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A novel analytical technique to obtain the solitary solutions for nonlinear evolution equation of fractional order

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Abstract

We investigate some solitary wave results of time fractional evolution equations. By employing the extended rational $\exp(-\frac{\psi'}{\psi})(\eta)$ -expansion method, a few different results including kink, singular-kink, multiple soliton, and periodic wave solutions are formally generated. It is worth mentioning that the solutions obtained are more general with more parameters. The exact solutions are constructed in the form of exponential, trigonometric, rational, and hyperbolic functions. With the choice of proper values of parameters, graphs to some of the obtained solutions are drawn. On comparing some special cases, our solutions are in good agreement with the results published previously and the remaining are new.

MSC: 35R11; 26A33; 74J35; 35C08

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1 Introduction

In the exceptional development of nonlinear sciences and engineering, during the last few decades, many researchers seem to be interested in obtaining exact and numerical solutions for nonlinear partial differential (NLPD) equations. One used the modified variational iteration algorithm-II [1], the direct algebraic method [2], the extended trial equation method [3], the enhanced $(\frac{G'}{G})$ -expansion method [4], the GERFM method [5, 6] using generalized fractional integral conditions [7], applying the non-compact measure and Monch's theorem [8–10], the modified simple equation [11], the function transformation method [12], the extended modified mapping method [13], the function transformation method [14] and the reductive perturbation method [15]. The study of exact solutions of nonlinear evolution equations plays a major role to explore the internal mechanism of nonlinear phenomena [3, 13]. Fractional calculus is a dominant tool in several nonlinear fields such as plasma physics, fluid mechanics, solid-state physics, optical fibers, quantum field theory, biophysics, chemical kinematics, electricity, chemistry, biology, geochemistry,

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propagation of shallow water waves and engineering [7, 10, 16]. For this purpose many techniques were used such as the homogeneous balance method [17], the exp-function method [18], the improved extended F-expansion method [19], and the homotopy perturbation method [20]. The Camassa–Holm (CH) equation is

$$u_t + 2ku_x - u_{xxt} + \beta uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1}$$

The above equation gained the fame as a model describing the unidirectional propagation of shallow water waves over a flat bottom [21] and in this perception, if β is positive, the solitary solutions are smooth solitons. In Eq. (1) k and β are some parameters, $u(x, t)$ is the dependent variable, while x and t are the spatial and temporal variables. It is also considered as an equation with a bi-Hamiltonian structure, particularly, when $\beta = 0$, the above stated equation has a peakon type solution. So, the CH equation has some other peakon solutions and smooth solutions. Song and Tian investigated the modified Camassa–Holm (MCHE) equation in [22],

$$u_t + 2ku_x - u_{xxt} + \beta u^n u_x = 2u_x u_{xx} + uu_{xxx}. \tag{2}$$

Further, one assumed that the soliton changes with $\eta = x - Vt$, where V is the speed of wave propagation and t is the temporal variable. For $V > 0$, the wave moves in the positive x -direction, whereas for $V < 0$, the wave moves in the negative x -direction, then the terms $2u_x u_{xx} + uu_{xxx}$ of Eq. (2) will be very small and the above equation becomes

$$u_t + 2ku_x - u_{xxt} + \beta uu_x = 0. \tag{3}$$

Consider Eq. (3), the modified Camassa–Holm (MCH) equation [22], which is known as the simplified modified Camassa–Holm (SMCH) equation

$$u_t + 2ku_x - u_{xxt} + \beta u^n u_x = 0. \tag{4}$$

In this paper, we consider $n = 2$,

$$u_t + 2ku_x - u_{xxt} + \beta u^2 u_x = 0. \tag{5}$$

With the help of He’s semi-inverse method [23] we have obtained the analytical solutions of the CH equation and the SMCH equation, Abbasbandy via the homotopy method [24] obtained traveling wave solutions of the SMCH equation. Recently many solitary solutions of SMCH equation were obtained [25, 26]. Here, we utilize the extended rational $\exp(-\frac{\psi'}{\psi})(\eta)$ -expansion method for solving the simplified modified Camassa–Holm (SMCH) equation.

2 Description of method

We will explain the extended rational $\exp(-\frac{\psi'}{\psi})(\eta)$ -expansion method for establishing solitary solutions. Suppose the fractional PDE is

$$\psi(u, D_t^\alpha u, D_t^{2\alpha} u, D_t^\alpha u_x, u_x, u_{xx}, \dots) = 0, \quad 0 \leq \alpha \leq 1, t > 0, \tag{6}$$

where $D_t^\alpha u, D_x^\alpha u, D_{xx}^\alpha u$, are fractional derivatives, u is any unknown function and ψ is a polynomial involving u and its different derivatives. So as to solve (6), the steps are as follows.

Step I: We assume the following equation:

$$\eta = x \pm V \frac{t^\alpha}{\Gamma(\alpha + 1)}, \tag{7}$$

where V is the speed of the wave propagation, $u = u(\eta)$.

Equations (6) and (7) yield the ODE

$$\psi(u, \pm Vu', ku', k^2 u'', V^2 u''', \dots) = 0. \tag{8}$$

Step II: Suppose the solution of Eq. (8) is expressed as

$$u(\eta) = \frac{a[0] + \frac{d}{d\eta} (\sum_{n=1}^M (a_n (e^{\psi(\eta)})^n))}{\sum_{n=0}^M (b_n (e^{\psi(\eta)})^n)}, \tag{9}$$

$$\psi'(\eta) = \mu \exp(\psi(\eta)) + \exp(-\psi(\eta)) + \lambda, \tag{10}$$

while, a_n, b_n are arbitrary constants and $\psi(\eta), a_n \neq 0, b_n \neq 0, \psi(\eta)$ satisfies Eq. (1), we got solutions from Eq. (10) as follows.

Family I: When $\lambda^2 - 4\mu > 0$,

$$\psi(\eta) = \ln \left\{ \left(-\sqrt{\lambda^2 - 4\mu} \operatorname{Tanh} \left((\eta + c) \frac{\sqrt{\lambda^2 - 4\mu}}{2} \right) - \lambda \right) \frac{1}{2\mu} \right\}.$$

Family II: When $-4\mu + \lambda^2 < 0$,

$$\psi(\eta) = \ln \left\{ \left(\sqrt{-4\mu + \lambda^2} \operatorname{Tan} \left((\eta + c) \frac{\sqrt{\lambda^2 - 4\mu}}{2} \right) - \lambda \right) \frac{1}{2\mu} \right\}.$$

Family III: When $\lambda \neq 0, \mu = 0$ and $\lambda^2 - 4\mu > 0$,

$$\psi(\eta) = -\ln \left\{ \frac{\lambda}{e^{\lambda(\eta+k_1)} - 1} \right\}.$$

Family IV: When $\mu \neq 0, \lambda \neq 0$ and $\lambda^2 - 4\mu = 0$,

$$\psi(\eta) = \ln \left\{ \frac{(2\lambda(\eta + k_1) + 4)}{(\lambda^2(\eta + k_1))} \right\}.$$

Family V: When $\mu = 0, \lambda^2 - 4\mu = 0$ and $\lambda = 0$,

$$\varphi(\eta) = \ln(\eta + k_1).$$

Step III: Exploring the value of M from Eq. (8). From Eq. (8) and Eq. (10), we have a polynomial of $e^{M\varphi(\eta)}$. By comparing, we get a system of equations with V, λ, μ, a_n , and putting in Eq. (9) all cases of Eq. (10), we obtain traveling wave results of Eq. (6).

