d}_qr + |{}_ib_0|, & i=1, \\ \frac{L_i}{\Gamma_q(\sigma_i)}\max_{t\in\overline{J}}\int_0^t(t-qr)^{(\sigma_i-1)}r^{-\alpha_i}\,\mathrm{d}_qr \\ \quad+\sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!} + \frac{L_i\Gamma_q(i-\zeta_{i-1})}{\Gamma_q(\sigma_i-\zeta_{i-1})} \\ \quad\times\int_0^1(1-qr)^{(\sigma_i-\zeta_{i-1}-1)}r^{-\alpha_i}\,\mathrm{d}_qr, & 2\le i\le m, \end{cases} \\
&\quad\le \begin{cases} \frac{L_i\Gamma_q(1-\alpha_i)}{\Gamma_q(\sigma_i+1-\alpha_i)}\max_{t\in\overline{J}}t^{\sigma_i-\alpha_i} + |{}_ib_0|, & i=1, \\ L_i[\frac{\Gamma_q(1-\alpha_i)}{\Gamma_q(\sigma_i+1-\alpha_i)}\max_{t\in\overline{J}}t^{\sigma_i-\alpha_i} \\ \quad+\frac{\Gamma_q(i-\zeta_{i-1})\Gamma_q(1-\alpha_i)}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1}+1-\alpha_i)}] + \sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!}, & 2\le i\le m, \end{cases} \\
&\quad\le \begin{cases} L_i\Lambda_i + |{}_ib_0|, & i=1, \\ L_i\Lambda_i + \sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!}, & 2\le i\le m. \end{cases}
\end{aligned} \tag{15}$$

On the other hand, by using (14) and (15), we get

$$\bigl\|\mathcal{N}[k_1,k_2,\dots,k_m](t)\bigr\| \le \max_{1\le i\le m}\left\{L_1\Lambda_1 + |{}_ib_0|, L_i\Lambda_i + \sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!}\right\}.$$

Thus $\mathcal{N}(K)$ is bounded. Let $(k_1,k_2,\dots,k_m)\in K$ and $t_1,t_2\in\overline{J}$ with $t_1<t_2$. Then we have

$$\begin{aligned}
&\bigl\|\mathcal{N}[k_1,k_2,\dots,k_m](t_2) - \mathcal{N}[k_1,k_2,\dots,k_m](t_1)\bigr\| \\
&\quad= \max_{1\le i\le m}\bigl\|N_i(k_1,k_2,\dots,k_i)(t_2) - N_i(k_1,k_2,\dots,k_i)(t_1)\bigr\|_\infty
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\bigl\|N_i(k_1,k_2,\dots,k_i)(t_2)-N_i(k_1,k_2,\dots,k_i)(t_1)\bigr\|_\infty \\
&\quad\le \begin{cases}
\max_{t\in\bar{J}} \bigl| \int_0^{t_2} \frac{(t_2-qr)^{(\sigma_i-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i)} r^{\alpha_i} w_i(r,k_i(r))\,\mathrm{d}_q r \\
\quad - \int_0^{t_1} \frac{(t_1-qr)^{(\sigma_i-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i)} r^{\alpha_i} w_i(r,k_i(r))\,\mathrm{d}_q r\bigr|, & i=1, \\
\max_{t\in\bar{J}} \bigl| \int_0^{t_2} \frac{(t_2-qr)^{(\sigma_i-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i)} r^{\alpha_i} w_i(r,k_1(r),\dots,k_i(r))\,\mathrm{d}_q r \\
\quad - \int_0^{t_1} \frac{(t_1-qr)^{(\sigma_i-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i)} r^{\alpha_i} w_i(r,k_1(r),\dots,k_i(r))\,\mathrm{d}_q r\bigr| \\
\quad + \sum_{j=0}^{i-2} \frac{|{}_ib_j|}{j!}(t_2^j - t_1^j) + \frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!}(t_2^{i-1}-t_1^{i-1}) \\
\quad \times \int_0^1 \frac{(1-qr)^{(\sigma_i-\zeta_{i-1}-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i-\zeta_{i-1})} \\
\quad \times |r^{\alpha_i} w_i(r,k_1(r),\dots,k_i(r))\,\mathrm{d}_q r|\,\mathrm{d}_q r, & 2\le i\le m.
\end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigl\|N_i(k_1,k_2,\dots,k_i)(t_2)-N_i(k_1,k_2,\dots,k_i)(t_1)\bigr\|_\infty \\
&\quad\le \begin{cases}
\frac{L_i\Gamma_q(1-\alpha_i)}{\Gamma_q(\sigma_i+1-\alpha_i)}(t_2^{\sigma_i-\alpha_i}-t_1^{\sigma_i-\alpha_i}), & i=1, \\
\frac{L_i\Gamma_q(1-\alpha_i)}{\Gamma_q(\sigma_i+1-\alpha_i)}(t_2^{\sigma_i-\alpha_i}-t_1^{\sigma_i-\alpha_i}) \\
\quad + \sum_{j=0}^{i-2} \frac{|{}_ib_j|}{j!}(t_2^j-t_1^j) \\
\quad + \frac{L_i\Gamma_q(i-\zeta_{i-1})\Gamma_q(1-\alpha_i)}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1}+1-\alpha_i)}(t_2^{i-1}-t_1^{i-1}), & 2\le i\le m.
\end{cases}
\end{aligned} \tag{17}$$

Now, by using (16) and (17), we obtain

$$\begin{aligned}
&\bigl\|\mathcal{N}[k_1,k_2,\dots,k_m](t_2)-\mathcal{N}[k_1,k_2,\dots,k_m](t_1)\bigr\| \\
&\quad= \max_{1\le i\le m}\Biggl\{ \frac{L_i\Gamma_q(1-\alpha_1)}{\Gamma_q(\sigma_1+1-\alpha_1)}\bigl(t_2^{\sigma_1-\alpha_1}-t_1^{\sigma_1-\alpha_1}\bigr), \\
&\qquad \frac{L_i\Gamma_q(1-\alpha_i)}{\Gamma_q(\sigma_i+1-\alpha_i)}\bigl(t_2^{\sigma_i-\alpha_i}-t_1^{\sigma_i-\alpha_i}\bigr) + \sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!}\bigl(t_2^j-t_1^j\bigr) \\
&\qquad + \frac{L_i\Gamma_q(i-\zeta_{i-1})\Gamma_q(1-\alpha_i)}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1}+1-\alpha_i)}\bigl(t_2^{i-1}-t_1^{i-1}\bigr)\Biggr\}.
\end{aligned} \tag{18}$$

The right-hand side of (18) is independent of $(k_1,k_2,\dots,k_m)$ and, by assumption $\sigma_1 > \alpha_1$ and the fact $\sigma_i > \alpha_i$, tends to zero as $t_1 \to t_2$. This implies that $\mathcal{N}(K)$ is equicontinuous. Now, by using the Arzelà–Ascoli theorem, we conclude that $\mathcal{N}$ is completely continuous. □

Theorem 6 *The m-dimensional system of singular fractional q-differential equations* (1) *has a unique solution on* $\bar{J}$ *whenever there exist nonnegative constants* ${}_i\eta_j$ $(j = 1,2,\dots,i$, $i = 1,2,\dots,m$, $m \ge 2)$ *satisfying*

$$t^{\alpha_i}\bigl|w_i(t,k_1,\dots,k_i)-w_i(t,l_1,\dots,l_i)\bigr| \le \sum_{j=1}^{i} {}_i\eta_j|k_j-l_j| \tag{19}$$

for all $t \in \overline{J}$ and $(k_1, \dots, k_i)$, $(l_1, \dots, l_i) \in \mathbb{R}^i$, and also

$$\Sigma = \max_{2 \le i \le m} \left\{ {}_1\eta_1 \Lambda_1, \sum_{j=1}^{i} {}_i\eta_j \Lambda_i \right\} < 1, \tag{20}$$

where the constants Λ_i are defined by (11).

Proof We prove that $\mathcal{N}$ is a contractive operator on S. Assume that $(k_1, k_2, \dots, k_m) \in S$ and $(l_1, l_2, \dots, l_m) \in S$. Then we have

$$\begin{aligned} &\bigl\| \mathcal{N}[k_1, k_2, \dots, k_m](t) - \mathcal{N}[l_1, l_2, \dots, l_m](t) \bigr\| \\ &\quad = \max_{1 \le i \le m} \bigl\| N_i(k_1, k_2, \dots, k_i)(t) - N_i(l_1, l_2, \dots, l_i)(t) \bigr\|_\infty \end{aligned} \tag{21}$$

for almost all $t \in \overline{J}$. Hence,

$$\begin{aligned} &\bigl\| N_i(k_1, k_2, \dots, k_i)(t) - N_i(l_1, l_2, \dots, l_i)(t) \bigr\|_\infty \\ &\quad \le \begin{cases} \max_{t \in \overline{J}} \int_0^t \frac{(t-qr)^{(\sigma_i - 1)} r^{-\alpha_i}}{\Gamma_q(\sigma_i)} \\ \quad \times r^{\alpha_i} |w_i(r, k_i(r)) - w_i(r, l_i(r))| \, \mathrm{d}_q r, & i = 1, \\ \max_{t \in \overline{J}} \int_0^t \frac{(t-qr)^{(\sigma_i - 1)} r^{-\alpha_i}}{\Gamma_q(\sigma_i)} \\ \quad \times r^{\alpha_i} |w_i(r, k_1(r), \dots, k_i(r)) \\ \quad - w_i(r, l_1(r), \dots, l_i(r))| \, \mathrm{d}_q r \\ \quad + \max_{t \in \overline{J}} \frac{\Gamma_q(i - \zeta_{i-1})}{(i-1)!} t^{i-1} \int_0^1 \frac{(1-qr)^{(\sigma_i - \zeta_{i-1} - 1)} r^{-\alpha_i}}{\Gamma_q(\sigma_i - \zeta_{i-1})} \\ \quad \times r^{\alpha_i} |w_i(r, k_1(r), \dots, k_i(r)) \\ \quad - w_i(r, l_1(r), \dots, l_i(r))| \, \mathrm{d}_q r, & 2 \le i \le m. \end{cases} \end{aligned}$$

