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Some modifications in conformable fractional integral inequalities

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Abstract
The prevalence of the use of integral inequalities has dramatically influenced the evolution of mathematical analysis. The use of these useful tools leads to faster advances in the presentation of fractional calculus. This article investigates the Hermite–Hadamard integral inequalities via the notion of $\mathcal{F}$-convexity. After that, we introduce the notion of $\mathcal{F}_\mu$-convexity in the context of conformable operators. In view of this, we establish some Hermite–Hadamard integral inequalities (both trapezoidal and midpoint types) and some special case of those inequalities as well. Finally, we present some examples on special means of real numbers. Furthermore, we offer three plot illustrations to clarify the results.

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1 Introduction

For any $\nu_1, \nu_2 \in [a,b]$ and $\ell \in [0,1]$, the real-valued function g on an interval $[a,b]$ is called a convex function if the following holds:

$$g\big(\ell\nu_1 + (1-\ell)\nu_2\big) \le \ell g(\nu_1) + (1-\ell)g(\nu_2). \tag{1.1}$$

The theory and application of convexity has a close relationship with theory and application of inequalities or integral inequalities. The convex function (1.1) has been extended and generalized in several directions, such as pseudo-convex [1], quasi-convex [2], strongly convex [3], ϵ-convex [4], s-convex [5], h-convex [6, 7], (α, m)-convex [8, 9], invex and preinvex [10–12], and other kinds of convex functions by a number of mathematicians; see [13–21] for more details.

Integral inequalities form an essential field of study among the field of mathematical analysis. They have been vital in providing bounds to solve some boundary value problems in fractional calculus, and in establishing the existence and uniqueness of solutions for certain fractional differential equations. Convexity plays an important role in the field of integral inequality due to the behavior of its definition. Also, there is a strong connection between convexity and integral inequality. For this reason, many known integral inequalities have been established in the literature. The Hermite–Hadamard (HH) inequality is the most well known one: for an L^1 convex function $g:\mathcal{I}\subseteq\mathbb{R}\to\mathbb{R}$ with $\nu_1,\nu_2\in\mathcal{I}$, $\nu_1<\nu_2$, the HH inequality is defined as follows:

$$g\left(\frac{\nu_1+\nu_2}{2}\right)\leq\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}g(x)\,dx\leq\frac{g(\nu_1)+g(\nu_2)}{2}. \tag{1.2}$$

A huge number of researchers in the field of applied mathematics have dedicated their interest to generalize, improve, refine, counterpart, and extend HH inequality (1.2) for various types of convex functions; see e.g. [22–30].

Recently, Samet [31] introduced a new notion of convexity for certain functions that depends on some axioms. This often generalizes various types of convexity e.g. ϵ-convex functions, α-convex functions, h-convex functions, and so on. Also, for further details, visit [16, 32].

Throughout our study, we suppose that $\mathcal{I}\subseteq\mathbb{R}$ ($\mathbb{R}$ the set of real numbers), $\mathcal{V}=\{(\eta_1,\eta_2);\eta_\ell\in[\nu_1,\nu_2],\ell=1,2\}$ and $\bar{\mathfrak{R}}=\{(\eta_1,\eta_2,\eta_3);\eta_i\in\mathbb{R},\ell=1,2,3\}$. Then the family of $\mathcal{F}$ of functions $F:\bar{\mathfrak{R}}\times[0,1]\to\mathbb{R}$ satisfies the major axioms [31]:

(Λ_1) If $\mathbf{y}_\ell\in L^1(0,1)$, $\ell=1,2,3$, then for every $\gamma\in[0,1]$ we have

$$\begin{aligned}&\int_0^1 F\bigl(\mathbf{y}_1(\eta),\mathbf{y}_2(\eta),\mathbf{y}_3(\eta),\gamma\bigr)\,d\eta\\&\quad=F\left(\int_0^1\mathbf{y}_1(\eta)\,d\eta,\int_0^1\mathbf{y}_2(\eta)\,d\eta,\int_0^1\mathbf{y}_3(\eta)\,d\eta,\gamma\right).\end{aligned}$$

(Λ_2) For every $u\in L^1(0,1)$, $w\in L^\infty(0,1)$, and $(z_1,z_2)\in\mathbb{R}^2$, we have

$$\int_0^1 F\bigl(w(\eta)u(\eta),w(\eta)z_1,w(\eta)z_2,\eta\bigr)\,d\eta=\mathcal{T}_{F,w}\left(\int_0^1 w(\eta)u(\eta)\,d\eta,z_1,z_2\right),$$

where $\mathcal{T}_{F,w}:\bar{\mathfrak{R}}\to\mathbb{R}$ is a function depending on (F,w). Moreover, it is a nondecreasing function according to the first variable.

(Λ_3) For any $(w,\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3)\in\mathbb{R}^4$, $\mathbf{y}_4\in[0,1]$, we have

$$wF(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3,\mathbf{y}_4)=F(w\mathbf{y}_1,w\mathbf{y}_2,w\mathbf{y}_3,\mathbf{y}_4)+L_w,$$

where $L_w\in\mathbb{R}$ is a constant (depending on w).

Definition 1.1 Let $g:[\nu_1,\nu_2]\subseteq\mathbb{R}\to\mathbb{R}$ with $\nu_1<\nu_2$ be a function, then we say that g is a convex function according to $F\in\mathcal{F}$ (or briefly F-convex function) iff

$$\bar{F}\bigl(g\bigl(\eta x+(1-\eta)y\bigr),g(x),g(y),\eta\bigr)\leq 0,\quad (x,y,\eta)\in\mathcal{V}\times[0,1].$$

Remark 1.1 Suppose that $(\nu_1,\nu_2)\in\mathbb{R}^2$ with $\nu_1<\nu_2$,

(i) if $g:[\nu_1,\nu_2]\subset\mathbb{R}\to\mathbb{R}$ is an ε-convex function, or equivalently [26]

$$g\bigl(\eta x+(1-\eta)y\bigr)\leq\eta g(x)+(1-\eta)g(y)+\varepsilon,\quad (x,y,\eta)\in\mathcal{V}\times[0,1],$$

then we define the functions $F : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as follows:

$$F(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \mathbf{y}_1 - \mathbf{y}_4\mathbf{y}_2 - (1 - \mathbf{y}_4)\mathbf{y}_3 - \varepsilon, \tag{1.3}$$

and $\mathcal{T}_{F,w} : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as

$$\mathcal{T}_{F,w}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \mathbf{y}_1 - \left(\int_0^1 tw(\eta)\,d\eta\right)\mathbf{y}_2 - \left(\int_0^1 (1-\eta)w(\eta)\,d\eta\right)\mathbf{y}_3 - \varepsilon. \tag{1.4}$$

For

$$L_w = (1 - w)\varepsilon, \tag{1.5}$$

we can observe that $F \in \mathcal{F}$ and

$$F\big(g\big(\eta x + (1-\eta)y\big), g(x), g(y), \eta\big) = g\big(\eta x + (1-\eta)y\big) - \eta g(x) - (1-\eta)g(y) - \varepsilon \leq 0,$$

and this tells us g is an F-convex function. In a particular case, we take $\varepsilon = 0$ to show that g is an F-convex function according to F when g is assumed to be a convex function.

(ii) If $g : [\nu_1, \nu_2] \subset \mathrm{R} \to \mathrm{R}$ is a μ-convex function with $\mu \in (0,1]$, or equivalently

$$g\big(\eta x + (1-\eta)y\big) \leq \eta^\mu g(x) + \big(1 - \eta^\mu\big)g(y), \quad (x, y, \eta) \in \mathcal{V} \times [0,1].$$

Then we define the function $F : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as follows:

$$F(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \mathbf{y}_1 - \mathbf{y}_4^\mu\mathbf{y}_2 - \big(1 - \mathbf{y}_4^\mu\big)\mathbf{y}_3, \tag{1.6}$$

and $\mathcal{T}_{F,w} : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as

$$\mathcal{T}_{F,w}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \mathbf{y}_1 - \left(\int_0^1 \eta^\mu w(\eta)\,d\eta\right)\mathbf{y}_2 - \left(\int_0^1 \big(1 - \eta^\mu\big)w(\eta)\,d\eta\right)\mathbf{y}_3. \tag{1.7}$$

For $L_w = 0$, we can observe that $F \in \mathcal{F}$ and

$$F\big(g\big(\eta x + (1-\eta)y\big), g(x), g(y), \eta\big) = g\big(\eta x + (1-\eta)y\big) - \eta^\mu g(x) - \big(1 - \eta^\mu\big)g(y) - \varepsilon \leq 0,$$

or g is an F-convex function.

(iii) If $h : \mathcal{I} \to \mathrm{R}$ is a function and it is not identically 0, where $(0,1) \subseteq \mathcal{I}$. Also, suppose that $g : [\nu_1, \nu_2] \subset \mathcal{I} \to [0, \infty)$ is an h-convex function, that is,

$$g\big(\eta x + (1-\eta)y\big) \leq h(\eta)g(x) + h(1-\eta)g(y), \quad (x, y, \eta) \in \mathcal{V} \times [0,1].$$

Then we define the functions $F : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as follows:

$$F(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \mathbf{y}_1 - h(\mathbf{y}_4)\mathbf{y}_3 - h(1 - \mathbf{y}_4)\mathbf{y}_2, \tag{1.8}$$

and $\mathcal{T}_{\digamma,w} : \bar{\mathfrak{R}} \times [0,1] \to \mathbb{R}$ as

$$\mathcal{T}_{\digamma,w}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3) = \mathbf{y}_1 - \left(\int_0^1 h(\eta)w(\eta)\,d\eta\right)\mathbf{y}_2 - \left(\int_0^1 h(1-\eta)w(\eta)\,d\eta\right)\mathbf{y}_3. \tag{1.9}$$

For $L_w = (1-w)\varepsilon$, we can observe that $\digamma \in \mathcal{F}$ and

$$\digamma\big(g\big(\eta x + (1-\eta)y\big), g(x), g(y), \eta\big) = g\big(\eta x + (1-\eta)y\big) - h(\eta)g(x) - h(1-\eta)g(y) - \varepsilon \le 0,$$

or we can say that g is an $\digamma$-convex function.

In recent years, many possible inequalities have been proposed in the context of fractional calculus including the midpoint and trapezoidal formula inequalities and inequalities for ε-convexity, α-convexity, (α,m)-convexity, and h-convexity; see [26, 31, 33] for more details.

2 Conformable fractional operators and $\digamma_\mu$-convexity

In the last fifteen years, the definition of fractional calculus has been more appropriate to describe historical dependence processes than the local limit definitions of integer ordinary differential equations (ODEs) or partial differential equations (PDEs), and has received more and more attention in many mathematical and physical fields, see for details [34–44]. Differential equations of fractional order are more accurate than differential equations of integer order in describing the nature of things and objective laws. In 1695, Leibnitz discovered fractional derivatives, and after that more and more researchers have dedicated themselves to the study of fractional calculus. The most commonly used fractional calculus definitions are Riemann–Liouville definition, Caputo definition, and conformable fractional definition in basic mathematical and engineering application research. In the present paper, we deal with the conformable fractional definition [45–47] in order to obtain our results.

In this section, we recall some preliminaries and properties on conformable fractional calculus. For further details and applications, see the previously published articles [33, 45–54].

Definition 2.1 ([47]) Let $g : [0,\infty) \to \mathbb{R}$, then the μth order conformable derivative of g at η is defined by

$$D_\mu(g)(\eta) = \lim_{\epsilon\to 0} \frac{g(\eta + \epsilon\eta^{1-\mu}) - g(\eta)}{\epsilon}, \quad \mu \in (0,1), \eta > 0. \tag{2.1}$$

For μ-differentiable function g in some $(0,\mu)$, $\mu > 0$, $\lim_{t\to 0^+} g^{(\mu)}(\eta)$ exist, define

$$g^{(\mu)}(0) = \lim_{t\to 0^+} g^{(\mu)}(\eta).$$

Furthermore, if g is differentiable, then we have

$$D_\mu(g)(\eta) = \eta^{1-\mu} g'(\eta), \quad \text{where } g'(\eta) = \lim_{\epsilon\to 0} \frac{g(\eta+\epsilon) - g(\eta)}{\epsilon}. \tag{2.2}$$

Observe that we can write $g^{(\mu)}(\eta)$ for $\frac{d_\mu}{d_\mu \eta}(g(\eta))$ or simply $D_\mu(g)(\eta)$ to denote a μth order conformable derivative of g at η. Furthermore, if the μth order conformable derivative of g exists, then we can simply say g is μ-differentiable.

