

Research Article

On the Fractional View Analysis of Keller–Segel Equations with Sensitivity Functions

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Received 2 April 2020; Revised 22 June 2020; Accepted 1 July 2020; Published 28 July 2020

Academic Editor: Dimitri Volchenkov

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In this paper, the fractional view analysis of the Keller–Segel equations with sensitivity functions is presented. The Caputo operator has been used to pursue the present research work. The natural transform is combined with the homotopy perturbation method, and a new scheme for implementation is derived. The modified established method is named as the homotopy perturbation transform technique. The derived results are compared with the solution of the Laplace Adomian decomposition technique by using the systems of fractional Keller–Segel equations. The solution graphs and the table have shown that the obtained results coincide with the solution of the Laplace Adomian decomposition method. Fractional-order solutions are determined to confirm the reliability of the current method. It is observed that the solutions at various fractional orders are convergent to an integer-order solution of the problems. The suggested procedure is very attractive and straight forward and therefore can be modified to solve high nonlinear fractional partial differential equations and their systems.

1. Introduction

Fractional differential equations (FDEs) are the generalizations of the standard integer-order differential equations. FDEs have gained much attention in the recent decades, as they are broadly used in several areas to analyze different processes, such as the processing of the signal, control theory, identification of the system, fluid flow, biomathematics, and other fields [1–3]. FDEs have been implemented in the modeling of the description of unification of diffusion, fractional random walk, and systems of both diffusive and subdiffusive. It is also an investigation that in many physical phenomena, experiments have proved that the solution of fractional-order DEs is in good agreement with experimental

data of any physical phenomena to integer-order DEs. For example, the noninteger differential equations (DEs) can model some physical phenomena more effectively such as delineates memory, heredity effects, properties of different materials, and internal friction process [4–9].

The importance of FDEs is found in the literature because it can model most of the physical phenomena in science and engineering more accurately as compared to integer-order models [10–12], and therefore the researchers have shown much interest to study fractional calculus and FDEs during the last decades. Because of the significance of FDEs, the mathematicians are working to develop many useful and powerful numerical and analytical techniques to determine the actual or approximate solutions of the

targeted problems [13–15]. Many other powerful and efficient techniques have been suggested to obtain the exact or analytical solutions for FDEs. For instance, these techniques include the Laplace Adomian decomposition method (LADM) [16–18], Chebyshev wavelet methods (CWM) [19], collocation-shooting method [20], power series methods (PSM) [21], fractional Bernstein polynomials along with shooting method [22], fractional-Legendre spectral Galerkin method [23], variational iterative transform method (VITM) [24], homotopy perturbation transform method (HPTM) [25–27], homotopy analysis transform method (HATM) [28, 29], reduced differential transform method (RDTM) [30, 31], finite element technique (FET) [32], finite difference technique (FDT) [33], and q -homotopy analysis transform method (q-HATM) [34]. Based on these techniques, a wide range of FDEs have been analyzed.

The most important cell motion which can be used as a response to the gradient of a chemical compound is known as chemotaxis. It has an important contribution in the population cell number and growth of biology. E. Keller and L. Segal suggested the first mathematical equation of chemotaxis in 1970. They displayed parabolic schemes to describe the method of aggregating a cell slime mold depending on a molecule attraction [35]. In this article, we considered the coupled time-fractional Keller–Segel equation [36–39]:

$$\begin{aligned} D_{\Psi}^{\gamma} \mu(\zeta, \Psi) - a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} + \frac{\partial}{\partial \zeta} \left\{ \mu(\zeta, \Psi) \frac{\partial [\chi \rho(\zeta, \psi)]}{\partial \zeta} \right\} &= 0, \\ D_{\Psi}^{\gamma} \rho(\zeta, \Psi) - b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} - c \mu(\zeta, \Psi) + d \rho(\zeta, \Psi) &= 0. \end{aligned} \quad (1)$$

With initial conditions,

$$\begin{cases} \mu(\zeta, \Psi) = \mu_0(\zeta), \\ \rho(\zeta, \Psi) = \rho_0(\zeta), \end{cases} \quad (2)$$

where the unidentified function $\mu(\zeta, \Psi)$ defines the concentrations of amoebae, the chemical concentration is denoted by $\rho(\zeta, \Psi)$, and the chemotactic is defined by the partial derivative of $\partial/\partial \zeta \{ \mu(\zeta, \Psi) \partial [\chi \rho(\zeta, \psi)] / \partial \zeta \}$, $\chi(\rho)$ is the sensitivity function. Some other types of sensitivity functions can be observed in the coupled time-fractional K-S chemotactic model in [40] and also in the K-S model with a logic sensitivity function and small diffusivity [41]. Also, singular sensitivity can be seen in the system of the two-dimensional K-S system in [36, 42, 43]. The K-S model has been extensively analyzed recently. For example, Kamel et al. have used LADM for the solution of the S-K equation [37]. Bournaveas [38] gave the one-dimensional Keller–Segel model with a fractional cell diffusion. Zayernouri [39] formed a fractional class of implicit Adams–Moulton and explicit Adams–Bashforth techniques.

The HPM is mixed with the natural transform method to generate a highly effective method to handle the solution

of several nonlinear problems and is known as the homotopy perturbation transform method (HPTM). HPTM generates a convergent series form solution that converges to the exact solution of the problems and provides closed-form solutions. The proposed method can combine two important techniques to achieve an effective solution for nonlinear equations. Gorbhani first proposed the use of He’s polynomial in the nonlinear terms [44, 45]. It should be remembered that the HPTM is implemented without any flexibility or restrictive assumptions or transforms and round off free error. Many authors have solved fractional-order diffusion equations [46], partial differential equations [47, 48], and wave-like equations [49] by using linear and nonlinear problems with the help of HPTM.

In the current research paper, HPTM is implemented to solve fractional-order Keller–Segel equations. The solutions achieved through the suggested technique are straightforward and simple. The quality of the current method is appropriate to provide the analytical results to the given examples. The HPTM solutions are shown to be in close contact with the solutions of other existing techniques.

2. Preliminary Concepts

Definition 1. Let $g \in C_{\beta}$ and $\beta \geq -1$, and then the Riemann–Liouville integral of order γ is given by [1–3]

$$J_{\Psi}^{\gamma} g(\zeta, \Psi) = \frac{1}{\Gamma(\gamma)} \int_0^{\Psi} (\Psi - \theta)^{\gamma-1} g(\zeta, \theta) d\theta, \quad \Psi > 0. \quad (3)$$

Definition 2. Let $g \in C_{\beta}$ and $\beta \geq -1$, and then the Caputo definition of the fractional derivative of order γ if $m - 1 < \gamma \leq m$ with $m \in \mathbb{N}$ is described as [1–3]

$$D_{\Psi}^{\gamma} g(\Psi) = \begin{cases} \frac{d^m g(\Psi)}{d\Psi^m}, & \gamma = m \in \mathbb{N}, \\ \frac{1}{\Gamma(m - \gamma)} \int_0^{\Psi} (\Psi - \theta)^{m-\gamma-1} g^{(m)}(\theta) d\theta, & m - 1 < \gamma < m, \quad m \in \mathbb{N}. \end{cases} \quad (4)$$