Now, by using (19), we obtain

$$\begin{aligned} &\bigl\| N_i(k_1, k_2, \dots, k_i)(t) - N_i(l_1, l_2, \dots, l_i)(t) \bigr\|_\infty \\ &\quad \le \begin{cases} \frac{{}_i\eta_1}{\Gamma_q(\sigma_i)} \|k_i - l_i\|_\infty \max_{t \in \overline{J}} \int_0^t (t - qr)^{(\sigma_i - 1)} r^{-\alpha_i}, & i = 1, \\ (\sum_{j=1}^i {}_i\eta_j \|k_i - l_j\|_\infty) [\max_{t \in \overline{J}} \int_0^t \frac{(t-qr)^{(\sigma_i - 1)} r^{-\alpha_i}}{\Gamma_q(\sigma_i)} \\ \quad + \frac{\Gamma_q(i - \zeta_{i-1})}{(i-1)! \Gamma_q(\sigma_i - \zeta_{i-1})} \\ \quad \times \int_0^1 (1 - qr)^{(\sigma_i - \zeta_{i-1} - 1)} r^{-\alpha_i} \, \mathrm{d}_q r], & 2 \le i \le m, \end{cases} \\ &\quad \le \begin{cases} \frac{{}_i\eta_1 B_q(\sigma_i, 1 - \alpha_i)}{\Gamma_q(\sigma_i)} \|k_i - l_i\|_\infty \max_{t \in \overline{J}} t^{\sigma_i - \alpha_i}, & i = 1, \\ \sum_{j=1}^i {}_i\eta_j \max_{1 \le i \le m} \|k_i - l_i\|_\infty [\frac{B_q(\sigma_i, 1 - \alpha_i)}{\Gamma_q(\sigma_i)} \max_{t \in \overline{J}} t^{\sigma_i - \alpha_i} \\ \quad + \frac{\Gamma_q(i - \zeta_{i-1}) B_q(\sigma_i - \zeta_{i-1}, 1 - \alpha_i)}{(i-1)! \Gamma_q(\sigma_i - \zeta_{i-1})}], & 2 \le i \le m, \end{cases} \\ &\quad \le \begin{cases} \frac{{}_i\eta_1 \Gamma_q(1 - \alpha_i)}{\Gamma_q(\sigma_i + 1 - \alpha_i)} \|k_i - l_i\|_\infty, & i = 1, \\ \sum_{j=1}^i {}_i\eta_j [\frac{\Gamma_q(1 - \alpha_i)}{\Gamma_q(\sigma_i + 1 - \alpha_i)} \\ \quad + \frac{\Gamma_q(i - \zeta_{i-1}) \Gamma_q(1 - \alpha_i)}{(i-1)! \Gamma_q(\sigma_i - \zeta_{i-1} + 1 - \alpha_i)}] \\ \quad \times \|(k_1 - l_1, k_2 - l_2, \dots, k_i - l_i)\|_\infty, & 2 \le i \le m. \end{cases} \end{aligned} \tag{22}$$

If we apply (21) and (22), then we get

$$\begin{aligned}&\big\|\mathcal{N}[k_1,k_2,\ldots,k_m](t)-\mathcal{N}[l_1,l_2,\ldots,l_m](t)\big\|\\&\quad\le\max_{2\le i\le m}\left\{{}_1\eta_1\Lambda_1,\sum_{j=1}^{i}{}_i\eta_j\Lambda_i\right\}\big\|(k_1-l_1,k_2-l_2,\ldots,k_i-l_i)(t)\big\|_\infty.\end{aligned}$$

Now, by using (20), we have

$$\Sigma=\max_{2\le i\le m}\left\{{}_1\eta_1\Lambda_1,\sum_{j=1}^{i}{}_i\eta_j\Lambda_i\right\}<1.$$

Hence, $\mathcal{N}$ is a contraction. By using the Banach contraction principle, $\mathcal{N}$ has a unique fixed point which is the unique solution for system (1). □

Now, we consider different conditions on system (1).

Theorem 7 *Let $m\ge 2$, $\sigma_1\in(0,1)$, $\sigma_1>\alpha_1$, $\sigma_i\in(i-1,i)$ for $i=2,\ldots,m$, $\alpha_i\in(0,1)$ for $i=1,2,\ldots,m$, $w_i:J\times\mathbb{R}^i\to\mathbb{R}$ be functions with $\lim_{t\to0^+}w_i(t,\ldots)=\infty$, and $t^{\alpha_i}w_i(t,\ldots)$ be continuous maps on $\overline{J}\times\mathbb{R}^i$. Then system (1) has a solution on $\overline{J}$.*

Proof Assume that

$$L_i=\max_{t\in\overline{J}}t^{\alpha_i}\big|w_i\big(t,k_1(t),\ldots,k_i(t)\big)\big|\tag{23}$$

and define the set $K_r\subset S$ by

$$K_r=\big\{(k_1,k_2,\ldots,k_m)\in S:\big\|(k_1,k_2,\ldots,k_m)\big\|\le r\big\},$$

where

$$r=\max_{2\le i\le m}\left\{L_1\Lambda_1+|{}_1b_0|,L_i\Lambda_i+\sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!}\right\}.\tag{24}$$

We show that $\mathcal{N}$ maps K_r into K_r. For $(k_1,k_2,\ldots,k_m)\in K_r$ and $t\in\overline{J}$, put

$$\mathcal{N}[k_1,k_2,\ldots,k_m](t)=\max_{1\le i\le m}\big\|N_i(k_1,k_2,\ldots,k_i)(t)\big\|_\infty.\tag{25}$$

Thus, we have

$$\begin{aligned}
&\bigl\|N_i(k_1,k_2,\dots,k_i)(t)-N_i(l_1,l_2,\dots,l_i)(t)\bigr\|_\infty\\
&\quad\le\begin{cases}
\max_{t\in\bar{J}}\int_0^t\frac{(t-qr)^{(\sigma_i-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i)}r^{\alpha_i}\\
\quad\times|w_i(r,k_i(r))-w_i(r,l_i(r))|\,\mathrm{d}_qr, & i=1,\\
\max_{t\in\bar{J}}\int_0^t\frac{(t-qr)^{(\sigma_i-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i)}r^{\alpha_i}\\
\quad\times|w_i(r,k_1(r),\dots,k_i(r))-w_i(r,l_1(r),\dots,l_i(r))|\,\mathrm{d}_qr\\
\quad+\max_{t\in\bar{J}}\frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!}t^{i-1}\\
\quad\times\int_0^1\frac{(1-qr)^{(\sigma_i-\zeta_{i-1}-1)}r^{-\alpha_i}}{\Gamma_q(\sigma_i-\zeta_{i-1})}r^{\alpha_i}\\
\quad\times|w_i(r,k_1(r),\dots,k_i(r))\\
-w_i(r,l_1(r),\dots,l_i(r))|\,\mathrm{d}_qr, & 2\le i\le m,
\end{cases}\\
&\quad\le\begin{cases}
\frac{L_i}{\Gamma_q(\sigma_i)}\max_{t\in\bar{J}}\int_0^t(t-qr)^{(\sigma_i-1)}r^{-\alpha_i}r^{\alpha_i}\,\mathrm{d}_qr+|{}_ib_0|, & i=1,\\
\frac{L_i}{\Gamma_q(\sigma_i)}\max_{t\in\bar{J}}\int_0^t(t-qr)^{(\sigma_i-1)}r^{-\alpha_i}r^{\alpha_i}\,\mathrm{d}_qr\\
\quad+[\sum_{j=0}^{i-2}\frac{{}_ib_j}{j!}]+\frac{L_i\Gamma_q(i-\zeta_{i-1})}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1})}\\
\quad\times\int_0^1(1-qr)^{(\sigma_i-\zeta_{i-1}-1)}r^{-\alpha_i}\,\mathrm{d}_qr, & 2\le i\le m.
\end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\bigl\|N_i(k_1,k_2,\dots,k_i)(t)\bigr\|_\infty\\
&\quad\le\begin{cases}
\frac{L_1\Gamma_q(1-\alpha_i)}{\Gamma_q(\sigma_i+1-\alpha_i)}\max_{t\in\bar{J}}t^{\sigma_i-\alpha_i}+|{}_ib_0|, & i=1,\\
L_i[\frac{\Gamma_q(1-\alpha_i)}{\Gamma_q(\sigma_i1-\alpha_i)}\max_{t\in\bar{J}}t^{\sigma_i-\alpha_i}\\
\quad+\frac{\Gamma_q(i-\zeta_{i-1})}{(i-1)!\Gamma_q(\sigma_i-\zeta_{i-1}1-\alpha_i)}]+\sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!}, & 2\le i\le m,
\end{cases}\\
&\quad\le\begin{cases}
L_i\Lambda_i+|{}_ib_0|, & i=1,\\
L_i\Lambda_i+\sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!}, & 2\le i\le m.
\end{cases}
\end{aligned}\tag{26}$$