Theorem 2.1 ([48]) *Assume that $\mu \in (0,1]$ and f, g are two μ-differentiable functions at a point $\eta > 0$. Then we have:*

1. $D_\mu(\nu_1 f + \nu_2 g) = \nu_1 D_\mu(f) + \nu_2 D_\mu(g)$ *for all* $\nu_1, \nu_2 \in \mathbb{R}$,
2. $D_\mu(fg) = fD_\mu(g) + gD_\mu(f)$,
3. $D_\mu(\frac{f}{g}) = \frac{gD_\mu(f)-fD_\mu(g)}{g^2}$,
4. $D_\mu(c) = 0$ *for each constant function, namely* $g(\eta) = c$,
5. $D_\mu(1) = 0$,
6. $D_\mu(\frac{1}{\mu}\eta^\mu) = 1$.

Some basic properties of conformable operator are now stated, which are useful in what follows.

Definition 2.2 ([47]) Assume that $\mu \in (0,1]$, $0 \leq \nu_1 \leq \nu_2$, and $g : [\nu_1, \nu_2] \subset \mathbb{R} \to \mathbb{R}$, then we say that a function g is μ-fractional integrable on the interval $[\nu_1, \nu_2]$ if the following integral

$$\int_{\nu_1}^{\nu_2} g(\eta)\, d_\mu \eta = \int_{\nu_1}^{\nu_2} g(\eta)\eta^{\mu-1}\, d\eta \tag{2.3}$$

exists and is finite.

Remark 2.1

(a) We indicate by $L_\mu^1([\nu_1, \nu_2])$ all μ-fractional integrable functions on an interval $[\nu_1, \nu_2]$.

(b) The usual Riemann improper integral has the form

$$I_\mu^{\nu_1}(g)(\eta) = I_1^{\nu_1}\left(\eta^{\mu-1} g\right) = \int_{\nu_1}^{t} x^{\mu-1} g(x)\, dx, \quad \mu \in (0,1]. \tag{2.4}$$

Theorem 2.2 ([47, 48]) *Let $g : (\nu_1, \nu_2) \to \mathbb{R}$ be differentiable and $\mu \in (0,1]$. Then, for all $\eta > \nu_1$, we have*

$$I_\mu^{\nu_1} D_\mu^{\nu_1}(g)(\eta) = g(\eta) - g(\nu_1).$$

Theorem 2.3 ([51]) *Suppose that $g : [\nu_1, \infty) \to \mathbb{R}$ such that $g^{(n)}$ is continuous. Then, for each $\eta > \nu_1$, we have*

$$D_\mu^{\nu_1} I_\mu^{\nu_1}(g)(\eta) = g(\eta), \quad \mu \in (n, n+1],$$

which is called the inverse property.

Theorem 2.4 ([47, 48]) *Let $g : [\nu_1, \nu_2] \subset \mathbb{R} \to \mathbb{R}$ be two functions with fg is differentiable. Then*

$$\int_{\nu_1}^{\nu_2} g(x) D_\mu^{\nu_1}(h)(x)\, d_\mu x = gh|_{\nu_1}^{\nu_2} - \int_{\nu_1}^{\nu_2} h(x) D_\mu^{\nu_1}(g)(x)\, d_\mu x.$$

Theorem 2.5 ([47, 48]) *Let $f, g : [\nu_1, \nu_2] \subset \mathrm{R} \to \mathrm{R}$ be a continuous function on $[\nu_1, \nu_2]$ and with $0 \le \nu_1 \le \nu_2$. Then*

$$\left|I_\mu^{\nu_1}(g)(\eta)\right| \le I_\mu^{\nu_1}|f|(\eta), \quad \mu \in (0,1].$$

It is time to define the concept of $\digamma_\mu$-convexity on conformable integrals, namely the family of $\digamma_\mu$.

The family of $\digamma_\mu$ of functions $\digamma_\mu : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ satisfies the major axioms:

$(\bar{\Lambda}_1)$ If $\mathbf{y}_\ell \in L^1(0,1)$, $\ell = 1,2,3$, then for every $\gamma \in [0,1]$ we have

$$\begin{aligned}&\int_0^1 \digamma_\mu\big(\mathbf{y}_1(\eta), \mathbf{y}_2(\eta), \mathbf{y}_3(\eta), \gamma\big)\, d\eta \\ &\quad = \digamma_\mu\left(\int_0^1 \mathbf{y}_1(\eta)\, d\eta, \int_0^1 \mathbf{y}_2(\eta)\, d\eta, \int_0^1 \mathbf{y}_3(\eta)\, d\eta, \gamma\right).\end{aligned}$$

$(\bar{\Lambda}_2)$ For every $u \in L^1(0,1)$, $w \in L^\infty(0,1)$, and $(z_1, z_2) \in \mathrm{R}^2$, we have

$$\int_0^1 \digamma_\mu\big(w(\eta)u(\eta), w(\eta)z_1, w(\eta)z_2, \eta\big)\, d\eta = \mathcal{T}_{\digamma_\mu, w}\left(\int_0^1 w(\eta)u(\eta)\, d\eta, z_1, z_2\right),$$

where $\mathcal{T}_{\digamma_\mu, w} : \bar{\mathfrak{R}} \to \mathrm{R}$ is a nondecreasing function according to the first variable which depends on $(\digamma_\mu, w)$.

$(\bar{\Lambda}_3)$ For any $(w, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) \in \mathrm{R}^4$, $\mathbf{y}_4 \in [0,1]$, we have

$$w\digamma_\mu(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \digamma_\mu(w\mathbf{y}_1, w\mathbf{y}_2, w\mathbf{y}_3, \mathbf{y}_4) + L_w,$$

where $L_w \in \mathrm{R}$ is a constant (depending on w).

Definition 2.3 Let $\mu \in (0,1]$ and $g : [\nu_1, \nu_2] \subset \mathrm{R} \to \mathbb{R}$ with $\nu_1 < \nu_2$ be a function, then we say g is a conformable convex function according to $\digamma_\mu \in \mathcal{F}$ (or briefly $\digamma_\mu$-conformable convex function) if

$$\digamma_\mu\big(g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big), g\big(x^\mu\big), g\big(y^\mu\big), \eta^\mu\big) \le 0, \quad (x, y, \eta) \in \mathcal{V} \times [0,1].$$

Remark 2.2 Suppose that $(\nu_1, \nu_2) \in \mathrm{R}^2$ with $\nu_1 < \nu_2$.

(i) Let $g : [\nu_1, \nu_2] \subset \mathrm{R} \to \mathrm{R}$ be an ε-conformable convex function, or equivalently

$$g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big) \le \eta^\mu g\big(x^\mu\big) + \big(1-\eta^\mu\big)g\big(y^\mu\big) + \varepsilon, \quad (x, y, \eta) \in \mathcal{V} \times [0,1].$$

Then we define the function $\digamma_\mu : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as follows:

$$\digamma_\mu(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \mathbf{y}_1 - \mathbf{y}_4^\mu \mathbf{y}_2 - \big(1-\mathbf{y}_4^\mu\big)\mathbf{y}_3 - \varepsilon, \tag{2.5}$$

and $\mathcal{T}_{\digamma_\mu, w} : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as

$$\mathcal{T}_{\digamma_\mu, w}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \mathbf{y}_1 - \left(\int_0^1 \eta^\mu w\big(\eta^\mu\big)\, d_\mu\eta\right)\mathbf{y}_2 - \left(\int_0^1 \big(1-\eta^\mu\big)w(\eta)\, d_\mu\eta\right)\mathbf{y}_3 - \varepsilon. \tag{2.6}$$

For

$$L_w = (1 - w)\varepsilon, \tag{2.7}$$

it can be observed that $\digamma \in \mathcal{F}$ and

$$\begin{aligned}&\digamma_\mu\big(g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big), g\big(x^\mu +\big), g\big(y^\mu\big), \eta^\mu\big)\\&\quad = g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big) - \eta^\mu g\big(x^\mu\big) - \big(1-\eta^\mu\big)g\big(y^\mu\big) - \varepsilon \le 0,\end{aligned}$$

or in another meaning g is an $\digamma$-conformable convex function. In particular, g is an $\digamma$-conformable convex function according to $\digamma$ for $\varepsilon = 0$ when g is a conformable convex function.

(ii) Let $g : [\nu_1, \nu_2] \subset \mathcal{I} \to \mathrm{R}$ be a μ-conformable convex function $\mu \in (0,1]$; that is,

$$g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big) \le \eta g\big(x^\mu\big) + (1-\eta)g\big(y^\mu\big), \quad (x,y,\eta) \in \mathcal{V} \times [0,1].$$

Then we define the functions $\digamma_\mu : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as follows:

$$\digamma_\mu(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \mathbf{y}_1 - \mathbf{y}_4\mathbf{y}_2 - (1-\mathbf{y}_4)\mathbf{y}_3, \tag{2.8}$$

and $\mathcal{T}_{\digamma_\mu, w} : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as

$$\mathcal{T}_{\digamma_\mu, w}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \mathbf{y}_1 - \bigg(\int_0^1 tw\big(\eta^\mu\big)\,d_\mu\eta\bigg)\mathbf{y}_2 - \bigg(\int_0^1 (1-\eta)w\big(\eta^\mu\big)\,d_\mu\eta\bigg)\mathbf{y}_3. \tag{2.9}$$

For $L_w = 0$, we can observe that $\digamma_\mu \in \mathcal{F}$ and

$$\begin{aligned}&\digamma_\mu\big(g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big), g\big(x^\mu\big), g\big(y^\mu\big), \eta^\mu\big)\\&\quad = g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big) - \eta g\big(x^\mu\big) - (1-\eta)g\big(y^\mu\big) - \varepsilon \le 0,\end{aligned}$$

or equivalently g is an $\digamma_\mu$-conformable convex function.

(iii) Let $h : \mathcal{I} \to \mathrm{R}$ be a function, which is not identically 0, where $(0,1) \subseteq \mathcal{I}$. Let $g : [\nu_1, \nu_2] \subset \mathcal{I} \to [0,\infty)$ be an h-conformable convex function, or let

$$g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big) \le h\big(\eta^\mu\big)g\big(x^\mu\big) + h\big(1-\eta^\mu\big)g\big(y^\mu\big), \quad (x,y,\eta) \in \mathcal{V} \times [0,1].$$

Then we define the functions $\digamma_\mu : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as follows:

$$\digamma_\mu(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4) = \mathbf{y}_1 - h(\mathbf{y}_4)\mathbf{y}_3 - h\big(1-\mathbf{y}_4^\mu\big)\mathbf{y}_2, \tag{2.10}$$

and $\mathcal{T}_{\digamma_\mu, w} : \bar{\mathfrak{R}} \times [0,1] \to \mathrm{R}$ as

$$\begin{aligned}\mathcal{T}_{\digamma_\mu, w}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \mathbf{y}_1 &- \bigg(\int_0^1 h\big(\eta^\mu\big)w\big(\eta^\mu\big)\,d_\mu\eta\bigg)\mathbf{y}_2\\&- \bigg(\int_0^1 h\big(1-\eta^\mu\big)w\big(\eta^\mu\big)\,d_\mu\eta\bigg)\mathbf{y}_3.\end{aligned} \tag{2.11}$$

For $L_w = (1-w)\varepsilon$, we can observe that $\digamma_\mu \in \mathcal{F}$ and

$$\begin{aligned}&\digamma_\mu\big(g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big), g\big(x^\mu\big), g\big(y^\mu\big), \eta^\mu\big)\\ &\quad = g\big(\eta^\mu x^\mu + \big(1-\eta^\mu\big)y^\mu\big) - h\big(\eta^\mu\big)g\big(x^\mu\big) - h\big(1-\eta^\mu\big)g\big(y^\mu\big) - \varepsilon \le 0,\end{aligned}$$

or equivalently we can say g is an $\digamma_\mu$-conformable convex function.