3 Caputo’s fractional derivative

Definition 3.1 ([27]) A real function, $h(x)$, $x > 0$, is supposed to be in space C_α if there exists any real number $p (> \alpha)$, such that

$$h(x) = x^p h_1(x), \quad \text{where } h_1(x) \in C[0, \infty).$$

Definition 3.2 ([27]) A real function $h(x)$, $x > 0$, is assumed to be in space if any real number C_α^m , $m \in \mathbb{N} \cup \{0\}$, if $h^{(m)} \in C_\alpha$.

Definition 3.3 ([27]) Let $h \in C_\alpha$ and $\alpha \geq -1$, then for the Riemann–Liouville (R–L) (left-sided) integral of left-sided order μ , $\mu > 0$, we get

$$I_t^\mu h(x, t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - T)^{\mu-1} h(x, T) dT, \quad t > 0,$$

where $\Gamma(\mu)$ is the Gamma function.

Definition 3.4 ([27]) The Caputo fractional derivative of the left-sided function $h(x)$ with respect to t , $h \in C_{-1}^m$, $m \in \mathbb{N} \cup \{0\}$, is given as

$$\begin{aligned} D_t^\mu h(x, t) &= \frac{\partial^m}{\partial t^m} h(x, t), \quad \mu = m \\ &= I_t^{m-\mu} \frac{\partial^m}{\partial t^m} h(x, t), \quad m - 1 \leq \mu < m, m \in \mathbb{N}, \\ I_t^\mu D_t^\mu h(x, t) &= h(x, t) - \sum_{k=0}^{m-1} \frac{\partial^k h}{\partial t^k}(x, 0) \frac{t^k}{k!}, \quad m - 1 < \mu \leq m, m \in \mathbb{N}, \\ I_t^\mu t^\nu &= \frac{\Gamma(\nu + 1)}{\Gamma(\mu + \nu + 1)} t^{\mu+\nu}. \end{aligned}$$

4 The simplified modified Camassa–Holm (SMCH) equation

Consider the simplified modified Camassa–Holm (SMCH) equation

$$D_t^\alpha u + 2ku_x - u_{xxt} + \beta u^2 u_x = 0, \tag{11}$$

where $k \in \mathfrak{R}$ and $\beta > 0$ are parameters. We obtain

$$\eta = x - V \frac{t^\alpha}{\Gamma(\alpha + 1)}, \quad u(x, t) = u(\eta),$$

we can convert Eq. (11) into an ODE,

$$(2k - V)u - Vu'' + \frac{1}{3}\beta u^3 + P = 0, \tag{12}$$

therefore the trial solution of suggested algorithm is

$$u(\eta) = \frac{\frac{d}{d\eta} (\sum_{n=1}^M (a_n (e^{Y(\eta)})^n)) + a[0]}{\sum_{n=0}^M (b_n (e^{Y(\eta)})^n)}. \tag{13}$$

From (10), (12) and (13), we get

$$H_0 + H_1 e^{\varphi(\eta)} + H_2 e^{2\varphi(\eta)} + H_3 e^{3\varphi(\eta)} + H_4 e^{4\varphi(\eta)} + H_5 e^{5\varphi(\eta)} + H_6 e^{6\varphi(\eta)} = 0, \tag{14}$$

where $H_1, H_2, H_3, H_4, H_5, H_6$ are given as follows:

$$\begin{aligned} H_1 &= 3V\lambda^3 a_1 b_0^2 - 3V\lambda^2 a_0 b_0 b_1 - 12V\lambda^2 a_1 b_0 b_1, \\ H_2 &= -3V\lambda^3 a_1 b_0 b_1 + 21V\lambda^2 \mu a_1 b_0^2 + 3V\lambda^2 a_0 b_1^2, \\ H_3 &= 6V\lambda^2 \mu a_1 b_0 b_1 + 36V\mu^2 \lambda a_1 b_0^2 + 3V\lambda^2 \mu a_0 b_1^2, \\ H_4 &= 3V\lambda^2 \mu a_1 b_1^2 + 27V\mu^2 \lambda a_1 b_0 b_1 + 18V\mu^3 \mu a_1 b_0^2, \\ H_5 &= 9V\mu^2 \lambda a_1 b_1^2 + 18V\mu^3 a_1 b_0 b_1 + 3\beta \mu^2 \lambda a_1^3, \\ H_6 &= 6V\mu^3 a_1 b_1^2 + \beta \mu^3 a_1^3. \end{aligned}$$

We obtained the coefficients of algebraic equations:

$$[H_0 = 0, H_1 = 0, H_2 = 0, H_3 = 0, H_4 = 0, H_5 = 0, H_6 = 0]. \tag{15}$$

Solving these algebraic equations by using Maple, we obtain the solution sets, and by choosing some specific cases we get the following.

Solution 1

$$\begin{aligned} a_0 &= \frac{a_1(\lambda b_0 - b_1)}{b_1}, & P &= -\frac{\lambda a_1(-\beta \lambda^2 a_1^2 + 3Vb_1^2 - 6kb_1^2)}{3b_1^3}, \\ \lambda &= \lambda, & b_0 &= b_0, & b_1 &= b_1, & a_1 &= a_1, & \beta &= \beta, & \mu &= 0. \end{aligned}$$

Case III: Whenever $\mu = 0$ and $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$u_1(\eta) = \frac{a_1(\lambda b_0 - b_1)\lambda}{b_1(b_1 e^{-\lambda(\frac{Vt^\alpha}{\Gamma(\alpha+1)} - x)} + \lambda b_0 - b_1)}.$$

Case IV: When, $\lambda \neq 0$, $\lambda^2 - 4\mu = 0$ and $\mu \neq 0$,

$$u_2(\eta) = \frac{a_1(\lambda b_0 - b_1)\lambda^2(\frac{Vt^\alpha}{\Gamma(\alpha+1)} - x)}{b_1(\lambda^2 t b_0 \frac{Vt^\alpha}{\Gamma(\alpha+1)} - 2\lambda b_1 \frac{Vt^\alpha}{\Gamma(\alpha+1)} - \lambda^2 x b_0 + 2\lambda x b_1 + 2b_1)}.$$

Case V: If $\mu = 0$, and $\lambda = 0$,

$$u_3(\eta) = \frac{a_1(\lambda b_0 - b_1)}{b_1(b_1 \frac{Vt^\alpha}{\Gamma(\alpha+1)} - x b_1 - b_0)}.$$

Solution 2

$$b_1 = b_1, \quad a_1 = 0, \quad a_0 = a_0, \quad P = 0, \quad \beta = 0,$$

$$\lambda = \frac{\sqrt{V(V-2k)}}{V}, \quad b_0 = \frac{b_1\sqrt{V(V-2k)}}{V-2k}, \quad \mu = 0.$$

Case III: Whenever $\mu = 0$ and $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$u_4(\eta) = \frac{\sqrt{V(V-2k)}a_0e^{\frac{(\frac{Vt^\alpha}{\Gamma(\alpha+1)}-x)\sqrt{V(V-2k)}}{V}}}{Vb_1}.$$