Remark 1. Some basic properties are given below [1–3]:

$$\begin{aligned} D_{\zeta}^{\gamma} I_{\zeta}^{\lambda} g(\zeta) &= g(\zeta), \quad I_{\zeta}^{\gamma} \zeta^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\gamma + \lambda + 1)} \zeta^{\gamma+\lambda}, \quad \gamma > 0, \lambda > -1, \zeta > 0, \\ D_{\zeta}^{\gamma} I_{\zeta}^{\gamma} g(\zeta) &= g(\zeta) - \sum_{k=0}^m g^{(k)}(0^+) \frac{\zeta^k}{k!}, \quad \text{for } \zeta > 0. \end{aligned} \quad (5)$$

Definition 3. The natural transform $f(\Psi)$ of the function $N[f(\Psi)]$ for $\Psi \in R$ is defined as [50]

$$N[f(\Psi)] = \mathcal{G}(s, u) = \int_{-\infty}^{\infty} e^{-s\Psi} f(\Psi) d\Psi;$$

$$s, u \in (-\infty, \infty),$$

$$N[f(\Psi)Q(\Psi)] = N^+[f(\Psi)] = \mathcal{G}^+(s, u)$$

$$= \int_0^{\infty} e^{-s\Psi} f(\Psi) d\Psi;$$

$$s, u \in (0, \infty), \quad \text{and} \quad \Psi \in R.$$

$$(6)$$

Theorem 1. *The natural transform of the fractional derivative of the function in the Riemann–Liouville sense is given as [50]*

$$N^+[D^\gamma f(\Psi)] = \mathcal{G}_\gamma(s, u) = \frac{s^\gamma}{u^\gamma} \mathcal{G}(s, u)$$

$$- \sum_{j=0}^{m-1} \frac{s^j}{u^{\gamma-j}} [D^{\gamma-j-1} f(\Psi)]_{\Psi=0}, \quad \ell - 1 \leq \gamma < \ell.$$

$$(7)$$

Theorem 2. *The natural transform of the fractional derivative of the function in the Caputo sense is given as [50]*

$$N^+[{}^c D^\gamma f(\Psi)] = \mathcal{G}_\gamma^c(s, u) = \frac{s^\gamma}{u^\gamma} \mathcal{G}(s, u)$$

$$- \sum_{j=0}^{\ell-1} \frac{s^{\gamma-(j+1)}}{u^{\gamma-j}} [D^j f(\Psi)]_{\Psi=0}, \quad \ell - 1 \leq \gamma < \ell.$$

$$(8)$$

3. Implementation of HPTM [51]

To explain the fundamental concept of this technique, we consider a particular fractional-order nonlinear partial differential nonhomogeneous equation:

$$D_\Psi^\gamma \mu(\zeta, \Psi) + \mathcal{R}\mu(\zeta, \Psi) + \mathcal{N}\mu(\zeta, \Psi) = g(\zeta, \Psi), \quad 1 < \gamma \leq 2.$$

$$(9)$$

With initial conditions,

$$\mu(\zeta, 0) = h(\zeta), \quad \mu_\Psi(\zeta, 0) = f(\zeta), \quad (10)$$

where the fractional derivative in equation (9) is defined in the Caputo sense. The operators \mathcal{R} and \mathcal{N} describe the linear and nonlinear operators, respectively, and $g(\zeta, \Psi)$ is the source term.

Using natural transformation in equation (9), we get [51]

$$N[D_\Psi^\gamma \mu(\zeta, \Psi)] + N[\mathcal{R}\mu(\zeta, \Psi)] + N[\mathcal{N}\mu(\zeta, \Psi)] = N[g(\zeta, \Psi)].$$

$$(11)$$

With the help of the fractional derivative natural property, we have

$$N[\mu(\zeta, \Psi)] = \frac{h(\zeta)}{s} + \frac{f(\zeta)}{s^2} + \frac{u^\gamma}{s^\gamma} N[g(\zeta, \Psi)]$$

$$- \frac{u^\gamma}{s^\gamma} N[\mathcal{R}\mu(\zeta, \Psi)] - \frac{u^\gamma}{s^\gamma} N[\mathcal{N}\mu(\zeta, \Psi)].$$

$$(12)$$

Taking the inverse natural transformation of equation (12), we get

$$\mu(\zeta, \Psi) = \mathcal{G}(\zeta, \Psi) - N^{-1} \left\{ \frac{u^\gamma}{s^\gamma} N[\mathcal{R}\mu(\zeta, \Psi) + \mathcal{N}\mu(\zeta, \Psi)] \right\}.$$

$$(13)$$

Using the HPTM procedure, the solution is expressed as

$$\mu(\zeta, \Psi) = \sum_{m=0}^{\infty} p^m \mu_m(\zeta, \Psi). \quad (14)$$

The nonlinear term can be decomposed as

$$\mathcal{N}\mu(\zeta, \Psi) = \sum_{m=0}^{\infty} p^m \mathcal{H}_m(\zeta, \Psi). \quad (15)$$

Few He's polynomials $\mathcal{H}_m(u)$ [35, 40] are described by

$$\mathcal{H}_\ell(\mu_0, \mu_1, \dots, \mu_\ell) = \frac{1}{\ell!} \frac{\partial^\ell}{\partial p^\ell} \left[N \left(\sum_{i=0}^{\infty} p^i \mu_i \right) \right]_{p=0}, \quad \ell = 0, 1, 2, 3, \dots$$

$$(16)$$

Utilizing equations (14) and (15) in equation (13), we have

$$\sum_{\ell=0}^{\infty} p^\ell \mu_\ell(\zeta, \Psi) = \mathcal{G}(\zeta, \Psi) - p \left(\left\{ \frac{u^\gamma}{s^\gamma} N \left[\mathcal{R} \sum_{\ell=0}^{\infty} p^\ell \mu_\ell(\zeta, \Psi) \right. \right. \right.$$

$$\left. \left. + \sum_{\ell=0}^{\infty} p^\ell \mathcal{H}_\ell(\zeta, \Psi) \right\} \right).$$

$$(17)$$

Using He's polynomials in HPTM and comparing the coefficient of power p , we get

$$p^0: \mu_0(\zeta, \Psi) = \mathcal{G}(\zeta, \Psi),$$

$$p^1: \mu_1(\zeta, \Psi) = -N^{-1} \left\{ \frac{u^\gamma}{s^\gamma} N[\mathcal{R}\mu_0(\zeta, \Psi) + \mathcal{H}_0\mu] \right\},$$

$$p^2: \mu_2(\zeta, \Psi) = -N^{-1} \left\{ \frac{u^\gamma}{s^\gamma} N[\mathcal{R}\mu_1(\zeta, \Psi) + \mathcal{H}_1\mu] \right\},$$

$$p^3: \mu_3(\zeta, \Psi) = -N^{-1} \left\{ \frac{u^\gamma}{s^\gamma} N[\mathcal{R}\mu_2(\zeta, \Psi) + \mathcal{H}_2\mu] \right\}.$$

$$(18)$$

The remaining of the $\mu_\ell(\zeta, \Psi)$ components can be totally obtained, and the sequence result is thus fully determined. Finally, we calculate the analytical result of $\mu(\zeta, \Psi)$:

$$\mu(\zeta, \Psi) = \lim_{N \rightarrow \infty} \sum_{\ell=0}^N \mu_\ell(\zeta, \Psi). \quad (19)$$

Generally, the solutions of the above series converge very quickly.