Now, by using (25) and (26), we conclude that

$$\bigl\|\mathcal{N}[k_1,k_2,\dots,k_m](t)\bigr\|\le\max_{2\le i\le m}\left\{L_1\Lambda_1+|{}_1b_0|,L_i\lambda_i+\sum_{j=0}^{i-2}\frac{|{}_ib_j|}{j!}\right\},\tag{27}$$

and so $\|\mathcal{N}[k_1,k_2,\dots,k_m](t)\|\le r$. By using Lemma 4, we get

$$\mathcal{N}[k_1,k_2,\dots,k_m](t)\in C(\bar{J}).$$

Moreover, $\mathcal{N}[k_1,k_2,\dots,k_m](t)\in K_r$ for $(k_1,k_2,\dots,k_m)\in K_r$. Thus $\mathcal{N}(K_r)\subset K_r$, and so $\mathcal{N}$ maps K_r into K_r. On the other hand, by using Lemma 5, $\mathcal{N}$ is completely continuous. Now, by using Lemma 2, the map $\mathcal{N}$ has a fixed point which is a solution for system (1). □

Now, we provide two examples to illustrate our main results. In this way, we give a computational technique for checking the m-dimensional system (1). We need to present a

Algorithm 4 The proposed method for calculated $\int_a^b f(r)\,d_q r$

```
function g = Iq(q, x, n, fun)
p=1;
for k=0:n
p=p+ q^k*fun(x*q^k);
end;
g=x* (1-q) * p;
end
```

Algorithm 5 The proposed method for calculated $I_q^{\sigma}[x]$

```
function g = Iq_sigma(q, sigma,  x, n, fun)
p=0;
for k=0:n
s1=1;
for i=0:k-1
s1=s1*(1-q^(sigma+i));
end
s2=1;
for i=0:k-1
s2=s2*(1-q^(i+1));
end
p=p + q^k*s1*fun(x*q^k)/s2;
end;
g=round((x^sigma)* ((1-q)^sigma)* p, 6);
end
```

simplified analysis which is able to execute the values of the q-gamma function. For this purpose, we provide a pseudo-code description of the method for calculation of the q-gamma function of order n in Algorithms 2, 3, 5, and 4.

Example 1 Consider the increasing variables singular 5-dimensional system of fractional q-differential equations

$$\begin{cases} {}^c\mathcal{D}_q^{\frac{7}{9}}[k_1](t) = w_1(t,k_1), \\ {}^c\mathcal{D}_q^{\frac{8}{7}}[k_2](t) = w_2(t,k_1,k_2), \\ {}^c\mathcal{D}_q^{\frac{11}{4}}[k_3](t) = w_3(t,k_1,k_2,k_3), \\ {}^c\mathcal{D}_q^{\frac{16}{5}}[k_4](t) = w_4(t,k_1,k_2,k_3,k_4), \\ {}^c\mathcal{D}_q^{\frac{31}{7}}[k_5](t) = w_5(t,k_1,k_2,k_3,k_4,k_5), \end{cases} \tag{28}$$

under the boundary value conditions $k_1(0) = \frac{7}{9}$, $k_2(0) = \frac{3}{5}$,

$$\begin{cases} k_3(0) = \frac{1}{2}, \quad k_3'(0) = 2\sqrt{3}, \\ k_4(0) = \sqrt{5}, \quad k_4'(0) = \frac{\sqrt{5}}{3}, \quad k_4''(0) = \frac{15}{7}, \\ k_5(0) = \frac{\sqrt{3}}{3}, \quad k_5'(0) = 1, \quad k_5''(0) = 0, \quad k_5'''(0) = \frac{13}{4}, \end{cases}$$

and ${}^c\mathcal{D}_q^{\frac{1}{2}}[k_2](1) = {}^c\mathcal{D}_q^{\frac{4}{3}}[k_3](1) = {}^c\mathcal{D}_q^{\frac{5}{2}}[k_4](1) = {}^c\mathcal{D}_q^{\frac{11}{3}}[k_5](1) = 0$, where $t \in (0,1]$. Put

$$w_1(t,k_1) = \frac{3\cos^2 k_1(t)}{20\sqrt{t}},$$

$$w_2(t,k_1,k_2) = \frac{2(|k_1(t)| + |k_2(t)|)}{25\pi \sqrt[3]{t}(1 + |k_1(t)| + |k_2(t)|)},$$

$$w_3(t,k_1,k_2,k_3) = \frac{\sin k_1(t) - \cos k_2(t) + \sin k_3(t)}{20\pi \sqrt[5]{t^3}},$$

$$w_4(t,k_1,k_2,k_3,k_4) = \frac{\cos^2 k_1(t) + \sin^2 k_2(t) + \cos^2 k_3(t) + \sin^2 k_4(t)}{25\pi \sqrt[7]{t^5}(1 + \cos^2 k_1(t) + \sin^2 k_2(t) + \cos^2 k_3(t) + \sin^2 k_4(t))},$$

$$w_5(t,k_1,k_2,k_3,k_4,k_5) = \frac{|k_1(t)| + |k_2(t)| + |k_3(t)| + |k_4(t)| - |k_5(t)|}{20\pi \sqrt[6]{t^5}(1 + \exp(|k_1(t)| + |k_2(t)| + |k_3(t)| + |k_4(t)| - |k_5(t)|))},$$

$m = 5$, $\sigma_1 = \frac{7}{9} \in (0,1)$, $\sigma_2 = \frac{8}{7} \in (1,2)$, $\sigma_3 = \frac{11}{4} \in (2,3)$, $\sigma_4 = \frac{16}{5} \in (3,4)$, $\sigma_5 = \frac{31}{7} \in (4,5)$, $\zeta_1 = \frac{1}{2} \in [0,1]$, $\zeta_2 = \frac{4}{3} \in [1,2]$, $\zeta_3 = \frac{5}{2} \in [2,3]$, $\zeta_4 = \frac{11}{3} \in [3,4]$, ${}_1b_0 = \frac{7}{9}$, ${}_2b_0 = \frac{3}{5}$, ${}_3b_0 = \frac{1}{2}$, ${}_4b_0 = \sqrt{5}$, ${}_5b_0 = \frac{\sqrt{3}}{3}$, ${}_3b_1 = 2\sqrt{3}$, ${}_4b_1 = \frac{\sqrt{5}}{3}$, ${}_5b_1 = 1$, ${}_4b_2 = \frac{15}{7}$, ${}_5b_2 = 0$, and ${}_5b_3 = \frac{13}{4}$. Now, we check inequalities (19) and (20). For each $t \in \bar{J}$, $(k_1,k_2,\dots,k_5)$, and $(l_1,l_2,\dots,l_5) \in \mathbb{R}^5$, we have

$$\begin{aligned} t^{\alpha_1}\bigl|w_1\bigl(t,k_1(t)\bigr) - w_1\bigl(t,l_1(t)\bigr)\bigr| &\le t^{\frac{4}{7}}\left|\frac{3\cos^2 k_1(t)}{20\pi\sqrt{t}} - \frac{3\cos^2 l_1(t)}{20\pi\sqrt{t}}\right| \\ &\le \frac{3t^{\frac{1}{14}}}{20}\bigl|\cos^2 k_1(t) - \cos^2 l_1(t)\bigr| \\ &\le \frac{3t^{\frac{1}{14}}}{10\pi}\bigl|\sin k_1(t) - \sin l_1(t)\bigr| \le \frac{3t^{\frac{1}{14}}}{10\pi}\bigl|k_1(t) - l_1(t)\bigr|, \end{aligned}$$

$\alpha_1 = \frac{4}{7}$, ${}_1\eta_1 = \frac{3}{10\pi}$,

$$\begin{aligned} &t^{\alpha_2}\bigl|w_2\bigl(t,k_1(t),k_2(t)\bigr) - w_2\bigl(t,l_1(t),l_2(t)\bigr)\bigr| \\ &\quad\le t^{\frac{2}{5}}\left|\frac{2(|k_1(t)| + |k_2(t)|)}{25\pi\sqrt[3]{t}(1 + |k_1(t)| + |k_2(t)|)} - \frac{2(|l_1(t)| + |l_2(t)|)}{25\pi\sqrt[3]{t}(1 + |l_1(t)| + |l_2(t)|)}\right| \\ &\quad\le \frac{2t^{\frac{1}{15}}}{25\pi}\bigl|\bigl|k_1(t)\bigr| + |k_2(t)| - \bigl(\bigl|l_1(t)\bigr| + \bigl|l_2(t)\bigr|\bigr)\bigr| \\ &\quad\le \frac{2t^{\frac{1}{15}}}{25\pi}\bigl[\bigl|k_1(t) - l_1(t)\bigr| + \bigl|k_2(t) - l_2(t)\bigr|\bigr], \end{aligned}$$

$\alpha_2 = \frac{2}{5}$, ${}_2\eta_1 = {}_2\eta_2 = \frac{2}{25\pi}$,

$$\begin{aligned} &t^{\alpha_3}\bigl|w_3\bigl(t,k_1(t),k_2(t),k_3(t)\bigr) - w_3\bigl(t,l_1(t),l_2(t),l_3(t)\bigr)\bigr| \\ &\quad\le t^{\frac{5}{8}}\left|\frac{\sin k_1(t) - \cos k_2(t) + \sin k_3(t)}{20\pi\sqrt[5]{t^3}} - \frac{\sin l_1(t) - \cos l_2(t) + \sin l_3(t)}{20\pi\sqrt[5]{t^3}}\right| \\ &\quad\le \frac{t^{\frac{1}{40}}}{20\pi}\bigl|\bigl(\sin k_1(t) - \cos k_2(t) + \sin k_3(t)\bigr) - \bigl(\sin l_1(t) - \cos l_2(t) + \sin l_3(t)\bigr)\bigr| \\ &\quad\le \frac{t^{\frac{1}{40}}}{20\pi}\bigl[\bigl|k_1(t) - l_1(t)\bigr| + \bigl|k_2(t) - l_2(t)\bigr| + \bigl|k_3(t) - l_3(t)\bigr|\bigr], \end{aligned}$$