Case IV: When $\lambda \neq 0$, $\lambda^2 - 4\mu = 0$, and $\mu \neq 0$,

$$u_5(\eta) = -\frac{a_0(V-2k)(\frac{Vt^\alpha}{\Gamma(\alpha+1)}-x)}{b_1(\sqrt{V(V-2k)}\frac{Vt^\alpha}{\Gamma(\alpha+1)}-\sqrt{V(V-2k)}x-2V)}.$$

Case V: While $\mu = 0$, and $\lambda = 0$,

$$u_6(\eta) = -\frac{a_0(V-2k)}{b_1(2k\frac{Vt^\alpha}{\Gamma(\alpha+1)}+Vx-\frac{V^2t^\alpha}{\Gamma(\alpha+1)}-2kx+\sqrt{V(V-2k)})}.$$

Solution 3

$$b_1 = b_1, \quad a_1 = 0, \quad a_0 = a_0, \quad b_0 = -\frac{b_1\sqrt{V(V-2k)}}{V-2k},$$

$$P = 0, \quad \lambda = -\frac{\sqrt{V(V-2k)}}{V}, \quad \beta = 0, \quad \mu = 0.$$

Case III: Whenever $\mu = 0$ and $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$u_7(\eta) = -\frac{a_0e^{-\frac{\sqrt{V(V-2k)}(\frac{Vt^\alpha}{\Gamma(\alpha+1)}-x)}{V}}(\sqrt{V(V-2k)})}{Vb_1}.$$

Case IV: When $\mu \neq 0$, and $\lambda \neq 0$, $\lambda^2 - 4\mu = 0$,

$$u_8(\eta) = \frac{a_0(V-2k)(\frac{Vt^\alpha}{\Gamma(\alpha+1)}-x)}{b_1(\sqrt{V(V-2k)}\frac{Vt^\alpha}{\Gamma(\alpha+1)}-\sqrt{V(V-2k)}x+2V)}.$$

Case V: When $\mu = 0$, and $\lambda = 0$,

$$u_9(\eta) = -\frac{a_0(V-2k)}{b_1(-Vx+2kx+\frac{V^2t^\alpha}{\Gamma(\alpha+1)}+\sqrt{V(V-2k)})-2k\frac{Vt^\alpha}{\Gamma(\alpha+1)}}.$$

Solution 4

$$b_1 = b_1, \quad a_1 = 0, \quad a_0 = a_0, \quad \lambda = \lambda, \quad b_0 = \frac{2\lambda b_1 V}{V\lambda^2 + 2V - 4k},$$

$$P = 0, \quad \beta = -\frac{24V(V^2 - 4kV + 4k^4)b_1^2}{(V\lambda^2 + 2V - 4k)^2 a_0^2}, \quad \mu = \frac{V\lambda^2 + 2V - 4k}{4V}.$$

Case II: Whenever $\mu \neq 0$, and $\lambda^2 - 4\mu < 0$,

$$u_{10}(\eta) = -\frac{a_0(V\lambda^2 + 2V - 4k)\sqrt{2}}{4b_1 V \tan(\frac{1}{2}(\frac{Vt^\alpha}{\Gamma(\alpha+1)}-x)\sqrt{2}\sqrt{\frac{V-2k}{V}}\sqrt{\frac{V-2k}{V}})}.$$

Case III: While $\lambda \neq 0$, and $\mu = 0$,

$$u_{11}(\eta) = -\frac{(V\lambda^2 + 2V - 4k)\lambda a_0}{b_1(e^{-\lambda(\frac{Vt^\alpha}{\Gamma(\alpha+1)} - x)} V\lambda^2 + V\lambda^2 + 2e^{-\lambda(\frac{Vt^\alpha}{\Gamma(\alpha+1)} - x)} V - 4e^{-\lambda(\frac{Vt^\alpha}{\Gamma(\alpha+1)} - x)} k - 2V + 4k)}$$

Case IV: When $\mu \neq 0$, $\lambda^2 - 4\mu = 0$, and $\lambda \neq 0$,

$$u_{12}(\eta) = -\frac{a_0(V\lambda^2 + 2V - 4k)\lambda^2(\frac{Vt^\alpha}{\Gamma(\alpha+1)} - x)}{2b_1(-V\lambda^2 - 2\lambda V - 4\lambda k\frac{Vt^\alpha}{\Gamma(\alpha+1)} - 4xk\lambda - 2V + 4k + 2\lambda^2\frac{Vt^\alpha}{\Gamma(\alpha+1)})}$$

Case V: When $\mu = 0$, and $\lambda = 0$,

$$u_{13}(\eta) = -\frac{(V\lambda^2 + 2V - 4k)a_0}{b_1(-xV\lambda^2 - 2V\lambda - 4k\frac{Vt^\alpha}{\Gamma(\alpha+1)} - 2Vx + 2\lambda^2\frac{t^\alpha}{\Gamma(\alpha+1)} + 4kx + \lambda^2\frac{V^2t^\alpha}{\Gamma(\alpha+1)})}$$

Solution 5

$$b_1 = \frac{b_0(V\lambda + \sqrt{V^2 - 2kV})}{2V}, \quad P = -\frac{a_0(\lambda(V\lambda + \sqrt{V^2 - 2kV}) - V\lambda^2 - V + 2k)}{2b_0}$$

$$\lambda = \lambda, \quad b_0 = b_0, \quad \beta = 0, \quad \mu = \frac{V\lambda^2 - V + 2k}{4V}, \quad a_1 = 0, \quad a_0 = a_0.$$

Case I: Whenever $\mu \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$u_{14}(\eta) = -\frac{a_0(V\lambda^2 - V + 2k)}{b_0\sqrt{\frac{V-2k}{V}} \tan(\frac{1}{2}(\frac{-x+Vt^\alpha}{\Gamma(\alpha+1)}))\sqrt{\frac{V-2k}{V}} V\lambda + \sqrt{V(V-2k)}\sqrt{-\frac{V-2k}{V}} \tan(\frac{1}{2}\sqrt{\frac{V-2k}{V}}(\frac{-x+Vt^\alpha}{\Gamma(\alpha+1)})) - \lambda\sqrt{V(V-2k)} - V + 2k}$$

Case II: If $\mu \neq 0$, and $\lambda^2 - 4\mu < 0$,

$$u_{15}(\eta) = -\frac{a_0(V\lambda^2 - V + 2k)}{b_0\sqrt{-\frac{V-2k}{V}} \tan(\frac{1}{2}(\frac{-x+Vt^\alpha}{\Gamma(\alpha+1)}))\sqrt{-\frac{V-2k}{V}} V\lambda + \sqrt{V(V-2k)}\sqrt{-\frac{V-2k}{V}} \tan(\frac{1}{2}\sqrt{-\frac{V-2k}{V}}(\frac{-x+Vt^\alpha}{\Gamma(\alpha+1)})) + \lambda\sqrt{V(V-2k)} + V - 2k}$$

