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On stability analysis and existence of positive solutions for a general non-linear fractional differential equations

Amita Devi[1], Anoop Kumar[1], Dumitru Baleanu[2,3,4*] and Aziz Khan[5]

*Correspondence:
dumitru@cankaya.edu.tr
[2]Department of Mathematics, Faculty of Art and Sciences, Cankaya University, Cankaya, Turkey
[3]Institute of Space Science, Magurele-Bucharest, Magurele-Bucharest, Romania
Full list of author information is available at the end of the article

Abstract

In this article, we deals with the existence and uniqueness of positive solutions of general non-linear fractional differential equations (FDEs) having fractional derivative of different orders involving p-Laplacian operator. Also we investigate the Hyers–Ulam (HU) stability of solutions. For the existence result, we establish the integral form of the FDE by using the Green function and then the existence of a solution is obtained by applying Guo–Krasnoselskii's fixed point theorem. For our purpose, we also check the properties of the Green function. The uniqueness of the result is established by applying the Banach contraction mapping principle. An example is offered to ensure the validity of our results.

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1 Introduction

Fractional calculus concerns the applications of derivatives and integrals of arbitrary order. During the last few decades, it received great attention because of its various applications in diverse scientific fields. Arbitrary-order models are more flexible than integer-order models. FDEs arise in numerous scientific and engineering fields such as physics, polymer rheology, geophysics, biophysics, aerodynamics, capacitor theory, biology, nonlinear oscillation of earthquake, control theory, blood flow phenomena, viscoelasticity, and electrical circuits. For the exhaustive study of its applications, we refer to extensive work [1–7]. The fundamental differences between exponential decay, the power law, the Mittag-Leffler law and some possible applications in nature are presented in [8, 9].

Nowadays, the existence and uniqueness (EU) of solutions for different type of FDEs is a field of intensive research. Here, we introduce some important and recent work of several researcher about the existence of a positive solution (EPS) of different classes of FDEs. For example, the EU results for Dirichlet and mixed problems of singular FDEs with the Riemann–Liouville sense of fractional derivative were investigated by Agarwal et al. [10, 11]. Baleanu et al. in [12] established the existence of a solution on partially ordered

Banach spaces for a non-linear FDEs. Vong studied the singular FDEs involving non-local type boundary conditions in [13] by using fixed point techniques. For more details of EU of different types of FDEs with different types of fractional derivatives, see [14–35]. Numerical solutions for the fractional Fisher's type equations involving the Atangana–Baleanu fractional derivative by methods of spectral collocation are in [26]. As of lately, some authors investigated the FDEs with p-Laplacian operator by diverse types of mathematical techniques. For instance, Khan et al. investigated the existence criterion for solutions for FDEs involving the ϕ_p-Laplacian operator in [36]. The EU of results for FDEs with ϕ_p-Laplacian operator are analyzed by Chuanzhi Bai in [37] via fixed point theorems. Also we present the Green function's properties and two examples to illustrate the results. The EPS for FDEs with the ϕ_p-Laplacian operator is studied by Tian et al. [38] and EPSs are obtained with the help of a monotone iterative method. For more EU results for FDEs with a p-Laplacian operator one may refer to [39–42]

Recently, a great interest has been shown in the study of HU stability of non-linear FDEs with different type of boundary conditions. By HU stability we mean that there exists an exact solution very close to the approximate solution of a FDE and that the error can be calculated. The EU of solutions and HU stability FDEs with p-Laplacian operator and ABC-fractional derivative involving a spatial singularity is derived by Khan et al. in [43] using the well-known Guo–Krasnoselskii theorem. Khan et al. [44] discussed the analytical study of existence and stability results of a singular non-linear FDEs with ϕ_p-operator involving fractional integral and differential boundary conditions. The EU and HU stability of solutions for a coupled system of FDEs involving the derivative in Caputo's sense are proved by Khan et al. [45] using a Leray–Schauder-type fixed point theorem and topological degree theory. Li et al. investigated the HU stability of FDEs in [46] and also presented an example to illustrate their result. Stability and EU of solutions for the fractional order HIV model were introduced by Khan et al. in [47]. Existence and stability of solutions for singular delay FDEs with fractional integral initial conditions by using the Green function and the fixed point theorem were established by the Khan et al. in [48]. For more details of stability analysis, see [49–57].