For the conformable operators, we recall some early findings in the earlier literature which may help us in finding our main results. For example in [55], Sarikaya et al. investigated new results for the conformable fractional operator, and their results are as follows.

Theorem 2.6 ([55, Theorem 11]) *Let $\mu \in (0,1]$ and $g : [\nu_1, \nu_2] \subset \mathrm{R} \to \mathbb{R}$ be a μ-fractional differentiable function on (ν_1, ν_2) with $0 \le \nu_1 < \nu_2$. Then we have*

$$g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right) \le \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g\big(x^\mu\big)\,d_\mu x \le \frac{g(\nu_1^\mu) + g(\nu_2^\mu)}{2}. \tag{2.12}$$

Lemma 2.1 ([55, Lemma 3]) *Let $\mu \in (0,1]$ and $g : [\nu_1, \nu_2] \subset \mathrm{R} \to \mathbb{R}$ be a μ-fractional differentiable function on (ν_1, ν_2) with $0 \le \nu_1 < \nu_2$. If $D_\mu(g)$ is a μ-fractional integrable function on $[\nu_1, \nu_2]$, then we have*

$$\begin{aligned}&\frac{g(\nu_1^\mu) + g(\nu_2^\mu)}{2} - \int_{\nu_1}^{\nu_2} g\big(x^\mu\big)\,d_\mu x\\ &\quad = \frac{\nu_2^\mu - \nu_1^\mu}{2}\int_0^1 \big(1 - 2\eta^\mu\big)D_\mu(g)\big(\eta^\mu \nu_1^\mu + \big(1-\eta^\mu\big)\nu_2^\mu\big)\,d_\mu \eta.\end{aligned} \tag{2.13}$$

Lemma 2.2 ([55, Lemma 4]) *Let $\mu \in (0,1]$ and $g : [\nu_1, \nu_2] \subset \mathrm{R} \to \mathbb{R}$ be a μ-fractional differentiable function on (ν_1, ν_2) with $0 \le \nu_1 < \nu_2$. If $D_\mu(g)$ is a μ-fractional integrable function on $[\nu_1, \nu_2]$, then we have*

$$\begin{aligned}&\frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g\big(x^\mu\big)\,d_\mu x - g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right)\\ &\quad = \big(\nu_2^\mu - \nu_1^\mu\big)\int_0^1 p(\eta)D_\mu(g)\big(\eta^\mu \nu_1^\mu + \big(1-\eta^\mu\big)\nu_2^\mu\big)\,d_\mu \eta,\end{aligned} \tag{2.14}$$

where

$$P(\eta) = \begin{cases}\eta^\mu, & 0 \le t \le \frac{1}{2^{1/\mu}},\\ \eta^\mu - 1, & \frac{1}{2^{1/\mu}} \le t \le 1.\end{cases}$$

In view of these indices, we investigate some new inequalities of HH type for the $\digamma$ and $\digamma_\mu$-convex functions involving conformable fractional operators in this attempt. Specifically, we investigate some inequalities of trapezoidal and midpoint type.

3 Hermite–Hadamard inequalities for $\digamma$-convex functions

This section deals with the investigation of HH-type inequalities for $\digamma$-convex functions.

Theorem 3.1 *Let* $g : [\nu_1, \nu_2] \subset \mathbb{R} \to \mathbb{R}$ *be a* μ*-fractional differentiable function on* (ν_1, ν_2) *with* $0 \le \nu_1 < \nu_2$. *If* g *is an* F*-convex function on* $[\nu_1, \nu_2]$ *for some* $F \in \mathcal{F}$, *then*

$$F\left(g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right), \frac{\mu}{\nu_2^\mu - \nu_1^\mu} \int_{\nu_1}^{\nu_2} g(x^\mu)\, d_\mu x, \frac{\mu}{\nu_2^\mu - \nu_1^\mu} \int_{\nu_1}^{\nu_2} g(x^\mu)\, d_\mu x, \frac{1}{2}\right) + \int_0^1 L_{w(\eta)}\, d\eta \le 0, \tag{3.1}$$

$$\mathcal{T}_{F,w}\left(\frac{2\mu}{\nu_2^\mu - \nu_1^\mu} \int_{\nu_1}^{\nu_2} g(x^\mu)\, d_\mu x, g(\nu_1^\mu) + g(\nu_2^\mu), g(\nu_1^\mu) + g(\nu_2^\mu)\right) + \int_0^1 L_{w(\eta)}\, d\eta \le 0. \tag{3.2}$$

Proof The F-convexity of g leads to

$$F\left(g\left(\frac{x+y}{2}\right), g(x), g(y), \frac{1}{2}\right), \quad x, y \in [\nu_1, \nu_2].$$

For the values $x = \eta^\mu \nu_1^\mu + (1-\eta^\mu)\nu_2^\mu$ and $y = (1-\eta^\mu)\nu_1^\mu + \eta^\mu \nu_2^\mu$, where $\eta \in [0,1]$, we obtain

$$F\left(g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right), g\big(\eta^\mu \nu_1^\mu + (1-\eta^\mu)\nu_2^\mu\big), g\big((1-\eta^\mu)\nu_1^\mu + \eta^\mu \nu_2^\mu\big), \frac{1}{2}\right) \le 0.$$

Multiplying this inequality $w(\eta) = 1$ and making use of axiom (Λ_3), we get

$$F\left(g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right), g\big(\eta^\mu \nu_1^\mu + (1-\eta^\mu)\nu_2^\mu\big), g\big((1-\eta^\mu)\nu_1^\mu + \eta^\mu \nu_2^\mu\big), \frac{1}{2}\right) + L_{w(\eta)} \le 0.$$

Integrating over $[0,1]$ according to η and making use of axiom (Λ_1), we get

$$\begin{aligned} F\Bigg(&\int_0^1 g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right) d_\mu \eta, \int_0^1 g\big(\eta^\mu \nu_1^\mu + (1-\eta^\mu)\nu_2^\mu\big)\, d_\mu \eta, \\ &\int_0^1 g\big((1-\eta^\mu)\nu_1^\mu + \eta^\mu \nu_2^\mu\big)\, d_\mu \eta, \frac{1}{2}\Bigg) + \int_0^1 L_{w(\eta)}\, d_\mu \eta \le 0, \end{aligned}$$

that is,

$$F\left(g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right), \frac{\mu}{\nu_2^\mu - \nu_1^\mu} \int_{\nu_1}^{\nu_2} g(x^\mu)\, d_\mu x, \frac{\mu}{\nu_2^\mu - \nu_1^\mu} \int_{\nu_1}^{\nu_2} g(x^\mu)\, d_\mu x, \frac{1}{2}\right) + \int_0^1 L_{w(\eta)}\, d\eta \le 0,$$

where we have used

$$\begin{aligned} \int_0^1 g\big(\eta^\mu \nu_1^\mu + (1-\eta^\mu)\nu_2^\mu\big)\, d_\mu \eta &= \int_0^1 g\big((1-\eta^\mu)\nu_1^\mu + \eta^\mu \nu_2^\mu\big)\, d_\mu \eta \\ &= \frac{\mu}{\nu_2^\mu - \nu_1^\mu} \int_{\nu_1}^{\nu_2} g(x^\mu)\, d_\mu x. \end{aligned}$$

This completely gives the proof of (3.1). On the other hand, since g is F-convex, we have

$$F\big(g\big(\eta^\mu \nu_1^\mu + (1-\eta^\mu)\nu_2^\mu\big), g(\nu_1^\mu), g(\nu_2^\mu), \eta\big) \le 0,$$
$$F\big(g\big((1-\eta^\mu)\nu_1^\mu + \eta^\mu \nu_2^\mu\big), g(\nu_2^\mu), g(\nu_1^\mu), \eta\big) \le 0.$$

We make use of the linearity of $\mathcal{F}$ to get

$$\mathcal{F}\big(g\big(\eta^{\mu}\nu_1^{\mu}+\big(1-\eta^{\mu}\big)\nu_2^{\mu}\big)+g\big(\big(1-\eta^{\mu}\big)\nu_1^{\mu}+\eta^{\mu}\nu_2^{\mu}\big),g\big(\nu_1^{\mu}\big)+g\big(\nu_2^{\mu}\big),g\big(\nu_1^{\mu}\big)+g\big(\nu_2^{\mu}\big),\eta\big)\leq 0.$$

Applying the axiom (Λ_3) for $w(\eta)=1$, we get

$$\begin{aligned}&\mathcal{F}\big(g\big(\eta^{\mu}\nu_1^{\mu}+\big(1-\eta^{\mu}\big)\nu_2^{\mu}\big)+g\big(\big(1-\eta^{\mu}\big)\nu_1^{\mu}+\eta^{\mu}\nu_2^{\mu}\big),g\big(\nu_1^{\mu}\big)+g\big(\nu_2^{\mu}\big),g\big(\nu_1^{\mu}\big)+g\big(\nu_2^{\mu}\big),\eta\big)\\&\quad+L_{w(\eta)}\leq 0.\end{aligned}$$

Integrating over $[0,1]$ according to η and making use of axiom (Λ_2) we get

$$\mathcal{T}_{\mathcal{F},w}\left(\frac{2\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2}g\big(x^{\mu}\big)\,d_{\mu}x,g\big(\nu_1^{\mu}\big)+g\big(\nu_2^{\mu}\big),g\big(\nu_1^{\mu}\big)+g\big(\nu_2^{\mu}\big)\right)+\int_0^1 L_{w(\eta)}\,d\eta\leq 0.$$

This completes the proof of (3.2). Thus, the proof of Theorem 3.1 is completed. □

Corollary 3.1 *Theorem 3.1 with g to be ε-convex leads to*

$$g\left(\frac{\nu_1^{\mu}+\nu_2^{\mu}}{2}\right)-\varepsilon\leq\frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2}g\big(x^{\mu}\big)\,d_{\mu}x\leq\frac{g(\nu_1^{\mu})+g(\nu_2^{\mu})}{2}+\frac{\varepsilon}{2}. \tag{3.3}$$

Proof By making use of $w(\eta)=1$ in (1.5), we get

$$\int_0^1 L_{w(\eta)}\,d\eta=0. \tag{3.4}$$

Making use of (1.3), (3.1), and (3.4), we get

$$\begin{aligned}0&\geq F\left(g\left(\frac{\nu_1^{\mu}+\nu_2^{\mu}}{2}\right),\frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2}g\big(x^{\mu}\big)\,d_{\mu}x,\frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2}g\big(x^{\mu}\big)\,d_{\mu}x,\frac{1}{2}\right)+\int_0^1 L_{w(\eta)}\,d\eta\\&=g\left(\frac{\nu_1^{\mu}+\nu_2^{\mu}}{2}\right)-\frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2}g\big(x^{\mu}\big)\,d_{\mu}x-\varepsilon,\end{aligned}$$

or equivalently,

$$g\left(\frac{\nu_1^{\mu}+\nu_2^{\mu}}{2}\right)-\varepsilon\leq\frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2}g\big(x^{\mu}\big)\,d_{\mu}x.$$

Making use of $w(\eta)=1$ in (1.4), we have

$$\begin{aligned}\mathcal{T}_{\mathcal{F},w}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3)&=\mathbf{y}_1-\left(\int_0^1 t\,d\eta\right)\mathbf{y}_2-\left(\int_0^1(1-\eta)\,d\eta\right)\mathbf{y}_3-\varepsilon\\&=\mathbf{y}_1-\frac{\mathbf{y}_2+\mathbf{y}_3}{2}-\varepsilon,\quad \mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\in\mathbb{R}.\end{aligned} \tag{3.5}$$

Now, from (3.2) and (3.5), we can deduce

$$\begin{aligned}0&\geq\mathcal{T}_{\mathcal{F},w}\left(\frac{2\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2}g\big(x^{\mu}\big)\,d_{\mu}x,g\big(\nu_1^{\mu}\big)+g\big(\nu_2^{\mu}\big),g\big(\nu_1^{\mu}\big)+g\big(\nu_2^{\mu}\big)\right)+\int_0^1 L_{w(\eta)}\,d\eta\\&=\frac{2\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2}g\big(x^{\mu}\big)\,d_{\mu}x-\big(g\big(\nu_1^{\mu}\big)+g\big(\nu_2^{\mu}\big)\big)-\varepsilon.\end{aligned}$$

This gives

$$\frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x \le \frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2} + \frac{\varepsilon}{2}.$$

This ends the proof of (3.3). □

Remark 3.1 Inequality (3.3) with $\varepsilon = 0$ becomes inequality (2.12).