Case III: Whenever $\mu = 0$ and $\lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$u_{16}(\eta) = \frac{2a_0V\lambda}{b_1(V\lambda e^{-\lambda(\frac{-x+Vt^\alpha}{\Gamma(\alpha+1)})} + \sqrt{V(V-2k)}e^{-\lambda(\frac{-x+Vt^\alpha}{\Gamma(\alpha+1)})} + V\lambda - \sqrt{V(V-2k)})}$$

Case IV: When $\mu \neq 0$, $\lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$u_{17}(\eta) = -\frac{a_0V\lambda^2(-x + \frac{Vt^\alpha}{\Gamma(\alpha+1)})}{b_1(\sqrt{(V-2k)}V\lambda\frac{Vt^\alpha}{\Gamma(\alpha+1)} - \sqrt{V(V-2k)}\lambda x - V\lambda - \sqrt{V(V-2k)})}$$

Case V: Whenever $\mu = 0$, and $\lambda = 0$,

$$u_{18}(\eta) = -\frac{2a_0V}{b_0(\lambda^2\frac{Vt^\alpha}{\Gamma(\alpha+1)} + \sqrt{V(V-2k)}\frac{Vt^\alpha}{\Gamma(\alpha+1)} - V\lambda x - 2V - \sqrt{V(V-2k)}x)}$$

Solution 6

$$\lambda = \frac{\sqrt{2}\sqrt{V(2V\mu - V + 2k)}}{V}, \quad b_0 = \frac{\sqrt{2}b_1\sqrt{V(2V\mu - V + 2k)}}{2\mu V},$$

$$b_1 = b_1, \quad a_1 = a_1, \quad a_0 = \frac{-(V - 2k)}{2V\mu}, \quad P = 0, \quad \beta = -\frac{6Vb_1^2}{a_1^2}, \quad \mu = \mu.$$

Case I: If $\mu \neq 0$ and $\lambda^2 - 4\mu > 0$,

$$u_{19}(\eta) = -\frac{(V - 2k)a_1}{Vb_1 \tanh\left(\frac{1}{2}\left(\frac{-x + Vt^\alpha}{\Gamma(\alpha + 1)}\right)\sqrt{-\frac{2V - 4k}{V}}\sqrt{-\frac{2V - 4k}{V}}\right)}.$$

Case II: If $\mu \neq 0$ and $\lambda^2 - 4\mu < 0$,

$$u_{20}(\eta) = \frac{(V - 2k)a_1\sqrt{2}}{2Vb_1\sqrt{\frac{V - 2k}{V}} \tan\left(\frac{1}{2}\left(\frac{-x + Vt^\alpha}{\Gamma(\alpha + 1)} - x\right)\sqrt{2}\sqrt{\frac{V - 2k}{V}}\right)}.$$

5 Analogy and analysis

Several authors have been discussed the simplified modified Camassa–Holm (SMCH) equation by using different techniques for finding exact traveling wave results. Particularly, Liu et al. [22] used the (G'/G) -expansion method, Najafi et al. [23], used He’s semi-inverse method applied $\exp(-\varphi(\eta))$ -expansion method and Redi et al. [19] applied an improved (G'/G) -expansion method; Gundogdu et al. [28] applied the elliptic function expansion method to get the traveling wave solutions. Akber et al. [26] obtained solutions $u_1(\phi)$, $u_3(\phi)$, $u_6(\phi)$, $u_7(\phi)$ and $u_8(\phi)$ that are equivalent to our solutions $u_{10}(\eta)$, $u_{19}(\eta)$ and $u_{20}(\eta)$. For instance, Akber et al. [26] obtained solutions $u_1(\phi)$ with suitable values of $m_2 = 0, d = 0, B = 0, \sqrt{\Omega} = 1, A = 1$ matching with our solution $u_{19}(\eta)$ (see Table 1). In the same manner Akber et al. [26] obtained solutions $u_3(\phi)$ with suitable values of $m_2 = 0, d = 0, \sqrt{\Delta} = 1, A = 2$ matching with our obtained solution $u_{10}(\eta)$ (see Table 1). Mohyud-Din et al. [29] obtained solutions $u_4(\eta)$ and $u_{10}(\eta)$, for $k = 1, b_0 = \frac{1}{2}, \beta = -1, \eta = 0, c_1 = 1$ that are equivalent to our obtained solution $u_1(\eta)$ for $\lambda = 1, \mu = -1, k = 1, a_1 = 2\sqrt{3}, b_0 = 2, b_1 = 1, t = 0, x = 0$. Similarly for different values of Mohyud-Din et al. [29] obtained solutions $u_2(\eta)$, $u_4(\eta)$, $u_8(\eta)$, $u_9(\eta)$, $u_{18}(\eta)$ and $u_{23}(\eta)$ that are equivalent to our solutions $u_1(\eta)$, $u_3(\eta)$ and $u_7(\eta)$ (see Table 2). In Table 3, our solutions and those of Lu et al. [25] are compared. For instance, Lu et al. [25] addressed solution u_4 for the particular choices of $\beta_1 = 1, \sqrt{\beta_1} = 5, \sqrt{\beta_3} = 1/2, \epsilon = 1, k = -1, \omega = 1$ and $\xi_0 = 0$ is equivalent to our obtained solution $u_{19}(\eta)$ (see Table 3). The rest of the solutions are new. Hence, our method produces more solutions than other methods existing in the current literature. In our work all the results obtained have been verified through Maple to be found to be correct.

6 Physical interpretation

Now we provide graphical representations of some results with different parametric values. Figure 1 shows the periodic wave profile of $u_1(\eta)$ for $\alpha = 0.2, a_1 = 11, \lambda = 1, V = 12, b_0 = 1, b_1 = 12$. Figure 2 represents the multiple soliton solution of $u_3(\eta)$ for $\alpha = 0.25, a_1 = 11, V = 12, b_1 = 11, \mu = 3$, while Fig. 3 displays the profile of solitary wave of $u_4(\eta)$ for the choice of $\alpha = 0.30, a_1 = 11, \lambda = 1, V = 12, b_1 = 11, \mu = 3$.

Table 1 Comparing the results of Akber et al. [26] and Liu et al. [22] with our results