4. Numerical Examples

Example 1. The Keller–Segal equations with a fractional derivative are given as [37–39]

$$D_{\Psi}^{\gamma} \mu(\zeta, \Psi) - a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} + \frac{\partial}{\partial \zeta} \left\{ \mu(\zeta, \Psi) \frac{\partial [\chi \rho(\zeta, \psi)]}{\partial \zeta} \right\} = 0,$$

$$D_{\Psi}^{\gamma} \rho(\zeta, \Psi) - b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} - c \mu(\zeta, \Psi) + d \rho(\zeta, \Psi) = 0. \quad (20)$$

Subjecting to the initial solutions, we get

$$\begin{aligned} \mu(\zeta, 0) &= m e^{-\zeta^2}, \\ \rho(\zeta, 0) &= n e^{-\zeta^2}. \end{aligned} \quad (21)$$

Taking natural transformation of equation (20), we get

$$\begin{aligned} N \left[\frac{\partial^{\gamma} \mu(\zeta, \Psi)}{\partial \Psi^{\gamma}} \right] &= N \left[a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} - \frac{\partial}{\partial \zeta} \left(\mu(\zeta, \Psi) \frac{\partial [\chi \rho(\zeta, \psi)]}{\partial \zeta} \right) \right], \\ N \left[\frac{\partial^{\gamma} \rho(\zeta, \Psi)}{\partial \Psi^{\gamma}} \right] &= N \left[b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} + c \mu(\zeta, \Psi) - d \rho(\zeta, \Psi) \right], \\ \frac{s^{\gamma}}{u^{\gamma}} N \{ \mu(\zeta, \Psi) \} - \frac{s^{\gamma-1}}{u^{\gamma}} \mu(\zeta, 0) &= N \left[a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} - \frac{\partial}{\partial \zeta} \left(\mu(\zeta, \Psi) \frac{\partial [\chi \rho(\zeta, \psi)]}{\partial \zeta} \right) \right], \\ \frac{s^{\gamma}}{u^{\gamma}} N \{ \rho(\zeta, \Psi) \} - \frac{s^{\gamma-1}}{u^{\gamma}} \rho(\zeta, 0) &= N \left[b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} + c \mu(\zeta, \Psi) - d \rho(\zeta, \Psi) \right]. \end{aligned} \quad (22)$$

Simplifying the above equation, we get

$$\begin{aligned} N[\mu(\zeta, \Psi)] &= \frac{1}{s} [\mu(\zeta, 0)] + \frac{u^{\gamma}}{s^{\gamma}} N \left(a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} - \frac{\partial}{\partial \zeta} \left\{ \mu(\zeta, \Psi) \frac{\partial [\chi \rho(\zeta, \psi)]}{\partial \zeta} \right\} \right), \\ N[\rho(\zeta, \Psi)] &= \frac{1}{s} [\rho(\zeta, 0)] + \frac{u^{\gamma}}{s^{\gamma}} N \left[b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} + c \mu(\zeta, \Psi) - d \rho(\zeta, \Psi) \right]. \end{aligned} \quad (23)$$

Using inverse natural transform, we have

$$\begin{aligned} \mu(\zeta, \Psi) &= \mu(\zeta, 0) - N^{-1} \left\{ \frac{u^{\gamma}}{s^{\gamma}} N \left[a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} - \frac{\partial}{\partial \zeta} \left(\mu(\zeta, \Psi) \frac{\partial [\chi \rho(\zeta, \psi)]}{\partial \zeta} \right) \right] \right\}, \\ \rho(\zeta, \Psi) &= \rho(\zeta, 0) - N^{-1} \left\{ \frac{u^{\gamma}}{s^{\gamma}} N \left[b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} + c \mu(\zeta, \Psi) - d \rho(\zeta, \Psi) \right] \right\}. \end{aligned} \quad (24)$$

Then, the chemotactic term $\partial/\partial\zeta\{\mu(\zeta, \Psi)\partial[\chi\rho(\zeta, \psi)]/\partial\zeta\} = 0$.

Now, implementing HPM, we get

$$\begin{aligned} \sum_{\ell=0}^{\infty} p^\ell \mu_\ell(\zeta, \Psi) &= m e^{-\zeta^2} + p \left(N^- \left\{ \frac{u^\gamma}{s^\gamma} N \left[a \sum_{\ell=0}^{\infty} p^\ell \frac{\partial^2 \mu_\ell(\zeta, \Psi)}{\partial \zeta^2} \right] \right\} \right), \\ \sum_{\ell=0}^{\infty} p^\ell \rho_\ell(\zeta, \Psi) &= n e^{-\zeta^2} + p \left(N^- \left\{ \frac{u^\gamma}{s^\gamma} N \left[\begin{array}{l} b \sum_{\ell=0}^{\infty} p^\ell \frac{\partial^2 \rho_\ell(\zeta, \Psi)}{\partial \zeta^2} + c \sum_{\ell=0}^{\infty} p^\ell \mu_\ell(\zeta, \Psi) \\ -d \sum_{\ell=0}^{\infty} p^\ell \rho_\ell(\zeta, \Psi) \end{array} \right] \right\} \right). \end{aligned} \quad (25)$$

By comparing the coefficients of powers p , we get

$$\begin{aligned} p^0: \mu_0(\zeta, \Psi) &= m e^{-\zeta^2}, \\ p^0: \rho_0(\zeta, \Psi) &= n e^{-\zeta^2}, \\ p^1: \mu_1(\zeta, \Psi) &= N^- \left\{ \frac{u^\gamma}{s^\gamma} N \left[a \frac{\partial^2 \mu_0(\zeta, \Psi)}{\partial \zeta^2} \right] \right\}, \\ \mu_1(\zeta, \Psi) &= 2 \operatorname{am}(2\zeta^2 - 1) e^{-\zeta^2} \frac{\Psi^\gamma}{\Gamma(\gamma + 1)}, \\ p^1: \rho_1(\zeta, \Psi) &= N^- \left\{ \frac{u^\gamma}{s^\gamma} N \left[b \frac{\partial^2 \rho_0(\zeta, \Psi)}{\partial \zeta^2} + c \mu_0(\zeta, \Psi) - d \rho_0(\zeta, \Psi) \right] \right\}, \\ \rho_1(\zeta, \Psi) &= \{2bn(2\zeta^2 - 1) + (cm - dn)\} e^{-\zeta^2} \frac{\Psi^\gamma}{\Gamma(\gamma + 1)}, \\ p^2: \mu_2(\zeta, \Psi) &= N^- \left\{ \frac{u^\gamma}{s^\gamma} N \left[a \frac{\partial^2 \mu_1(\zeta, \Psi)}{\partial \zeta^2} \right] \right\}, \\ \mu_2(\zeta, \Psi) &= 4a^2(3m - 6m\zeta^2 - 6 \operatorname{am}\zeta^2 + 4m\zeta^3) e^{-\zeta^2} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)}, \\ p^2: \rho_2(\zeta, \Psi) &= N^- \left\{ \frac{u^\gamma}{s^\gamma} N \left[b \frac{\partial^2 \rho_1(\zeta, \Psi)}{\partial \zeta^2} + c \mu_1(\zeta, \Psi) - d \rho_1(\zeta, \Psi) \right] \right\}, \\ \rho_2(\zeta, \Psi) &= b[-24bn - 4bn(2m^2 - 1) + (8b\zeta^2 - 2b \operatorname{dn})(2\zeta^2 - 1)] e^{-\zeta^2} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)} \\ &\quad + [(2 - 4\zeta^2 - 1)(cm + dn) - 2ca(2m\zeta^2 - m)] e^{-\zeta^2} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)}. \end{aligned} \quad (26)$$