Corollary 3.2 *Theorem 3.1 with g to be h-convex leads to*

$$\frac{1}{2h(\frac{1}{2})}g\left(\frac{\nu_1^\mu+\nu_2^\mu}{2}\right) \le \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x \le \frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2}\int_0^1 h(\eta)\,d\eta. \tag{3.6}$$

Proof Making use (1.5) and (3.1) with $L_{w(\eta)} = 0$, we have

$$\begin{aligned} 0 &\ge F\left(g\left(\frac{\nu_1^\mu+\nu_2^\mu}{2}\right), \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x, \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x, \frac{1}{2}\right) + \int_0^1 L_{w(\eta)}\,d\eta \\ &= g\left(\frac{\nu_1^\mu+\nu_2^\mu}{2}\right) - h\left(\frac{1}{2}\right)\frac{2\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x, \end{aligned}$$

or equivalently,

$$\frac{1}{2h(\frac{1}{2})}g\left(\frac{\nu_1^\mu+\nu_2^\mu}{2}\right) \le \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x.$$

Now, by making use of $w(\eta) = 1$ in (1.4) and (3.2), we get

$$\begin{aligned} 0 &\ge \mathcal{T}_{F,w}\left(\frac{2\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x, g(\nu_1^\mu)+g(\nu_2^\mu), g(\nu_1^\mu)+g(\nu_2^\mu)\right) + \int_0^1 L_{w(\eta)}\,d\eta \\ &= \frac{2\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x - \left(g(\nu_1^\mu)+g(\nu_2^\mu)\right)\left(\int_0^1 h(\eta)\,d\eta\right). \end{aligned}$$

This gives

$$\frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x \le \left(\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2}\right)\left(\int_0^1 h(\eta)\,d\eta\right).$$

Thus, the proof of (3.6) is completed. □

4 Hermite–Hadamard inequalities for $\mathcal{F}_\mu$-convex functions

Here, we deal with the investigation of HH-type inequalities for $\mathcal{F}_\mu$-convex functions. This section is separated into two subsections: a section for the trapezoidal formula inequality and the other one for the midpoint formula inequality of HH type, respectively.

4.1 Trapezoidal inequalities for $\mathcal{F}_\mu$-convex functions

Theorem 4.1 *Let* $g : [\nu_1, \nu_2] \subset \mathrm{R} \to \mathrm{R}$ *be a μ-fractional differentiable function on* (ν_1, ν_2) *and* $D_\mu(g)$ *be a μ-fractional integrable function on* $[\nu_1, \nu_2]$ *with* $0 \le \nu_1 < \nu_2$. *If* $|D_\mu(g)|$ *is* *an $\digamma_\mu$-convex function on $[\nu_1,\nu_2]$ for some $\digamma_\mu \in \mathcal{F}$ and the function $\eta \in [0,1] \to L_{w(\eta^\mu)}$ belongs to $L_1[0,1]$, where $w(\eta^\mu) = |1-2\eta^\mu|$, then we have the inequality*

$$\mathcal{T}_{\digamma_\mu,w}\left(\frac{2}{\nu_2^\mu-\nu_1^\mu}\left|\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2}-\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x\right|, \left|D_\mu(g)(\nu_1^\mu)\right|, \left|D_\mu(g)(\nu_2^\mu)\right|, \eta\right) + \int_0^1 L_{w(\eta)}\,d_\mu\eta \le 0. \tag{4.1}$$

Proof The $\digamma_\mu$-convexity of $|D_\mu(g)|$ leads to

$$\digamma_\mu\left(\left|D_\mu(g)\left(\eta^\mu\nu_1^\mu+(1-\eta^\mu)\nu_2^\mu\right)\right|, \left|D_\mu(g)(\nu_1^\mu)\right|, \left|D_\mu(g)(\nu_2^\mu)\right|, \eta\right) \le 0.$$

By applying axiom $(\bar{\Lambda}_3)$ for $w(\eta^\mu) = |1-2\eta^\mu|$, $\eta \in [0,1]$, we can deduce

$$\digamma_\mu\left(w(\eta^\mu)\left|D_\mu(g)\left(\eta^\mu\nu_1^\mu+(1-\eta^\mu)\nu_2^\mu\right)\right|, w(\eta^\mu)\left|D_\mu(g)(\nu_1^\mu)\right|, w(\eta^\mu)\left|D_\mu(g)(\nu_2^\mu)\right|, \eta\right) \le 0.$$

Integrating over $[0,1]$ according to η and by making use of axiom $(\bar{\Lambda}_2)$, we obtain

$$\mathcal{T}_{\digamma_\mu,w}\left(\int_0^1 w(\eta^\mu)\left|D_\mu(g)\left(\eta^\mu\nu_1^\mu+(1-\eta^\mu)\nu_2^\mu\right)\right|\,d_\mu\eta, \left|D_\mu(g)(\nu_1^\mu)\right|, \left|D_\mu(g)(\nu_2^\mu)\right|, \eta\right) + \int_0^1 L_{w(\eta^\mu)}\,d_\mu t \le 0.$$

From Lemma 2.1, we have

$$\left|\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2}-\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x\right| \le \frac{\nu_2^\mu-\nu_1^\mu}{2}\int_0^1 w(\eta^\mu)\left|D_\mu(g)\left(\eta^\mu\nu_1^\mu+(1-\eta^\mu)\nu_2^\mu\right)\right|\,d_\mu\eta.$$

Since $\mathcal{T}_{\digamma_\mu,w}$ is nondecreasing according to the first variable, then we can deduce

$$\mathcal{T}_{\digamma_\mu,w}\left(\frac{2}{\nu_2^\mu-\nu_1^\mu}\left|\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2}-\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x\right|, \left|D_\mu(g)(\nu_1^\mu)\right|, \left|D_\mu(g)(\nu_2^\mu)\right|, \eta\right) + \int_0^1 L_{w(\eta^\mu)}\,d_\mu\eta \le 0,$$

which ends the proof of (4.1). □

Corollary 4.1 *Theorem 4.1 with $|D_\mu(g)|$ to be ε conformable convex leads to*

$$\left|\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2}-\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x\right| \le \frac{\nu_2^\mu-\nu_1^\mu}{2}\left(\frac{2^{3\mu^2}+6\times 2^{\mu^2}-8}{6\mu\times 2^{3\mu^2}}\right)\left(\left|D_\mu(g)(\nu_1^\mu)\right|+\left|D_\mu(g)(\nu_2^\mu)\right|+\frac{2\mu-1}{2\mu}\varepsilon\right). \tag{4.2}$$

Proof We know that any ε-convex is $\mathcal{F}_{\mu}$-convex. So, by making use of $w(\eta^{\mu}) = |1-2\eta^{\mu}|$ in (2.7) and by using Definition 2.2, we get

$$\int_0^1 L_{w(\eta)}\,d_{\mu}\eta = \varepsilon\int_0^1 \bigl(1-w(\eta)\bigr)\,d_{\mu}\eta = \frac{1}{2\mu}\varepsilon.$$

By making use of $w(\eta^{\mu}) = |1-2\eta^{\mu}|$ in (2.6), we get

$$\begin{aligned}\mathcal{T}_{\mathcal{F}_{\mu},w}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3) &= \mathbf{y}_1 - \biggl(\int_0^1 \eta^{\mu}\bigl|1-2\eta^{\mu}\bigr|\,d_{\mu}\eta\biggr)\mathbf{y}_2 - \biggl(\int_0^1 \bigl(1-\eta^{\mu}\bigr)\bigl|1-2\eta^{\mu}\bigr|\,d_{\mu}\eta\biggr)\mathbf{y}_3 - \varepsilon\\ &= \mathbf{y}_1 - \biggl(\frac{2^{3\mu^2}+6\times 2^{\mu^2}-8}{6\mu\times 2^{3\mu^2}}\biggr)(\mathbf{y}_2+\mathbf{y}_3) - \varepsilon\end{aligned}$$

for $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\in\mathbb{R}$. By making use of Theorem 4.1, we get

$$\begin{aligned}0 &\geq \mathcal{T}_{\mathcal{F}_{\mu},w}\biggl(\frac{2}{\nu_2^{\mu}-\nu_1^{\mu}}\biggl|\frac{g(\nu_1^{\mu})+g(\nu_2^{\mu})}{2} - \frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2} g\bigl(x^{\mu}\bigr)\,d_{\mu}x\biggr|,\\ &\quad \bigl|D_{\mu}(g)\bigl(\nu_1^{\mu}\bigr)\bigr|,\bigl|D_{\mu}(g)\bigl(\nu_2^{\mu}\bigr)\bigr|,\eta\biggr) + \int_0^1 L_{w(\eta^{\mu})}\,d_{\mu}\eta\\ &= \frac{2}{\nu_2^{\mu}-\nu_1^{\mu}}\biggl|\frac{g(\nu_1^{\mu})+g(\nu_2^{\mu})}{2} - \frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2} g\bigl(x^{\mu}\bigr)\,d_{\mu}x\biggr|\\ &\quad - \biggl(\frac{2^{3\mu^2}+6\times 2^{\mu^2}-8}{6\mu\times 2^{3\mu^2}}\biggr)\bigl(\bigl|D_{\mu}(g)\bigl(\nu_1^{\mu}\bigr)\bigr| + \bigl|D_{\mu}(g)\bigl(\nu_2^{\mu}\bigr)\bigr|\bigr) - \varepsilon + \frac{1}{2\mu}\varepsilon.\end{aligned}$$

This rearranges to the required inequality (4.2). □

Remark 4.1 Corollary 4.1 with $\varepsilon = 0$ becomes Theorem 13 in [55].

Corollary 4.2 *Theorem 4.1 with $|D_{\mu}(g)|$ to be μ-conformable convex leads to*

$$\begin{aligned}&\biggl|\frac{g(\nu_1^{\mu})+g(\nu_2^{\mu})}{2} - \frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2} g\bigl(x^{\mu}\bigr)\,d_{\mu}x\biggr|\\ &\quad\leq \frac{\nu_2^{\mu}-\nu_1^{\mu}}{2(\mu+1)(2\mu+1)}\biggl(1+\frac{\mu}{2^{1/\mu}}\biggr)\bigl(\bigl|D_{\mu}(g)\bigl(\nu_1^{\mu}\bigr)\bigr| + \bigl|D_{\mu}(g)\bigl(\nu_2^{\mu}\bigr)\bigr|\bigr).\end{aligned}\tag{4.3}$$

Proof We know that any μ-convex is $\mathcal{F}_{\mu}$-convex. So, by making use of $w(\eta^{\mu}) = |1-2\eta^{\mu}|$ in (2.9), we get

$$\begin{aligned}\mathcal{T}_{\mathcal{F}_{\mu},w}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3) &= \mathbf{y}_1 - \biggl(\int_0^1 t\bigl|1-2\eta^{\mu}\bigr|\,d_{\mu}\eta\biggr)\mathbf{y}_2 - \biggl(\int_0^1 (1-\eta)\bigl|1-2\eta^{\mu}\bigr|\,d_{\mu}\eta\biggr)\mathbf{y}_3\\ &= \mathbf{y}_1 - \frac{1}{(\mu+1)(2\mu+1)}\biggl(1+\frac{\mu}{2^{1/\mu}}\biggr)(\mathbf{y}_2+\mathbf{y}_3)\end{aligned}$$

for $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\in\mathbb{R}$. Then, by applying Theorem 4.1, we have

$$\begin{aligned}0 &\geq \mathcal{T}_{\mathcal{F}_{\mu},w}\biggl(\frac{2}{\nu_2^{\mu}-\nu_1^{\mu}}\biggl|\frac{g(\nu_1^{\mu})+g(\nu_2^{\mu})}{2} - \frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2} g\bigl(x^{\mu}\bigr)\,d_{\mu}x\biggr|,\\ &\quad \bigl|D_{\mu}(g)\bigl(\nu_1^{\mu}\bigr)\bigr|,\bigl|D_{\mu}(g)\bigl(\nu_2^{\mu}\bigr)\bigr|,\eta\biggr) + \int_0^1 L_{w(\eta^{\mu})}\,d_{\mu}\eta\end{aligned}$$

$$= \frac{2}{\nu_2^\mu - \nu_1^\mu}\left|\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2} - \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x\right|$$
$$- \frac{1}{(\mu+1)(2\mu+1)}\left(1+\frac{\mu}{2^{1/\mu}}\right)\left(\left|D_\mu(g)(\nu_1^\mu)\right| + \left|D_\mu(g)(\nu_2^\mu)\right|\right).$$