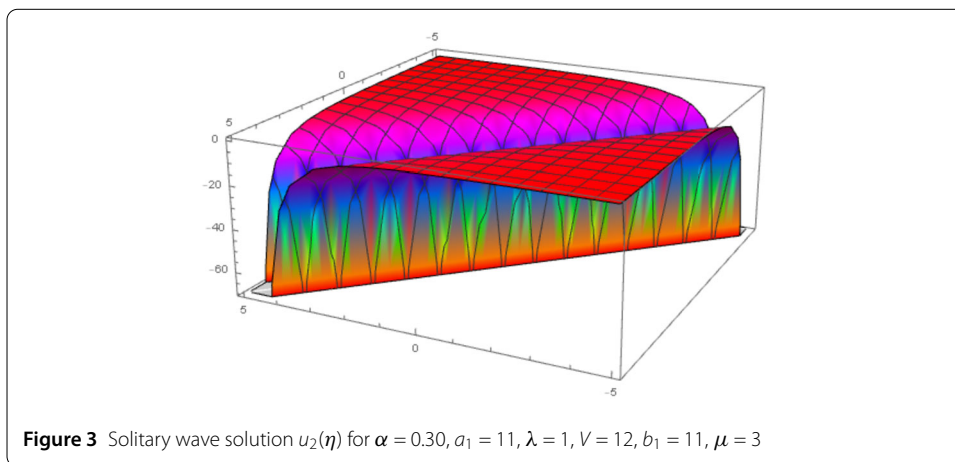
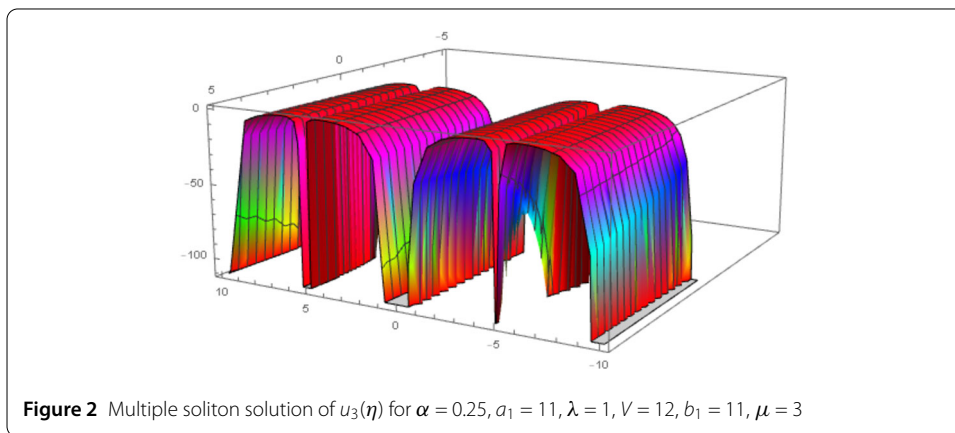
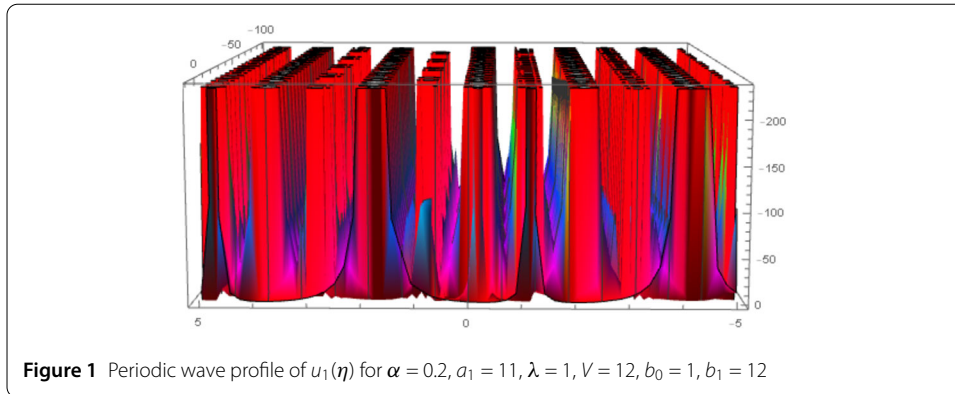
Obtained results	Results of Liu et al. and Akber et al.
<p>If $m_1 = \frac{(V-2k)a_1}{vb_1} \sqrt{-\frac{2V-4k}{V}} = 1$, then the solution is $u_{19} = m_1 \coth\left(\frac{x - \frac{Vt}{\Gamma(\alpha+1)}}{2}\right)$</p>	<p>If $m_2 = 0, d = 0, \sqrt{\Omega} = 1, \Phi = x - \frac{Vt}{\Gamma(\alpha+1)}, A = 1$, then the solution is $u_1 = m_1 \coth\left(\frac{x - \frac{Vt}{\Gamma(\alpha+1)}}{2}\right)$</p>
<p>If $m_1 = \frac{(V-2k)a_1}{vb_1} \sqrt{-\frac{2V-4k}{V}} = 1$, then the solution is $u_{19} = m_2 \coth\left(\frac{x - \frac{Vt}{\Gamma(\alpha+1)}}{2}\right)$</p>	<p>If $m_2 = 0, d = 0, \sqrt{\Omega} = 1, \Phi = x - \frac{Vt}{\Gamma(\alpha+1)}, A = 2$, then the solution is $u_6 = m_2 \coth\left(\frac{x - \frac{Vt}{\Gamma(\alpha+1)}}{2}\right)$</p>
<p>If $\sqrt{-\frac{2V-4k}{V}} = \frac{1}{\sqrt{2}}$ and $m_1 = \frac{a_0(V\lambda^2+2V+4K)}{2vb_1}$, then the solution is $u_{10} = m_1 \cot\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>	<p>If $m_2 = 0, d = 0, B = 0, \iota\sqrt{\Omega} = 1, \Phi = \frac{Vt}{\Gamma(\alpha+1)} - x, A = 1$, then the solution is $u_3 = m_1 \cot\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>
<p>If $\sqrt{-\frac{2V-4k}{V}} = \frac{1}{\sqrt{2}}$ and $m_2 = \frac{a_0(V\lambda^2+2V+4K)}{2vb_1}$, then the solution is $u_{10} = m_2 \cot\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>	<p>If $m_1 = \frac{1}{2}, d = 0, \iota\sqrt{\Omega} = 1, \Phi = \frac{Vt}{\Gamma(\alpha+1)} - x, A = 2$, then the solution is $u_8 = m_1 \cot\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>
<p>If $\sqrt{-\frac{2V-4k}{V}} = 1$ and $m_2 = \frac{a_1(2K+V)}{vb_1}$ then the solution is $u_{19} = m_2 \coth\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>	<p>If $m_1 = \frac{1}{2}, d = 0, \sqrt{\Delta} = 1, \Phi = \frac{Vt}{\Gamma(\alpha+1)} - x, A = 2$, then the solution is $u_7 = m_2 \coth\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>
<p>If $\sqrt{-\frac{2V-4k}{V}} = 1, a_1 = 1$ and $m_2 = \frac{2K+V}{vb_1} = \frac{2}{5}$ then the solution is $u_{20} = \frac{2}{5} \coth\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>	<p>If $m_1 = \frac{1}{5}, m_2 = 0, d = 0, \sqrt{\Delta} = 1, \Phi = \frac{Vt}{\Gamma(\alpha+1)} - x, A = 2$, then the solution is $u_6 = \frac{2}{5} \coth\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>
<p>If $\sqrt{-\frac{2V-4k}{V}} = \frac{1}{\sqrt{2}}$ and $m_1 = \frac{a_1(-2K+V)}{2vb_1} = 1$ then we get $u_{20} = \cot\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>	<p>If $m_1 = \frac{1}{2}, m_2 = 0, d = 0, \sqrt{\Delta} = 1, \Phi = \frac{Vt}{\Gamma(\alpha+1)} - x, A = 2$, then the solution is $u_8 = \cot\left(\frac{\frac{Vt}{\Gamma(\alpha+1)} - x}{2}\right)$</p>

Table 2 Comparing the results of Mohyud-Din et al. [17] and Najafi et al. [30] with our results

