Baleanu et al. *Advances in Difference Equations* (2020) 2020:350
https://doi.org/10.1186/s13662-020-02813-8

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On a strong-singular fractional differential equation

Dumitru Baleanu[1,2], Khadijeh Ghafarnezhad[5], Shahram Rezapour[3,4,5*] and Mehdi Shabibi[6]

*Correspondence:
shahramrezapour@duytan.edu.vn;
sh.rezapour@mail.cmuh.org.tw;
sh.rezapour@azaruniv.ac.ir;
rezapourshahram@yahoo.ca
[3]Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam
[4]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan
Full list of author information is available at the end of the article

Abstract

It is important we try to solve complicate differential equations specially strong singular ones. We investigate the existence of solutions for a strong-singular fractional boundary value problem under some conditions. In this way, we provide a new technique for our study. We provide an example to illustrate our main result.

MSC: Primary 34A08; secondary 37C25

Keywords: Boundary value conditions; Integro-differential equation; Strong-singularity; The fractional Caputo derivative

1 Introduction

Fractional arithmetic theory has gained a special place in various sciences. In recent years, numerous works have been published in the field of fractional integro-differential equations such as q-differences [1–6], positive solutions [7, 8], fractional integro-differential equations [9–13], approximate solutions [14–16], hybrid problems [17, 18], and applied modelings [19–23]. It has been showed that one of the best methods for mathematical describing of complicate phenomena is modeling of the problems as singular fractional integro-differential equations (see [24–26]) which have been studied by some researchers (see, for example, [27–30]). Note that most published works on singular fractional equations have studied weak singularities, while it is important we try to review strong singular fractional integro-differential equations. There are a few works on strong singularities [31–33].

In 2014, Jleli et al. studied the existence of a positive solution for the singular fractional boundary value problem $D^\alpha u(t) + f(t, u(t)) = 0$ with boundary value conditions $u(0) = u'(0) = 0$ and $u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i)$, where $0 < t < 1$, $2 < \alpha \leq 3$, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, $f : (0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $f(t, x)$ is singular at $t = 0$, and D^α is the Caputo derivative [8]. In 2016, Shabibi et al. reviewed the multi-singular pointwise defined fractional integro-differential equation $D^\mu x(t) + f(t, x(t), x'(t), D^\beta x(t), I^p x(t)) = 0$ under different boundary conditions, where $\mu \in [2, 3)$ or $\mu \in [3, \infty)$, $0 \leq t \leq 1$, $x \in C^1[0, 1]$, $\beta, \xi, \eta \in (0, 1)$, $p > 1$, D^μ is the Caputo fractional derivative of order μ and $f : [0, 1] \times \mathbb{R}^5 \to \mathbb{R}$ is a function such that $f(t, \cdot, \cdot, \cdot, \cdot)$ is singular at some points $t \in [0, 1]$ [28].

In 2018, Baleanu et al. investigated the existence of solutions for the pointwise defined problem $D^\alpha x(t) + f(t, x(t), x'(t), D^\beta x(t), \int_0^t h(\xi) x(\xi)\, d\xi, \phi(x(t))) = 0$ with boundary value conditions $x(1) = x(0) = x''(0) = x^n(0) = 0$, where $\alpha \geq 2$, $\lambda, \mu, \beta \in (0,1)$, $\phi : X \to X$ is a mapping such that $\|\phi(x) - \phi(y)\| \leq \theta_0\|x - y\| + \theta_1\|x' - y'\|$ for some nonnegative real numbers θ_0 and $\theta_1 \in [0,\infty)$ and all $x, y \in X$, D^α is the Caputo fractional derivative of order α, $f(t, x_1(t), \dots, x_5(t)) = f_1(t, x_1(t), \dots, x_5(t))$ for all $t \in [0,\lambda)$, $f(t, x_1(t), \dots, x_5(t)) = f_2(t, x_1(t), \dots, x_5(t))$ for all $t \in [\lambda, \mu]$ and $f(t, x_1(t), \dots, x_5(t)) = f(t, x_1(t), \dots, x_5(t))$ for all $t \in (\mu, 1]$, $f_1(t, \cdot, \cdot, \cdot, \cdot)$ and $f_3(t, \cdot, \cdot, \cdot, \cdot)$ are continuous on $[0,\lambda)$ and $(\mu, 1]$, and $f_2(t, \cdot, \cdot, \cdot, \cdot)$ is multi-singular [25]. They published another work on a three-step crisis integro-differential equation [26]. In 2020, Talaee et al. reviewed the existence of solutions for the fractional differential pointwise defined problem $D^\alpha x(t) = f(t, x(t), x'(t), D^\beta x(t), \int_0^t g(\xi)x(\xi)\,d\xi)$ with boundary value conditions $x(\mu) = \int_0^1 h(z)x(z)\,dz$ and $x(0) = x^{(j)}(0) = 0$ for $2 \leq j \leq n-1$, where $\alpha \geq 2$, $n = [\alpha] + 1$, $\mu, \beta \in (0,1)$, $g, h : [0,1] \to \mathbb{R}$ are mappings such that $g, zh \in L^1[0,1]$ and $f \in L^1$ is singular at some points $[0,1]$ [30].

By using the main idea of these works, we investigate the existence of solutions for the strong singular fractional differential equation

$$D^\alpha x(t) = f\big(t, x(t), I^{p_1}x(t), \dots, I^{p_m}x(t)\big), \tag{1}$$

with some boundary value conditions, where $\alpha \geq 1$, $p_1, \dots, p_m > 0$, $m \geq 1$, D^α is the fractional Caputo derivative of order α and $f(t, \cdot, \dots, \cdot)$ is strong singular at some points $[0,1]$.

The Riemann–Liouville integral of order p with the lower limit $a \geq 0$ for a function $f : (a,\infty) \to \mathbb{R}$ is defined by $I^p_{a^+}f(t) = \frac{1}{\Gamma(p)}\int_a^t (t-s)^{p-1}f(s)\,ds$ provided that the right-hand side is pointwise define on (a,∞) [34]. We denote $I^p_{0^+}f(t)$ by $I^p f(t)$. The Caputo fractional derivative of order $\alpha > 0$ is defined by ${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}\,ds$, where $n = [\alpha] + 1$ and $f : (a,\infty) \to \mathbb{R}$ is a function [34].

Let Ψ be the family of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^\infty \psi^n(t) < \infty$ for all $t > 0$ [35]. One can check that $\psi(t) < t$ for all $t > 0$ [35]. Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two maps. Then T is called an α-admissible map whenever $\alpha(x,y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ [35]. Let (X,d) be a metric space, $\psi \in \Psi$, and $\alpha : X \times X \to [0,\infty)$ be a map. A self-map $T : X \to X$ is called an α-ψ-contraction whenever $\alpha(x,y)\,d(Tx, Ty) \leq \psi(d(x,y))$ for all $x, y \in X$ [35]. We need the next results.

Lemma 1 ([35]) *Let (X,d) be a complete metric space, $\psi \in \Psi$, $\alpha : X \times X \to [0,\infty)$ be a map, and $T : X \to X$ be an α-admissible α-ψ-contraction. If T is continuous and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.*

Lemma 2 ([34]) *Let $n-1 \leq \alpha < n$ and $x \in C(0,1)$. Then $I^\alpha D^\alpha x(t) = x(t) + \sum_{i=0}^{n-1} c_i t^i$ for some real constants $c_0, \dots, c_{n-1}$.*

Lemma 3 ([36]) *For all $z > 0$ and $\omega > -1$, we have $\int_0^t (t-s)^{\omega-1}s^z\,ds = B(z+1,\omega)t^{\omega+z}$, where $B(z,\omega) = \frac{\Gamma(\omega)\Gamma(z)}{\Gamma(\omega+z)}$.*

2 Main results

Now, we are ready for preparing our main results. For the next key result, we use the main idea of [25] to conclude that it is valid on $L^1[0,1]$.

Lemma 4 *Let $\alpha \geq 1$, $[\alpha] = n-1$, k be a natural number, $\mu \in (0,1)$, $\gamma_1, \dots, \gamma_k \in (0,1)$, $\lambda_1, \dots, \lambda_k \geq 0$ and $q_1, \dots, q_k > 0$ be such that $\sum_{i=1}^k \frac{\lambda_i\gamma_i^{q_i}}{\Gamma(q_i+1)} < 1$ and $f \in L^1[0,1]$. Then the* *solution of the problem* $D^{\alpha}x(t) = f(t)$ *with boundary conditions* $x^{(2)}(0) = \cdots = x^{(n-1)}(0) = 0$, $x(0) = \int_0^1 x(\xi)\,d\xi$, *and* $x(\mu) = \sum_{i=1}^{k} \lambda_i I^{q_i} x(\gamma_i)$ *is* $x(t) = \int_0^1 G(t,s)f(s)\,ds$, *where the Green function* $G(t,s)$ *is defined by*