$\alpha_3 = \frac{5}{8}$, ${}_3\eta_1 = {}_3\eta_2 = {}_3\eta_3 = \frac{1}{20\pi}$,

$$\begin{aligned}
&t^{\alpha_4}\bigl|w_4\bigl(t,k_1(t),k_2(t),k_3(t)k_4(t)\bigr)-w_4\bigl(t,l_1(t),l_2(t),l_3(t),l_4(t)\bigr)\bigr|\\
&\quad\le t^{\frac{7}{9}}\biggl|\frac{\cos^2 k_1(t)+\sin^2 k_2(t)+\cos^2 k_3(t)+\sin^2 k_4(t)}{25\pi\sqrt[7]{t^5}(1+\cos^2 k_1(t)+\sin^2 k_2(t)+\cos^2 k_3(t)+\sin^2 k_4(t))}\\
&\qquad-\frac{\cos^2 l_1(t)+\sin^2 l_2(t)+\cos^2 l_3(t)+\sin^2 l_4(t)}{25\pi\sqrt[7]{t^5}(1+\cos^2 l_1(t)+\sin^2 l_2(t)+\cos^2 l_3(t)+\sin^2 l_4(t))}\biggr|\\
&\quad\le\frac{t^{\frac{4}{63}}}{25\pi}\bigl|\bigl(\cos^2 k_1(t)+\sin^2 k_2(t)+\cos^2 k_3(t)+\sin^2 k_4(t)\bigr)\\
&\qquad-\bigl(\cos^2 l_1(t)+\sin^2 l_2(t)+\cos^2 l_3(t)+\sin^2 l_4(t)\bigr)\bigr|\\
&\quad\le\frac{t^{\frac{4}{63}}}{25\pi}\bigl[\bigl|\cos^2 k_1(t)-\cos^2 l_1(t)\bigr|+\bigl|\sin^2 k_2(t)-\sin^2 l_2(t)\bigr|\\
&\qquad+\bigl|\cos^2 k_3(t)-\cos^2 l_3(t)\bigr|+\bigl|\sin^2 k_4(t)-\sin^2 l_4(t)\bigr|\bigr]\\
&\quad\le\frac{2t^{\frac{4}{63}}}{25\pi}\bigl[\bigl|\sin k_1(t)-\sin l_1(t)\bigr|+\bigl|\sin k_2(t)-\sin l_2(t)\bigr|\\
&\qquad+\bigl|\sin k_3(t)-\sin l_3(t)\bigr|+\bigl|\sin k_4(t)-\sin l_4(t)\bigr|\bigr],\\
&\quad\le\frac{2t^{\frac{4}{63}}}{25\pi}\bigl[\bigl|k_1(t)-l_1(t)\bigr|+\bigl|k_2(t)-l_2(t)\bigr|+\bigl|k_3(t)-l_3(t)\bigr|+\bigl|k_4(t)-l_4(t)\bigr|\bigr],
\end{aligned}$$

$\alpha_4 = \frac{7}{9}$, ${}_4\eta_1 = {}_4\eta_2 = {}_4\eta_3 = {}_4\eta_4 = \frac{2}{25\pi}$,

$$\begin{aligned}
&t^{\alpha_5}\bigl|w_5\bigl(t,k_1(t),k_2(t),k_3(t)k_4(t),k_5(t)\bigr)-w_5\bigl(t,l_1(t),l_2(t),l_3(t),l_4(t),l_5(t)\bigr)\bigr|\\
&\quad\le t^{\frac{10}{11}}\biggl|\frac{|k_1(t)|+|k_2(t)|+|k_3(t)|+|k_4(t)|-|k_5(t)|}{20\sqrt[6]{t^5}(1+\exp(|k_1(t)|+|k_2(t)|+|k_3(t)|+|k_4(t)|-|k_5(t)|))}\\
&\qquad-\frac{|l_1(t)|+|l_2(t)|+|l_3(t)|+|l_4(t)|-|l_5(t)|}{20\pi\sqrt[6]{t^5}(1+\exp(|l_1(t)|+|l_2(t)|+|l_3(t)|+|l_4(t)|-|l_5(t)|))}\biggr|\\
&\quad\le\frac{t^{\frac{5}{66}}}{20\pi}\bigl|\bigl|k_1(t)\bigr|+\bigl|k_2(t)\bigr|+\bigl|k_3(t)\bigr|+|k_4(t)|-|k_5(t)|\\
&\qquad-\bigl(\bigl|l_1(t)\bigr|+\bigl|l_2(t)\bigr|+\bigl|l_3(t)\bigr|+\bigl|l_4(t)\bigr|-\bigl|l_5(t)\bigr|\bigr)\bigr|\\
&\quad\le\frac{t^{\frac{5}{66}}}{20\pi}\bigl[\bigl|k_1(t)-l_1(t)\bigr|+\bigl|k_2(t)-l_2(t)\bigr|+\bigl|k_3(t)-l_3(t)\bigr|\\
&\qquad+\bigl|k_4(t)-l_4(t)\bigr|+\bigl|k_5(t)-l_5(t)\bigr|\bigr],
\end{aligned}$$

and $\alpha_5 = \frac{10}{11}$, ${}_5\eta_1 = {}_5\eta_2 = {}_5\eta_3 = {}_5\eta_4 = {}_5\eta_5 = \frac{1}{20\pi}$. On the other hand, by using (11), we obtain

$$\Lambda_1 = \frac{\Gamma_q(1-\alpha_1)}{\Gamma_q(\sigma_1+1-\alpha_1)} = \frac{\Gamma_q(1-\frac{4}{7})}{\Gamma_q(\frac{7}{9}+1-\frac{4}{7})} = \frac{\Gamma_q(\frac{3}{7})}{\Gamma_q(\frac{76}{63})},$$

$$\begin{aligned}
\Lambda_2 &= \frac{\Gamma_q(1-\alpha_2)}{\Gamma_q(\sigma_2+1-\alpha_2)} + \frac{\Gamma_q(2-\zeta_1)\Gamma_q(1-\alpha_2)}{\Gamma_q(\sigma_2-\zeta_1+1-\alpha_2)} \\
&= \frac{\Gamma_q(1-\frac{6}{7})}{\Gamma_q(\frac{8}{7}+1-\frac{6}{7})} + \frac{\Gamma_q(2-\frac{1}{2})\Gamma_q(1-\frac{6}{7})}{\Gamma_q(\frac{8}{7}-\frac{1}{2}+1-\frac{6}{7})} \\
&= \frac{\Gamma_q(\frac{1}{7})}{\Gamma_q(\frac{9}{7})} + \frac{\Gamma_q(\frac{3}{2})\Gamma_q(\frac{1}{7})}{\Gamma_q(\frac{11}{14})}, \\
\Lambda_3 &= \frac{\Gamma_q(1-\alpha_3)}{\Gamma_q(\sigma_3+1-\alpha_3)} + \frac{\Gamma_q(3-\zeta_2)\Gamma_q(1-\alpha_3)}{2!\Gamma_q(\sigma_3-\zeta_2+1-\alpha_3)} \\
&= \frac{\Gamma_q(1-\frac{5}{8})}{\Gamma_q(\frac{11}{4}+1-\frac{5}{8})} + \frac{\Gamma_q(3-\frac{4}{3})\Gamma_q(1-\frac{5}{8})}{2!\Gamma_q(\frac{11}{4}-\frac{4}{3}+1-\frac{5}{8})} \\
&= \frac{\Gamma_q(\frac{3}{8})}{\Gamma_q(\frac{25}{8})} + \frac{\Gamma_q(\frac{5}{3})\Gamma_q(\frac{3}{8})}{2!\Gamma_q(\frac{43}{24})}, \\
\Lambda_4 &= \frac{\Gamma_q(1-\alpha_4)}{\Gamma_q(\sigma_4+1-\alpha_4)} + \frac{\Gamma_q(4-\zeta_3)\Gamma_q(1-\alpha_4)}{3!\Gamma_q(\sigma_4-\zeta_3+1-\alpha_4)} \\
&= \frac{\Gamma_q(1-\frac{7}{9})}{\Gamma_q(\frac{16}{5}+1-\frac{7}{9})} + \frac{\Gamma_q(4-\frac{5}{2})\Gamma_q(1-\frac{7}{9})}{3!\Gamma_q(\frac{16}{5}-\frac{5}{2}+1-\frac{7}{9})} \\
&= \frac{\Gamma_q(\frac{2}{9})}{\Gamma_q(\frac{154}{45})} + \frac{\Gamma_q(\frac{3}{2})\Gamma_q(\frac{2}{9})}{3!\Gamma_q(\frac{83}{90})}, \\
\Lambda_5 &= \frac{\Gamma_q(1-\alpha_5)}{\Gamma_q(\sigma_5+1-\alpha_5)} + \frac{\Gamma_q(5-\zeta_4)\Gamma_q(1-\alpha_5)}{4!\Gamma_q(\sigma_5-\zeta_4+1-\alpha_5)} \\
&= \frac{\Gamma_q(1-\frac{10}{11})}{\Gamma_q(\frac{31}{7}+1-\frac{10}{11})} + \frac{\Gamma_q(5-\frac{11}{3})\Gamma_q(1-\frac{10}{11})}{4!\Gamma_q(\frac{31}{7}-\frac{11}{3}+1-\frac{10}{11})} \\
&= \frac{\Gamma_q(\frac{1}{11})}{\Gamma_q(\frac{348}{77})} + \frac{\Gamma_q(\frac{4}{3})\Gamma_q(\frac{1}{11})}{4!\Gamma_q(\frac{197}{231})}.
\end{aligned}$$