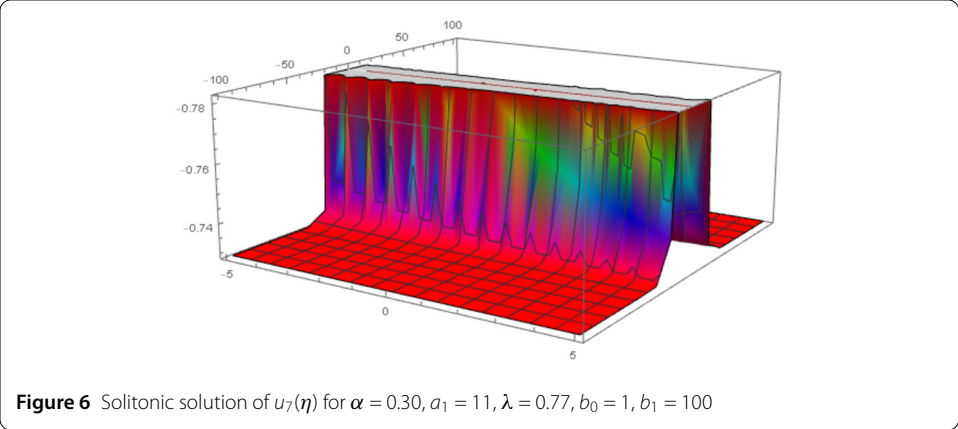
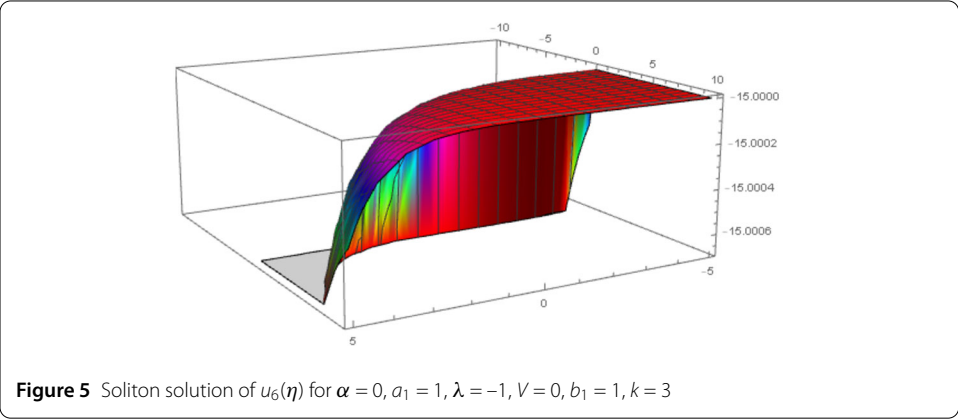
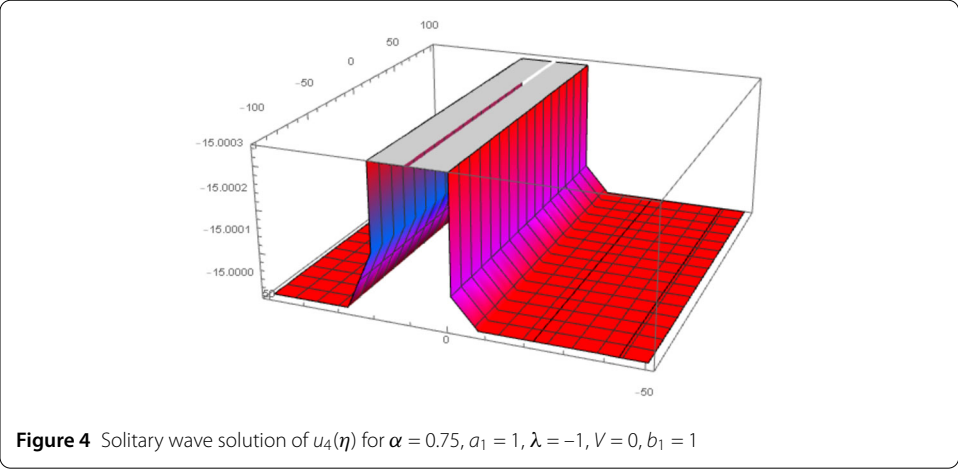
Obtained results	Results of Najafi et al. and Mohyud-Din et al.
<p>If $\lambda = 1, \mu = -1, k = 1, a_1 = 2\sqrt{3}, b_0 = 2, b_1 = 1, t = 0, x = 0$ then the solution is $u_1(\eta) = -2\sqrt{3}$</p>	<p>If $k = 1, b_1 = \frac{1}{2}, \beta = -1, \eta = 0, c_1 = 1$ then the solution is $u_4(\eta) = -2\sqrt{3}$</p>
<p>If $\lambda = 1, \mu = -1, k = 1, a_1 = 2\sqrt{3}, b_0 = 2, b_1 = 1, t = 0, x = 0$ then the solution is $u_1(\eta) = -2\sqrt{3}$</p>	<p>If $k = 1, b_1 = \frac{1}{2}, \beta = -1, \eta = 0, c_1 = 1$ then the solution is $u_{10}(\eta) = -2\sqrt{3}$</p>
<p>If $\lambda = \frac{2}{\sqrt{2}}, \mu = -1, k = 1, a_1 = 3\sqrt{6}, b_0 = \sqrt{2}, b_1 = 1, V = 6\sqrt{2}, t = 1, x = 0$ then the solution is $u_3(\eta) = \frac{3\sqrt{6}}{5\sqrt{2}}$</p>	<p>If $\lambda = 1, k = 1, b_0 = 1, \beta = -1, \eta = 0, c_1 = 1, \mu = 1$ then the solution is $u_4(\eta) = \frac{3\sqrt{6}}{5\sqrt{2}}$</p>
<p>If $\lambda = \frac{2}{\sqrt{2}}, \mu = -1, k = 1, a_1 = 3\sqrt{6}, b_0 = \sqrt{2}, b_1 = 1, V = 6\sqrt{2}, t = 1, x = 0$ then the solution is $u_3(\eta) = \frac{3\sqrt{6}}{5\sqrt{2}}$</p>	<p>If $\lambda = 1, k = 1, b_0 = 1, \beta = -1, \eta = 0, c_1 = 1, \mu = 1$ then the solution is $u_9(\eta) = \frac{3\sqrt{6}}{5\sqrt{2}}$</p>
<p>If $V = 6, a_1 = 1, t = 0, x = 0, b_1 = \frac{-1}{6}, k = \frac{5}{2}$ then the solution is $u_7(\eta) = \sqrt{6}$</p>	<p>If $\lambda = \sqrt{-1}, k = 1, b_0 = 0, \beta = 1, k = 1$ then the solution is $u_2(\eta) = \sqrt{6}$</p>
<p>If $V = 6, a_1 = 1, t = 0, x = 0, b_1 = \frac{-1}{6}, k = \frac{5}{2}$ then the solution is $u_7(\eta) = \sqrt{6}$</p>	<p>If $\lambda = \sqrt{-1}, k = 1, b_0 = 0, \beta = 1, k = 1$ then the solution is $u_8(\eta) = \sqrt{6}$</p>
	<p>If $k = -6, b_1 = \frac{12}{\sqrt{6}}, \beta = -1, k = -6, c_1 = \frac{\sqrt{6}}{12}$ then the solutions are $u_{18}(\eta) = u_{23}(\eta) = -2\sqrt{3}$. It matches with our obtained solution (i) in this table</p>