Motivated by the above work, we introduce the EU and HU stability results, for nonliner FDEs involving Caputo fractional derivatives of distinct orders with ϕ_p^* Laplacian operator:

$$\begin{cases} {}^c\mathscr{D}^{\zeta}\phi_p^*[{}^c\mathscr{D}^{\sigma}(\mathfrak{z}(t)-\sum_{i=1}^m\lambda_i(t))] = -\psi^*(t,\mathfrak{z}(t)), \quad t\in[0,1], \\ \phi_p^*[{}^c\mathscr{D}^{\sigma}\mathfrak{z}(t)-\sum_{i=1}^m\lambda_i(t)]|_{t=0}=0, \qquad \mathfrak{z}(0)=\sum_{i=1}^m\lambda_i(0), \\ \mathfrak{z}'(1)=\sum_{i=1}^m\lambda_i'(1), \qquad \mathfrak{z}^j(0)=\sum_{i=1}^m\lambda_i^j(0) \quad \text{for } j=2,3,4,\dots,n-1, \end{cases} \tag{1.1}$$

where ${}^c\mathscr{D}^{\zeta}$, ${}^c\mathscr{D}^{\sigma}$ denotes the derivative of fractional order ζ and σ in Caputo's sense, respectively, and ψ^*, $\lambda_i(t)$ are continuous functions. The orders $n-1<\sigma\leq n$, $0<\zeta\leq 1$ where $n\geq 4$, $\psi^*\in\mathscr{L}[0,1]$ and $\phi_p^*(\mathfrak{z})=|\mathfrak{z}|^{p-1}\mathfrak{z}$ denotes the p-Laplacian operator and satisfies $\frac{1}{p}+\frac{1}{q}=1$, $(\phi_p^*)^{-1}=\phi_q^*$. The rest of article is divided in four parts. Basic definitions and desired lemmas are presented in Sect. 1 and properties of the Green functions are discussed in Sect. 2. The EU results are given in Sect. 3. HU stability is discussed in Sect. 4. In Sect. 5, we introduce an example.

Here, we introduce certain definitions, desired lemmas and theorems, which are essential to find the main result.

Definition 1.1 ([1]) For an integrable and real valued continuous function ψ^* defined on $(0,+\infty)$, the Riemann–Liouville integral of fractional order $\delta \in \mathbb{R}$ is defined as

$$I^{\delta}\psi^*(y) = \frac{1}{\Gamma(\delta)}\int_0^y (y-x)^{\delta-1}\psi^*(x)\,dx, \quad \delta > 0.$$

Definition 1.2 ([1]) For an n-times continuously differentiable real valued function ψ^* defined on $(0,+\infty)$, the Caputo derivative of fractional order $\delta \in \mathbb{R}$ $(\delta > 0)$ is defined as

$${}^c\mathcal{D}^{\delta}\psi^*(y) = \frac{1}{\Gamma(\mathsf{n}-\delta)}\int_0^y (y-x)^{\mathsf{n}-\delta-1}\big(\psi^*\big)^n(x)\,dx, \quad \mathsf{n}-1 < \delta < \mathsf{n}, \mathsf{n} = [\delta]+1,$$

where $[\delta]$ represents the greatest integer and the integral exists on the $(0,+\infty)$ interval.

Lemma 1.1 ([2]) *Let $\sigma \in (k-1,k]$ and $\psi^*(t) \in C^{k-1}$, then*

$$\mathcal{J}^{\sigma}\mathcal{D}^{\sigma}\psi^*(y) = \psi^*(y) + a_0 + a_1 y + a_2 y^2 + a_3 y^3 + \cdots + a_{k-1}y^{k-1},$$

for the $a_j \in \mathbb{R}$ for $j = 0,1,2,\ldots,k-1$.

Theorem 1.2 ([58, 59], Guo–Krasnoselskii theorem) *Consider Ω^* to be a Banach space and let a cone $\mathcal{K}^* \in \Omega^*$. Assume that $\mathcal{A}_1^*, \mathcal{A}_2^*$ are two bounded subsets of Ω^* such that $0 \in \mathcal{A}_1^*$, $\overline{\mathcal{A}_1^*} \subset \mathcal{A}_2^*$. Then an operator $\mathcal{G}^* : \mathcal{K}^* \cap (\overline{\mathcal{A}_2^*} \setminus \mathcal{A}_1^*) \longrightarrow \mathcal{K}^*$, which is completely continuous and satisfies*

$$(\mathcal{P}_1^*) \quad \|\mathcal{G}^*\mathfrak{z}\| \le \|\mathfrak{z}\| \quad \text{if } \mathfrak{z} \in \mathcal{K}^* \cap \partial\mathcal{A}_1^* \quad \text{and} \quad \|\mathcal{G}^*\mathfrak{z}\| \ge \|\mathfrak{z}\| \quad \text{if } \mathfrak{z} \in \mathcal{K}^* \cap \partial\mathcal{A}_2^*,$$