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j(\gamma_j - s)^{\alpha+q_j-1}}{\theta_q \Gamma(\alpha+q_j)} + \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)(1-s)^{\alpha}}{\theta_q \Gamma(\alpha+1)} - \frac{(\mu-s)^{\alpha-1}}{\theta_q \Gamma(\alpha)},$$

whenever $s \leq \mu$, $s \leq t$, $s \leq \gamma_1 < \cdots < \gamma_k < 1$,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{j=j_0}^{k} \frac{\lambda_j(\gamma_j - s)^{\alpha+q_j-1}}{\theta_q \Gamma(\alpha+q_j)} + \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)(1-s)^{\alpha}}{\theta_q \Gamma(\alpha+1)} - \frac{(\mu-s)^{\alpha-1}}{\theta_q \Gamma(\alpha)},$$

whenever $s \leq \mu$, $s \leq t$, $\gamma_1 < \gamma_2 < \cdots \gamma_{j_0-1} \leq s \leq \gamma_{j_0} < \cdots < \gamma_k < 1$,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{j=j_0}^{k} \frac{\lambda_j(\gamma_j - s)^{\alpha+q_j-1}}{\theta_q \Gamma(\alpha+q_j)} + \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)(1-s)^{\alpha}}{\theta_q \Gamma(\alpha+1)},$$

whenever $s \geq \mu$, $s \leq t$, $\gamma_1 < \gamma_2 < \cdots < \gamma_{j_0-1} \leq s \leq \gamma_{j_0} < \cdots < \gamma_k < 1$,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)(1-s)^{\alpha}}{\theta_q \Gamma(\alpha+1)},$$

whenever $s \geq \mu$, $s \leq t$, $\gamma_1 < \gamma_2 < \cdots \gamma_k < s < 1$,

$$G(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)(1-s)^{\alpha}}{\theta_q \Gamma(\alpha+1)},$$

whenever $s \geq \mu$, $s \leq t$, $\gamma_1 < \gamma_2 < \cdots < \gamma_{j_0-1} \leq s \leq \gamma_{j_0} < \cdots < \gamma_k < 1$,

$$G(t,s) = \sum_{j=j_0}^{k} \frac{\lambda_j(\gamma_j - s)^{\alpha+q_j-1}}{\theta_q \Gamma(\alpha+q_j)} \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)(1-s)^{\alpha}}{\theta_q \Gamma(\alpha+1)},$$

whenever $s \geq \mu$, $s \geq t$, $\gamma_1 < \gamma_2 < \cdots < \gamma_{j_0-1} \leq s \leq \gamma_{j_0} < \cdots < \gamma_k < 1$, *and*

$$G(t,s) = \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)(1-s)^{\alpha}}{\theta_q \Gamma(\alpha+1)},$$

whenever $s \geq \mu$, $s \geq t$, $\gamma_1 < \gamma_2 < \cdots < \gamma_k < s < 1$. *Here,* $\theta_{q+1} := 1 - \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i+1}}{\Gamma(q_i+2)}$ *and* $\theta_q := 1 - \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i}}{\Gamma(q_i+1)}$.

Proof Let x be a solution for the problem. By using Lemma 2, we have

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds + c_0 + c_1 t + \cdots + c_n t^n$$

for some real constants $c_0, \ldots, c_n$. Since $x^{(2)}(0) = \cdots = x^{(n-1)}(0) = 0$, we get $c_2 = \cdots = c_n = 0$, and so $x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds + c_0 + c_1 t$. Thus, $x(0) = c_0$ and

$$\begin{aligned} \int_0^1 x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 \int_0^t (t-s)^{\alpha-1} f(s)\,ds + c_0 + \frac{c_1}{2} \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f(s)\,ds + c_0 + \frac{c_1}{2}. \end{aligned}$$

Now, by using the condition $x(0) = \int_0^1 x(\xi)\,d\xi$, we obtain $\frac{1}{\Gamma(\alpha+1)} \int_0^1 (t-s)^{\alpha} f(s)\,ds + c_0 + \frac{c_1}{2} = c_0$, and so $c_1 = \frac{-2}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f(s)\,ds$. Hence,

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds + c_0 - \frac{2t}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f(s)\,ds, \tag{2}$$

and so $x(\mu) = \frac{1}{\Gamma(\alpha)} \int_0^{\mu} (\mu-s)^{\alpha-1} f(s)\,ds + c_0 - \frac{2\mu}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f(s)\,ds$. On the other hand, we have $I^{q_i}(t) = \frac{1}{\Gamma(q_i)} \int_0^t (t-s)^{q_i-1}\,ds = \frac{t^{q_i}}{\Gamma(q_i+1)}$ for all $1 \le i \le k$. By using Lemma 3, we get $I^{q_i}(t) = \frac{1}{\Gamma(q_i)} \int_0^t t(t-s)^{q_i-1}\,ds = \frac{1}{\Gamma(q_i)} B(2,q_i) t^{2+q_i-1} = \frac{1}{\Gamma(q_i)} \cdot \frac{\Gamma(2)\Gamma(q_i)}{\Gamma(2+q_i)} t^{q_i+1} = \frac{t^{q_i+1}}{\Gamma(2+q_i)}$. Since $I^{q_i} I^{\alpha} f(t) = I^{q_i+\alpha} f(t)$, by using (2) we obtain

$$\begin{aligned} I^{q_i} x(t) = {} & \frac{1}{\Gamma(\alpha+q_i)} \int_0^t (t-s)^{\alpha+q_i-1} f(s)\,ds + c_0 \frac{t^{q_i}}{\Gamma(q_i+1)} \\ & - \frac{2t^{q_i+1}}{\Gamma(\alpha+1)\Gamma(q_i+2)} \int_0^1 (1-s)^{\alpha} f(s)\,ds. \end{aligned}$$

Hence,

$$\begin{aligned} \lambda_i I^{q_i} x(\gamma_i) = {} & \frac{\lambda_i}{\Gamma(\alpha+q_i)} \int_0^{\gamma_i} (\gamma_i-s)^{\alpha+q_i-1} f(s)\,ds + \frac{\lambda_i c_0 \gamma_i^{q_i}}{\Gamma(q_i+1)} \\ & - \frac{2\lambda_i \gamma_i^{q_i+1}}{\Gamma(\alpha+1)\Gamma(q_i+2)} \int_0^1 (1-s)^{\alpha} f(s)\,ds \end{aligned}$$

for all $1 \le i \le k$, and so

$$\begin{aligned} \sum_{i=1}^k \lambda_i I^{q_i} x(\gamma_i) = {} & \sum_{i=1}^k \frac{\lambda_i}{\Gamma(\alpha+q_i)} \int_0^{\gamma_i} (\gamma_i-s)^{\alpha+q_i-1} f(s)\,ds \\ & + c_0 \sum_{i=1}^k \frac{\lambda_i \gamma_i^{q_i}}{\Gamma(q_i+1)} - \frac{2}{\Gamma(\alpha+1)} \sum_{i=1}^k \frac{\lambda_i \gamma_i^{q_i+1}}{\Gamma(q_i+2)} \int_0^1 (1-s)^{\alpha} f(s)\,ds. \end{aligned}$$

Since $x(\mu) = \sum_{i=1}^k \lambda_i I^{q_i} x(\gamma_i)$, we get

$$\frac{1}{\Gamma(\alpha)} \int_0^{\mu} (\mu-s)^{\alpha-1} f(s)\,ds + c_0 - \frac{2\mu}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f(s)\,ds$$

$$= \sum_{i=1}^{k} \frac{\lambda_i}{\Gamma(\alpha + q_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha+q_i-1} f(s)\,ds + c_0 \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i}}{\Gamma(q_i + 1)} - \frac{2}{\Gamma(\alpha+1)} \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i+1}}{\Gamma(q_i + 2)} \int_0^1 (1-s)^{\alpha} f(s)\,ds,$$

and so

$$c_0 = \sum_{i=1}^{k} \frac{\lambda_i}{\theta_q \Gamma(\alpha + q_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha+q_i-1} f(s)\,ds + \frac{2\mu}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f(s)\,ds - \frac{1}{\theta_q \Gamma(\alpha)} \int_0^{\mu} (\mu - s)^{\alpha-1} f(s)\,ds - \frac{2}{\theta_q \Gamma(\alpha+1)} \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i+1}}{\Gamma(q_i + 2)} \int_0^1 (1-s)^{\alpha} f(s)\,ds,$$

where $\theta_q := 1 - \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i}}{\Gamma(q_i+1)}$ and $\theta_{q+1} := 1 - \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i+1}}{\Gamma(q_i+2)}$. Thus,