The series form of equation (20) is given by

The HPTM solution is given as

$$\begin{aligned}\mu(\zeta, \Psi) &= \sum_{\ell=0}^{\infty} \mu_{\ell}(\zeta, \Psi) = \mu_0(\zeta, \Psi) + \mu_1(\zeta, \Psi) + \mu_2(\zeta, \Psi) \\ &\quad + \mu_3(\zeta, \Psi) + \dots, \\ \rho(\zeta, \Psi) &= \sum_{\ell=0}^{\infty} \rho_{\ell}(\zeta, \Psi) = \rho_0(\zeta, \Psi) + \rho_1(\zeta, \Psi) + \rho_2(\zeta, \Psi) \\ &\quad + \rho_3(\zeta, \Psi) + \dots.\end{aligned}\tag{27}$$

$$\begin{aligned}\mu(\zeta, \Psi) &= m e^{-\zeta^2} + 2am(2\zeta^2 - 1)e^{-\zeta^2} \frac{\Psi^{\gamma}}{\Gamma(\gamma + 1)} + 4a^2(3m - 6m\zeta^2 - 6am\zeta^2 + 4m\zeta^3)e^{-\zeta^2} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)} + \dots \\ \rho(\zeta, \Psi) &= n e^{-\zeta^2} + \{2bn(2\zeta^2 - 1) + (cm - dn)\} e^{-\zeta^2} \frac{\Psi^{\gamma}}{\Gamma(\gamma + 1)} + b\{-24bn - 4bn(2m^2 - 1) + (8b\zeta^2 - 2b\,dn)(2\zeta^2 - 1)\} e^{-\zeta^2} \\ &\quad \cdot \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)} + \{(2 - 4\zeta^2 - 1)(cm + dn) - 2ca(2m\zeta^2 - m)\} e^{-\zeta^2} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)} + \dots.\end{aligned}\tag{28}$$

The series solutions are obtained by using the numerical values $d = 0.8$, $m = 160$, $b = 3$, $n = 120$, $a = 0.5$, and $c = 1$:

$$\begin{aligned}\mu(\zeta, \Psi) &= 160e^{-\zeta^2} + 160(2\zeta^2 - 1)e^{-\zeta^2} \frac{\Psi^{\gamma}}{\Gamma(\gamma + 1)} + 5(96 - 384\zeta^2 + 128\zeta^3)e^{-\zeta^2} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)} \\ &\quad + 5(252 - 24\zeta^2 + 128\zeta^3 + 175\zeta^4)e^{-\zeta^2} \frac{\Psi^{3\gamma}}{\Gamma(3\gamma + 1)} + \dots \\ \rho(\zeta, \Psi) &= 120e^{-\zeta^2} + [320(2\zeta^2 - 1) - 64]e^{-\zeta^2} \frac{\Psi^{\gamma}}{\Gamma(\gamma + 1)} + [3450 - 7680\zeta^2 - 1920(2\zeta^2 - 1) \\ &\quad - 7680\zeta + 12\zeta^2(320(2\zeta^2 - 1) + 64)] \\ &\quad + 5(192\zeta - 128\zeta^3) - 2\{320 - \zeta[160(2\zeta^2 - 1) + 32]\} \\ &\quad \cdot e^{-\zeta^2} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)} \{5(1360 - 215\zeta^2 - 28(8\zeta^2 + 7\zeta^3 - 2)) - 3045\zeta + 8\zeta^2(8\zeta^2 + 7\zeta^3 - 2)\} \\ &\quad + (123\zeta - 840\zeta^3) - 0.9(1280\zeta - 2\zeta[160(8\zeta^2 + 7\zeta^3 - 2) + 64])\} \\ &\quad \cdot e^{-\zeta^2} \frac{\Psi^{3\gamma}}{\Gamma(3\gamma + 1)} + \dots.\end{aligned}\tag{29}$$

Example 2. The Keller–Segal equations with a fractional derivative are given as [37–39]

$$D_{\Psi}^y \mu(\zeta, \Psi) - a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} + \frac{\partial \mu(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho(\zeta, \Psi)}{\partial \zeta} - \mu(\zeta, \Psi) \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} = 0, \quad (30)$$

$$D_{\Psi}^y \rho(\zeta, \Psi) - b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} - c \mu(\zeta, \Psi) + d \rho(\zeta, \Psi) = 0.$$

With the initial conditions:

$$\begin{aligned} \mu(\zeta, 0) &= m e^{-\zeta^2}, \\ \rho(\zeta, 0) &= n e^{-\zeta^2}. \end{aligned} \quad (31)$$

Applying natural transformation to equation (30), we get

$$\begin{aligned} N \left\{ \frac{\partial^y \mu(\zeta, \Psi)}{\partial \Psi^y} \right\} &= N \left[a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} - \frac{\partial \mu(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho(\zeta, \Psi)}{\partial \zeta} + \mu(\zeta, \Psi) \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} \right], \\ N \left\{ \frac{\partial^y \rho(\zeta, \Psi)}{\partial \Psi^y} \right\} &= N \left[b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} + c \mu(\zeta, \Psi) - d \rho(\zeta, \Psi) \right], \end{aligned} \quad (32)$$

$$\frac{s^y}{u^y} N \{ \mu(\zeta, \Psi) \} - \frac{s^{y-1}}{u^y} \mu(\zeta, 0) = N \left[a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} - \frac{\partial \mu(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho(\zeta, \Psi)}{\partial \zeta} + \mu(\zeta, \Psi) \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} \right],$$

$$\frac{s^y}{u^y} N \{ \rho(\zeta, \Psi) \} - \frac{s^{y-1}}{u^y} \rho(\zeta, 0) = N \left[b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} + c \mu(\zeta, \Psi) - d \rho(\zeta, \Psi) \right].$$

Simplifying the above equation, we get

$$\begin{aligned} N \{ \mu(\zeta, \Psi) \} &= \frac{1}{s} \{ \mu(\zeta, 0) \} + \frac{u^y}{s^y} N \left[a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} - \frac{\partial \mu(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho(\zeta, \Psi)}{\partial \zeta} + \mu(\zeta, \Psi) \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} \right], \\ N \{ \rho(\zeta, \Psi) \} &= \frac{1}{s} \{ \rho(\zeta, 0) \} + \frac{u^y}{s^y} N \left[b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} + c \mu(\zeta, \Psi) - d \rho(\zeta, \Psi) \right]. \end{aligned} \quad (33)$$

Using inverse natural transform, we have

$$\begin{aligned} \mu(\zeta, \Psi) &= \mu(\zeta, 0) - N^{-1} \left[\frac{u^y}{s^y} N \left[a \frac{\partial^2 \mu(\zeta, \Psi)}{\partial \zeta^2} - \frac{\partial \mu(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho(\zeta, \Psi)}{\partial \zeta} + \mu(\zeta, \Psi) \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} \right] \right], \\ \rho(\zeta, \Psi) &= \rho(\zeta, 0) - N^{-1} \left[\frac{u^y}{s^y} N \left[b \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2} + c \mu(\zeta, \Psi) - d \rho(\zeta, \Psi) \right] \right]. \end{aligned} \quad (34)$$