or

$$(\mathcal{P}_2^*) \quad \|\mathcal{G}^*\mathfrak{z}\| \ge \|\mathfrak{z}\| \quad \text{if } \mathfrak{z} \in \mathcal{K}^* \cap \partial\mathcal{A}_1^* \quad \text{and} \quad \|\mathcal{G}^*\mathfrak{z}\| \le \|\mathfrak{z}\| \quad \text{if } \mathfrak{z} \in \mathcal{K}^* \cap \partial\mathcal{A}_2^*,$$

has a fixed point in $\mathcal{K}^* \cap (\overline{\mathcal{A}_2^*} \setminus \mathcal{A}_1^*)$.

Lemma 1.3 ([44, 45]) *For the p-Laplacian operator ϕ_p^*, the following conditions hold true:*

(1) *If $|\gamma_1|, |\gamma_2| \ge \sigma > 0$, $1 < p \le 2$, $\gamma_1\gamma_2 > 0$, then*

$$\big|\phi_p^*(\gamma_1) - \phi_p^*(\gamma_2)\big| \le (p-1)\sigma^{p-2}|\gamma_1 - \gamma_2|.$$

(2) *If $p > 2$, $|\gamma_1|, |\gamma_2| \le \sigma^* > 0$, then*

$$\big|\phi_p^*(\gamma_1) - \phi_p^*(\gamma_2)\big| \le (p-1)\big(\sigma^*\big)^{p-2}|\gamma_1 - \gamma_2|.$$

2 Green function and properties

Theorem 2.1 *Consider $\psi^* \in \mathcal{C}[0,1]$ satisfying the FDE with ϕ_p^* (1.1). Then, for $\zeta \in (0,1]$ and $\sigma \in (n-1,n]$, the FDEs (1.1) involving the ϕ_p^* Laplacian operator has a solution equivalent to*

$$\mathfrak{z}(t) = \sum_{i=1}^{m}\lambda_i(t) + \int_0^1 \mathcal{H}^{\sigma}(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,\mathfrak{z}(\epsilon)\big)\,d\epsilon\right)ds, \tag{2.1}$$

where the Green function $\mathcal{H}^{\sigma}(t,\mathfrak{s})$ *is defined by*

$$\mathcal{H}^{\sigma}(t,\mathfrak{s}) = \begin{cases} \frac{-(t-\mathfrak{s})^{\sigma-1}}{\Gamma(\sigma)} + t\frac{(1-\mathfrak{s})^{\sigma-2}}{\Gamma(\sigma-1)}, & 0 < \mathfrak{s} \le t < 1, \\ t\frac{(1-\mathfrak{s})^{\sigma-2}}{\Gamma(\sigma-1)}, & 0 < t \le \mathfrak{s} < 1. \end{cases} \tag{2.2}$$

Proof Taking the integral operator I^{ζ} on both sides (1.1) and using Lemma 1.1, Eq. (1.1) becomes

$$\phi_p^*\left({}^c\mathcal{D}^{\sigma}\left[\mathfrak{z}(t) - \sum_{i=1}^{m}\lambda_i(t)\right]\right) = -I^{\zeta}\left[\psi^*\left(t,\mathfrak{z}(t)\right)\right] + C_0. \tag{2.3}$$

From the condition $\phi_p^*({}^c\mathcal{D}^{\sigma}[\mathfrak{z}(t) - \sum_{i=1}^{m}\lambda_i(t)])|_{t=0} = 0, \implies C_0 = 0$.

Using the value of $C_0 = 0$, then (2.3) becomes

$$\phi_p^*\left({}^c\mathcal{D}^{\sigma}\left[\mathfrak{z}(t) - \sum_{i=1}^{m}\lambda_i(t)\right]\right) = -I^{\zeta}\left[\psi^*\left(t,\mathfrak{z}(t)\right)\right]. \tag{2.4}$$

Applying the q-Laplacian operator further on (2.4) we get the form

$${}^c\mathcal{D}^{\sigma}\left[\mathfrak{z}(t) - \sum_{i=1}^{m}\lambda_i(t)\right] = -\phi_q^*\left(I^{\zeta}\left[\psi^*\left(t,\mathfrak{z}(t)\right)\right]\right). \tag{2.5}$$

Again taking the integral operator I^{σ} to both sides of (2.5) and using Lemma 1.1, then (2.5) becomes

$$\mathfrak{z}(t) - \sum_{i=1}^{m}\lambda_i(t) = -I^{\sigma}\left(\phi_q^*\left(I^{\zeta}\left[\psi^*\left(t,\mathfrak{z}(t)\right)\right]\right)\right) + a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_{n-1}t^{n-1}, \tag{2.6}$$

where $a_j \in \mathbb{R}$ for $j = 0,1,2,\ldots,n-1$.