Tables 1, 2, and 3 show $\Lambda_i \approx 1.4269, 6.1292, 2.1068, 2.2574, 3.8301$, $\Lambda_i \approx 1.9041, 9.5549, 2.2455, 2.2349, 2.4713$, $\Lambda_i \approx 2.1668, 11.5144, 2.2172, 2.0036, 1.4726$ for $1 \le i \le 5$ and for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively. It is clear that $\sum_{j=1}^{2} {}_2\eta_j = \frac{4}{25\pi}$, $\sum_{j=1}^{3} {}_3\eta_j = \frac{3}{20\pi}$, $\sum_{j=1}^{4} {}_2\eta_j = \frac{8}{25\pi}$, and $\sum_{j=1}^{5} {}_2\eta_j = \frac{1}{4\pi}$. In Tables 4, 5, and 6, we can see that $\Sigma = 0.3122, 0.4866$, and 0.5864, indeed

$$\Sigma = \max_{2\le i\le m}\left\{ {}_1\eta_1\Lambda_1, \sum_{j=1}^{i} {}_i\eta_j\Lambda_i \right\} < 1,$$

for $q = \frac{1}{10}, \frac{1}{2}$, and $\frac{6}{7}$, respectively (Fig. 1). Thus, the assumptions and conditions of Theorem 6 hold. Hence the singular 5-dimensional system of fractional q-differential equations (28) has a unique solution on $(0,1]$. Note that Algorithm 6 shows us how we can obtain the parameters of Example 1.

Table 1 Some numerical results of Λ_i in Example 1 for $q = \frac{1}{10}$

n	$q = \frac{1}{10}$				
	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
1	1.4220	6.0895	2.0978	2.2431	3.7960
2	1.4264	6.1253	2.1059	2.2559	3.8267
3	1.4268	6.1288	2.1067	2.2572	3.8298
4	1.4269	6.1292	2.1068	2.2573	3.8301
5	1.4269	6.1292	2.1068	2.2574	3.8301
6	1.4269	6.1292	2.1068	2.2574	3.8301
7	1.4269	6.1292	2.1068	2.2574	3.8301

Table 2 Some numerical results of Λ_i in Example 1 for $q = \frac{1}{2}$

n	$q = \frac{1}{2}$				
	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
1	1.6027	7.8053	1.7179	1.589	1.5801
2	1.7551	8.689	1.9738	1.8957	1.9947
3	1.8300	9.1239	2.1076	2.0611	2.225
⋮	⋮	⋮	⋮	⋮	⋮
10	1.9035	9.5516	2.2445	2.2336	2.4696
11	1.9038	9.5533	2.245	2.2343	2.4706
12	1.9039	9.5541	2.2453	2.2346	2.4711
13	1.9040	9.5545	2.2454	2.2348	2.4713
14	1.9040	9.5547	2.2455	2.2349	2.4714
15	1.9041	9.5548	2.2455	2.2349	2.4715
16	1.9041	9.5549	2.2455	2.2349	2.4715
17	1.9041	9.5549	2.2455	2.2349	2.4715
18	1.9041	9.5549	2.2455	2.2349	2.4715
19	1.9041	9.5549	2.2455	2.235	2.4716
20	1.9041	9.5549	2.2455	2.235	2.4716
21	1.9041	9.5549	2.2455	2.235	2.4716

Table 3 Some numerical results of Λ_i in Example 1 for $q = \frac{6}{7}$

n	$q = \frac{6}{7}$				
	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
1	0.8523	5.0338	0.5331	0.4502	0.1978
2	1.0644	6.0573	0.7033	0.5582	0.2622
3	1.2371	6.8796	0.8621	0.6681	0.3312
⋮	⋮	⋮	⋮	⋮	⋮
57	2.1668	11.5134	2.2168	2.0031	1.4721
58	2.1668	11.5136	2.2168	2.0032	1.4722
59	2.1668	11.5137	2.2169	2.0033	1.4722
60	2.1668	11.5138	2.2169	2.0033	1.4723
61	2.1668	11.5139	2.217	2.0034	1.4723
⋮	⋮	⋮	⋮	⋮	⋮
68	2.1668	11.5143	2.2171	2.0035	1.4725
69	2.1668	11.5144	2.2171	2.0036	1.4725
70	2.1668	11.5144	2.2172	2.0036	1.4725
71	2.1668	11.5144	2.2172	2.0036	1.4726
72	2.1668	11.5144	2.2172	2.0036	1.4726
73	2.1668	11.5144	2.2172	2.0036	1.4726

Table 4 Some numerical results of Σ in Example 1 for $q = \frac{1}{10}$

n	$q = \frac{1}{10}$					Σ
	${}_1\eta_1\Lambda_1$	$\sum_{j=1}^{2} {}_2\eta_j\Lambda_2$	$\sum_{j=1}^{3} {}_3\eta_j\Lambda_3$	$\sum_{j=1}^{4} {}_4\eta_j\Lambda_4$	$\sum_{j=1}^{5} {}_5\eta_j\Lambda_5$	
1	0.1358	0.3101	0.1002	0.2285	0.3021	0.3101
2	0.1362	0.312	0.1006	0.2298	0.3045	0.312
3	0.1363	0.3121	0.1006	0.2299	0.3048	0.3121
4	0.1363	0.3122	0.1006	0.2299	0.3048	0.3122
5	0.1363	0.3122	0.1006	0.2299	0.3048	0.3122
6	0.1363	0.3122	0.1006	0.2299	0.3048	0.3122
7	0.1363	0.3122	0.1006	0.2299	0.3048	0.3122
8	0.1363	0.3122	0.1006	0.2299	0.3048	0.3122

Table 5 Some numerical results of Σ in Example 1 for $q = \frac{1}{2}$

n	$q = \frac{1}{2}$					Σ
	${}_1\eta_1\Lambda_1$	$\sum_{j=1}^{2} {}_2\eta_j\Lambda_2$	$\sum_{j=1}^{3} {}_3\eta_j\Lambda_3$	$\sum_{j=1}^{4} {}_4\eta_j\Lambda_4$	$\sum_{j=1}^{5} {}_5\eta_j\Lambda_5$	
1	0.1531	0.3975	0.082	0.1619	0.1257	0.3975
2	0.1676	0.4425	0.0942	0.1931	0.1587	0.4425
3	0.1748	0.4647	0.1006	0.2099	0.1771	0.4647
⋮	⋮	⋮	⋮	⋮	⋮	⋮
8	0.1816	0.4859	0.107	0.2271	0.196	0.4859
9	0.1817	0.4863	0.1071	0.2274	0.1964	0.4863
10	0.1818	0.4865	0.1072	0.2275	0.1965	0.4865
11	0.1818	0.4865	0.1072	0.2276	0.1966	0.4865
12	0.1818	0.4866	0.1072	0.2276	0.1966	0.4866
13	0.1818	0.4866	0.1072	0.2276	0.1967	0.4866
14	0.1818	0.4866	0.1072	0.2276	0.1967	0.4866
15	0.1818	0.4866	0.1072	0.2276	0.1967	0.4866

Table 6 Some numerical results of Σ in Example 1 for $q = \frac{6}{7}$

n	$q = \frac{6}{7}$					Σ
	${}_1\eta_1\Lambda_1$	$\sum_{j=1}^{2} {}_2\eta_j\Lambda_2$	$\sum_{j=1}^{3} {}_3\eta_j\Lambda_3$	$\sum_{j=1}^{4} {}_4\eta_j\Lambda_4$	$\sum_{j=1}^{5} {}_5\eta_j\Lambda_5$	
1	0.0814	0.2564	0.0255	0.0459	0.0157	0.2564
2	0.1016	0.3085	0.0336	0.0569	0.0209	0.3085
3	0.1181	0.3504	0.0412	0.068	0.0264	0.3504
⋮	⋮	⋮	⋮	⋮	⋮	⋮
43	0.2068	0.5859	0.1057	0.2036	0.1168	0.586
44	0.2068	0.586	0.1057	0.2037	0.1169	0.5861
45	0.2068	0.5861	0.1057	0.2038	0.1169	0.5861
46	0.2068	0.5861	0.1058	0.2038	0.117	0.5862
47	0.2068	0.5862	0.1058	0.2038	0.117	0.5862
48	0.2068	0.5862	0.1058	0.2039	0.117	0.5862
49	0.2068	0.5862	0.1058	0.2039	0.117	0.5863
50	0.2068	0.5863	0.1058	0.2039	0.1171	0.5863
51	0.1696	0.5863	0.1058	0.204	0.1171	0.5863
52	0.1696	0.5863	0.1058	0.204	0.1171	0.5863
53	0.1696	0.5863	0.1058	0.204	0.1171	0.5863
54	0.1696	0.5863	0.1058	0.204	0.1171	0.5863
55	0.1696	0.5864	0.1058	0.204	0.1171	0.5864
56	0.1696	0.5864	0.1058	0.204	0.1171	0.5864
57	0.1696	0.5864	0.1058	0.204	0.1171	0.5864