This rearranges to the required inequality (4.3). □

Corollary 4.3 *Theorem 4.1 with $|D_\mu(g)|$ to be h-conformable convex leads to*

$$\left|\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2} - \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x\right|$$
$$\leq \frac{\nu_2^\mu - \nu_1^\mu}{2}\left(\int_0^1 h(\eta^\mu)\left|1-2\eta^\mu\right|d_\mu\eta\right)\left(\left|D_\mu(g)(\nu_1^\mu)\right| + \left|D_\mu(g)(\nu_2^\mu)\right|\right). \tag{4.4}$$

Proof It is known that every μ-convex is F_μ-convex. So, by making use of $w(\eta^\mu) = |1-2\eta^\mu|$ in (2.11), we get

$$\mathcal{T}_{F_\mu,w}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3) = \mathbf{y}_1 - \left(\int_0^1 h(\eta^\mu)\left|1-2\eta^\mu\right|d_\mu\eta\right)\mathbf{y}_2$$
$$- \left(\int_0^1 h(1-\eta^\mu)\left|1-2\eta^\mu\right|d_\mu\eta\right)\mathbf{y}_3$$
$$= \mathbf{y}_1 - \left(\int_0^1 h(\eta^\mu)\left|1-2\eta^\mu\right|d_\mu\eta\right)(\mathbf{y}_2+\mathbf{y}_3)$$

for $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in \mathrm{R}$. Then, by using Theorem 4.1, we get

$$0 \geq \mathcal{T}_{F_\mu,w}\left(\frac{2}{\nu_2^\mu - \nu_1^\mu}\left|\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2} - \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x\right|,\right.$$
$$\left.\left|D_\mu(g)(\nu_1^\mu)\right|, \left|D_\mu(g)(\nu_2^\mu)\right|, \eta\right) + \int_0^1 L_{w(\eta^\mu)}\,d_\mu\eta$$
$$= \frac{2}{\nu_2^\mu - \nu_1^\mu}\left|\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2} - \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x\right|$$
$$- \left(\int_0^1 h(\eta^\mu)\left|1-2\eta^\mu\right|d_\mu\eta\right)\left(\left|D_\mu(g)(\nu_1^\mu)\right| + \left|D_\mu(g)(\nu_2^\mu)\right|\right).$$

This completes the proof of (4.4). □

Theorem 4.2 *Let $g : [\nu_1,\nu_2] \subset \mathrm{R} \to \mathrm{R}$ be a μ-fractional differentiable function on (ν_1,ν_2) and $D_\mu(g)$ be a μ-fractional integrable function on $[\nu_1,\nu_2]$ with $0 \leq \nu_1 < \nu_2$. If $|D_\mu(g)|^{\frac{p}{p-1}}$ is an F_μ-convex function on $[\nu_1,\nu_2]$ for some $F_\mu \in \mathcal{F}$, then we have*

$$\mathcal{T}_{F_\mu,1}\left(\nu_1(g,p), \left|D_\mu(g)(\nu_1^\mu)\right|^{\frac{p}{p-1}}, \left|D_\mu(g)(\nu_2^\mu)\right|^{\frac{p}{p-1}}\right) \leq 0, \tag{4.5}$$

where

$$\nu_1(g,p) = \left(\frac{2}{\nu_2^\mu - \nu_1^\mu}\right)^{\frac{p}{p-1}} \left(\frac{1}{2\mu(p+1)}\left\{2 - \left(1 - \frac{1}{2^{\mu^2-1}}\right)^{p+1} - \left(\frac{1}{2^{\mu^2-1}} - 1\right)^{p+1}\right\}\right)^{\frac{-1}{p-1}}$$
$$\times \left|\frac{g(\nu_1^\mu) + g(\nu_2^\mu)}{2} - \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\, d_\mu x\right|^{\frac{p}{p-1}}.$$

Proof By using the $\digamma_\mu$-convexity of $|D_\mu(g)|^{\frac{p}{p-1}}$, we have

$$\digamma_\mu\left(\left|D_\mu(g)\left(\eta^\mu\nu_1^\mu + (1-\eta^\mu)\nu_2^\mu\right)\right|^{\frac{p}{p-1}}, \left|D_\mu(g)(\nu_1^\mu)\right|^{\frac{p}{p-1}}, \left|D_\mu(g)(\nu_2^\mu)\right|^{\frac{p}{p-1}}, \eta\right) \le 0.$$

By making use of $w(\eta^\mu) = 1$ in axiom $(\bar{\varLambda}_3)$, we obtain

$$\mathcal{T}_{\digamma_\mu,1}\left(\int_0^1 \left|D_\mu(g)\left(\eta^\mu\nu_1^\mu + (1-\eta^\mu)\nu_2^\mu\right)\right|^{\frac{p}{p-1}} d_\mu\eta, \left|D_\mu(g)(\nu_1^\mu)\right|^{\frac{p}{p-1}}, \left|D_\mu(g)(\nu_2^\mu)\right|^{\frac{p}{p-1}}\right) \le 0.$$

Then, by making use Lemma of 2.2, we have

$$\left|\frac{g(\nu_1^\mu) + g(\nu_2^\mu)}{2} - \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\, d_\mu x\right|$$
$$\le \frac{\nu_2^\mu - \nu_1^\mu}{2}\left(\frac{1}{2\mu(p+1)}\left\{2 - \left(1 - \frac{1}{2^{\mu^2-1}}\right)^{p+1} - \left(\frac{1}{2^{\mu^2-1}} - 1\right)^{p+1}\right\}\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^1 \left|D_\mu(g)\left(\eta^\mu\nu_1^\mu + (1-\eta^\mu)\nu_2^\mu\right)\right|^{\frac{p}{p-1}} d_\mu\eta\right)^{\frac{p-1}{p}}.$$

Since $\mathcal{T}_{\digamma_\mu,w}$ is nondecreasing according to the first variable, then we can deduce

$$\mathcal{T}_{\digamma_\mu,1}\left(\nu_1(g,p), \left|D_\mu(g)(\nu_1^\mu)\right|^{\frac{p}{p-1}}, \left|D_\mu(g)(\nu_2^\mu)\right|^{\frac{p}{p-1}}\right) \le 0.$$

This completes the proof of (4.5). □

Corollary 4.4 *Theorem 4.2 with* $|D_\mu(g)|^{\frac{p}{p-1}}$ *to be* ε*-conformable convex leads to*

$$\left|\frac{g(\nu_1^\mu) + g(\nu_2^\mu)}{2} - \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\, d_\mu x\right|$$
$$\le \frac{\nu_2^\mu - \nu_1^\mu}{2}\left(\frac{1}{2\mu(p+1)}\left\{2 - \left(1 - \frac{1}{2^{\mu^2-1}}\right)^{p+1} - \left(\frac{1}{2^{\mu^2-1}} - 1\right)^{p+1}\right\}\right)^{\frac{1}{p}}$$
$$\times \left(\frac{|D_\mu(g)(\nu_1^\mu)|^{\frac{p}{p-1}} + |D_\mu(g)(\nu_2^\mu)|^{\frac{p}{p-1}}}{2\mu} + \varepsilon\right)^{\frac{p-1}{p}}. \tag{4.6}$$

Proof By making use of $w(\eta^\mu) = |1 - 2\eta^\mu|$ in (2.6) and by Definition 2.3, we get

$$\mathcal{T}_{\digamma_\mu,w}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \mathbf{y}_1 - \left(\int_0^1 \eta^\mu\left|1 - 2\eta^\mu\right| d_\mu\eta\right)\mathbf{y}_2 - \left(\int_0^1 (1-\eta^\mu)\left|1 - 2\eta^\mu\right| d_\mu\eta\right)\mathbf{y}_3 - \varepsilon$$
$$= \mathbf{y}_1 - \frac{\mathbf{y}_2 + \mathbf{y}_3}{2\mu} - \varepsilon$$

for $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in \mathbb{R}$. Then, by using Theorem 4.2, we have

$$\begin{aligned}
0 &\geq \mathcal{T}_{F_{\mu},1}\left(\nu_1(g,p), \left|D_\mu(g)\left(\nu_1^\mu\right)\right|^{\frac{p}{p-1}}, \left|D_\mu(g)\left(\nu_2^\mu\right)\right|^{\frac{p}{p-1}}\right) - \epsilon \\
&= \nu_1(g,p) - \frac{|D_\mu(g)(\nu_1^\mu)|^{\frac{p}{p-1}} + |D_\mu(g)(\nu_2^\mu)|^{\frac{p}{p-1}}}{2\mu} - \varepsilon \\
&= \left(\frac{2}{\nu_2^\mu - \nu_1^\mu}\right)^{\frac{p}{p-1}} \left(\frac{1}{2\mu(p+1)}\left\{2 - \left(1 - \frac{1}{2^{\mu^2-1}}\right)^{p+1} - \left(\frac{1}{2^{\mu^2-1}} - 1\right)^{p+1}\right\}\right)^{\frac{-1}{p-1}} \\
&\quad \times \left|\frac{g(\nu_1^\mu) + g(\nu_2^\mu)}{2} - \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g\left(x^\mu\right) d_\mu x\right|^{\frac{p}{p-1}} \\
&\quad - \frac{|D_\mu(g)(\nu_1^\mu)|^{\frac{p}{p-1}} + |D_\mu(g)(\nu_2^\mu)|^{\frac{p}{p-1}}}{2\mu} - \varepsilon.
\end{aligned}$$

This completes our proof. □

Remark 4.2 Corollary 4.4 with $\varepsilon = 0$ becomes Theorem 13 in [55].