Using the boundary conditions $\mathfrak{z}^j(0) = \sum_{i=1}^{m}\lambda_i^j(0)$ for $j = 0,2,3,4,\ldots,n-1$, in (2.6), $\implies a_j = 0$ for $j = 0,2,3,4,\ldots,n-1$. and $\mathfrak{z}'(1) = \sum_{i=1}^{m}\lambda_i'(1)$, implies that $a_1 = I^{\sigma-1}(\phi_q^*(I^{\zeta}[\psi^*(t,\mathfrak{z}(t))]))|_{t=1}$

Putting the values of the constants a_i in (2.6), we get

$$\mathfrak{z}(t) = \sum_{i=1}^{m}\lambda_i(t) + \int_0^1 \mathcal{H}^{\sigma}(t,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\epsilon)^{\zeta-1}\psi^*\left(\epsilon,\mathfrak{z}(\epsilon)\right)d\epsilon\right)d\mathfrak{s}, \tag{2.7}$$

where $\mathcal{H}^{\sigma}(t,\mathfrak{s})$ is defined in (2.2). □

Lemma 2.2 *The Green function* $\mathcal{H}^{\sigma}(t,\mathfrak{s})$ *defined in* (2.2) *satisfies the following conditions*:

($\mathcal{B}_1$) $\mathcal{H}^{\sigma}(t,\mathfrak{s}) > 0\ \forall \mathfrak{s}, t \in (0,1)$;

($\mathcal{B}_2$) *the function* $\mathcal{H}^{\sigma}(t,\mathfrak{s})$ *is increasing and* $\mathcal{H}^{\sigma}(1,\mathfrak{s}) = \max_{t\in[0,1]}\mathcal{H}^{\sigma}(t,\mathfrak{s})$;

($\mathcal{B}_3$) $\mathcal{H}^{\sigma}(1,\mathfrak{s}) \ge t^{\sigma-1}\max_{t\in[0,1]}\mathcal{H}^{\sigma}(t,\mathfrak{s})$ *for* $t,\mathfrak{s} \in (0,1)$.

Proof To prove $(\mathcal{B}_1)$, we take two cases $\forall t, s \in (0,1)$.

Case 1. For $s \leq t$. As $\sigma > 3$, then

$$\begin{aligned}\mathcal{H}^{\sigma}(t,s) &= \frac{-(t-s)^{\sigma-1}}{\Gamma(\sigma)} + t\frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} \\ &= \frac{-t^{\sigma-1}(1-\frac{s}{t})^{\sigma-1}}{\Gamma(\sigma)} + t\frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} \\ &\geq \frac{-t^{\sigma-1}(1-s)^{\sigma-1}}{\Gamma(\sigma)} + t^{\sigma-1}\frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} > 0. \end{aligned} \tag{2.8}$$

Case 2. When $t \leq s$, we evaluate

$$\mathcal{H}^{\sigma}(t,s) = t\frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} > 0. \tag{2.9}$$

From (2.8) and (2.9), it is proved that $\mathcal{H}^{\sigma}(t,s) > 0$ $\forall s, t \in (0,1)$.

To prove the condition $(\mathcal{B}_2)$, we assume that $\forall s, t \in (0,1)$.

Case 1. For $s \leq t$. As $\sigma > 3$, then

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{H}^{\sigma}(t,s) &= \frac{-(t-s)^{\sigma-2}}{\Gamma(\sigma-1)} + \frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} \\ &= \frac{-t^{\sigma-2}(1-\frac{s}{t})^{\sigma-2}}{\Gamma(\sigma-1)} + \frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} \\ &\geq \frac{-t^{\sigma-2}(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} + t^{\sigma-2}\frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} > 0. \end{aligned} \tag{2.10}$$

Case 2. When $t \leq s$, we find that

$$\frac{\partial}{\partial t}\mathcal{H}^{\sigma}(t,s) = \frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} > 0. \tag{2.11}$$