Now, implementing HPM, we get

$$\begin{aligned} \sum_{\ell=0}^{\infty} p^{\ell} \mu_{\ell}(\zeta, \Psi) &= m e^{-\zeta^2} + p \left[N^{-1} \left[\frac{u^y}{s^y} N \left[a \sum_{\ell=0}^{\infty} p^{\ell} \frac{\partial^2 \mu_{\ell}(\zeta, \Psi)}{\partial \zeta^2} - \sum_{\ell=0}^{\infty} p^{\ell} \mathcal{H}_{\ell} + \sum_{\ell=0}^{\infty} p^{\ell} \mathcal{H}'_{\ell} \right] \right] \right], \\ \sum_{\ell=0}^{\infty} p^{\ell} \rho_{\ell}(\zeta, \Psi) &= n e^{-\zeta^2} + p \left[N^{-1} \left[\frac{u^y}{s^y} N \left[b \sum_{\ell=0}^{\infty} p^{\ell} \frac{\partial^2 \rho_{\ell}(\zeta, \Psi)}{\partial \zeta^2} + c \sum_{\ell=0}^{\infty} p^{\ell} \mu_{\ell}(\zeta, \Psi) - d \sum_{\ell=0}^{\infty} p^{\ell} \rho_{\ell}(\zeta, \Psi) \right] \right] \right], \end{aligned} \quad (35)$$

where \mathcal{H}_ℓ and \mathcal{H}'_ℓ are He's polynomials that show the nonlinear terms. So, representing He's polynomials for $\mathcal{H}_\ell(\mu)$, we find that

$$\begin{aligned}\sum_{\ell=0}^{\infty} p^\ell \mathcal{H}_\ell &= \frac{\partial \mu(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho(\zeta, \Psi)}{\partial \zeta}, \\ \mathcal{H}_0 &= \frac{\partial \mu_0(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho_0(\zeta, \Psi)}{\partial \zeta}, \\ \mathcal{H}_1 &= \frac{\partial \mu_0(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho_1(\zeta, \Psi)}{\partial \zeta} + \frac{\partial \mu_1(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho_0(\zeta, \Psi)}{\partial \zeta}, \\ \mathcal{H}_2 &= \frac{\partial \mu_0(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho_2(\zeta, \Psi)}{\partial \zeta} + \frac{\partial \mu_1(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho_1(\zeta, \Psi)}{\partial \zeta} + \frac{\partial \mu_2(\zeta, \Psi)}{\partial \zeta} \frac{\partial \rho_0(\zeta, \Psi)}{\partial \zeta}.\end{aligned}\tag{36}$$

For $\mathcal{H}'_1(\mu)$, we find that

$$\begin{aligned}\sum_{\ell=0}^{\infty} p^\ell \mathcal{H}'_\ell &= \mu(\zeta, \Psi) \frac{\partial^2 \rho(\zeta, \Psi)}{\partial \zeta^2}, \\ \mathcal{H}'_0 &= \mu_0(\zeta, \Psi) \frac{\partial^2 [\rho_0(\zeta, \Psi)]}{\partial \zeta^2}, \\ \mathcal{H}'_1 &= \mu_0(\zeta, \Psi) \frac{\partial^2 [\rho_0(\zeta, \Psi)]}{\partial \zeta^2} + \mu_1(\zeta, \Psi) \frac{\partial^2 [\rho_0(\zeta, \Psi)]}{\partial \zeta^2}, \\ \mathcal{H}'_2 &= \mu_0(\zeta, \Psi) \frac{\partial^2 [\rho_2(\zeta, \Psi)]}{\partial \zeta^2} + \mu_1(\zeta, \Psi) \frac{\partial^2 [\rho_1(\zeta, \Psi)]}{\partial \zeta^2} + \mu_2(\zeta, \Psi) \frac{\partial^2 [\rho_0(\zeta, \Psi)]}{\partial \zeta^2}.\end{aligned}\tag{37}$$

The series solutions are obtained by using the numerical values $d = 0.8$, $m = 160$, $b = 3$, $n = 120$, $a = 0.5$, and $c = 1$.

By comparing the coefficients of powers p , we get

$$\begin{aligned}p^0: \mu_0(\zeta, \Psi) &= 160e^{-\zeta^2}, \\ p^0: \rho_0(\zeta, \Psi) &= 120e^{-\zeta^2}, \\ p^1: \mu_1(\zeta, \Psi) &= N^- \left[\frac{u^\gamma}{s^\gamma} N \left[a \frac{\partial^2 \mu_0(\zeta, \Psi)}{\partial \zeta^2} - \mathcal{H}_0 + \mathcal{H}'_0 \right] \right], \\ \mu_1(\zeta, \Psi) &= 4 \left\{ (9600\zeta^2 - 40)e^{-\zeta^2} - 60 \right\} e^{-\zeta^2} \frac{\Psi^\gamma}{\Gamma(\gamma + 1)}, \\ \rho_1(\zeta, \Psi) &= 8 \left\{ 3(60\zeta^2 - 30) + 8 \right\} e^{-\zeta^2} \frac{\Psi^\gamma}{\Gamma(\gamma + 1)},\end{aligned}\tag{38}$$

$$\begin{aligned}
p^2: \mu_2(\zeta, \Psi) &= N^- \left\{ \frac{u^\gamma}{s^\gamma} N \left[a \frac{\partial^2 \mu_1(\zeta, \Psi)}{\partial \zeta^2} - \mathcal{H}_1 + \mathcal{H}'_1 \right] \right\}, \\
\mu_2(\zeta, \Psi) &= 160 \left\{ 120 - 240 - (240\zeta^2 - 1)e^{-\zeta^2} - 2[240\zeta + \zeta^2(240\zeta^2 - 1)]e^{-\zeta^2} \right\} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)} \\
&\quad + 7680 \{ 120 - 4\zeta [15(2\zeta^2 - 1) + 2] \} \zeta e^{-2\zeta^2} \\
&\quad - \left\{ (38400(480 + 160(2\zeta^2 - 1)) - 960)e^{-2\zeta^2} + 500(307 - 614.4\zeta)\zeta e^{-3\zeta^2} \right\} \\
&\quad \cdot \frac{\Psi^{3\gamma}}{\Gamma(3\gamma + 1)} + \left\{ \begin{array}{l} 1280(720 - 2(90(2\zeta^2 - 1) + 8) + 4\zeta^2(90(2\zeta^2 - 1) + 8)) \\ + 160((240\zeta^2 - 1) - 480\zeta^2 + 478.125)120(4\zeta^2 - 1) \end{array} \right\} e^{-\zeta^2} \frac{\Psi^{3\gamma}}{\Gamma(3\gamma + 1)}, \\
p^2: \rho_2(\zeta, \Psi) & \\
\rho_2(\zeta, \Psi) &= \left\{ 25956 + 6(2\zeta^2 - 1)^2 + 1320(1 - 4\zeta^2) + 3(2\zeta^2 - 1) \right\} e^{-\zeta^2} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)}.
\end{aligned} \tag{39}$$

The series form of equation (30) is given by

$$\begin{aligned}
\mu(\zeta, \Psi) &= \sum_{\ell=0}^{\infty} \mu_\ell(\zeta, \Psi) = \mu_0(\zeta, \Psi) + \mu_1(\zeta, \Psi) + \mu_2(\zeta, \Psi) + \mu_3(\zeta, \Psi) + \dots, \\
\rho(\zeta, \Psi) &= \sum_{\ell=0}^{\infty} \rho_\ell(\zeta, \Psi) = \rho_0(\zeta, \Psi) + \rho_1(\zeta, \Psi) + \rho_2(\zeta, \Psi) + \rho_3(\zeta, \Psi) + \dots.
\end{aligned} \tag{40}$$