Corollary 4.5 *Theorem* 4.2 *with* $|D_\mu(g)|$ *to be* μ*-convex leads to*

$$\begin{aligned}
&\left|\frac{g(\nu_1^\mu) + g(\nu_2^\mu)}{2} - \frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g\left(x^\mu\right) d_\mu x\right| \\
&\quad \leq \frac{\nu_2^\mu - \nu_1^\mu}{2}\left(\frac{1}{2\mu(p+1)}\left\{2 - \left(1 - \frac{1}{2^{\mu^2-1}}\right)^{p+1} - \left(\frac{1}{2^{\mu^2-1}} - 1\right)^{p+1}\right\}\right)^{\frac{1}{p}} \\
&\quad\quad \times \left(\frac{\mu|D_\mu(g)(\nu_1^\mu)|^{\frac{p}{p-1}} + |D_\mu(g)(\nu_2^\mu)|^{\frac{p}{p-1}}}{\mu(\mu+1)}\right)^{\frac{p-1}{p}}.
\end{aligned} \tag{4.7}$$

Proof By making use of $w(\eta^\mu) = 1$ in (2.9), we get

$$\begin{aligned}
\mathcal{T}_{F_{\mu},1}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) &= \mathbf{y}_1 - \left(\int_0^1 \eta \, d_\mu \eta\right)\mathbf{y}_2 - \left(\int_0^1 (1-\eta) \, d_\mu \eta\right)\mathbf{y}_3 \\
&= \mathbf{y}_1 - \frac{\mu \mathbf{y}_2 + \mathbf{y}_3}{\mu(\mu+1)}
\end{aligned}$$

for $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in \mathbb{R}$. Then, by using Theorem 4.2, we have

$$\begin{aligned}
0 &\geq \mathcal{T}_{F_{\mu},1}\left(\nu_1(g,p), \left|D_\mu(g)\left(\nu_1^\mu\right)\right|^{\frac{p}{p-1}}, \left|D_\mu(g)\left(\nu_2^\mu\right)\right|^{\frac{p}{p-1}}\right) \\
&= \nu_1(g,p) - \frac{\mu|D_\mu(g)(\nu_1^\mu)|^{\frac{p}{p-1}} + |D_\mu(g)(\nu_2^\mu)|^{\frac{p}{p-1}}}{\mu(\mu+1)}.
\end{aligned}$$

This rearranges to the required inequality (4.7). □

Corollary 4.6 *Theorem 4.2 with $|D_\mu(g)|$ to be h-convex leads to*

$$\begin{aligned}
&\left|\frac{g(\nu_1^\mu)+g(\nu_2^\mu)}{2}-\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x\right| \\
&\quad\le \frac{\nu_2^\mu-\nu_1^\mu}{2}\left(\frac{1}{2\mu(p+1)}\left\{2-\left(1-\frac{1}{2^{\mu^2-1}}\right)^{p+1}-\left(\frac{1}{2^{\mu^2-1}}-1\right)^{p+1}\right\}\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_0^1 h(\eta^\mu)\,d_\mu\eta\right)^{\frac{p-1}{p}}\left(\left|D_\mu(g)(\nu_1^\mu)\right|^{\frac{p}{p-1}}+\left|D_\mu(g)(\nu_2^\mu)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}.
\end{aligned} \tag{4.8}$$

Proof By making use of (2.11) with $w(\eta^\mu)=1$, we have

$$\begin{aligned}
\mathcal{T}_{\mathcal{F}_\mu,1}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3) &= \mathbf{y}_1-\left(\int_0^1 h(\eta^\mu)\,d_\mu\eta\right)\mathbf{y}_2-\left(\int_0^1 h(1-\eta^\mu)\,d_\mu\eta\right)\mathbf{y}_3 \\
&= \mathbf{y}_1-\left(\int_0^1 h(\eta^\mu)\,d_\mu\eta\right)(\mathbf{y}_2+\mathbf{y}_3)
\end{aligned}$$

for $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\in\mathbb{R}$. Then, by making use of Theorem 4.2, we get

$$\begin{aligned}
&0\ge \mathcal{T}_{\mathcal{F}_\mu,1}\left(\nu_1(g,p),\left|D_\mu(g)(\nu_1^\mu)\right|^{\frac{p}{p-1}},\left|D_\mu(g)(\nu_2^\mu)\right|^{\frac{p}{p-1}}\right) \\
&\quad= \nu_1(g,p)-\left(\int_0^1 h(\eta^\mu)\,d_\mu\eta\right)\left(\left|D_\mu(g)(\nu_1^\mu)\right|^{\frac{p}{p-1}}+\left|D_\mu(g)(\nu_2^\mu)\right|^{\frac{p}{p-1}}\right).
\end{aligned}$$

This rearranges to the required inequality (4.8). □

4.2 Midpoint formula inequalities for $\mathcal{F}_\mu$-convex functions

Theorem 4.3 *Let $g:[\nu_1,\nu_2]\subset\mathbb{R}\to\mathbb{R}$ be a μ-fractional differentiable function on (ν_1,ν_2) and $D_\mu(g)$ be a μ-fractional integrable function on $[\nu_1,\nu_2]$ with $0\le\nu_1<\nu_2$. If $|D_\mu(g)|$ is an $\mathcal{F}_\mu$-convex function on $[\nu_1,\nu_2]$ for some $\mathcal{F}_\mu\in\mathcal{F}$ and the function $\eta\in[0,1]\to L_{w(\eta)}$ belongs to $L_1[0,1]$, where $w(\eta)=|P(\eta)|$ ($P(\eta)$ is given in Lemma 2.2, then we have the inequality*

$$\begin{aligned}
&\mathcal{T}_{\mathcal{F}_\mu,w}\left(\frac{1}{\nu_2^\mu-\nu_1^\mu}\left|\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x-g\left(\frac{\nu_1^\mu+\nu_2^\mu}{2}\right)\right|,\left|D_\mu(g)(\nu_1^\mu)\right|,\left|D_\mu(g)(\nu_2^\mu)\right|,\eta\right) \\
&\quad+\int_0^1 L_{w(\eta)}\,d_\mu\eta\le 0.
\end{aligned} \tag{4.9}$$

Proof By using the $\mathcal{F}_\mu$-convexity of $|D_\mu(g)|$, we have

$$\mathcal{F}_\mu\left(\left|D_\mu(g)\left(\eta^\mu\nu_1^\mu+(1-\eta^\mu)\nu_2^\mu\right)\right|,\left|D_\mu(g)(\nu_1^\mu)\right|,\left|D_\mu(g)(\nu_2^\mu)\right|,\eta\right)\le 0.$$

Making use of axiom $(\bar{A}_3)$ for $w(\eta)=|P(\eta)|$, $\eta\in[0,1]$, we obtain

$$\mathcal{F}_\mu\left(w(\eta)\left|D_\mu(g)\left(\eta^\mu\nu_1^\mu+(1-\eta^\mu)\nu_2^\mu\right)\right|,w(\eta)\left|D_\mu(g)(\nu_1^\mu)\right|,w(\eta)\left|D_\mu(g)(\nu_2^\mu)\right|,\eta\right)\le 0.$$

Integrating over $[0,1]$ according to η and by making use of axiom $(\bar{\Lambda}_2)$, we can obtain

$$\mathcal{T}_{F_{\mu},w}\Big(\int_0^1 w(\eta)\big|D_\mu(g)\big(\eta^\mu\nu_1^\mu+\big(1-\eta^\mu\big)\nu_2^\mu\big)\big|\,d_\mu\eta, \big|D_\mu(g)\big(\nu_1^\mu\big)\big|, \big|D_\mu(g)\big(\nu_2^\mu\big)\big|,\eta\Big) + \int_0^1 L_{w(\eta)}\,d_\mu\eta \le 0.$$

From Lemma 2.2, we have

$$\left|\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_{\nu_1}^{\nu_2} g\big(x^\mu\big)\,d_\mu x - g\left(\frac{\nu_1^\mu+\nu_2^\mu}{2}\right)\right| \le \big(\nu_2^\mu-\nu_1^\mu\big)\int_0^1 w(\eta)\big|D_\mu(g)\big(\eta^\mu\nu_1^\mu+\big(1-\eta^\mu\big)\nu_2^\mu\big)\big|\,d_\mu\eta.$$

Since $\mathcal{T}_{F_{\mu},w}$ is nondecreasing according to the first variable, then we can deduce

$$\mathcal{T}_{F,w}\left(\frac{1}{\nu_2^\mu-\nu_1^\mu}\left|\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_{\nu_1}^{\nu_2} g\big(x^\mu\big)\,d_\mu x - g\left(\frac{\nu_1^\mu+\nu_2^\mu}{2}\right)\right|, \big|D_\mu(g)\big(\nu_1^\mu\big)\big|, \big|D_\mu(g)\big(\nu_2^\mu\big)\big|,\eta\right) + \int_0^1 L_{w(\eta)}\,d_\mu\eta \le 0.$$

This completes the proof of (4.9). □

Corollary 4.7 *Theorem 4.3 with $|D_\mu(g)|$ to be ε-convex leads to*

$$\left|\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_{\nu_1}^{\nu_2} g\big(x^\mu\big)\,d_\mu x - g\left(\frac{\nu_1^\mu+\nu_2^\mu}{2}\right)\right| \le \big(\nu_2^\mu-\nu_1^\mu\big)\left(\frac{|D_\mu(g)(\nu_1^\mu)|+|D_\mu(g)(\nu_2^\mu)|}{8\mu}+\frac{4\mu-3}{\mu}\varepsilon\right). \tag{4.10}$$

Proof By making use of $w(\eta^\mu)=|P(\eta)|$ in (2.7) as well as Definition 2.3, we get

$$\int_0^1 L_{w(\eta)}\,d_\mu\eta = \varepsilon\int_0^1\big(1-\big|P(\eta)\big|\big)\,d_\mu\eta = \frac{3}{4\mu}\varepsilon.$$

Then, by making use (2.6) with $w(\eta^\mu)=|P(\eta)|$, we get

$$\begin{aligned}\mathcal{T}_{F_{\mu},w}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3) &= \mathbf{y}_1-\left(\int_0^1\eta^\mu\big|P(\eta)\big|\,d_\mu\eta\right)\mathbf{y}_2-\left(\int_0^1\big(1-\eta^\mu\big)\big|P(\eta)\big|\,d_\mu\eta\right)\mathbf{y}_3-\varepsilon\\ &= \mathbf{y}_1-\frac{\mathbf{y}_2+\mathbf{y}_3}{8\mu}-\varepsilon,\end{aligned}$$

for $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3\in\mathbb{R}$. Thus, by using Theorem 4.3, we have

$$0\ge\mathcal{T}_{F,w}\left(\frac{1}{\nu_2^\mu-\nu_1^\mu}\left|\frac{\mu}{\nu_2^\mu-\nu_1^\mu}\int_a^{\nu_2} g\big(x^\mu\big)\,d_\mu x - g\left(\frac{\nu_1^\mu+\nu_2^\mu}{2}\right)\right|, \big|D_\mu(g)\big(\nu_1^\mu\big)\big|, \big|D_\mu(g)\big(\nu_2^\mu\big)\big|,\eta\right) + \int_0^1 L_{w(\eta)}\,d_\mu\eta$$

$$= \frac{1}{\nu_2^\mu - \nu_1^\mu}\left|\frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_a^{\nu_2} g(x^\mu)\,d_\mu x - g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right)\right| - \frac{|D_\mu(g)(\nu_1^\mu)| + |D_\mu(g)(\nu_2^\mu)|}{8\mu} - \varepsilon + \frac{3}{4\mu}\varepsilon.$$

This completes the proof of (4.10). □

Remark 4.3 Corollary 4.7 with $\varepsilon = 0$ becomes Theorem 14 in [55].

Corollary 4.8 *Theorem 4.3 with $|D_\mu(g)|$ to be μ-convex leads to*

$$\begin{aligned}&\left|\frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x - g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right)\right| \\ &\quad\le (\nu_2^\mu - \nu_1^\mu)\left\{\frac{\mu}{(\mu+1)(2\mu+1)}\left(1 - \frac{1}{2^{1/\mu}+1}\right)\left|D_\mu(g)(\nu_1^\mu)\right| \right.\\ &\qquad\left.+ \left[\frac{1}{4\mu} - \frac{\mu}{(\mu+1)(2\mu+1)}\left(1 - \frac{1}{2^{1/\mu}+1}\right)\right]\left|D_\mu(g)(\nu_2^\mu)\right|\right\}.\end{aligned} \tag{4.11}$$

Proof By making use of $w(\eta^\mu) = |P(\eta)|$ in (2.9), we get

$$\begin{aligned}\mathcal{T}_{F_{\mu,w}}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3) &= \mathbf{y}_1 - \left(\int_0^1 t\left|P(\eta)\right|d_\mu\eta\right)\mathbf{y}_2 - \left(\int_0^1 (1-\eta)\left|P(\eta)\right|d_\mu\eta\right)\mathbf{y}_3 \\ &= \mathbf{y}_1 - \frac{\mu}{(\mu+1)(2\mu+1)}\left(1 - \frac{1}{2^{1/\mu}+1}\right)\mathbf{y}_2 \\ &\quad - \left[\frac{1}{4\mu} - \frac{\mu}{(\mu+1)(2\mu+1)}\left(1 - \frac{1}{2^{1/\mu}+1}\right)\right]\mathbf{y}_3\end{aligned}$$

for $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3 \in \mathbb{R}$. It follows from Theorem 4.3 that

$$\begin{aligned}0 &\ge \mathcal{T}_{F_{\mu,w}}\left(\frac{1}{\nu_2^\mu - \nu_1^\mu}\left|\frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x - g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right)\right|,\right.\\ &\qquad\left.\left|D_\mu(g)(\nu_1^\mu)\right|,\left|D_\mu(g)(\nu_2^\mu)\right|,\eta\right) + \int_0^1 L_{w(\eta^\mu)}\,d_\mu\eta \\ &= \frac{1}{\nu_2^\mu - \nu_1^\mu}\left|\frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x - g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right)\right| \\ &\quad - \frac{\mu}{(\mu+1)(2\mu+1)}\left(1 - \frac{1}{2^{1/\mu}+1}\right)\left|D_\mu(g)(\nu_1^\mu)\right| \\ &\quad - \left[\frac{1}{4\mu} - \frac{\mu}{(\mu+1)(2\mu+1)}\left(1 - \frac{1}{2^{1/\mu}+1}\right)\right]\left|D_\mu(g)(\nu_2^\mu)\right|.\end{aligned}$$