From Eqs. (2.10) and (2.11), it is shown that $\frac{\partial}{\partial t}\mathcal{H}^{\sigma}(t,s) > 0$ $\forall s, t \in (0,1)$, consequently, $\frac{\partial}{\partial t}\mathcal{H}^{\sigma}(t,s)$ is an increasing function. Thus, we have for $t \geq s$

$$\max_{t\in[0,1]} \mathcal{H}^{\sigma}(t,s) = \frac{-(1-s)^{\sigma-1}}{\Gamma(\sigma)} + \frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} = \mathcal{H}^{\sigma}(1,s), \tag{2.12}$$

and for $s \geq t$

$$\max_{t\in[0,1]} \mathcal{H}^{\sigma}(t,s) = \frac{(1-s)^{\sigma-2}}{\Gamma(\sigma-1)} = \mathcal{H}^{\sigma}(1,s). \tag{2.13}$$

To prove the condition $(\mathcal{B}_3)$, we assume that

Case 1. For $\mathscr{s} \le t$. As $\sigma > 3$, then

$$\begin{aligned}
\mathscr{H}^{\sigma}(t,\mathscr{s}) &= \frac{-(t-\mathscr{s})^{\sigma-1}}{\Gamma(\sigma)} + t\frac{(1-\mathscr{s})^{\sigma-2}}{\Gamma(\sigma-1)} \\
&= \frac{-t^{\sigma-1}(1-\frac{\mathscr{s}}{t})^{\sigma-1}}{\Gamma(\sigma)} + t\frac{(1-\mathscr{s})^{\sigma-2}}{\Gamma(\sigma-1)} \\
&\ge \frac{-t^{\sigma-1}(1-\mathscr{s})^{\sigma-1}}{\Gamma(\sigma)} + t^{\sigma-1}\frac{(1-\mathscr{s})^{\sigma-2}}{\Gamma(\sigma-1)} \\
&= t^{\sigma-1}\left(\frac{-(1-\mathscr{s})^{\sigma-1}}{\Gamma(\sigma)} + \frac{(1-\mathscr{s})^{\sigma-2}}{\Gamma(\sigma-1)}\right) = t^{\sigma-1}\mathscr{H}^{\sigma}(1,\mathscr{s}).
\end{aligned} \tag{2.14}$$

Case 2. For $t \le \mathscr{s}$, we evaluate

$$\begin{aligned}
\mathscr{H}^{\sigma}(t,\mathscr{s}) &= t\frac{(1-\mathscr{s})^{\sigma-2}}{\Gamma(\sigma-1)} \\
&\ge t^{\sigma-1}\frac{(1-\mathscr{s})^{\sigma-2}}{\Gamma(\sigma-1)} = t^{\sigma-1}\mathscr{H}^{\sigma}(1,\mathscr{s}).
\end{aligned} \tag{2.15}$$

Thus, by Eqs. (2.14) and (2.15), condition $\mathscr{B}_3$ is proved. □

3 Existence result

Now we prove our existence result by introducing the following conditions.

Let $\Omega^* = \mathscr{C}[0,1]$ be the Banach space containing all real valued functions defined on $[0,1]$, which are continuous and endowed with the $\|\mathfrak{z}\| = \max_{t\in[0,1]}\{|\mathfrak{z}(t)| : \mathfrak{z} \in \Omega^*\}$. Suppose that $\mathscr{K}^* \in \Omega^*$ is a cone of functions, which are positive and of the type $\mathscr{K}^* = \{\mathfrak{z} \in \Omega^* : \mathfrak{z}(t) \ge t^{\sigma}\|\mathfrak{z}\|, t \in [0,1]\}$. Let $\mathscr{A}^*(r) = \{\mathfrak{z} \in \mathscr{K}^* : \|\mathfrak{z}\| < r\}$, $\partial\mathscr{A}^*(r) = \{\mathfrak{z} \in \mathscr{K}^* : \|\mathfrak{z}\| = r\}$. By using Theorem 2.1. Equation (1.1) is equivalent to

$$\mathfrak{z}(t) = \sum_{i=1}^{m}\lambda_i(t) + \int_0^1 \mathscr{H}^{\sigma}(t,\mathscr{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathscr{s}}(\mathscr{s}-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,\mathfrak{z}(\epsilon)\big)\,d\epsilon\right)d\mathscr{s}. \tag{3.1}$$

Let us define an operator $\mathscr{G}^* : \mathscr{K}^* \setminus \{0\} \to \Omega^*$ associated with problem (1.1), such that

$$\mathscr{G}^*\mathfrak{z}(t) = \sum_{i=1}^{m}\lambda_i(t) + \int_0^1 \mathscr{H}^{\sigma}(t,\mathscr{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathscr{s}}(\mathscr{s}-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,\mathfrak{z}(\epsilon)\big)\,d\epsilon\right)d\mathscr{s}. \tag{3.2}$$