The HPTM solution is given as

$$\begin{aligned}
\mu(\zeta, \Psi) &= 160e^{-\zeta^2} + 4 \left[(9600\zeta^2 - 40)e^{-\zeta^2} - 60 \right] e^{-\zeta^2} \frac{\Psi^\gamma}{\Gamma(\gamma + 1)} \\
&\quad + 160 \left\{ 120 - 240 - (240\zeta^2 - 1)e^{-\zeta^2} - 2[240\zeta + \zeta^2(240\zeta^2 - 1)]e^{-\zeta^2} \right\} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)} \\
&\quad + 7680 \{ 120 - 4\zeta [15(2\zeta^2 - 1) + 2] \} \zeta e^{-2\zeta^2} \\
&\quad - \left\{ (38400[480 + 160(2\zeta^2 - 1)] - 960)e^{-2\zeta^2} + 500(307 - 614.4\zeta)\zeta e^{-3\zeta^2} \right\} \\
&\quad \cdot \frac{\Psi^{3\gamma}}{\Gamma(3\gamma + 1)} + \left\{ 1280(720 - 2(90(2\zeta^2 - 1) + 8) + 4\zeta^2(90(2\zeta^2 - 1) + 8)) \right. \\
&\quad \left. + 160[(240\zeta^2 - 1) - 480\zeta^2 + 478.125](120(4\zeta^2 - 1)) \right\} e^{-\zeta^2} \frac{\Psi^{3\gamma}}{\Gamma(3\gamma + 1)} + \dots \\
\rho(\zeta, \Psi) &= 120e^{-\zeta^2} + 8 \{ 3(60\zeta^2 - 30) + 8 \} e^{-\zeta^2} \frac{\Psi^\gamma}{\Gamma(\gamma + 1)} \\
&\quad + \left\{ 25956 + 6(2\zeta^2 - 1)^2 + 1320(1 - 4\zeta^2) + 3(2\zeta^2 - 1) \right\} e^{-\zeta^2} \frac{\Psi^{2\gamma}}{\Gamma(2\gamma + 1)} + \dots.
\end{aligned} \tag{41}$$

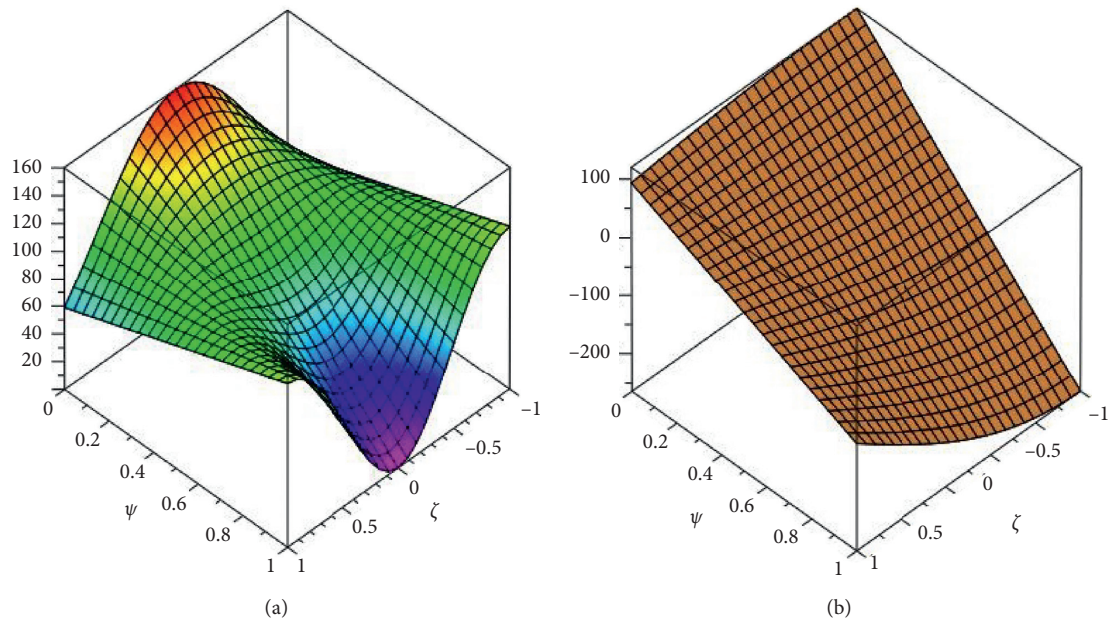


FIGURE 1: HPTM solutions of $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$ at $\gamma = 1$.

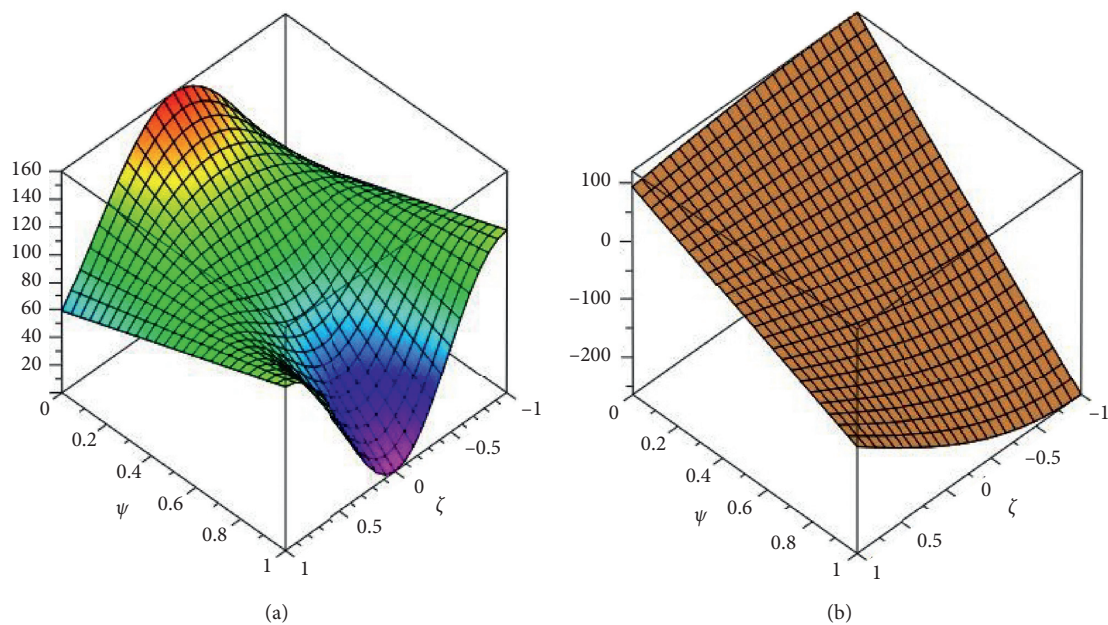


FIGURE 2: LADM solutions of $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$ at $\gamma = 1$.

5. Results and Discussion

In Figures 1 and 2, the comparative study of HPTM and LADM solutions has been made for variables $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$ at $\gamma = 1$ for Example 1. The graphical representation has shown the close relation between HPTM and LADM solutions. In Figures 3 and 4, the graphs of $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$ versus ζ are plotted for fixed $\Psi = 1$ for both fractional and integer orders of Example 2. The graphical representation has confirmed the convergence of fractional-order

solutions towards integer-order solutions. In Table 1, the LADM and HPTM solutions of Example 1 are compared at $\zeta = 1$ and $\gamma = 1$. It is observed that LADM and HPTM solutions are identical and justify the reliability of the proposed techniques. In Figures 5 and 6, the HPTM solutions, and in Figure 7, the HPTM solutions at fractional orders $\gamma = 0.5, 0.6, 0.8$, and 1, are discussed for variables $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$. It is investigated that both the methods have a higher degree of accuracy and provide the closed-form solution to Example 2.