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds \\ & + \sum_{i=1}^{k} \frac{\lambda_i}{\theta_q \Gamma(\alpha + q_i)} \int_0^{\gamma_i} (\gamma_i - s)^{\alpha+q_i-1} f(s)\,ds \\ & + \frac{2\mu}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f(s)\,ds \\ & - \frac{1}{\theta_q \Gamma(\alpha)} \int_0^{\mu} (\mu - s)^{\alpha-1} f(s)\,ds \\ & - \frac{2}{\theta_q \Gamma(\alpha+1)} \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i+1}}{\Gamma(q_i + 2)} \int_0^1 (1-s)^{\alpha} f(s)\,ds \\ & - \frac{2t}{\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f(s)\,ds \end{aligned}$$

or

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds \\ & + \frac{\lambda_1}{\theta_q \Gamma(\alpha + q_1)} \int_0^{\gamma_1} (\gamma_1 - s)^{\alpha+q_1-1} f(s)\,ds \\ & + \frac{\lambda_2}{\theta_q \Gamma(\alpha + q_2)} \int_0^{\gamma_2} (\gamma_2 - s)^{\alpha+q_2-1} f(s)\,ds \\ & + \cdots + \frac{\lambda_k}{\theta_q \Gamma(\alpha + q_k)} \int_0^{\gamma_k} (\gamma_k - s)^{\alpha+q_k-1} f(s)\,ds \\ & + \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)}{\theta_q \Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} f(s)\,ds \\ & - \frac{1}{\theta_q \Gamma(\alpha)} \int_0^{\mu} (\mu - s)^{\alpha-1} f(s)\,ds = \int_0^1 G(t,s) f(s), \end{aligned}$$

where $G(t,s)$ is the Green function. This completes the proof. □

Note that in the last result, it remains only the boundary value conditions $x(0) = \int_0^1 x(\xi)\,d\xi$ and $x(\mu) = \sum_{i=1}^{k} \lambda_i I^{q_i} x(\gamma_i)$ whenever $1 \leq \alpha < 2$. It is easy to see

$$\begin{aligned}\bigl|G(t,s)\bigr| &\leq \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j(\gamma_j - s)^{\alpha+q_j-1}}{\theta_q \Gamma(\alpha+q_j)} \\ &\quad + \frac{2|\mu - \theta_{q+1} - t\theta_q - 1|(1-s)^{\alpha}}{\theta_q \Gamma(\alpha+1)} + \frac{(\mu-s)^{\alpha-1}}{\theta_q \Gamma(\alpha)}\end{aligned}$$

and G is continuous with respect to t. Consider the Banach space $X = C[0,1]$ with the sup norm. Let $g : [0,1] \times X^{m+1} \to \mathbb{R}$ be singular at the points $\gamma_1 < \gamma_2 < \cdots < \gamma_k$ in $[0,1]$. Define the map $F : X \to X$ by

$$\begin{aligned}Fx(t) &= \int_0^1 G(t,s) g\bigl(s, x(s), I^{p_1}x(s), \ldots, I^{p_m}x(s)\bigr)\,ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\bigl(s, x(s), I^{p_1}x(s), \ldots, I^{p_m}x(s)\bigr)\,ds \\ &\quad + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha+q_j)} \int_0^{\gamma_j} (\gamma_j - s)^{\alpha+q_j-1} g\bigl(s, x(s), I^{p_1}x(s), \ldots, I^{p_m}x(s)\bigr)\,ds \\ &\quad + \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)}{\theta_q \Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha} g\bigl(s, x(s), I^{p_1}x(s), \ldots, I^{p_m}x(s)\bigr)\,ds \\ &\quad - \frac{1}{\theta_q \Gamma(\alpha)} \int_0^{\mu} (\mu-s)^{\alpha-1} f\bigl(s, x(s), I^{p_1}x(s), \ldots, I^{p_m}x(s)\bigr)\,ds.\end{aligned}$$

Let $0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = 1$ and $f(s, \cdot, \ldots, \cdot)$ is singular at each t_i for $1 \leq i \leq r$. Put $n_0 = [\frac{2}{\min_{0 \leq i \leq r}(t_{i+1} - t_i)}] + 1$. For $n \geq n_0$, define $F^n : X \to X$ by

$$\begin{aligned}F^n x(t) &= \sum_{i=0}^{r-1} \int_{t_i + \frac{1}{n}}^{t_{i+1} - \frac{1}{n}} G(t,s) f\bigl(s, x(s), I^{p_1}x(s), \ldots, I^{p_m}x(s)\bigr)\,ds \\ &= \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{r-1} \int_{[0,t] \cap [t_i + \frac{1}{n}, t_{i+1} - \frac{1}{n}]} (t-s)^{\alpha-1} f\bigl(s, x(s), I^{p_1}x(s), \ldots, I^{p_m}x(s)\bigr)\,ds \\ &\quad + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha+q_j)} \\ &\quad \times \Biggl(\sum_{i=0}^{r-1} \int_{[0,\gamma_j] \cap [t_i + \frac{1}{n}, t_{i+1} - \frac{1}{n}]} (\gamma_j - s)^{\alpha+q_j-1} f\bigl(s, x(s), I^{p_1}x(s), \ldots, I^{p_m}x(s)\bigr)\,ds \Biggr) \\ &\quad + \frac{2(\mu - \theta_{q+1} - t\theta_q - 1)}{\theta_q \Gamma(\alpha+1)} \\ &\quad \times \sum_{i=0}^{r-1} \int_{[t_i + \frac{1}{n}, t_{i+1} - \frac{1}{n}]} (1-s)^{\alpha} f\bigl(s, x(s), I^{p_1}x(s), \ldots, I^{p_m}x(s)\bigr)\,ds \\ &\quad - \frac{1}{\theta_q \Gamma(\alpha)}\end{aligned}$$

$$\times \sum_{i=0}^{r-1} \int_{[0,\mu]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]} (\mu - s)^{\alpha-1} f\big(s, x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\, ds. \tag{3}$$

Note that each fixed point of F is a solution for problem (1).

Theorem 5 *Assume that $\alpha \geq 1$, $[\alpha] = n - 1$, $r, k, m \geq 1$, $\mu \in (0,1)$, $\gamma_1, \dots, \gamma_k \in (0,1)$, $\lambda_1, \dots, \lambda_k \geq 0$, $q_1, \dots, q_k > 0$, $p_1, \dots, p_m > 0$, $a_1, \dots .a_{m+1}$, and $\Lambda_1, \dots, \Lambda_{m+1} : \mathbb{R} \to [0,\infty)$ are some functions such that $\hat{a}_i(t) = (1 - t)^{\alpha-1} a_i(t) \in L^1(K_j)$ for every compact subset $K_j \subseteq (t_j, t_{j+1})$ for $i = 1, \dots, m + 1$ and $j = 1, \dots, r - 1$, $\lim_{z\to 0^+} \frac{\Lambda_i(z)}{z} = B_i \geq 0$, and we have $|f(t, x_1, \dots, x_{m+1}) - f(t, y_1, \dots, y_{m+1})| \leq \sum_{i=1}^{m+1} a_i(t)\Lambda_i(|x_i - y_i|)$ for all $(x_1, \dots, x_{m+1})$ and $(y_1, \dots, y_{m+1})$ in X^{m+1} and almost all $t \in [0,1]$. Suppose that*

$$\Delta \sum_{i=0}^{r-1} \left(\left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha + q_j)} + \frac{2(\theta_{q+1} - \mu + 1)}{\theta_q \Gamma(\alpha + 1)} + \frac{1}{\theta_q \Gamma(\alpha)} \right] \sum_{j'=1}^{m+1} B_{j'} \|\hat{a}_{j',i,n}\| \right) < 1,$$

where $\|\hat{a}_{j',i,n}\| := \int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} (1-s)^{\alpha-1} a_{j'}(s)\, ds$ and $\Delta := \max\{1, \frac{1}{\Gamma(p_1+1)}, \dots, \frac{1}{\Gamma(p_m+1)}\}$. Assume that there are two maps b and $N : X^{m+1} \to [0,\infty)$ such that $(1-t)^{\alpha-1} b(t) \in L^1(K_j)$ for every compact subset $K_j \subseteq (t_j, t_{j+1})$ for $j = 1, \dots, r-1$ and N is nondecreasing with respect to all its components and $\lim_{z\to 0^+} \frac{N(z,\dots,z)}{z} = \eta \geq 0$. Suppose that $|f(t, x_1, \dots, x_{m+1})| \leq b(t)N(x_1, \dots, x_{m+1})$ for all $(x_1, \dots, x_{m+1}) \in X^{m+1}$ and almost all $t \in [0,1]$. If

$$\eta \sum_{i=0}^{r-1} \|\hat{b}_{i,n}\| \left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha + q_j)} + \frac{2(\theta_{q+1} - \mu + 1)}{\theta_q \Gamma(\alpha + 1)} + \frac{1}{\theta_q \Gamma(\alpha)} \right] \in \left[0, \frac{1}{\Delta}\right),$$

then the singular problem (1) has a solution.