By using Theorem 2.1, the solution of FDE given by Eq. (1.1) is a fixed point $\mathfrak{z}(t)$ of $\mathscr{G}^*$ i.e.,

$$\mathfrak{z}(t) = \mathscr{G}^*\mathfrak{z}(t). \tag{3.3}$$

To obtain the existence result we need the following assumptions:

($\mathscr{R}_1$) $\psi^*(t,\mathfrak{z}(t)) : [0,1] \times (0,+\infty) \longrightarrow \mathbb{R}^+$ is a continuous function.

($\mathscr{R}_2$) $\lambda_i(t) : [0,1] \longrightarrow \mathbb{R}^+$ are also continuous functions for each $i = 1,2,3,\dots,m$, with

$$\sum_{i=1}^{m}\lambda_i(t)\| \le \overline{\Delta} < +\infty.$$

($\mathcal{R}_3$) $|\psi^*(t,\mathfrak{z}(t))| \le \phi_p^*(\Lambda_1|\mathfrak{z}(t)|^{l_1} + \Lambda_2)$ $\forall t \in [0,1]$, $\mathfrak{z} \in \Omega^*$ where Λ_1, Λ_2 are positive constants and $l_1 \in [0,1]$.

($\mathcal{R}_4$) $|\psi^*(t,\mathfrak{z}) - \psi^*(t,\upsilon)| \le \mathcal{L}(|\mathfrak{z} - \upsilon|)$ $\forall \mathcal{L} > 0$, $t \in [0,1]$, $\mathcal{L} > 0$, $\mathfrak{z},\upsilon \in \Omega^*$.

Theorem 3.1 *Let us assume that conditions* ($\mathcal{R}_1$)–($\mathcal{R}_3$) *are satisfied. Then* $\mathcal{G}^*$ *is a completely continuous operator.*

Proof For any $\mathfrak{z} \in (\overline{\mathcal{A}^*{}_2(r)}) \setminus \mathcal{A}_1^*(r))$, using Lemma 2.2 and (3.2), we have

$$\begin{aligned} \mathcal{G}^*\mathfrak{z}(t) &= \sum_{i=1}^{m} \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\bigl(\epsilon,\mathfrak{z}(\epsilon)\bigr)\,d\epsilon\right)ds \\ &\le \sum_{i=1}^{m} \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(1,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\bigl(\epsilon,\mathfrak{z}(\epsilon)\bigr)\,d\epsilon\right)ds, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \mathcal{G}^*\mathfrak{z}(t) &= \sum_{i=1}^{m} \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\bigl(\epsilon,\mathfrak{z}(\epsilon)\bigr)\,d\epsilon\right)ds \\ &\ge \sum_{i=1}^{m} \lambda_i(t) + t^{\sigma-1}\int_0^1 \mathcal{H}^\sigma(1,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\bigl(\epsilon,\mathfrak{z}(\epsilon)\bigr)\,d\epsilon\right)ds. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we arrive at

$$\mathcal{G}^*\mathfrak{z}(t) \ge t^{\sigma-1}\bigl\|\mathcal{G}^*\mathfrak{z}(t)\bigr\|, \quad t \in [0,1]. \tag{3.6}$$

This implies that $\mathcal{G}^* : (\overline{\mathcal{A}^*{}_2(r)}) \setminus \mathcal{A}_1^*(r)) \to \mathcal{K}^*$.

Next, to show that $\mathcal{G}^*$ is a continuous map, we prove that $\|\mathcal{G}^*\mathfrak{z}_n(t) - \mathcal{G}^*\mathfrak{z}(t)\| \longrightarrow 0$ as $n \longrightarrow \infty$; let us address

$$\begin{aligned} &\bigl\|\mathcal{G}^*\mathfrak{z}_n(t) - \mathcal{G}^*\mathfrak{z}(t)\bigr\| \\ &\quad= \left|\sum_{i=1}^{m} \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\bigl(\epsilon,\mathfrak{z}_n(\epsilon)\bigr)\,d\epsilon\right)ds \right. \\ &\qquad \left. - \sum_{i=1}^{m} \lambda_i(t) - \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\bigl(\epsilon,\mathfrak{z}(\epsilon)\bigr)\,d\epsilon\right)ds\right| \\ &\quad\le \int_0^1 \bigl|\mathcal{H}^\sigma(t,s)\bigr|\left|\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\bigl(\epsilon,\mathfrak{z}_n(\epsilon)\bigr)\,d\epsilon\right)ds \right. \\ &\qquad \left. - \phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\bigl(\epsilon,\mathfrak{z}(\epsilon)\bigr)\,d\epsilon\right)ds\right|. \end{aligned} \tag{3.7}$$

By continuity of the function ψ^*, we have $\|\mathcal{G}^*\mathfrak{z}_n(t) - \mathcal{G}^*\mathfrak{z}(t)\| \longrightarrow 0$ as $n \longrightarrow \infty$. This implies that $\mathcal{G}^*$ is a continuous map.