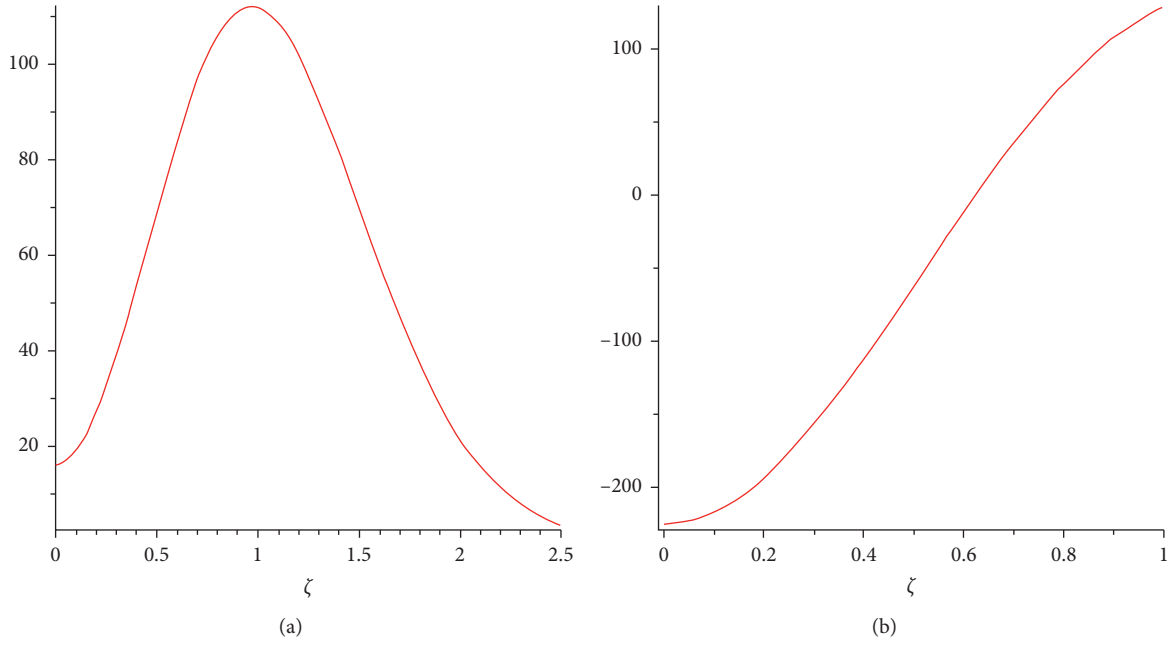


FIGURE 3: HPTM solutions of $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$ at $\gamma = 1$.

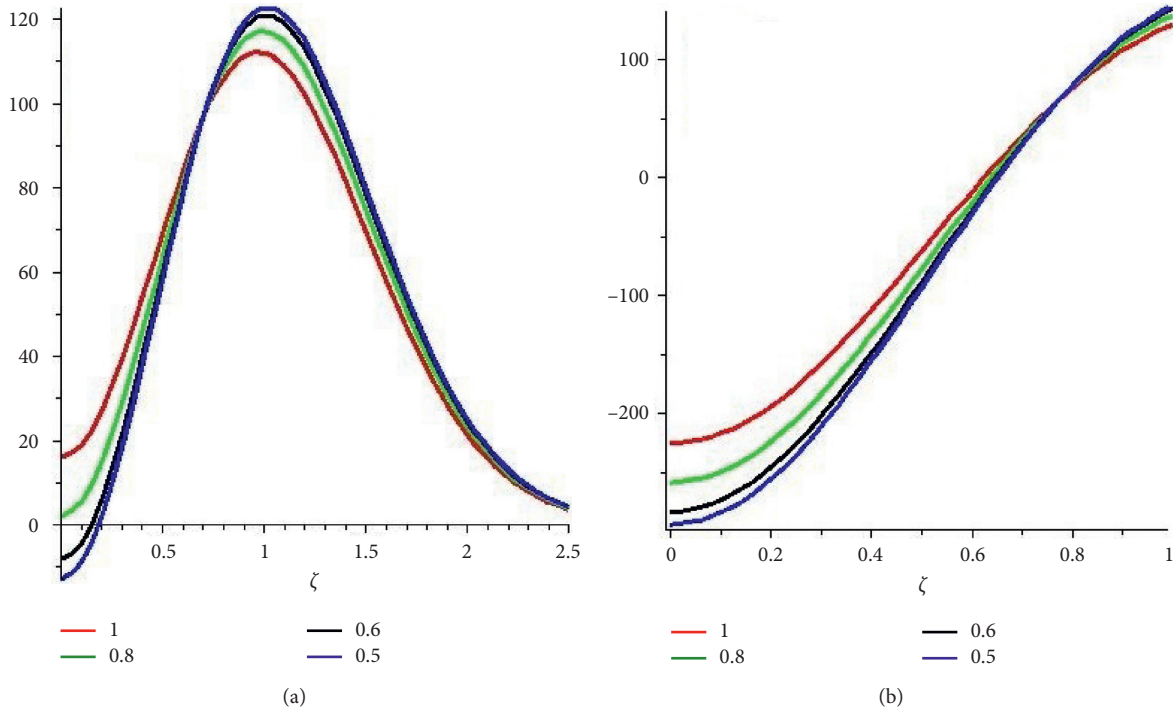


FIGURE 4: HPTM solutions of $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$ at different fractional orders $\gamma = 0.5, 0.6, 0.8, 1$ and $\Psi = 1$.

TABLE 1: Comparison of HDM [36], LADM [37], and HPTM at $\zeta = 1$ and $\gamma = 1$.

Ψ	HDM $\mu(\zeta, \psi)$	HDM $\rho(\zeta, \psi)$	LADM $\mu(\zeta, \psi)$	LADM $\rho(\zeta, \psi)$	HPTM $\mu(\zeta, \psi)$	HPTM $\rho(\zeta, \psi)$
0.2	2.70×10^5	8.0×10^2	2.70×10^5	8.0×10^2	2.70×10^5	8.0×10^2
0.4	3.0×10^6	2.0×10^3	3.0×10^6	2.0×10^3	3.0×10^6	2.0×10^3
0.6	5.90×10^6	4.0×10^3	5.90×10^6	4.0×10^3	5.90×10^6	4.0×10^3
0.8	8.50×10^6	7.0×10^3	8.50×10^6	7.0×10^3	8.50×10^6	7.0×10^3
1.00	1.50×10^7	1.20×10^4	1.50×10^7	1.20×10^4	1.50×10^7	1.20×10^4