Algorithm 6 The proposed method for solving problem (28) in Example 1 for which we use the conditions of Theorem 6

```
function  [Lambdai, sumeta_iLambda_i, Sigma]= ...
    systemproblem1(q, sigma, zeta, alpha, m, k, eta)
[xq yq]=size(q);
[xsigma ysigma]=size(sigma);
for n=1:k
Lambdai(n,1)=n;
D(n,1)=n;
Sigma(n,1)=n;
end;
column=2;
for s=1:yq
for n=1:k
Lambdai(n, column)=qGamma(q(s), 1-alpha(1), n)/qGamma(q(s), ...
    sigma(1)+1-alpha(1), n);
end;
column=column+1;
end;
column=2+yq;
for i=2:m
for s=1:yq
for n=1:k
Lambdai(n, column)=qGamma(q(s), 1-alpha(i), n)/qGamma(q(s), ...
    sigma(i)+1-alpha(i), n)+ qGamma(q(s), i-zeta(i-1), ...
    n)*qGamma(q(s), 1-alpha(i), ...
    n)/(factorial(i-1)*qGamma(q(s), sigma(i) - zeta(i-1) + 1 ...
    - alpha(i), n));
end;
column=column+1;
end;
end;
column=2;
for s=1:yq
for n=1:k
D(n, column)= Lambdai(n, column)*eta(1);
end;
column=column+1;
end;
column=2+yq;
for s=1:yq
for i=2:m
for n=1:k
D(n, column)= Lambdai(n, column)*eta(i);
end;
column=column+yq;
end;
column=2+yq+s;
end;
for s=1:yq
for n=1:k
maxrow=D(n, s+1);
column=s+1+yq;
for i=2:m
if D(n, column) > maxrow
maxrow=D(n, column);
end;
column=column+yq;
end;
Sigma(n, s+1)=maxrow;
end;
end;
sumeta_iLambda_i=D;
end
```

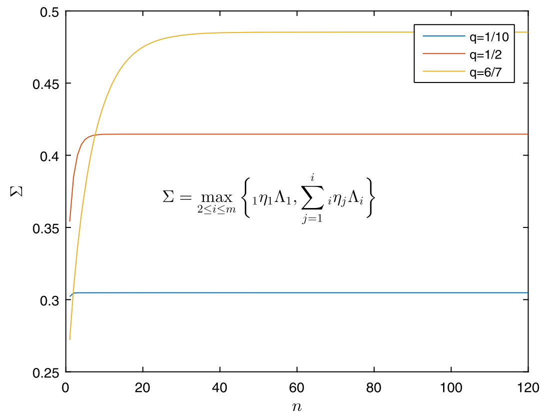

Figure 1 Numerical results of Σ for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$ in Example 1

Example 2 Consider the singular system of fractional q-differential equations

$$\begin{cases} {}^c\mathcal{D}_q^{\frac{9}{10}}[k_1](t) = w_1(t, k_1), \\ {}^c\mathcal{D}_q^{\frac{9}{5}}[k_2](t) = w_2(t, k_1, k_2), \\ {}^c\mathcal{D}_q^{\frac{17}{6}}[k_3](t) = w_3(t, k_1, k_2, k_3), \\ {}^c\mathcal{D}_q^{\frac{24}{7}}[k_4](t) = w_4(t, k_1, k_2, k_3, k_4), \\ {}^c\mathcal{D}_q^{\frac{13}{3}}[k_5](t) = w_5(t, k_1, k_2, k_3, k_4, k_5), \end{cases} \tag{29}$$

with boundary value conditions $k_1(0) = \frac{2}{3}$,

$$\begin{cases} k_2(0) = -1, \\ k_3(0) = 1, \quad k_3'(0) = \frac{2}{3}, \\ k_4(0) = \sqrt{7}, \quad k_4'(0) = \frac{\sqrt{7}}{3}, \quad k_4''(0) = \frac{\sqrt{5}}{3}, \\ k_5(0) = \frac{2}{3}, \quad k_5'(0) = \frac{6}{5}, \quad k_5''(0) = \frac{3}{8}, \quad k_5'''(0) = \frac{2\sqrt{2}}{5}, \end{cases}$$

${}^c\mathcal{D}_q^{\frac{1}{7}}[k_2](1) = {}^c\mathcal{D}_q^{\frac{8}{5}}[k_3](1) = {}^c\mathcal{D}_q^{\frac{11}{4}}[k_4](1) = {}^c\mathcal{D}_q^{\frac{7}{2}}[k_5](1) = 0$, where $t \in (0, 1]$. Put

$$\begin{aligned} w_1(t, k_1) &= \frac{\cos k_1(t)}{8\pi\sqrt{t}\exp(t)}, \\ w_2(t, k_1, k_2) &= \frac{2\cos(k_1(t) + k_2(t))}{15\pi\sqrt[3]{t}(1 + \sin(k_1(t) + k_2(t))}, \\ w_3(t, k_1, k_2, k_3) &= \frac{5(1 + \sin k_1(t) + \sin k_2(t) + \sin k_3(t))}{21\pi\sqrt[4]{t}}, \\ w_4(t, k_1, k_2, k_3, k_4) &= \frac{3\exp(2t)\cos^2(k_1(t) + k_3(t))}{8\pi\sqrt[5]{t}(1 + \cos^2(k_2(t) + k_4(t)))}, \\ w_5(t, k_1, k_2, k_3, k_4, k_5) &= \frac{\exp(-t)\sin(k_1(t) + k_2(t) + k_3(t) + k_4(t))}{9\pi\sqrt[4]{t^3}(1 + \sin(k_5(t)))}, \end{aligned}$$

Table 7 Some numerical results of Λ_i in Example 2 for $q = \frac{1}{10}$

n	$q = \frac{1}{10}$				
	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
1	2.0316	3.2152	3.3277	1.2641	3.1978
2	2.0417	3.2286	3.3477	1.2679	3.2225
3	2.0427	3.2299	3.3497	1.2682	3.2249
4	2.0428	3.2300	3.3499	1.2683	3.2252
5	2.0428	3.2300	3.3499	1.2683	3.2252
6	2.0428	3.2300	3.3499	1.2683	3.2252
7	2.0428	3.2300	3.3499	1.2683	3.2252

Table 8 Some numerical results of Λ_i in Example 2 for $q = \frac{1}{2}$

n	$q = \frac{1}{2}$				
	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
1	2.4734	3.3933	3.0493	0.6763	1.3697
2	2.7790	3.8511	3.6053	0.7758	1.7202
3	2.9305	4.0847	3.8985	0.8287	1.9145
⋮	⋮	⋮	⋮	⋮	⋮
13	3.0810	4.3213	4.202	0.8836	2.1219
14	3.0811	4.3214	4.2022	0.8837	2.122
15	3.0811	4.3215	4.2023	0.8837	2.1221
16	3.0811	4.3215	4.2023	0.8837	2.1221
17	3.0811	4.3215	4.2023	0.8837	2.1221
18	3.0812	4.3215	4.2023	0.8837	2.1221
19	3.0812	4.3215	4.2023	0.8837	2.1221
20	3.0812	4.3215	4.2023	0.8837	2.1222
21	3.0812	4.3215	4.2023	0.8837	2.1222
22	3.0812	4.3215	4.2023	0.8837	2.1222

Table 9 Some numerical results of Λ_i in Example 2 for $q = \frac{6}{7}$

n	$q = \frac{6}{7}$				
	Λ_1	Λ_2	Λ_3	Λ_4	Λ_5
1	1.2003	1.2924	0.8904	0.1853	0.1871
2	1.5754	1.7291	1.2336	0.2256	0.2434
3	1.8892	2.122	1.5593	0.2641	0.3038
⋮	⋮	⋮	⋮	⋮	⋮
62	3.6790	4.8818	4.4531	0.6682	1.2983
63	3.6791	4.8819	4.4532	0.6683	1.2983
64	3.6791	4.8819	4.4532	0.6683	1.2984
65	3.6791	4.8819	4.4533	0.6683	3.6791
66	3.6791	4.882	4.4533	0.6683	1.2984
67	3.6791	4.882	4.4533	0.6683	1.2984
68	3.6791	4.882	4.4534	0.6683	1.2984
69	3.6791	4.882	4.4534	0.6683	1.2984
70	3.6791	4.8821	4.4534	0.6683	1.2985

$m = 5$, $\sigma_1 = \frac{9}{10} \in (0,1)$, $\sigma_2 = \frac{9}{5} \in (1,2)$, $\sigma_3 = \frac{17}{6} \in (2,3)$, $\sigma_4 = \frac{24}{7} \in (3,4)$, $\sigma_5 = \frac{13}{3} \in (4,5)$, $\zeta_1 = \frac{2}{11} \in [0,1]$, $\zeta_2 = \frac{5}{3} \in [1,2]$, $\zeta_3 = \frac{7}{3} \in [2,3]$, $\zeta_4 = \frac{13}{4} \in [3,4]$, ${}_1b_0 = \frac{2}{3}$, ${}_2b_0 = -1$, ${}_3b_0 = 1$, ${}_4b_0 = \sqrt{7}$, ${}_5b_0 = \frac{2}{3}$, ${}_3b_1 = \frac{2}{3}$, ${}_4b_1 = \frac{\sqrt{7}}{3}$, ${}_5b_1 = 1$, ${}_4b_2 = \frac{\sqrt{5}}{3}$, ${}_5b_2 = \frac{3}{8}$, and ${}_5b_3 = \frac{2\sqrt{2}}{5}$. Now, we check (23) and (24). For each $t \in \overline{J}$ and $(k_1, k_2, \ldots, k_5) \in \mathbb{R}^5$, we have

$$L_1 = \max_{t\in[0,1]} t^{\alpha_1} \left| w_1\big(t, k_1(t)\big) \right| \leq \max_{t\in[0,1]} t^{\frac{3}{4}} \left| \frac{\cos k_1(t)}{8\pi\sqrt{t}\exp(t)} \right| \leq \frac{1}{8\pi}$$