Now, we have to prove $\mathcal{G}^*$ is uniformly bounded on $(\overline{\mathcal{A}^*{}_2(r)}) \setminus \mathcal{A}^*{}_1(r)$.

By (3.2) and using $(\mathcal{R}_2\mathcal{R}_3)$, for any $t \in [0,1]$, we get

$$\begin{aligned}
&\|\mathcal{G}^*\mathfrak{z}\| \\
&= \left|\sum_{i=1}^{m}\lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,\mathfrak{z}(\epsilon)\big)\,d\epsilon\right)d\mathfrak{s}\right| \\
&\leq \left|\sum_{i=1}^{m}\lambda_i(t)\right| + \int_0^1 \left|\mathcal{H}^\sigma(1,\mathfrak{s})\right|\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\epsilon)^{\zeta-1}\phi_p^*\big(\Lambda_1\|\mathfrak{z}(\epsilon)\|^{l_1}+\Lambda_2\big)\,d\epsilon\right)d\mathfrak{s} \\
&\leq \overline{\Delta} + \left[\frac{1}{\Gamma(\sigma+1)} + \frac{1}{\Gamma(\sigma)}\right]\left[\frac{1}{\Gamma(\zeta+1)}\right]^{q-1}\big(\Lambda_1\|\mathfrak{z}(\epsilon)\|^{l_1}+\Lambda_2\big) \\
&= \overline{\Delta} + \Theta\big(\Lambda_1\|\mathfrak{z}(\epsilon)\|^{l_1}+\Lambda_2\big),
\end{aligned} \tag{3.8}$$

where $\Theta = [\frac{1}{\Gamma(\sigma+1)} + \frac{1}{\Gamma(\sigma)}][\frac{1}{\Gamma(\zeta+1)}]^{q-1}$.

This proves that $\mathcal{G}^*$ is uniformly bounded.

In order to show that the operator $\mathcal{G}^*$ is compact, we show the equicontinuity of the operator $\mathcal{G}^*$.

For $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}
&\left|\mathcal{G}^*\mathfrak{z}(t_2) - \mathcal{G}^*\mathfrak{z}(t_1)\right| \\
&\leq \left|\sum_{i=1}^{m}\lambda_i(t_2) - \sum_{i=1}^{m}\lambda_i(t_1)\right| \\
&\quad + \left|\int_0^1 \mathcal{H}^\sigma(t_2,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,\mathfrak{z}(\epsilon)\big)\,d\epsilon\right)d\mathfrak{s}\right. \\
&\quad \left. - \int_0^1 \mathcal{H}^\sigma(t_1,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,\mathfrak{z}(\epsilon)\big)\,d\epsilon\right)d\mathfrak{s}\right| \\
&\leq \left|\sum_{i=1}^{m}\lambda_i(t_2) - \sum_{i=1}^{m}\lambda_i(t_1)\right| \\
&\quad + \int_0^1 \left|\mathcal{H}^\sigma(t_2,\mathfrak{s}) - \mathcal{H}^\sigma(t_1,\mathfrak{s})\right|\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\epsilon)^{\zeta-1}\phi_p^*\big(\Lambda_1\|\mathfrak{z}(\epsilon)\|^{l_1}+\Lambda_2\big)\,d\epsilon\right)d\mathfrak{s} \\
&\leq \left(\frac{|{t_2}^\sigma - {t_1}^\sigma|}{\Gamma(\sigma+1)} + \frac{|t_2-t_1|}{\Gamma(\sigma)}\right)\left[\frac{1}{\Gamma(\zeta+1)}\right]^{q-1}\big(\Lambda_1\|\mathfrak{z}(\epsilon)\|^{l_1}+\Lambda_2\big) \\
&\quad \times \left|\mathcal{G}^*\mathfrak{z}(t_2) - \mathcal{G}^*\mathfrak{z}(t_1)\right| \\
&\longrightarrow 0 \quad \text{as } (t_2-t_1) \longrightarrow 0.
\end{aligned} \tag{3.9}$$