Proof Let $x, y \in X$ and $t \in [0,1]$. Then we have

$$\begin{aligned}
&\big|F^n x(t) - F^n y(t)\big| \\
&\quad \leq \sum_{i=0}^{r-1} \int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} \big|G(t,s)\big| \big|f\big(s, x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big) \\
&\qquad - f\big(s, x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\big|\, ds \\
&\quad \leq \left| \sum_{i=0}^{r-1} \int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} \right| G(t,s) \big| \big[a_1(s)\Lambda_1\big(\big|x(s) - y(s)\big|\big) + a_2(s)\Lambda_2\big(\big|I^{p_1}\big(x(s) - y(s)\big)\big|\big) \\
&\qquad + \dots + a_{m+1}(s)\Lambda_{m+1}\big(\big|I^{p_m}\big(x(s) - y(s)\big)\big|\big)\big]\, ds.
\end{aligned}$$

For $1 \leq i \leq m$ and $t \in [0,1]$, we obtain

$$\big|I^{p_i}x(t)\big| \leq \frac{1}{p_i} \int_0^t (t-s)^{p_i-1} \big|x(s)\big|\, ds \leq \frac{\|x\|}{p_i} \int_0^t (t-s)^{p_i-1}\, ds = \frac{\|x\|}{\Gamma(p_i+1)} t^{p_i},$$

and so $|I^{p_i}x(t)| \leq \frac{\|x\|}{p_i+1}$. Hence,

$$\big|F^n x(t) - F^n y(t)\big|$$

$$\leq \sum_{i=0}^{r-1} \int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} \bigl|G(t,s)\bigr| \biggl[a_1(s)\Lambda_1\bigl(\|x-y\|\bigr) + a_2(s)\Lambda_2\biggl(\frac{\|x-y\|}{\Gamma(p_1+1)}\biggr) + \cdots + a_{m+1}(s)\Lambda_{m+1}\biggl(\frac{\|x-y\|}{\Gamma(p_m+1)}\biggr)\biggr]\,ds.$$

Since $\Delta := \max\{1, \frac{1}{\Gamma(p_1+1)}, \ldots, \frac{1}{\Gamma(p_m+1)}\}$, we get

$$\bigl|F^n x(t) - F^n y(t)\bigr| \leq \sum_{i=0}^{r-1} \biggl(\int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} \bigl|G(t,s)\bigr| \biggl[\sum_{j=1}^{m+1} a_j(s)\Lambda_j\bigl(\Delta\|x-y\|\bigr)\biggr]\,ds\biggr). \tag{4}$$

Let $\epsilon > 0$ be given. Since $\lim_{z\to 0^+} \frac{\Lambda_j(z)}{z} = B_j$ for all $1 \leq j \leq m+1$, there exists $\delta(\epsilon) > 0$ such that $|z| \leq \delta'$ implies $|\frac{\Lambda_j(z)}{z} - B_j| \leq \epsilon$, where $\delta' \leq \delta(\epsilon)$. Thus, $\Lambda_j(z) \leq (\epsilon + B_j)z$ for all $|z| \leq \delta'$. Let $\delta_0' := \min\{\epsilon, \delta(\epsilon)\}$ and $|z| \leq \delta_0'$. Then we have $\Lambda_j(z) \leq (\epsilon + B_j)z$ for all $1 \leq j \leq m+1$. If $\Delta\|x-y\| \leq \delta_0'$, then $\Lambda_j(\Delta\|x-y\|) \leq (\epsilon + B_j)\Delta\|x-y\| \leq (\epsilon + B_j)\delta_0' \leq (\epsilon + B_j)\epsilon$ for all $1 \leq j \leq m+1$. Also, $\Delta\|x-y\| \leq \delta_0'$ implies $\|x-y\| \leq \frac{\epsilon}{\Delta}$. Let $t \in [0,1]$. By using (4), we conclude that

$$\begin{aligned}
&\bigl|F^n x(t) - F^n y(t)\bigr| \\
&\quad\leq \sum_{i=0}^{r-1} \biggl(\int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} \biggl[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j (1-s)^{\alpha+q_j-1}}{\theta_q \Gamma(\alpha+q_j)} \\
&\qquad + \frac{2|\mu - \theta_{q+1} - t\theta_q - 1|(1-s)^{\alpha}}{\theta_q \Gamma(\alpha+1)} + \frac{(1-s)^{\alpha-1}}{\theta_q \Gamma(\alpha)}\biggr] \times \biggl[\sum_{j'=1}^{m+1} a_{j'}(s)(\epsilon + B_{j'})\biggr]\,ds\biggr)\epsilon \\
&\quad\leq \epsilon \sum_{j'=1}^{m+1} \epsilon + B_{j'} \biggl(\sum_{i=0}^{r-1} \biggl(\biggl[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha+q_j)} \\
&\qquad + \frac{2|\mu - \theta_{q+1} - t\theta_q - 1|}{\theta_q \Gamma(\alpha+1)} + \frac{1}{\theta_q \Gamma(\alpha)}\biggr] \times \biggl[\int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} (1-s)^{\alpha-1} a_{j'}(s)\biggr]\,ds\biggr)\biggr).
\end{aligned}$$

If $\|\hat{a}_{j',i,n}\| := \int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} (1-s)^{\alpha-1} a_{j'}(s)\,ds$, then

$$\begin{aligned}
\bigl|F^n x(t) - F^n y(t)\bigr| &\leq \epsilon \sum_{i=0}^{r-1} \biggl(\biggl[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha+q_j)} \\
&\quad + \frac{2|\mu - \theta_{q+1} - t\theta_q - 1|}{\theta_q \Gamma(\alpha+1)} + \frac{1}{\theta_q \Gamma(\alpha)}\biggr] \sum_{j'=1}^{m+1} (\epsilon + B_{j'})\|\hat{a}_{j',i,n}\|\biggr).
\end{aligned}$$

Since $1 - \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i+1}}{\Gamma(q_i+2)} > 1 - \sum_{i=1}^{k} \frac{\lambda_i \gamma_i^{q_i}}{\Gamma(q_i+1)}$, $\theta_{q+1} > \theta_q \geq t\theta_q > 0$ for all $t \in [0,1]$, and so

$$\sup_{t\in[0,1]} |\mu - \theta_{q+1} - t\theta_q - 1| \leq \sup_{t\in[0,1]} \bigl(|\mu - 1| + |\theta_{q+1} - t\theta_q|\bigr) = 1 - \mu + \theta_{q+1}.$$

Thus, we find

$$\begin{aligned}\|F^n x - F^n y\| &\le \epsilon \sum_{i=0}^{r-1}\left(\left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\right.\right.\\ &\quad\left.\left. + \frac{2(\theta_{q+1}-\mu+1)}{\theta_q\Gamma(\alpha+1)} + \frac{1}{\theta_q\Gamma(\alpha)}\right]\sum_{j'=1}^{m+1}(\epsilon + B_{j'})\|\hat{a}_{j',i,n}\|\right),\end{aligned}$$

and so $\|F^n x - F^n y\| \le \epsilon M_n$, where

$$\begin{aligned}M_n &= \sum_{i=0}^{r-1}\left(\left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\right.\right.\\ &\quad\left.\left. + \frac{2(\theta_{q+1}-\mu+1)}{\theta_q\Gamma(\alpha+1)} + \frac{1}{\theta_q\Gamma(\alpha)}\right]\sum_{j'=1}^{m+1}(\epsilon + B_{j'})\|\hat{a}_{j',i,n}\|\right).\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $F^n x \to F^n y$ as $x \to y$ for all $n \ge n_0$. Thus, F^n is continuous. Since $\lim_{z\to 0^+}\frac{N(\Delta z,\dots,\Delta z)}{\Delta z} = \eta$, there exists $r(\epsilon) > 0$ such that $\frac{N(\Delta z,\dots,\Delta z)}{\Delta z} \le \eta + \epsilon$ for all $z \in (0, r(\epsilon)]$, and so $N(\Delta z,\dots,\Delta z) \le (\eta+\epsilon)\Delta z$. Since

$$\eta\sum_{i=0}^{r-1}\|\hat{b}_{i,n}\|\left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)} + \frac{2(\theta_{q+1}-\mu+1)}{\theta_q\Gamma(\alpha+1)} + \frac{1}{\theta_q\Gamma(\alpha)}\right] \in \left[0, \frac{1}{\Delta}\right),$$

there is $\epsilon_0 > 0$ such that

$$(\eta+\epsilon_0)\sum_{i=0}^{r-1}\|\hat{b}_{i,n}\|\left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)} + \frac{2(\theta_{q+1}-\mu+1)}{\theta_q\Gamma(\alpha+1)} + \frac{1}{\theta_q\Gamma(\alpha)}\right] \in \left[0, \frac{1}{\Delta}\right).$$

On the other hand, we have $\lim_{z\to 0^+}\frac{\Lambda_{j'}(\Delta z)}{\Delta z} = B_{j'} \ge 0$ for all $1 \le j' \le m+1$. Let $\epsilon > 0$ be given. Choose $\delta(\epsilon) > 0$ such that $\frac{\Lambda_{j'}(\Delta z)}{\Delta z} < B_{j'} + \epsilon$ for all $0 \le z \le \delta(\epsilon)$. Hence, $\Lambda_{j'}(\Delta z) < (B_{j'}+\epsilon)\Delta z$ for $1 \le j' \le m+1$. Since

$$\Delta\sum_{i=0}^{r-1}\left(\left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)} + \frac{2(\theta_{q+1}-\mu+1)}{\theta_q\Gamma(\alpha+1)} + \frac{1}{\theta_q\Gamma(\alpha)}\right]\sum_{j'=1}^{m+1}B_{j'}\|\hat{a}_{j',i,n}\|\right) < 1,$$

there is $\epsilon_1 > 0$ such that

$$\begin{aligned}&\Delta\sum_{i=0}^{r-1}\left(\left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)} + \frac{2(\theta_{q+1}-\mu+1)}{\theta_q\Gamma(\alpha+1)} + \frac{1}{\theta_q\Gamma(\alpha)}\right]\sum_{j'=1}^{m+1}(B_{j'}+\epsilon_1)\|\hat{a}_{j',i,n}\|\right)\\ &\quad < 1.\end{aligned}$$