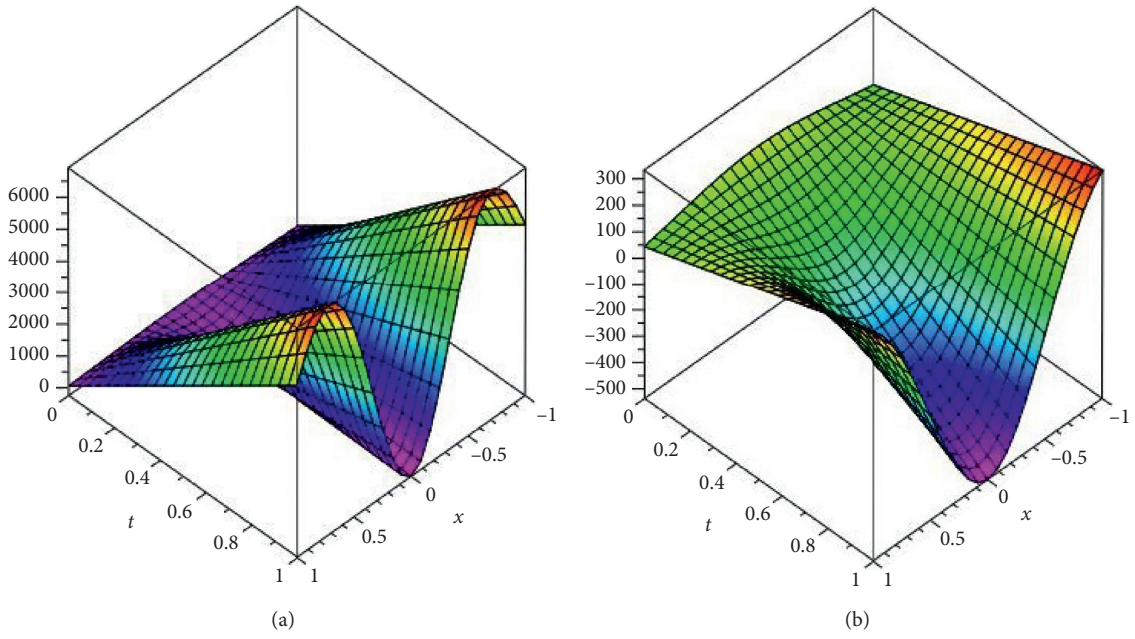


FIGURE 5: HPTM solutions of $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$ at $\gamma = 1$.

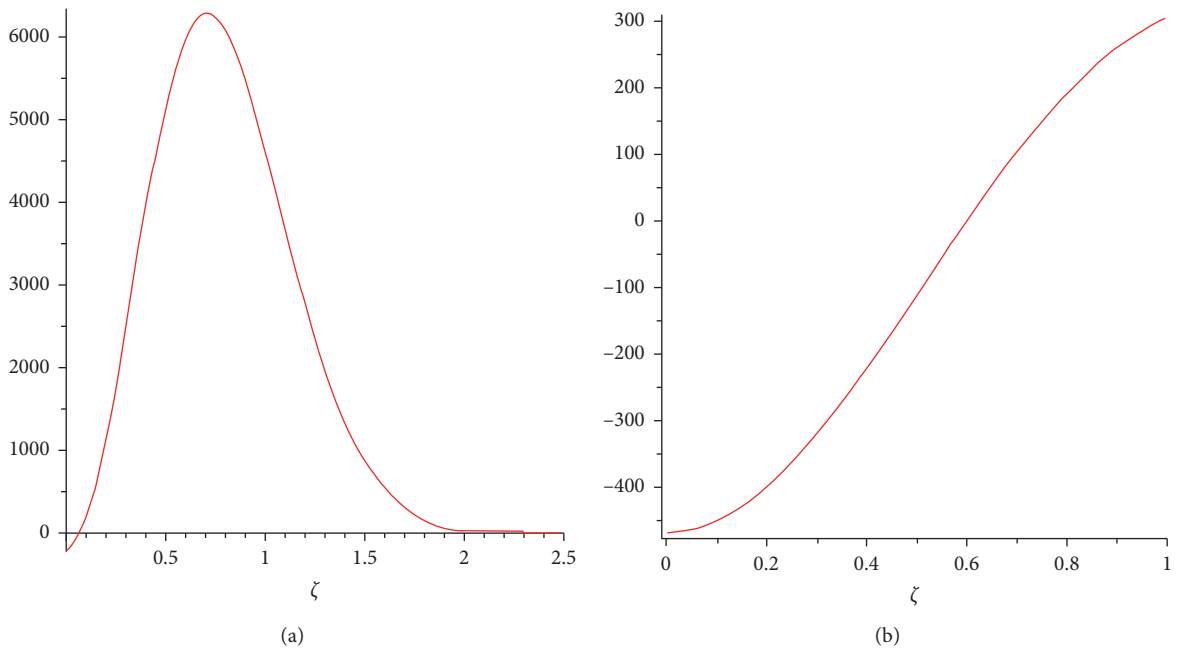


FIGURE 6: HPTM solutions of $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$ at $\gamma = 1$.

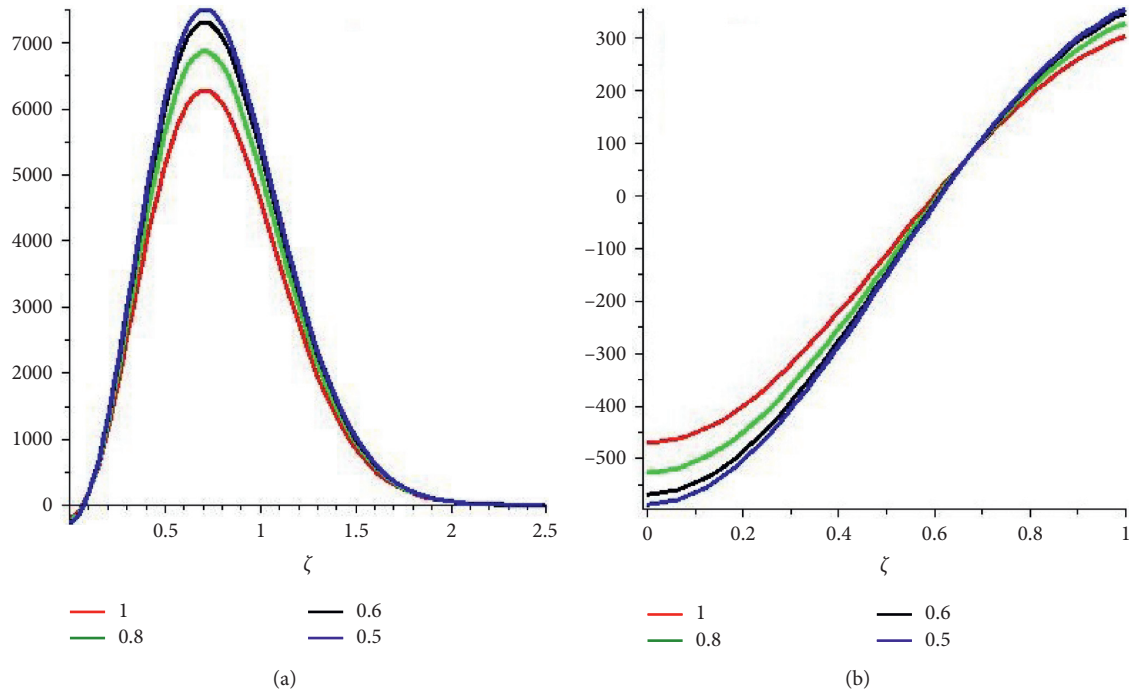


FIGURE 7: HPTM solutions of $\mu(\zeta, \Psi)$ and $\rho(\zeta, \Psi)$ at different fractional orders $\gamma = 0.5, 0.6, 0.8, 1$ and $\Psi = 1$.

6. Conclusion

In the current article, an effective technique which is known as the Laplace homotopy transform method is implemented to solve the systems of Keller–Segal equations for both fractional and integer orders of the derivatives within the Caputo operator. Two numerical examples of fractional Keller–Segal equations are presented to verify the reliability of the suggested method. The graphical and tabular representation has confirmed that the derived results are in close agreement with the solution of the Laplace Adomian decomposition method. Moreover, the current technique needs very small calculations and has a higher degree of accuracy for the targeted problems. In conclusion, the present technique is found to be an accurate and effective analytical technique to solve high nonlinear fractional systems of partial differential equations.

Data Availability

The numerical data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University, Saudi Arabia, for funding this work through the research groups program under the grant number (R.G.P2./99/41).

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