Table 3 Comparing the results of Lu et al. [25] with our results

Obtained results	Results of Lu et al.
<p>If $H_1 = \frac{a_1(V-2k)}{b_1\sqrt{V(V-2k)}}, a = 0$ and $V = 5$ then the solution is $u_{19} = H_1 \coth\left(\frac{5}{2} - x\right)$</p>	<p>If $H_1 = \frac{k\sqrt{6}\beta}{\beta_2\sqrt{-\delta(\beta_1 k^2+2)}}, \epsilon = 1, k = -1, \omega = 1, \beta_1 = 1, \sqrt{\beta_1} = 5, \sqrt{\beta_3} = \frac{1}{2}$ and $\zeta_0 = 0$ then the solution is $u_4 = H_1 \coth\left(\frac{5}{2} - x\right)$</p>
<p>If $H_1 = \frac{a_1(V-2k)}{b_1\sqrt{V(V-2k)}}, a = 0$ and $V = 5$ then the solution is $u_{19} = H_1 \coth\left(\frac{5}{2} - x\right)$</p>	<p>If $H_1 = \frac{2\sqrt{3}k\beta_1\sqrt{\beta_3\beta}}{\beta_2\sqrt{-\delta(\beta_1 k^2+2)}}, \epsilon = 1, k = -1, \omega = 1, \beta_1 = 1, \sqrt{\beta_1} = 5, \sqrt{\beta_3} = \frac{1}{2}$ and $\zeta_0 = 0$, then the solution is $u_7 = H_1 \coth\left(\frac{5}{2} - x\right)$</p>

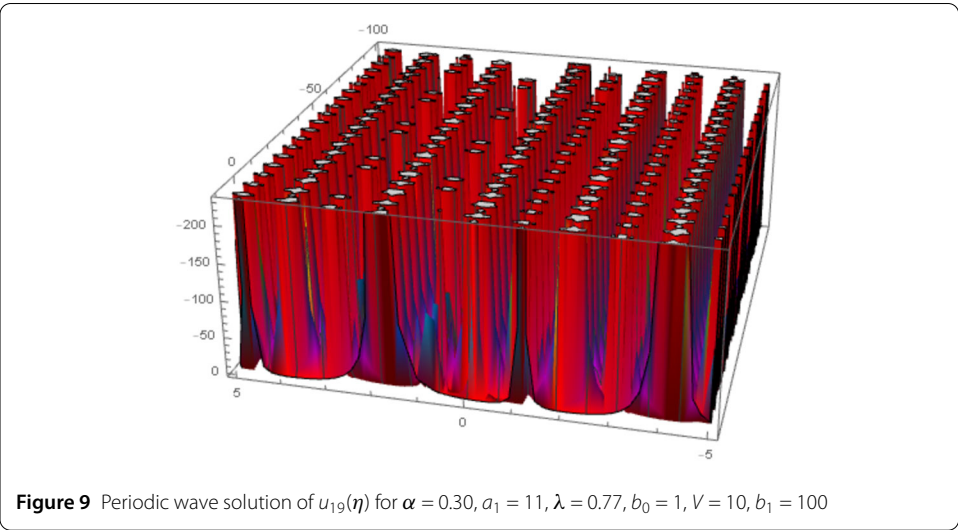
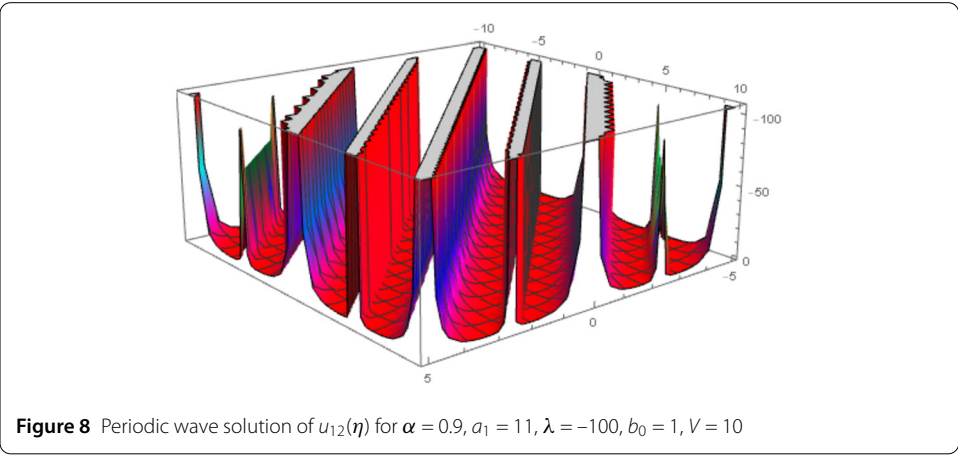
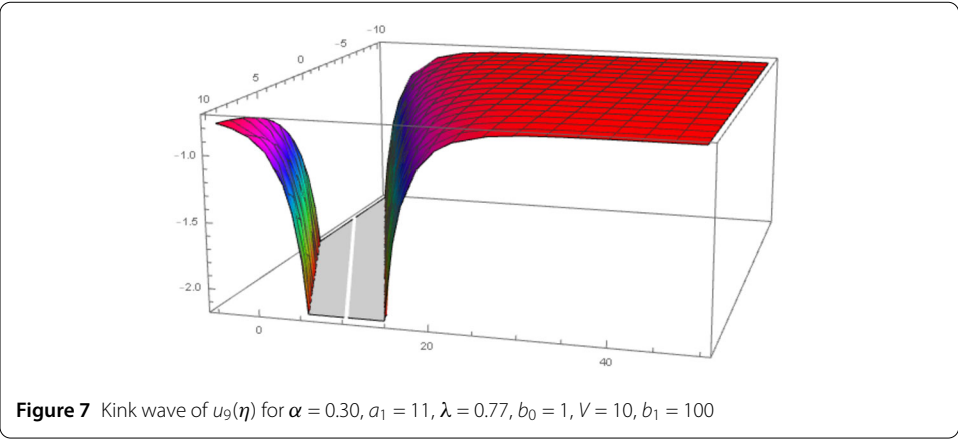


Again the solitary wave solution of $u_4(\eta)$ for $\alpha = 0.75, a_1 = 1, \lambda = -1, V = 0, b_1 = 1$ is obtained in Fig. 4. Figures 5 and 6 represent soliton solutions of $u_4(\eta)$ and $u_{14}(\eta)$ for different values of parameters. Solitons are special types of solitary waves that retain the identity upon interaction with other solitons and have implications in the field of cosmology of the universe. Figure 7 shows the kink wave solution of $u_9(\eta)$ for $\alpha = 0.30, a_1 = 11, \lambda = 0.77, b_0 = 1, V = 10, b_1 = 100$. The kink waves are the solitary waves that are rise or descend from one asymptote state to another. Figure 8 displays the periodic solution of $u_{12}(\eta)$ for $\alpha = 0.9, a_1 = 11, \lambda = -100, b_0 = 1, V = 10$. The periodic solutions are traveling wave results