Let $\delta_1 = \delta(\epsilon_1)$, $z \in (0, \delta_1]$ and $1 \le j' \le m+1$. Then we have

$$\Lambda_{j'}(\Delta z) \le (B_{j'}+\epsilon_1)\Delta z. \tag{5}$$

If $r_0 = \min\{r(\epsilon_0), \frac{\delta_1}{2}\}$, then $N(\Delta z, \dots, \Delta z) \le (\eta + \epsilon_0)\Delta z$ for all $z \in (o, r_0]$. Specially for $z = r_0$, we have $N(\Delta r_0, \dots, \Delta r_0) \le (\eta + \epsilon_0)\Delta r_0$. Put $C = \{x \in X : \|x\| \le r_0\}$. Define the map $\alpha : X^2 \to [0,\infty)$ by $\alpha(x,y) = 1$ whenever $x, y \in C$ and $\alpha(x,y) = 0$ elsewhere. If $\alpha(x,y) \ge 1$, then $\|x\| \le r_0$ and $\|y\| \le r_0$, and so

$$
\begin{aligned}
&\left|F^n x(t)\right| \\
&\quad\le \left|\sum_{i=0}^{r-1}\int_{t_i+\frac{1}{n}}^{t_{i+1}-\frac{1}{n}} G(t,s) f\big(s, x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\,ds\right| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,t]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(t-s)^{\alpha-1}\left|f\big(s, x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\right|ds \\
&\qquad + \sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\left(\sum_{i=0}^{r-1}\left(\int_{[0,\gamma_j]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\gamma_j-s)^{\alpha+q_j-1}\right.\right. \\
&\qquad \left.\left.\times\left|f\big(s, x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\right|ds\right)\right) \\
&\qquad + \frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha} \\
&\qquad \times\left|f\big(s, x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\right|ds \\
&\qquad + \frac{1}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,\mu]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\mu-s)^{\alpha} \\
&\qquad \times\left|f\big(s, x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\right|ds \\
&\quad\le \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,t]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(t-s)^{\alpha-1}b(s)N\big(x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\,ds \\
&\qquad + \sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\left(\sum_{i=0}^{r-1}\left(\int_{[0,\gamma_j]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\gamma_j-s)^{\alpha+q_j-1}\right.\right. \\
&\qquad \left.\left.\times b(s)N\big(x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\,ds\right)\right) \\
&\qquad + \frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha} \\
&\qquad \times b(s)N\big(x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\,ds \\
&\qquad + \frac{1}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,\mu]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\mu-s)^{\alpha} \\
&\qquad \times b(s)N\big(x(s), I^{p_1}x(s), \dots, I^{p_m}x(s)\big)\,ds \\
&\quad\le \frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,t]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(t-s)^{\alpha-1}b(s)N\left(\|x\|, \frac{\|x\|}{\Gamma(p_1+1)}, \dots, \frac{\|x\|}{\Gamma(p_m+1)}\right)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}(\sum_{i=0}^{r-1}\biggl(\int_{[0,\gamma_j]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\gamma_j-s)^{\alpha+q_j-1}\\
&\times b(s)N\biggl(\|x\|,\frac{\|x\|}{\Gamma(p_1+1)},\dots,\frac{\|x\|}{\Gamma(p_m+1)}\biggr)\,ds\biggr)\\
&+\frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha}\\
&\times b(s)N\biggl(\|x\|,\frac{\|x\|}{\Gamma(p_1+1)},\dots,\frac{\|x\|}{\Gamma(p_m+1)}\biggr)\,ds\\
&+\frac{1}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,\mu]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\mu-s)^{\alpha}\\
&b(s)N\biggl(\|x\|,\frac{\|x\|}{\Gamma(p_1+1)},\dots,\frac{\ll\times Vertx\|}{\Gamma(p_m+1)}\biggr)\,ds\\
\le{}&\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,t]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(t-s)^{\alpha-1}b(s)N\bigl(\Delta\|x\|,\dots,\Delta\|x\|\bigr)\,ds\\
&+\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\\
&\times\biggl(\sum_{i=0}^{r-1}\biggl(\int_{[0,\gamma_j]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\gamma_j-s)^{\alpha+q_j-1}b(s)N\bigl(\Delta\|x\|,\dots,\Delta\|x\|\bigr)\,ds\biggr)\biggr)\\
&+\frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha}b(s)N\bigl(\Delta\|x\|,\dots,\Delta\|x\|\bigr)\,ds\\
&+\frac{1}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,\mu]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\mu-s)^{\alpha}b(s)N\bigl(\Delta\|x\|,\dots,\Delta\|x\|\bigr)\,ds\\
\le{}&\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha-1}b(s)N\bigl(\Delta\|x\|,\dots,\Delta\|x\|\bigr)\,ds\\
&+\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\biggl(\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha+q_j-1}b(s)N\bigl(\Delta\|x\|,\dots,\Delta\|x\|\bigr)\,ds\biggr)\\
&+\frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha}b(s)N\bigl(\Delta\|x\|,\dots,\Delta\|x\|\bigr)\,ds\\
&+\frac{1}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha}b(s)N\bigl(\Delta\|x\|,\dots,\Delta\|x\|\bigr)\,ds\\
\le{}&\frac{N(\Delta r_0,\dots,\Delta r_0)}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha-1}b(s)\,ds\\
&+N(\Delta r_0,\dots,\Delta r_0)\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha-1}b(s)\,ds\\
&+\frac{2|\mu-\theta_{q+1}-t\theta_q-1|N(\Delta r_0,\dots,\Delta r_0)}{\theta_q\Gamma(\alpha+1)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha-1}b(s)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{N(\Delta r_0,\ldots,\Delta r_0)}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha-1}b(s)\,ds\\
&\leq \frac{(\eta+\epsilon_0)\Delta r_0}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\|\hat{b}_{i,n}\|+(\eta+\epsilon_0)\Delta r_0\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\sum_{i=0}^{r-1}\|\hat{b}_{i,n}\|\\
&+\frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}(\eta+\epsilon_0)\Delta r_0\sum_{i=0}^{r-1}\|\hat{b}_{i,n}\|+\frac{(\eta+\epsilon_0)\Delta r_0}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\|\hat{b}_{i,n}\|\\
&=(\eta+\epsilon_0)\Delta\|\hat{b}_{i,n}\|\Bigg(\frac{1}{\Gamma(\alpha)}+\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\\
&+\frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}+\frac{1}{\theta_q\Gamma(\alpha)}\Bigg)r_0\\
&\leq(\eta+\epsilon_0)\Delta\|\hat{b}_{i,n}\|\Bigg(\frac{1}{\Gamma(\alpha)}+\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\\
&+\frac{2(|\mu-1|+|\theta_{q+1}-t\theta_q|)}{\theta_q\Gamma(\alpha+1)}+\frac{1}{\theta_q\Gamma(\alpha)}\Bigg)r_0.
\end{aligned}
$$

Let $t\in[0,1]$ and $n\geq n_0$. Then

$$
\begin{aligned}
\|F^n x\|\leq(\eta+\epsilon_0)\Delta\|\hat{b}_{i,n}\|\Bigg[&\frac{1}{\Gamma(\alpha)}+\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\\
&+\frac{2(1-\mu+\theta_{q+1})}{\theta_q\Gamma(\alpha+1)}+\frac{1}{\theta_q\Gamma(\alpha)}\Bigg]r_0\leq r_0,
\end{aligned}
$$

and so $F^n x\in C$ for $n\geq n_0$. By using the same reasons, one can conclude that $F^n y\in C$ for $y\in C$. Thus, $\alpha(F^n x,F^n y)\geq 1$. Since $F^n x_0\in C$ for $x_0\in C$, $\alpha(x_0,F^n x_0)\geq 1$ for all $n\geq n_0$. Now, let $x,y\in X$, $t\in[0,1]$ and $n\geq n_0$. Then we have