Table 10 Numerical results of $(1) = L_1 \Lambda_1 + |_1 b_0|$ and $(i) = L_i \Lambda_i + \sum_{j=0}^{i-2} \frac{|_i b_j|}{j!}$ with $2 \le i \le 5$ in Example 2 for $q = \frac{1}{10}$

n	$q = \frac{1}{10}$					r
	(1)	(2)	(3)	(4)	(5)	
1	0.0808	1.1365	2.6755	5.0153	2.0615	5.0153
2	0.0812	1.137	2.6815	5.0186	2.0624	5.0186
3	0.0813	1.1371	2.6821	5.0189	2.0625	5.0189
4	0.0813	1.1371	2.6822	5.0190	2.0625	5.0190
5	0.0813	1.1371	2.6822	5.0190	2.0625	5.0190
6	0.0813	1.1371	2.6822	5.0190	2.0625	5.0190
7	0.0813	1.1371	2.6822	5.0190	2.0625	5.0190

Table 11 Numerical results of $(1) = L_1 \Lambda_1 + |_1 b_0|$ and $(i) = L_i \Lambda_i + \sum_{j=0}^{i-2} \frac{|_i b_j|}{j!}$ with $2 \le i \le 5$ in Example 2 for $q = \frac{1}{10}$

n	$q = \frac{1}{2}$					r
	(1)	(2)	(3)	(4)	(5)	
1	0.0984	1.144	2.5911	4.4968	1.9969	4.4968
2	0.1106	1.1634	2.7596	4.5846	2.0093	4.5846
3	0.1166	1.1734	2.8485	4.6312	2.0162	4.6312
⋮	⋮	⋮	⋮	⋮	⋮	⋮
7	0.1222	1.1828	2.9348	4.6767	2.023	4.6767
8	0.1224	1.1831	2.9377	4.6782	2.0233	4.6782
9	0.1225	1.1833	2.9392	4.679	2.0234	4.679
10	0.1225	1.1833	2.9399	4.6794	2.0234	4.6794
11	0.1226	1.1834	2.9403	4.6796	2.0235	4.6796
12	0.1226	1.1834	2.9404	4.6797	2.0235	4.6797
13	0.1226	1.1834	2.9405	4.6797	2.0235	4.6797
14	0.1226	1.1834	2.9406	4.6797	2.0235	4.6797
15	0.1226	1.1834	2.9406	4.6798	2.0235	4.6798
16	0.1226	1.1834	2.9406	4.6798	2.0235	4.6798
17	0.1226	1.1834	2.9406	4.6798	2.0235	4.6798

Table 12 Numerical results of $(1) = L_1 \Lambda_1 + |_1 b_0|$ and $(i) = L_i \Lambda_i + \sum_{j=0}^{i-2} \frac{|_i b_j|}{j!}$ with $2 \le i \le 5$ in Example 2 for $q = \frac{7}{6}$

n	$q = \frac{6}{7}$					r
	(1)	(2)	(3)	(4)	(5)	
1	0.0498	1.0549	1.9366	4.0638	1.9551	4.0638
2	0.0627	1.0734	2.0406	4.0993	1.9571	4.0993
3	0.0752	1.0901	2.1394	4.1333	1.9592	4.1333
⋮	⋮	⋮	⋮	⋮	⋮	⋮
37	0.1460	1.2065	3.0105	4.4869	1.994	4.4869
38	0.1461	1.2066	3.0114	4.4873	1.9941	4.4873
39	0.1461	1.2067	3.0122	4.4877	1.1461	4.4877
40	0.1462	1.2068	3.0128	4.488	1.9942	4.488
41	0.1462	1.2068	3.0134	4.4882	1.9942	4.4882
⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	0.1463	1.2071	3.0159	4.4894	1.9943	4.4894
51	0.1463	1.2071	3.016	4.4895	1.9943	4.4895
52	0.1463	1.2071	3.0161	4.4895	1.9943	4.4895
53	0.1463	1.2071	3.0162	4.4896	1.9943	4.4896
54	0.1463	1.2072	3.0163	4.4896	1.9943	4.4896
55	0.1463	1.2072	3.0164	4.4896	1.9944	4.4896
56	0.1463	1.2072	3.0164	4.4896	1.9944	4.4896

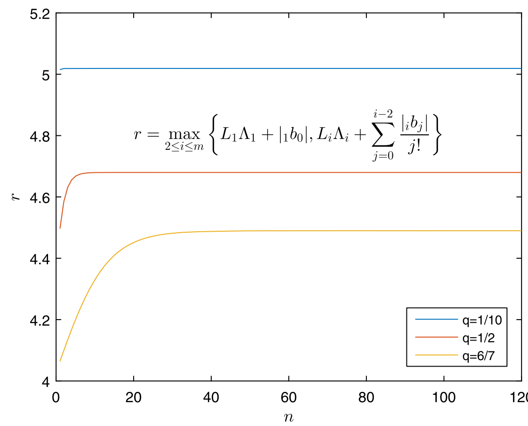

Figure 2 Numerical results of r for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$ in Example 2

Table 13 Numerical results of r in Example 2 for $q = \frac{1}{10}, \frac{1}{2}, \frac{7}{6}$

n	r		
	$q = \frac{1}{10}$	$q = \frac{1}{2}$	$q = \frac{6}{7}$
1	5.0153	4.4968	4.0638
2	5.0186	4.5846	4.0993
3	5.0189	4.6312	4.1333
4	<u>5.019</u>	4.6553	4.1663
5	5.019	4.6675	4.1981
6	5.019	4.6736	4.2285
⋮	⋮	⋮	⋮
13	5.019	4.6797	4.3835
14	5.019	4.6797	4.3975
15	5.019	<u>4.6798</u>	4.4098
16	5.019	4.6798	4.4205
17	5.019	4.6798	4.4299
⋮	⋮	⋮	⋮
61	5.019	4.6798	4.4897
62	5.019	4.6798	4.4897
63	5.019	4.6798	4.4897
64	5.019	4.6798	<u>4.4898</u>
65	5.019	4.6798	4.4898
66	5.019	4.6798	4.4898
67	5.019	4.6798	4.4898

for $\alpha_1 = \frac{3}{4}$,

$$\begin{aligned} L_2 &= \max_{t\in[0,1]} t^{\alpha_2}\left|w_2\big(t, k_1(t), k_2(t)\big)\right| \\ &\le \max_{t\in[0,1]} t^{\frac{2}{3}}\left|\frac{2\cos(k_1(t)+k_2(t))}{15\pi\sqrt[3]{t}(1+\sin(k_1(t)+k_2(t))}\right| \le \frac{2}{15\pi} \end{aligned}$$

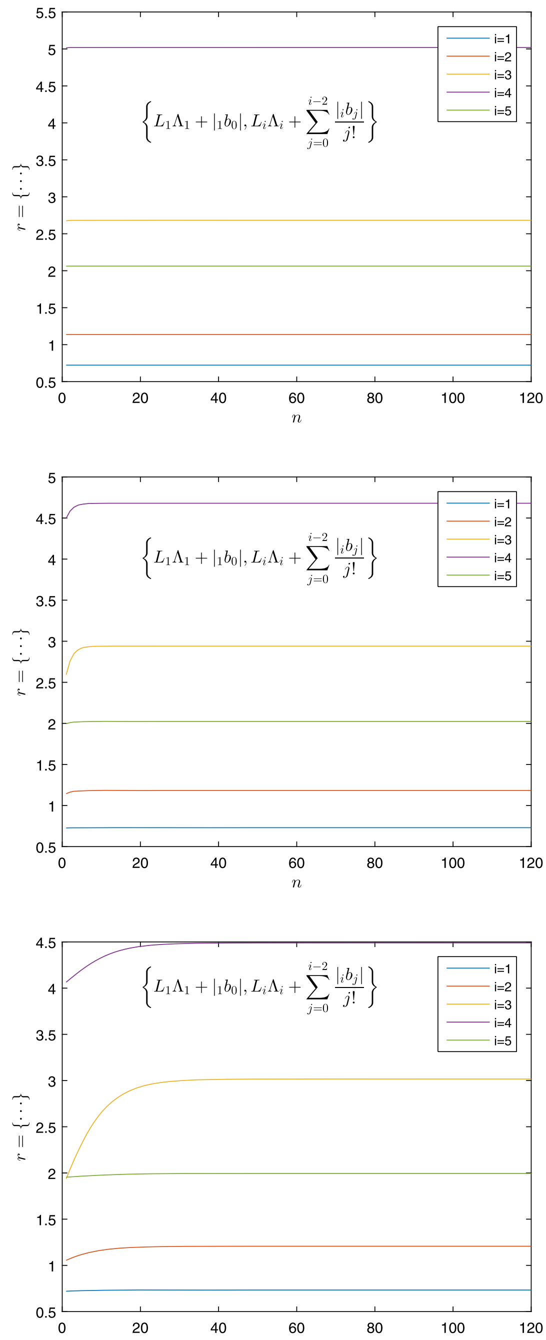

Figure 3 Numerical results of r for $q = \frac{1}{10}, \frac{1}{2}$, and $\frac{6}{7}$ in Example 2

for $\alpha_2 = \frac{2}{3}$,

$$\begin{aligned} L_3 &= \max_{t\in[0,1]} t^{\alpha_3} \bigl|w_3\bigl(t, k_1(t), k_2(t), k_3(t)\bigr)\bigr| \\ &\leq \max_{t\in[0,1]} t^{\frac{4}{5}} \left| \frac{5(1 + \sin k_1(t) + \sin k_2(t) + \sin k_3(t))}{21\pi \sqrt[4]{t}} \right| \leq \frac{20}{21\pi} \end{aligned}$$