This rearranges to the required inequality (4.11). □

Corollary 4.9 *Theorem 4.3 with $|D_\mu(g)|$ to be h-convex leads to*

$$\begin{aligned}&\left|\frac{\mu}{\nu_2^\mu - \nu_1^\mu}\int_{\nu_1}^{\nu_2} g(x^\mu)\,d_\mu x - g\left(\frac{\nu_1^\mu + \nu_2^\mu}{2}\right)\right| \\ &\quad\le (\nu_2^\mu - \nu_1^\mu)\left(\int_0^1 h(\eta^\mu)\left|1 - 2\eta^\mu\right|d_\mu\eta\right)\left(\left|D_\mu(g)(\nu_1^\mu)\right| + \left|D_\mu(g)(\nu_2^\mu)\right|\right).\end{aligned} \tag{4.12}$$

Proof By making use of $w(\eta^{\mu}) = |P(\eta)|$ in (2.11), we get

$$\begin{aligned}\mathcal{T}_{F_{\mu},w}(\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3) &= \mathbf{y}_1 - \left(\int_0^1 h(\eta^{\mu})\left|P(\eta)\right|\,d_{\mu}\eta\right)\mathbf{y}_2 - \left(\int_0^1 h(1-\eta^{\mu})\left|P(\eta)\right|\,d_{\mu}\eta\right)\mathbf{y}_3 \\ &= \mathbf{y}_1 - \left(\int_0^1 h(\eta^{\mu})\left|P(\eta)\right|\,d_{\mu}\eta\right)(\mathbf{y}_2+\mathbf{y}_3)\end{aligned}$$

for $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3 \in \mathbb{R}$. Then, by using Theorem 4.3, we get

$$\begin{aligned}0 &\geq \mathcal{T}_{F_{\mu},w}\left(\frac{1}{\nu_2^{\mu}-\nu_1^{\mu}}\left|\frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2} g(x^{\mu})\,d_{\mu}x - g\left(\frac{\nu_1^{\mu}+\nu_2^{\mu}}{2}\right)\right|,\right. \\ &\quad \left.\left|D_{\mu}(g)(\nu_1^{\mu})\right|,\left|D_{\mu}(g)(\nu_2^{\mu})\right|,\eta\right) + \int_0^1 L_{w(\eta^{\mu})}\,d_{\mu}\eta \\ &= \frac{1}{\nu_2^{\mu}-\nu_1^{\mu}}\left|\frac{\mu}{\nu_2^{\mu}-\nu_1^{\mu}}\int_{\nu_1}^{\nu_2} g(x^{\mu})\,d_{\mu}x - g\left(\frac{\nu_1^{\mu}+\nu_2^{\mu}}{2}\right)\right| \\ &\quad - \left(\int_0^1 h(\eta^{\mu})\left|P(\eta)\right|\,d_{\mu}\eta\right)\left(\left|D_{\mu}(g)(\nu_1^{\mu})\right| + \left|D_{\mu}(g)(\nu_2^{\mu})\right|\right),\end{aligned}$$

which rearranges to the required inequality (4.12). □

5 Application test

In this section we give some applications of our theorems to the special means for the positive numbers $\nu_1 > 0$ and $\nu_2 > 0$:

- Arithmetic mean:

$$\mathcal{A}(\nu_1,\nu_2) = \frac{\nu_1+\nu_2}{2}.$$

- Harmonic mean:

$$\mathcal{H} = \mathcal{H}(\nu_1,\nu_2) = \frac{2\nu_1\nu_2}{\nu_1+\nu_2}, \quad \nu_1,\nu_2 > 0.$$

- Logarithmic mean:

$$\mathcal{L}(\nu_1,\nu_2) = \frac{\nu_2-\nu_1}{\ln|\nu_2| - \ln|\nu_1|}, \quad |\nu_1| \neq |\nu_2|, \nu_1,\nu_2 \neq 0, \nu_1,\nu_2 \in \mathbb{R}.$$

- Generalized log-mean:

$$\mathcal{L}_p(\nu_1,\nu_2) = \left[\frac{{\nu_2}^{p+1} - {\nu_1}^{p+1}}{(p+1)(\nu_2-\nu_1)}\right]^{\frac{1}{p}}, \quad p \in \mathbb{Z}\setminus\{-1,0\}, \nu_1,\nu_2 \in \mathbb{R}, \nu_1 \neq \nu_2.$$

Proposition 5.1 *Let* $\mu \in (0,1]$, $\nu_1,\nu_2 \in \mathbb{R}$ *with* $0 < \nu_1 < \nu_2$. *Then we have*

$$F\left(\mathcal{A}^{-1}(\nu_1^{\mu},\nu_2^{\mu}), \mathcal{L}^{-1}(\nu_1^{\mu},\nu_2^{\mu}), \mathcal{L}^{-1}(\nu_1^{\mu},\nu_2^{\mu}), \frac{1}{2}\right) \leq 0, \tag{5.1}$$

$$\mathcal{T}_{F,w}\left(2\mathcal{L}^{-1}(\nu_1^{\mu},\nu_2^{\mu}), \frac{1}{2}\mathcal{H}^{-1}(\nu_1^{\mu},\nu_2^{\mu}), \frac{1}{2}\mathcal{H}^{-1}(\nu_1^{\mu},\nu_2^{\mu})\right) \leq 0. \tag{5.2}$$

Proof The assertion follows from Theorem 3.1 and a simple computation applied to $g(x) = \frac{1}{x}$, $x \in [\nu_1, \nu_2]$, where g is convex and therefore F-convex function on $[\nu_1, \nu_2]$ according to F defined in (1.3) with $\varepsilon = 0$. □

Proposition 5.2 *Let* $\mu \in (0,1]$, $\nu_1, \nu_2 \in \mathbb{R}$ *with* $0 < \nu_1 < \nu_2$. *Then we have*

$$\mathcal{A}^{-1}(\nu_1^\mu, \nu_2^\mu) \leq \mathcal{L}^{-1}(\nu_1^\mu, \nu_2^\mu) \leq \mathcal{H}^{-1}(\nu_1^\mu, \nu_2^\mu). \tag{5.3}$$

Proof The assertion follows from Corollary 3.1 and a simple computation applied to $g(x) = \frac{1}{x}$, $x \in [\nu_1, \nu_2]$, where it is easy to check that g is convex and therefore ε-convex with $\varepsilon = 0$. □

Proposition 5.3 *Let* $\mu \in (0,1]$, $\nu_1, \nu_2 \in \mathbb{R}$ *with* $0 < \nu_1 < \nu_2$. *Then we have*

$$F\left(\mathcal{A}^n(\nu_1^\mu, \nu_2^\mu), \mathcal{L}_n^n(\nu_1^\mu, \nu_2^\mu), \mathcal{L}_n^n(\nu_1^\mu, \nu_2^\mu), \frac{1}{2}\right) \leq 0, \tag{5.4}$$

$$\mathcal{T}_{F,w}\left(2\mathcal{L}_n^n(\nu_1^\mu, \nu_2^\mu), \nu_1^{n\mu} + \nu_2^{n\mu}, \nu_1^{n\mu} + \nu_2^{n\mu}\right). \tag{5.5}$$

Proof The assertion follows from Theorem 3.1 and a simple computation applied to $g(x) = x^n$, $x \in [\nu_1, \nu_2]$ with $n \geq 2$, where g is convex and therefore F-convex function on $[\nu_1, \nu_2]$ according to F defined in (1.3) with $\varepsilon = 0$. □

Proposition 5.4 *Let* $\mu \in (0,1]$, $\nu_1, \nu_2 \in \mathbb{R}$ *with* $0 < \nu_1 < \nu_2$. *Then we have*

$$\left|\mathcal{H}^{-1}(\nu_1^\mu, \nu_2^\mu) - \mathcal{L}^{-1}(\nu_1^\mu, \nu_2^\mu)\right| \leq \frac{\nu_2^\mu - \nu_1^\mu}{2}\left(\frac{2^{3\mu^2} + 6 \times 2^{\mu^2} - 8}{6\mu \times 2^{3\mu^2}}\right)\left(\nu_1^{-\mu(1+\mu)} + \nu_2^{-\mu(1+\mu)}\right). \tag{5.6}$$

Proof The assertion follows from Corollary 4.1 and a simple computation applied to $g(x) = -\frac{1}{x}$, $x \in [\nu_1, \nu_2]$, where it is easy to check that $|D_\mu(g)|$ is convex and therefore ε-convex with $\varepsilon = 0$. □

Proposition 5.5 *Let* $\mu \in (0,1]$, $\nu_1, \nu_2 \in \mathbb{R}$ *with* $0 < \nu_1 < \nu_2$. *Then we have*

$$\left|\mathcal{L}^{-1}(\nu_1^\mu, \nu_2^\mu) - \mathcal{A}^{-1}(\nu_1^\mu, \nu_2^\mu)\right| \leq (\nu_2^\mu - \nu_1^\mu)\left(\frac{\nu_1^{-\mu(1+\mu)} + \nu_2^{-\mu(1+\mu)}}{8\mu}\right). \tag{5.7}$$

Proof The assertion follows from Corollary 4.7 and a simple computation applied to $g(x) = -\frac{1}{x}$, $x \in [\nu_1, \nu_2]$, where it is easy to check that $|D_\mu(g)|$ is convex and therefore ε-convex with $\varepsilon = 0$. □

6 Three illustrative plots

In this section, we give three plots of three dimensions to the above propositions in the previous section.

- Fig. 1 represents Proposition 5.2 with $\mu = \frac{1}{2}$, $\nu_1 = x$, $\nu_2 = y$.
- Fig. 2 represents Proposition 5.4 $\mu = \frac{1}{2}$, $\nu_1 = x$, $\nu_2 = y$.
- Fig. 3 represents Proposition 5.5 $\mu = \frac{1}{2}$, $\nu_1 = x$, $\nu_2 = y$.