$$
\begin{aligned}
&\left|F^n x(t)-F^n y(t)\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,t]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(t-s)^{\alpha-1}\big|f\big(s,x(s),I^{p_1}x(s),\ldots,I^{p_m}x(s)\big)\\
&-f\big(s,y(s),I^{p_1}y(s),\ldots,I^{p_m}y(s)\big)\big|\,ds\\
&+\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\sum_{i=0}^{r-1}\Bigg(\int_{[0,\gamma_j]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\gamma_j-s)^{\alpha+q_j-1}\\
&\times\big|f\big(s,x(s),I^{p_1}x(s),\ldots,I^{p_m}x(s)\big)-f\big(s,y(s),I^{p_1}y(s),\ldots,I^{p_m}y(s)\big)\big|\,ds\Bigg)\\
&+\frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha}\\
&\times\big|f\big(s,x(s),I^{p_1}x(s),\ldots,I^{p_m}x(s)\big)-f\big(s,y(s),I^{p_1}y(s),\ldots,I^{p_m}y(s)\big)\big|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,\mu]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\mu-s)^{\alpha}\\
&\times\big|f\big(s,x(s),I^{p_1}x(s),\dots,I^{p_m}x(s)\big)-f\big(s,y(s),I^{p_1}y(s),\dots,I^{p_m}y(s)\big)\big|\,ds\\
\le{}&\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,t]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(t-s)^{\alpha-1}\big[a_1(s)\Lambda_1\big(\big|x(s)-y(s)\big|\big)\\
&+a_2(s)\Lambda_2\big(\big|I^{p_1}\big(x(s)-y(s)\big)\big|\big)+\cdots+a_{m+1}(s)\Lambda_{m+1}\big(\big|I^{p_m}\big(x(s)-y(s)\big)\big|\big)\big]\,ds\\
&+\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\sum_{i=0}^{r-1}\int_{[0,\gamma_j]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\gamma_j-s)^{\alpha+q_j-1}\big[a_1(s)\Lambda_1\big(\big|x(s)-y(s)\big|\big)\\
&+a_2(s)\Lambda_2\big(\big|I^{p_1}\big(x(s)-y(s)\big)\big|\big)+\cdots+a_{m+1}(s)\Lambda_{m+1}\big(\big|I^{p_m}\big(x(s)-y(s)\big)\big|\big)\big]\,ds\\
&+\frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha}\big[a_1(s)\Lambda_1\big(\big|x(s)-y(s)\big|\big)\\
&+a_2(s)\Lambda_2\big(\big|I^{p_1}\big(x(s)-y(s)\big)\big|\big)+\cdots+a_{m+1}(s)\Lambda_{m+1}\big(\big|I^{p_m}\big(x(s)-y(s)\big)\big|\big)\big]\,ds\\
&+\frac{1}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[0,\mu]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(\mu-s)^{\alpha}\big[a_1(s)\Lambda_1\big(\big|x(s)-y(s)\big|\big)\\
&+a_2(s)\Lambda_2\big(\big|I^{p_1}\big(x(s)-y(s)\big)\big|\big)+\cdots+a_{m+1}(s)\Lambda_{m+1}\big(\big|I^{p_m}\big(x(s)-y(s)\big)\big|\big)\big]\,ds\\
\le{}&\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha-1}\bigg[a_1(s)\Lambda_1\big(\|x-y\|\big)\\
&+a_2(s)\Lambda_2\bigg(\frac{\|x-y\|}{\Gamma(p_1+1)}\bigg)+\cdots+a_{m+1}(s)\Lambda_2\bigg(\frac{\|x-y\|}{\Gamma(p_m+1)}\bigg)\bigg]\,ds\\
&+\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\sum_{i=0}^{r-1}\int_{[0,\gamma_j]\cap[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha+q_j-1}\bigg[a_1(s)\Lambda_1\big(\|x-y\|\big)\\
&+a_2(s)\Lambda_2\bigg(\frac{\|x-y\|}{\Gamma(p_1+1)}\bigg)+\cdots+a_{m+1}(s)\Lambda_2\bigg(\frac{\|x-y\|}{\Gamma(p_m+1)}\bigg)\bigg]\,ds\\
&+\frac{2|\mu-\theta_{q+1}-t\theta_q-1|}{\theta_q\Gamma(\alpha+1)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha}\bigg[a_1(s)\Lambda_1\big(\|x-y\|\big)\\
&+a_2(s)\Lambda_2\bigg(\frac{\|x-y\|}{\Gamma(p_1+1)}\bigg)+\cdots+a_{m+1}(s)\Lambda_2\bigg(\frac{\|x-y\|}{\Gamma(p_m+1)}\bigg)\bigg]\,ds\\
&+\frac{1}{\theta_q\Gamma(\alpha)}\sum_{i=0}^{r-1}\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha}\bigg[a_1(s)\Lambda_1\big(\|x-y\|\big)\\
&+a_2(s)\Lambda_2\bigg(\frac{\|x-y\|}{\Gamma(p_1+1)}\bigg)+\cdots+a_{m+1}(s)\Lambda_2\bigg(\frac{\|x-y\|}{\Gamma(p_m+1)}\bigg)\bigg]\,ds\\
\le{}&\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{r-1}\Bigg[\sum_{j=1}^{m+1}\Lambda_j\big(\Delta\|x-y\|\big)\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha-1}a_j(s)\,ds\Bigg]\\
&+\Bigg(\sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)}\Bigg)\sum_{i=0}^{r-1}\Bigg[\sum_{j=1}^{m+1}\Lambda_j\big(\Delta\|x-y\|\big)\int_{[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]}(1-s)^{\alpha-1}a_j(s)\,ds\Bigg]
\end{aligned}
$$

$$+ \frac{2|\mu - \theta_{q+1} - t\theta_q - 1|}{\theta_q \Gamma(\alpha+1)} \sum_{i=0}^{r-1} \left[\sum_{j=1}^{m+1} \Lambda_j\big(\Delta\|x-y\|\big) \int_{[t_i+\frac{1}{n}, t_{i+1}-\frac{1}{n}]} (1-s)^{\alpha-1} a_j(s)\, ds \right]$$
$$+ \frac{1}{\theta_q \Gamma(\alpha)} \sum_{i=0}^{r-1} \left[\sum_{j=1}^{m+1} \Lambda_j\big(\Delta\|x-y\|\big) \int_{[t_i+\frac{1}{n}, t_{i+1}-\frac{1}{n}]} (1-s)^{\alpha-1} a_j(s)\, ds \right].$$

If $x, y \notin C$, then $\alpha(x,y) = 0$, and so $\alpha(x,y)\, d(F^n x, F^n y) = 0 \le d(x,y)$ for $x, y \notin C$. Hence, $\|x - y\| \le 2r_0 \le 2\frac{\delta_1}{2} = \delta_1$. Now, by using (5), $\Lambda_j(\Delta\|x-y\|) \le (B_j + \epsilon_1)\Delta\|x-y\|$. Thus,

$$\begin{aligned}
&\big|F^n x(t) - F^n y(t)\big| \\
&\quad\le \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{r-1} \left[\sum_{j=1}^{m+1} (B_j + \epsilon_1)\Delta\|x-y\| \|\hat{a}_{j,i,n}\| \right] \\
&\qquad+ \left(\sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha+q_j)} \right) \sum_{i=0}^{r-1} \left[\sum_{j=1}^{m+1} (B_j + \epsilon_1)\Delta\|x-y\| \|\hat{a}_{j,i,n}\| \right] \\
&\qquad+ \frac{2|\mu - \theta_{q+1} - t\theta_q - 1|}{\theta_q \Gamma(\alpha+1)} \sum_{i=0}^{r-1} \left[\sum_{j=1}^{m+1} (B_j + \epsilon_1)\Delta\|x-y\| \|\hat{a}_{j,i,n}\| \right] \\
&\qquad+ \frac{1}{\theta_q \Gamma(\alpha)} \sum_{i=0}^{r-1} \left[\sum_{j=1}^{m+1} (B_j + \epsilon_1)\Delta\|x-y\| \|\hat{a}_{j,i,n}\| \right],
\end{aligned}$$

and so

$$\begin{aligned}
&\big\|F^n x - F^n y\big\| \\
&\quad\le \Delta \left(\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha+q_j)} + \frac{2(1-\mu+\theta_{q+1})}{\theta_q \Gamma(\alpha+1)} \frac{1}{\theta_q \Gamma(\alpha)} \right) \\
&\qquad\times \left(\sum_{j=1}^{m+1} \left[(B_j + \epsilon_1) \sum_{i=0}^{r-1} \|\hat{a}_{j,i,n}\| \right] \right) \|x-y\| = \lambda\|x-y\|,
\end{aligned}$$

where $\lambda := \Delta(\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)} + \frac{2(1-\mu+\theta_{q+1})}{\theta_q\Gamma(\alpha+1)} \frac{1}{\theta_q\Gamma(\alpha)}) \times (\sum_{j=1}^{m+1}[(B_j + \epsilon_1)\sum_{i=0}^{r-1}\|\hat{a}_{j,i,n}\|])$. Hence, $\|F^n x - F^n y\| \le \lambda\|x-y\|$ for all $x, y \in X$. Now consider the map $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = \lambda t$. Then we have $\sum_{i=1}^{\infty} \psi^i(t) = \lambda t + \lambda^2 t + \cdots = \frac{\lambda}{1-\lambda} t < \infty$ for all $t \in [0,\infty)$. Thus, $\alpha(x,y)\, d(F^n x, F^n y) \le \psi(d(x,y))$ for all $x, y \in X$. Now, by using Lemma 1, we conclude that F^n has a fixed point x_n for each $n \ge n_0$, that is, $x_n(t) = F^n x_n(t)$ for all $t \in [0,1]$. Here, the map F^n is defined by (3). Let $\{x_n\}$ be a sequence of the fixed points. By using the proof, $\{x_n\} \subset C$ and so $\{x_n\}$ is bounded in X. In fact, we have

$$x_n(t) = \int_{[0,1]\backslash\{\bigcup_{i=0}^{r-1}[t_i+\frac{1}{n}, t_{i+1}-\frac{1}{n}]\}} G(t,s) f\big(s, x_n(s), I^{p_1}x_n(s), \ldots, I^{p_m}x_n(s)\big)\, ds$$

for all $t \in [0,1]$. Note that G is continuous with respect to t on $[0,1]\backslash\{\bigcup_{i=0}^{r-1}[t_i + \frac{1}{n}, t_{i+1} - \frac{1}{n}]\}$ as well as the maps $\frac{\partial G}{\partial t}, \ldots, \frac{\partial^{[\alpha]+1} G}{\partial t^{[\alpha]+1}}$. Hence,