for $\alpha_3 = \frac{4}{5}$,

$$L_4 = \max_{t\in[0,1]} t^{\alpha_4}\bigl|w_4\bigl(t,k_1(t),k_2(t),k_3(t),k_4(t)\bigr)\bigr| \leq \max_{t\in[0,1]} t^{\frac{1}{2}}\left|\frac{3\exp(2t)\cos^2(k_1(t)+k_3(t))}{8\pi\sqrt[5]{t}(1+\cos^2(k_2(t)+k_4(t)))}\right| \leq \frac{3e^2}{8\pi}$$

for $\alpha_4 = \frac{1}{2}$,

$$L_5 = \max_{t\in[0,1]} t^{\alpha_5}\bigl|w_5\bigl(t,k_1(t),k_2(t),k_3(t),k_4(t),k_5(t)\bigr)\bigr| \leq \max_{t\in[0,1]} t^{\frac{8}{9}}\left|\frac{\exp(-t)\sin(k_1(t)+k_2(t)+k_3(t)+k_4(t))}{9\pi\sqrt[4]{t^3}(1+\sin(k_5(t)))}\right| \leq \frac{1}{9\pi}$$

for $\alpha_5 = \frac{8}{9}$. Now, by using (11), we get

$$\begin{aligned}
\Lambda_1 &= \frac{\Gamma_q(1-\alpha_1)}{\Gamma_q(\sigma_1+1-\alpha_1)} = \frac{\Gamma_q(1-\frac{3}{4})}{\Gamma_q(\frac{9}{10}+1-\frac{3}{4})} = \frac{\Gamma_q(\frac{1}{4})}{\Gamma_q(\frac{23}{20})},\\
\Lambda_2 &= \frac{\Gamma_q(1-\alpha_2)}{\Gamma_q(\sigma_2+1-\alpha_2)} + \frac{\Gamma_q(2-\zeta_1)\Gamma_q(1-\alpha_2)}{\Gamma_q(\sigma_2-\zeta_1+1-\alpha_2)}\\
&= \frac{\Gamma_q(1-\frac{2}{3})}{\Gamma_q(\frac{9}{5}+1-\frac{2}{3})} + \frac{\Gamma_q(2-\frac{1}{7})\Gamma_q(1-\frac{2}{3})}{\Gamma_q(\frac{9}{5}-\frac{1}{7}+1-\frac{2}{3})}\\
&= \frac{\Gamma_q(\frac{1}{3})}{\Gamma_q(\frac{32}{15})} + \frac{\Gamma_q(\frac{13}{7})\Gamma_q(\frac{1}{3})}{\Gamma_q(\frac{209}{105})},\\
\Lambda_3 &= \frac{\Gamma_q(1-\alpha_3)}{\Gamma_q(\sigma_3+1-\alpha_3)} + \frac{\Gamma_q(3-\zeta_2)\Gamma_q(1-\alpha_3)}{2!\Gamma_q(\sigma_3-\zeta_2+1-\alpha_3)}\\
&= \frac{\Gamma_q(1-\frac{4}{5})}{\Gamma_q(\frac{17}{6}+1-\frac{4}{5})} + \frac{\Gamma_q(3-\frac{8}{5})\Gamma_q(1-\frac{4}{5})}{2!\Gamma_q(\frac{17}{6}-\frac{8}{5}+1-\frac{4}{5})}\\
&= \frac{\Gamma_q(\frac{1}{5})}{\Gamma_q(\frac{79}{30})} + \frac{\Gamma_q(\frac{7}{5})\Gamma_q(\frac{1}{5})}{2!\Gamma_q(\frac{43}{30})},\\
\Lambda_4 &= \frac{\Gamma_q(1-\alpha_4)}{\Gamma_q(\sigma_4+1-\alpha_4)} + \frac{\Gamma_q(4-\zeta_3)\Gamma_q(1-\alpha_4)}{3!\Gamma_q(\sigma_4-\zeta_3+1-\alpha_4)}\\
&= \frac{\Gamma_q(1-\frac{1}{2})}{\Gamma_q(\frac{24}{7}+1-\frac{1}{2})} + \frac{\Gamma_q(4-\frac{11}{4})\Gamma_q(1-\frac{1}{2})}{3!\Gamma_q(\frac{24}{7}-\frac{11}{4}+1-\frac{1}{2})}\\
&= \frac{\Gamma_q(\frac{1}{2})}{\Gamma_q(\frac{55}{14})} + \frac{\Gamma_q(\frac{5}{4})\Gamma_q(\frac{1}{2})}{3!\Gamma_q(\frac{5}{4})},\\
\Lambda_5 &= \frac{\Gamma_q(1-\alpha_5)}{\Gamma_q(\sigma_5+1-\alpha_5)} + \frac{\Gamma_q(5-\zeta_4)\Gamma_q(1-\alpha_5)}{4!\Gamma_q(\sigma_5-\zeta_4+1-\alpha_5)}\\
&= \frac{\Gamma_q(1-\frac{8}{9})}{\Gamma_q(\frac{13}{3}+1-\frac{8}{9})} + \frac{\Gamma_q(5-\frac{7}{2})\Gamma_q(1-\frac{8}{9})}{4!\Gamma_q(\frac{13}{3}-\frac{7}{2}+1-\frac{8}{9})}\\
&= \frac{\Gamma_q(\frac{1}{9})}{\Gamma_q(\frac{40}{9})} + \frac{\Gamma_q(\frac{3}{2})\Gamma_q(\frac{1}{9})}{4!\Gamma_q(\frac{17}{18})}.
\end{aligned}$$

Algorithm 7 The proposed method for solving problem (29) in Example 2 for which we use the conditions of Theorem 7

```
function  [Lambdai, LLambda, rmax, Maxr] = ...
    systemproblem2(q, sigma, zeta, alpha, m, k, ibj, Lmax)
[xq yq]=size(q);
[xsigma ysigma]=size(sigma);
for n=1:k
Lambdai(n,1)=n;
LLambda(n,1)=n;
temp(n,1)=n;
rmax(n,1)=n;
Maxr(n,1)=n;
end;
column=2;
for i=1:m
for s=1:yq
for n=1:k
if i==1
Lambdai(n, column)=qGamma(q(s), 1-alpha(i), ...
    n)/qGamma(q(s), sigma(i) +1-alpha(i), n);
else
Lambdai(n, column)=qGamma(q(s), 1-alpha(i), ...
    n)/qGamma(q(s), sigma(i) +1-alpha(i), n)+ ...
    qGamma(q(s), i-zeta(i-1), n)*qGamma(q(s), 1-alpha(i), ...
    n)/(factorial(i-1) * qGamma(q(s), sigma(i)-zeta(i-1) ...
    +1 - alpha(i), n));
end;
end;
column=column+1;
end;
end;
% reset column
column=2;
for i=1:m
for s=1:yq
for n=1:k
LLambda(n, column)= Lambdai(n, column)*Lmax(i);
end;
column=column+1;
end;
end;
% reset column
column=2;
for i=1:m
for s=1:yq
for n=1:k
if i==1
rmax(n, column)=LLambda(n, column) + ibj(i,i);
else
t=0;
for j=0:(i-2)
t=t+abs(ibj(i,j+1))/factorial(j);
end;
rmax(n, column)=LLambda(n, column)+t;
end;
end;
column=column+1;
end;
end;
for s=1:yq
for n=1:k
maxrow=rmax(n, s+1);
column=s+1+yq;
for i=2:m
if rmax(n, column)>maxrow
maxrow=rmax(n, column);
end;
column=column+yq;
end;
Maxr(n, s+1)=maxrow;
end;
end;
end
```

Tables 7, 8, and 9 show $\varLambda_i \approx 2.0428, 3.2300, 3.3499, 1.2683, 3.2252$, $\varLambda_i \approx 3.812, 4.3215, 4.2023, 0.8837, 2.1222$, $\varLambda_i \approx 3.6791, 4.8820, 4.4534, 0.6683, 1.2984$ for $1 \le i \le 5$ and $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively. Now, by using (24) and Algorithm 7, we conclude next results. According to Tables 10, 11, and 12, consider the set $K_r \subset S$ as

$$K_r = \{(k_1, k_2, \dots, k_m) \in S : \|(k_1, k_2, \dots, k_m)\| \le 5.0190\},$$
$$K_r = \{(k_1, k_2, \dots, k_m) \in S : \|(k_1, k_2, \dots, k_m)\| \le 4.6798\},$$
$$K_r = \{(k_1, k_2, \dots, k_m) \in S : \|(k_1, k_2, \dots, k_m)\| \le 4.4896\},$$

for $q = \frac{1}{10}, \frac{1}{2}$, and $\frac{6}{7}$, respectively. Table 10 shows that $L_1 \varLambda_1 + |{}_1b_0| \approx 0.0812$, $L_i \varLambda_i + \sum_{j=0}^{i-2} \frac{|{}_ib_j|}{j!} \approx 1.1371, 2.6822, 5.0190, 2.0625, 5.0190$. Table 11 shows $L_1 \varLambda_1 + |{}_1b_0| \approx 0.1226$, $L_i \varLambda_i + \sum_{j=0}^{i-2} \frac{|{}_ib_j|}{j!} \approx 1.1834, 2.9406, 4.6798, 2.0235$, Table 12 shows that $L_1 \varLambda_1 + |{}_1b_0| \approx 0.1463$, $L_i \varLambda_i + \sum_{j=0}^{i-2} \frac{|{}_ib_j|}{j!} \approx 1.2072, 3.0164, 4.4898, 1.9944$ for $2 \le i \le 5$ and $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively. Also, Table 13 shows us $r \approx 5.0190, 4.6798, 4.4898$ for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively (Figs. 3 and 2). Now, by using Theorem 7, the singular system of fractional q-differential equations (29) has a solution.

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Ethics approval and consent to participate

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Competing interests

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Consent for publication

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Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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