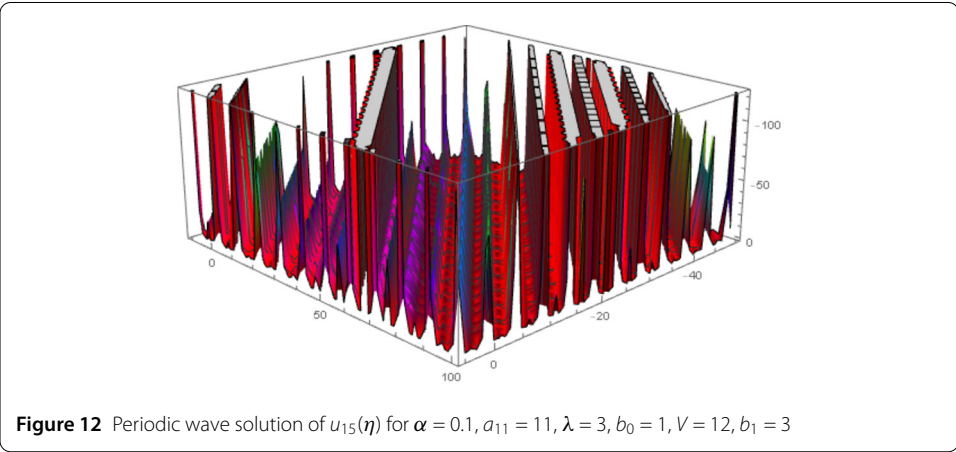
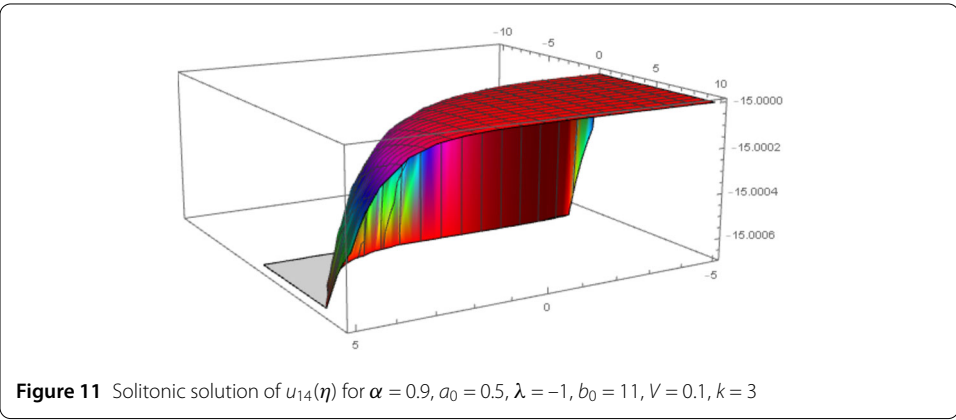
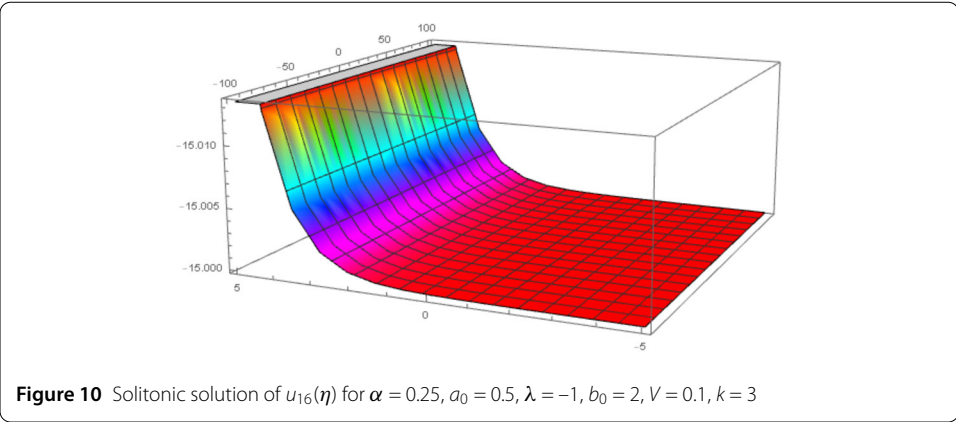
that are periodic. Finally, Fig. 9 represents the periodic solution of $u_{19}(\eta)$ for $\alpha = 0.30, a_1 = 11, \lambda = 0.77, b_0 = 1, V = 10, b_1 = 100$.

Figure 10 shows the soliton solution of $u_{16}(\eta)$ for the choice of $\alpha = 0.25, a_0 = 0.5, \lambda = -1, V = 0.1, b_0 = 2, k = 3$. Kink wave solution is obtained in Fig. 11 of $u_{14}(\eta)$ for $\alpha = 0.9, a_0 = 0.5, \lambda = -1, V = 0.1, b_0 = 11, k = 3$. Figure 12 displays the periodic solution of $u_{15}(\eta)$ for different values of $\alpha = 0.1, a_1 = 11, \lambda = 3, V = 12, b_0 = 1, b_1 = 3$. Finally in Fig. 13 we have found solitonic solution of $u_{19}(\eta)$ for $\alpha = 0.25, k = 5, V = 12, a_1 = 1, b_1 = 1$.

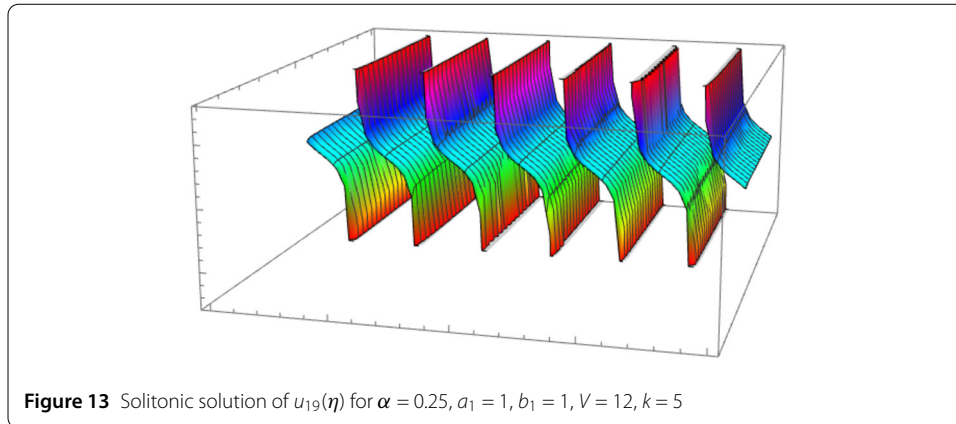


7 Conclusions

We have successfully employed the proposed technique to gain the generalized solitary solutions of the simplified modified Camassa–Holm (SMCH) equation in the article. The solitary solutions are periodic, trigonometric, hyperbolic, and rational functions. The



gained results may express a variety of new features of waves. By comparing our results, we concluded that some of the results are similar to the current literature, while the others are newly discovered, and were not explored elsewhere. The method is reliable in handling to initiate new results and we selected a new class of exact solutions. It is investigated that the physical parameters significantly modify the wave dynamics. The solutions formed in this paper can be beneficial in the study of wave breaking. Wave breaking is used in atmospheric gravity waves and in plasma physics. It is also utilized in the analysis for discussing local well-posedness and global existence in non-peaked solutions. The suggested



procedure fully approved the dependability of our computational work and could be implemented to study other physical problems.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed to writing the draft and the software and all reviewed and approved the final version of the manuscript.

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