$$\lim_{t_k \to t} \frac{\partial^m x_n(t_k)}{\partial t_k^m}$$

$$= \lim_{t_k \to t} \int_{[0,1]\setminus\{\bigcup_{i=0}^{r-1}[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]\}} \frac{\partial^m G}{\partial t_k^m}(t_k,s) f\big(s,x_n(s),I^{p_1}x_n(s),\dots,I^{p_m}x_n(s)\big)\,ds$$
$$= \int_{[0,1]\setminus\{\bigcup_{i=0}^{r-1}[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]\}} \frac{\partial^m G}{\partial t^m}(t,s) f\big(s,x_n(s),I^{p_1}x_n(s),\dots,I^{p_m}x_n(s)\big)\,ds = \frac{\partial^m x_n(t)}{\partial t^m}$$

for all $1 \le m \le [\alpha]+1$ and $t \in [0,1]\setminus\{\bigcup_{i=0}^{r-1}[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]\}$. Thus, the fixed points x_n belong to the space $X^\alpha = \{x : D^\alpha x \in C[0,1]\}$. This implies that the sequence $\{x_n'\}$ is equicontinuous, and so $\{x_n\}$ is relatively compact in X. Now, by using the Arzela–Ascoli theorem, there exists $x_0 \in X$ such that $\lim_{n\to\infty} x_n = x_0$. One can check that x_0 satisfies the boundary value conditions of problem (1). Since $x_n \in C$ for all n, we have

$$\chi_{[0,1]\setminus\{\bigcup_{i=0}^{r-1}[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]\}}(s)\big|G(t,s)f\big(s,x_n(s),I^{p_1}x_n(s),\dots,I^{p_m}x_n(s)\big)\big|$$
$$\le (\eta+\epsilon_0)\Delta\left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k}\frac{\lambda_j}{\theta_q\Gamma(\alpha+q_j)} + \frac{2(1-\mu+\theta_{q+1})}{\theta_q\Gamma(\alpha+1)} + \frac{1}{\theta_q\Gamma(\alpha)}\right]r_0$$
$$\times\, \chi_{[0,1]\setminus\{\bigcup_{i=0}^{r-1}[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]\}}(s)(1-s)^{\alpha-1}b(s),$$

where $\chi_E(s) = 1$ whenever $s \in E$ and $\chi_E(s) = 0$ whenever $s \notin E$. Note that the map $(1-s)^{\alpha-1}b(s)$ belongs to $L^1(K_j)$ for every compact subset $K_j \subseteq (t_j,t_{j+1})$ for $j = 1,\dots,r-1$, and so $\chi_{[0,1]\setminus\{\bigcup_{i=0}^{r-1}[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]\}}(s)(1-s)^{\alpha-1}b(s) \in L^1[0,1]$. Now, by using the Lebesgue dominated theorem, we conclude that

$$x_0(t) = \lim_{n\to\infty} x_n(t)$$
$$= \lim_{n\to\infty}\int_0^1 \chi_{[0,1]\setminus\{\bigcup_{i=0}^{r-1}[t_i+\frac{1}{n},t_{i+1}-\frac{1}{n}]\}}(s)G(t,s)f\big(s,x_n(s),I^{p_1}x_n(s),\dots,I^{p_m}x_n(s)\big)\,ds$$
$$= \int_0^1 G(t,s)f\big(s,x_0(s),I^{p_1}x_0(s),\dots,I^{p_m}x_0(s)\big)\,ds = Fx_0(t).$$

In fact, by using a similar method in (4), we have

$$\big|f\big(s,x_n(s),I^{p_1}x_n(s),\dots,I^{p_m}x_n(s)\big) - f\big(s,x_0(s),I^{p_1}x_0(s),\dots,I^{p_m}x_0(s)\big)\big|$$
$$\le \sum_{j=1}^{m+1} a_j(s)\Lambda_j\big(\Delta\|x_n - x\_0\|\big).$$

Let $\epsilon > 0$ be given. Choose $\delta(\epsilon) > 0$ such that $\Lambda_j(\Delta\|x_n - x_0\|) \le (\epsilon + B_j)\epsilon$ for all $n \ge n_0$ with $\|x_n - x_0\| < \delta(\epsilon)$. Hence,

$$\big|f\big(s,x_n(s),I^{p_1}x_n(s),\dots,I^{p_m}x_n(s)\big) - f\big(s,x_0(s),I^{p_1}x_0(s),\dots,I^{p_m}x_0(s)\big)\big|$$
$$\le \epsilon\sum_{j=1}^{m+1}(\epsilon + B_j)a_j(s),$$

and so $f(s,x_n(s),I^{p_1}x_n(s),\dots,I^{p_m}x_n(s)) \to f(s,x_0(s),I^{p_1}x_0(s),\dots,I^{p_m}x_0(s))$ as $x_n \to x_0$. This implies that F has the fixed point x_0 which is a solution for problem (1). □

Now, we provide an example to illustrate our main result.

Example 1 Consider the strong singular problem

$$D^{\frac{3}{2}}x(t) = f\big(t, x(t), I^{\frac{1}{2}}x(t)\big), \tag{6}$$

with boundary conditions $x(0) = \int_0^1 x(\xi)\,d\xi$ and $x(\frac{1}{3}) = I^{\frac{5}{2}}x(\frac{1}{2})$, where

$$f(t, x_1, x_2) = \frac{0.1}{(1-t)}\big(|x_1| + |x_2|\big).$$

Put $m = 1, k = 1, t_0 = 0, t_1 = 1, \mu = \frac{1}{3}, \lambda_1 = 1, q_1 = \frac{5}{2}, \gamma_1 = \frac{1}{2}, \Lambda_1(x) = \Lambda_2(x) = x, a_1(t) = a_2(t) = b(t) = \frac{0.1}{1-t}$, and $N(x_1, x_2) = |x_1| + |x_2|$. Then $B_1 = B_2 = 1$, where $B_i = \lim_{z\to 0^+} \frac{\Lambda_i(z)}{z}$. Note that $(1-t)^{\alpha-1}a_i(t) \in L^1(K_j)$ for all compact subsets $K_j \in (t_j, t_{j+1})$ $(j = 0, 1)$, $\theta_q := 1 - \sum_{i=1}^k \frac{\lambda_i \gamma_i^{q_i}}{\Gamma(q_i+1)} = 1 - \frac{(\frac{1}{2})^{\frac{5}{2}}}{\Gamma(\frac{7}{2})} = 1 - \frac{2}{15\sqrt{2\pi}}$, $\theta_{q+1} := 1 - \sum_{i=1}^k \frac{\lambda_i \gamma_i^{q_i+1}}{\Gamma(q_i+2)} = 1 - \frac{(\frac{1}{2})^{\frac{7}{2}}}{\Gamma(\frac{9}{2})} = 1 - \frac{2}{105\sqrt{2\pi}}$,

$$\Delta := \max\left\{1, \frac{1}{\Gamma(p_1+1)}, \dots, \frac{1}{\Gamma(p_m+1)}\right\} = \max\left\{1, \frac{1}{\Gamma(\frac{3}{2})}\right\} = \frac{2}{\sqrt{\pi}},$$

$\|\hat{b}_{i,n}\| = \|\hat{a}_{j',i,n}\| \le \int_{\frac{1}{n}}^{1-\frac{1}{n}} (1-s)^{\frac{1}{2}} \frac{0.1}{1-s}\,ds = 0.2,$

$$\big|f(t, x_1, \dots, x_{m+1})\big| \le b(t)N(x_1, \dots, x_{m+1}),$$

$N(x_1, x_2) = |x_1| + |x_2|$, $\eta := \lim_{z\to 0^+} \frac{N(z,z)}{z} = 2 \in [0, \infty)$,

$$\begin{aligned} &\Delta \sum_{i=0}^{r-1} \left(\left[\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha + q_j)} + \frac{2(\theta_{q+1} - \mu + 1)}{\theta_q \Gamma(\alpha+1)} \right.\right. \\ &\quad \left.\left. + \frac{1}{\theta_q \Gamma(\alpha)} \right] \sum_{j'=1}^{m+1} B_{j'} \|\hat{a}_{j',i,n}\| \right) \frac{2}{\sqrt{\pi}} \left(\left[\frac{1}{\Gamma(\frac{3}{2})} + \frac{1}{\theta_q \Gamma(\frac{7}{2})} \right.\right. \\ &\quad \left.\left. + \frac{2(\theta_{q+1} - \frac{1}{3} + 1)}{\theta_q \Gamma(\alpha+1)} + \frac{1}{\theta_q \Gamma(\alpha)} \right] \times 0.2 \right) < 1, \end{aligned}$$

and $\eta \sum_{i=0}^{r-1} \|\hat{b}_{i,n}\| [\frac{1}{\Gamma(\alpha)} + \sum_{j=1}^{k} \frac{\lambda_j}{\theta_q \Gamma(\alpha+q_j)} + \frac{2(\theta_{q+1}-\mu+1)}{\theta_q \Gamma(\alpha+1)} + \frac{1}{\theta_q \Gamma(\alpha)}] \in [0, \frac{1}{\Delta})$. Now, by using Theorem (5), we conclude that problem (6) has a solution.

3 Conclusion

There are some phenomena that can be modeled by fractional differential equations. But most singular fractional differential equations studied by researchers have simple singularity. In this work, by providing a new technique we review a strong singular fractional differential equation under some boundary value conditions.

Acknowledgements

The second and third authors were supported by Azarbaijan Shahid Madani University. Also, the fourth author was supported by Islamic Azad University, Mehran Branch. The authors express their gratitude dear unknown referees for their helpful suggestions which improved the final version of this paper.