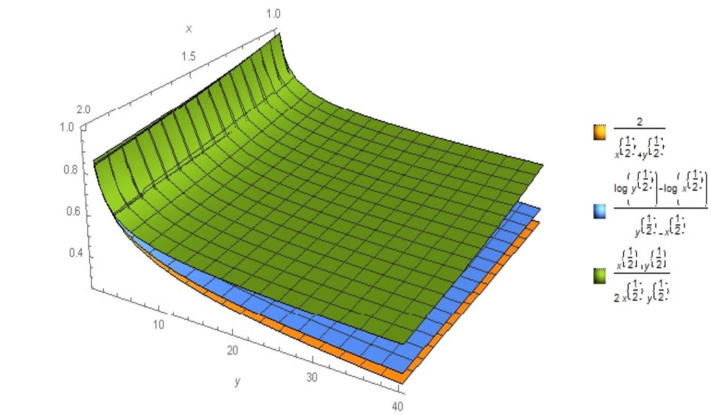

Figure 1 Figure representation for Proposition 5.2

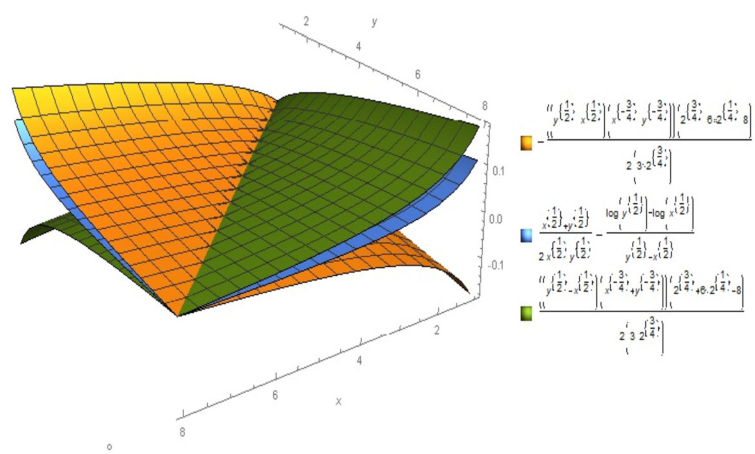

Figure 2 Figure representation for Proposition 5.4

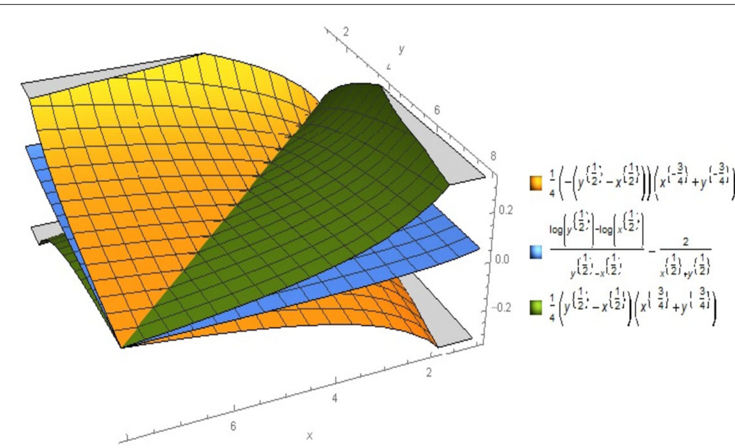

Figure 3 Figure representation for Proposition 5.5

7 Conclusion

Introducing new definitions in the calculus will always open new doors in the field of science and technology. The use of these new definitions in mathematical analysis always requires the presentation of integral inequalities related to them in order to find the existence and uniqueness of such problems. One of the new definitions presented for local fractional calculus is conformable fractional operator. In this study, we have considered the Hermite–Hadamard integral inequalities in the context of conformable fractional calculus. Also, we have introduced the notion of $\mathcal{F}_\mu$-convexity. For this, we have established some Hermite–Hadamard inequalities and related results in the contexts of fractional calculus and conformable fractional calculus.

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References

1. Mangasarian, O.L.: Pseudo-convex functions. SIAM J. Control Optim. **3**, 281–290 (1965)
2. Defnetti, B.: Sulla strati cazioni convesse. Ann. Math. Pures Appl. **30**, 173–183 (1949)
3. Polyak, B.T.: Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. Sov. Math. Dokl. **7**, 72–75 (1966)
4. Hyers, D.H., Ulam, S.M.: Approximately convex functions. Proc. Am. Math. Soc. **3**, 821–828 (1952)
5. Hudzik, H., Maligranda, L.: Some remarks on *s*-convex functions. Aequ. Math. **48**, 100–111 (1994)
6. Varosanec, S.: On *h*-convexity. J. Math. Anal. Appl. **326**(1), 303–311 (2007)
7. Mohammed, P.O.: On new trapezoid type inequalities for *h*-convex functions via generalized fractional integral. Turk. J. Anal. Number Theory **6**(4), 125–128 (2018)
8. Deng, J., Wang, J.: Fractional Hermite–Hadamard inequalities for (α, m)-logarithmically convex functions. J. Inequal. Appl. **2013**, 364 (2013)
9. Qi, F., Mohammed, P.O., Yao, J.C., Yao, Y.H.: Generalized fractional integral inequalities of Hermite–Hadamard type for (α, m)-convex functions. J. Inequal. Appl. **2019**, 135 (2019)
10. Hanson, M.A.: On sufficiency of the Kuhn–Tucker conditions. J. Math. Anal. Appl. **80**, 545–550 (1981)
11. Weir, A., Mond, B.: Preinvex functions in multiple objective optimization. J. Math. Anal. Appl. **136**, 29–38 (1988)
12. Mohammed, P.O.: New integral inequalities for preinvex functions via generalized beta function. J. Interdiscip. Math. **22**(4), 539–549 (2019)
13. Mohammed, P.O.: Inequalities of type Hermite–Hadamard for fractional integrals via differentiable convex functions. Turk. J. Anal. Number Theory **4**(5), 135–139 (2016)
14. Mohammed, P.O.: Inequalities of (k, s), (k, h)-type for Riemann–Liouville fractional integrals. Appl. Math. E-Notes **17**, 199–206 (2017)

15. Mohammed, P.O.: Some new Hermite–Hadamard type inequalities for *MT*-convex functions on differentiable coordinates. J. King Saud Univ., Sci. **30**, 258–262 (2018)
16. Mohammed, P.O., Sarikaya, M.Z.: Hermite–Hadamard type inequalities for *F*-convex function involving fractional integrals. J. Inequal. Appl. **2018**, 359 (2018)
17. Alomari, M., Darus, M., Kirmaci, U.S.: Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means. Comput. Math. Appl. **59**, 225–232 (2010)
18. Mohammed, P.O., Abdeljawad, T.: Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel. Adv. Differ. Equ. **2020**, 363 (2020)
19. İşcana, I., Turhan, S.: Generalized Hermite–Hadamard-Fejer type inequalities for GA-convex functions via fractional integral. Moroccan J. Pure Appl. Anal. **2**(1), 34–46 (2016)
20. Vivas-Cortez, M., Abdeljawad, T., Mohammed, P.O., Rangel-Oliveros, Y.: Simpson's integral inequalities for twice differentiable convex functions. Math. Probl. Eng. **2020**, Article ID 1936461 (2020)
21. Niculescu, C., Persson, L.E.: Convex Functions and Their Application. Springer, Berlin (2004)
22. Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N.: Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. **57**, 2403–2407 (2013)
23. Fernandez, A., Mohammed, P.: Hermite–Hadamard inequalities in fractional calculus defined using Mittag-Leffler kernels. Math. Methods Appl. Sci., 1–18 (2020). https://doi.org/10.1002/mma.6188
24. Mohammed, P.O.: Hermite–Hadamard inequalities for Riemann–Liouville fractional integrals of a convex function with respect to a monotone function. Math. Methods Appl. Sci., 1–11 (2019). https://doi.org/10.1002/mma.5784
25. Mohammed, P.O., Hamasalh, F.K.: New conformable fractional integral inequalities of Hermite–Hadamard type for convex functions. Symmetry **11**(2), 263 (2019). https://doi.org/10.3390/sym11020263
26. Mohammed, P.O., Sarikaya, M.Z.: On generalized fractional integral inequalities for twice differentiable convex functions. J. Comput. Appl. Math. **372**, 112740 (2020)
27. Mohammed, P.O., Abdeljawad, T.: Modification of certain fractional integral inequalities for convex functions. Adv. Differ. Equ. **2020**, 69 (2020)
28. Mohammed, P.O., Brevik, I.: A new version of the Hermite–Hadamard inequality for Riemann–Liouville fractional integrals. Symmetry **12**, 610 (2020). https://doi.org/10.3390/sym12040610
29. Mohammed, P.O., Sarikaya, M.Z., Baleanu, D.: On the generalized Hermite–Hadamard inequalities via the tempered fractional integrals. Symmetry **12**, 595 (2020). https://doi.org/10.3390/sym12040595
30. Baleanu, D., Mohammed, P.O., Zeng, S.: Inequalities of trapezoidal type involving generalized fractional integrals. Alex. Eng. J. (2020). https://doi.org/10.1016/j.aej.2020.03.039
31. Samet, B.: On an implicit convexity concept and some integral inequalities. J. Inequal. Appl. **2016**, 308 (2016)
32. Abdeljawad, T., Mohammed, P.O., Kashuri, A.: New modified conformable fractional integral inequalities of Hermite–Hadamard type with applications. J. Funct. Spaces **2020**, Article ID 4352357 (2020)
33. Iyiola, O.S., Nwaeze, E.R.: Some new results on the new conformable fractional calculus with application using D'Alambert approach. Prog. Fract. Differ. Appl. **2**(2), 115–122 (2016)
34. Mohammed, P.O.: A generalized uncertain fractional forward difference equations of Riemann–Liouville type. J. Math. Res. **11**(4), 43–50 (2019)
35. Arqub, O.A.: Application of residual power series method for the solution of time-fractional Schrödinger equations in one-dimensional space. Fundam. Inform. **166**, 87–110 (2019)
36. Singh, J., Kumar, D., Hammouch, Z., Atangana, A.: A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. Appl. Math. Comput. **316**, 504–515 (2018)
37. Gao, W., Veeresha, P., Prakasha, D.G., Baskonus, H.M., Yel, G.: A powerful approach for fractional Drinfeld–Sokolov–Wilson equation with Mittag-Leffler law. Alex. Eng. J. **58**, 1301–1311 (2019)
38. Yang, A.-M., et al.: Application of local fractional series expansion method to solve Klein–Gordon equations on Cantor sets. Abstr. Appl. Anal. **2014**, Article ID 372741 (2014)
39. Zhang, Z., Cattani, C., Yang, X.-J.: Local fractional homotopy perturbation method for solving non-homogeneous heat conduction equations in fractal domains. Entropy **17**, 6753–6764 (2015)
40. Yang, Y.-J., Baleanu, D., Yang, X.-J.: A local fractional variational iteration method for Laplace equation within local fractional operators. Abstr. Appl. Anal. **2013**, Article ID 202650 (2013)
41. Singh, J., Kumar, D., Kumar, S.: An efficient computational method for local fractional transport equation occurring in fractal porous media. Comput. Appl. Math. **39**, 137 (2020)
42. Singh, J., Jassim, H.K., Kumar, D.: An efficient computational technique for local fractional Fokker Planck equation. Physica A **555**, 124525 (2020)
43. Singh, J., Kumar, D., Baleanu, D.: A new analysis of fractional fish farm model associated with Mittag-Leffler-type kernel. Int. J. Biomath. **13**, 2050010 (2020)
44. Veeresha, P., Prakasha, D.G., Singh, J., et al.: Analytical approach for fractional extended Fisher–Kolmogorov equation with Mittag-Leffler kernel. Adv. Differ. Equ. **2020**, 174 (2020)
45. Khalil, R., Al Horani, M., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
46. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
47. Katugampola, U.: A new fractional derivative with classical properties. arXiv:1410.6535v2
48. Zhao, D., Luo, M.: General conformable fractional derivative and its physical interpretation. Calcolo **54**, 903–917 (2017)
49. Al-Rifae, M., Abdeljawad, T.: Fundamental results of conformable Sturm–Liouville eigenvalue problems. Complexity **2017**, Article ID 3720471 (2017)
50. Abdeljawad, T., Alzabut, J., Jarad, F.: A generalized Lyapunov-type inequality in the frame of conformable derivatives. Adv. Differ. Equ. **2017**, 321 (2017)
51. Atangana, A., Baleanu, D., Alsaedi, A.: New properties of conformable derivative. Open Math. **13**, 889–898 (2015)
52. Arqub, O.A., Al-Smadi, M.: Fuzzy conformable fractional differential equations: novel extended approach and new numerical solutions. Soft Comput. (2020). https://doi.org/10.1007/s00500-020-04687-0
53. Bulut, H., Sulaiman, T.A., Baskonus, H.M., Rezazadeh, H., Eslami, M., Mirzazadeh, M.: Optical solitons and other solutions to the conformable space-time fractional Fokas–Lenells equation. Optik **172**, 20–27 (2018)

54. Yavuz, M.: Novel solution methods for initial boundary value problems of fractional order with conformable differentiation. Int. J. Optim. Control Theor. Appl. **8**(1), 1–7 (2018)
55. Sarikaya, M.Z., Akkurt, A., Budak, H., Yildirim, M.E., Yildirim, H.: Hermite–Hadamard's inequalities for conformable fractional integrals. Int. J. Optim. Control Theor. Appl. **9**(1), 49–59 (2019)