Funding

Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Cankaya University, Ogretmenler Cad. 14 06530, Ankara, Turkey. [2]Institute of Space Sciences, P.O. Box, MG-23, R 76900, Magurele, Bucharest, Romania. [3]Institute of Research and Development, Duy Tan University, Da Nang 550000, Vietnam. [4]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan. [5]Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran. [6]Department of Mathematics, Islamic Azad University, Mehran Branch, Mehran, Ilam, Iran.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 14 April 2020 Accepted: 1 July 2020 Published online: 11 July 2020

References

1. Ntouyas, S.K., Samei, M.E.: Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus. Adv. Differ. Equ. **2019**, 475 (2019). https://doi.org/10.1186/s13662-019-2414-8
2. Liang, S., Samei, M.E.: New approach to solutions of a class of singular fractional q-differential problem via quantum calculus. Adv. Differ. Equ. **2020**, 14 (2020). https://doi.org/10.1186/s13662-019-2489-2
3. Samei, M.E.: Existence of solutions for a system of singular sum fractional q-differential equations via quantum calculus. Adv. Differ. Equ. **2020**, 23 (2020). https://doi.org/10.1186/s13662-019-2480-y
4. Samei, M.E., Yang, W.: Existence of solutions for k-dimensional system of multi-term fractional q-integro-differential equations under anti-periodic boundary conditions via quantum calculus. Math. Methods Appl. Sci. **43**(7), 4360–4382 (2020). https://doi.org/10.1002/mma.6198
5. Etemad, S., Rezapour, S., Samei, M.E.: α-ψ-contractions and solutions of a q-fractional differential inclusion with three-point boundary value conditions via computational results. Adv. Differ. Equ. **2020**, 218 (2020). https://doi.org/10.1186/s13662-020-02679-w
6. Rezapour, S., Samei, M.E.: On the existence of solutions for a multi-singular pointwise defined fractional q-integro-differential equation. Bound. Value Probl. **2020**, 38 (2020). https://doi.org/10.1186/s13661-020-01342-3
7. Hedayati, V., Samei, M.E.: Positive solutions of fractional differential equation with two pieces in chain interval and simultaneous Dirichlet boundary conditions. Bound. Value Probl. **2019**, 141 (2019). https://doi.org/10.1186/s13661-019-1251-8
8. Jleli, M., Karapinar, E., Samet, B.: Positive solutions for multi-points boundary value problems for singular fractional differential equations. J. Appl. Math. **2014**, Article ID 596123 (2014). https://doi.org/10.1186/s13662-020-02614-z
9. Aydogan, M.S., Baleanu, D., Mousalou, A., Rezapour, S.: On high order fractional integro-differential equations including the Caputo–Fabrizio derivative. Bound. Value Probl. **2018**, 90 (2018). https://doi.org/10.1186/s13661-018-1008-9
10. Baleanu, D., Rezapour, S., Mohammadi, H.: Some existence results on nonlinear fractional differential equations. Philos. Trans. A Math. Phys. Eng. Sci. **371**, 20120144 (2013). https://doi.org/10.1098/rsta.2012.0144
11. Baleanu, D., Mousalou, A., Rezapour, S.: The extended fractional Caputo–Fabrizio derivative of order $0 \leq \sigma < 1$ on $c_{\mathbb{R}}[0,1]$ and the existence of solutions for two higher-order series-type differential equations. Adv. Differ. Equ. **2018**, 255 (2018). https://doi.org/10.1186/s13662-018-1696-6
12. Baleanu, D., Mousalou, A., Rezapour, S.: On the existence of solutions for some infinite coefficient-symmetric Caputo–Fabrizio fractional integro-differential equations. Bound. Value Probl. **2017**, 145 (2017). https://doi.org/10.1186/s13661-017-0867-9
13. Baleanu, D., Rezapour, S., Saberpour, Z.: On fractional integro-differential inclusions via the extended fractional Caputo–Fabrizio derivation. Bound. Value Probl. **2019**, 79 (2019). https://doi.org/10.1186/s13661-019-1194-0
14. Kojabad, E.A., Rezapour, S.: Approximate solutions of a sum-type fractional integro-differential equation by using Chebyshev and Legendre polynomials. Adv. Differ. Equ. **2017**, 351 (2017). https://doi.org/10.1186/s13662-017-1404-y
15. Aydogan, S.M., Baleanu, D., Mousalou, A., Rezapour, S.: On approximate solutions for two higher-order Caputo–Fabrizio fractional integro-differential equations. Adv. Differ. Equ. **2017**, 221 (2017). https://doi.org/10.1186/s13662-017-1258-3
16. Baleanu, D., Mousalou, A., Rezapour, S.: A new method for investigating approximate solutions of some fractional integro-differential equations involving the Caputo–Fabrizio derivative. Adv. Differ. Equ. **2017**, 51 (2017). https://doi.org/10.1186/s13662-017-1088-3
17. Etemad, S., Rezapour, S., Samei, M.E.: On fractional hybrid and non-hybrid multi-term integro-differential inclusions with three-point integral hybrid boundary conditions. Adv. Differ. Equ. **2020**, 161 (2020). https://doi.org/10.1186/s13662-020-02627-8

18. Baleanu, D., Etemad, S., Rezapour, S.: On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators. Alex. Eng. J. (2020). https://doi.org/10.1016/j.aej.2020.04.053
19. Alizadeh, S., Baleanu, D., Rezapour, S.: Analyzing transient response of the parallel RCL circuit by using the Caputo–Fabrizio fractional derivative. Adv. Differ. Equ. **2020**, 55 (2020). https://doi.org/10.1186/s13662-020-2527-0
20. Baleanu, D., Jajarmi, A., Mohammadi, H., Rezapour, S.: A new study on the mathematical modeling of human liver with Caputo–Fabrizio fractional derivative. Chaos Solitons Fractals **134**, 109705 (2020). https://doi.org/10.1016/j.chaos.2020.109705
21. Baleanu, D., Mohammadi, H., Rezapour, S.: Analysis of the model of HIV-1 infection of CD4+ T-cell with a new approach of fractional derivative. Adv. Differ. Equ. **2020**, 71 (2020). https://doi.org/10.1186/s13662-020-02544-w
22. Baleanu, D., Etemad, S., Rezapour, S.: A hybrid Caputo fractional modeling for thermostat with hybrid boundary value conditions. Bound. Value Probl. **2020**, 64 (2020). https://doi.org/10.1186/s13661-020-01361-0
23. Baleanu, D., Mohammadi, H., Rezapour, S.: Mathematical theoretical study of a particular system of Caputo–Fabrizio fractional differential equations for the rubella disease model. Adv. Differ. Equ. **2020**, 184 (2020). https://doi.org/10.1186/s13662-020-02614-z
24. Almeida, R., Bastos, B.R.O., Monteiro, M.T.T.: Modeling some real phenomena by fractional differential equations. Math. Methods Appl. Sci. **39**(16), 4846–4855 (2016). https://doi.org/10.1002/mma.3818
25. Baleanu, D., Ghafarnezhad, K., Rezapour, S., Shabibi, M.: On the existence of solutions of a three steps crisis integro-differential equation. Adv. Differ. Equ. **2018**, 135 (2018). https://doi.org/10.1186/s13662-018-1583-1
26. Baleanu, D., Ghafarnezhad, K., Rezapour, S.: On a three steps crisis integro-differential equation. Adv. Differ. Equ. **2019**, 153 (2019). https://doi.org/10.1186/s13662-019-2088-2
27. Shabibi, M., Rezapour, S., Vaezpour, S.M.: A singular fractional integro- differential equation. UPB Sci. Bull. Ser. A **79**(1), 109–118 (2017)
28. Shabibi, M., Postolache, M., Rezapour, S., Vaezpour, S.M.: Investigation of a multi-singular pointwise defined fractional integro-differential equation. J. Math. Anal. **7**(5), 61–77 (2016)
29. Shabibi, M., Postolache, M., Rezapour, S.: Positive solutions for a singular sum fractional differential system. Int. J. Anal. Appl. **13**(1), 108–118 (2016)
30. Talaee, M., Shabibi, M., Gilani, A., Rezapour, S.: On the existence of solutions for a pointwise defined multi-singular integro-differential equation with integral boundary condition. Adv. Differ. Equ. **2020**, 41 (2020). https://doi.org/10.1186/s13662-020-2517-2
31. Gu, Y., Gao, H., Wang, H., Zhang, G.: A general algorithm for evaluating nearly strong-singular (and beyond) integrals in three-dimensional boundary element analysis. Comput. Mech. **59**, 779–793 (2017)
32. Wang, L., Cheng, K., Zhang, B.: A uniqueness result for strong singular Kirchhoff-type fractional Laplacian problems. Appl. Math. Optim. (2019). https://doi.org/10.1007/s00245-019-09612-y
33. Wei, L., Du, Y.: Positive solutions of elliptic equations with a strong singular potential. Bull. Lond. Math. Soc. **51**(2), 251–266 (2019)
34. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
35. Samet, B., Vetro, C., Vetro, P.: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal. **75**, 2154–2165 (2012)
36. Tatar, N.: An impulsive nonlinear singular version of the Gronwall–Bihari inequality. J. Inequal. Appl. **2006**, Article ID 84561 (2006).