Thus, $\mathcal{G}^*$ is an equicontinuous operator on $(\overline{\mathcal{A}^*{}_2(r)}) \setminus \mathcal{A}_1^*(r)$ and by the Arzela–Ascoli theorem $\mathcal{G}^*$ is compact on $(\overline{\mathcal{A}^*{}_2(r)}) \setminus \mathcal{A}_1^*(r)$. In fact, all the conditions of Theorem 2.1 [58] are satisfied. Thus $\mathcal{G}^* : (\overline{\mathcal{A}^*{}_2(r)}) \setminus \mathcal{A}_1^*(r)) \to \mathcal{K}^*$ is a completely continuous operator.

Now here, let us determine the hight functions for $\psi^*(t,\mathfrak{z}(t))$ for $r > 0$, $\forall t \in [0,1]$

$$\begin{cases}
\phi_{\min}^*(t,r) = \min_{t\in[0,1]}\{\psi^*(t,\mathfrak{z}(t)) : t^{\sigma-1}r \leq \mathfrak{z} \leq r\} \geq \overline{\mathsf{m}} > -\infty, \\
\phi_{\max}^*(t,r) = \max_{t\in[0,1]}\{\psi^*(t,\mathfrak{z}(t)) : t^{\sigma-1}r \leq \mathfrak{z} \leq r\} \leq \overline{\mathsf{M}} < +\infty.
\end{cases} \tag{3.10}$$

□

Theorem 3.2 *Suppose that assumptions* $(\mathcal{R}_1)$–$(\mathcal{R}_3)$, *are satisfied and* $\exists a, b \in \mathbb{R}^+$ *such that any of the following condition is satisfied*:

$(\mathcal{S}_1)$ $a \leq \|\sum_{i=1}^m \lambda_i(t)\| + \int_0^1 |\mathcal{H}^\sigma(1,s)|\phi_q^*(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\phi_{\min}^*(\epsilon,a)\,d\epsilon)\,ds < +\infty$ *and* $\|\sum_{i=1}^m \lambda_i(t)\| + \int_0^1 |\mathcal{H}^\sigma(1,s)|\phi_q^*(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\phi_{\max}^*(\epsilon,b)\,d\epsilon)\,ds \leq b$, *or*

$(\mathcal{S}_2)$ $\|\sum_{i=1}^m \lambda_i(t)\| + \int_0^1 |\mathcal{H}^\sigma(1,s)|\phi_q^*(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\phi_{\max}^*(\epsilon,a)\,d\epsilon)\,ds < a$ *and* $b \leq \|\sum_{i=1}^m \lambda_i(t)\| + \int_0^1 |\mathcal{H}^\sigma(1,s)|\phi_q^*(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\phi_{\min}^*(\epsilon,b)\,d\epsilon)\,ds < +\infty$.

Then Eq. (1.1) *has a positive solution* $z \in \mathcal{K}^*$ *and* $a \leq \|z\| \leq b$.

Proof Firstly, we are considering the case $(\mathcal{S}_1)$. If $z \in \partial\mathcal{A}^*(a)$ then $\|z\| = a$ and $\forall t \in [0,1]$, $t^{\sigma-1}a \leq z \leq a$. Using (3.10), $\phi_{\min}^*(t,a) \leq \psi^*(t,z(t))$, we write

$$\begin{aligned}
\|\mathcal{G}^* z\| &= \max_{t\in[0,1]}\left|\sum_{i=1}^m \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,z(\epsilon)\big)\,d\epsilon\right)ds\right| \\
&\geq \left\|\sum_{i=1}^m \lambda_i(t)\right\| + t^{\sigma-1}\int_0^1 \left|\mathcal{H}^\sigma(1,s)\right|\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,z(\epsilon)\big)\,d\epsilon\right)ds \\
&\geq \left\|\sum_{i=1}^m \lambda_i(t)\right\| + t^{\sigma-1}\int_0^1 \left|\mathcal{H}^\sigma(1,s)\right|\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\phi_{\min}^*(\epsilon,a)\,d\epsilon\right)ds \geq a \\
&= \|z\|.
\end{aligned} \tag{3.11}$$

Now, for all $t \in [0,1]$, $t^{\sigma-1}b \leq z \leq b$.
If $z \in \partial\mathcal{A}^*(b)$ then $\|z\| = b$ and, using (3.10), we have $\phi_{\max}^*(t,b) \geq \psi^*(t,z(t))$; we find

$$\begin{aligned}
\|\mathcal{G}^* z\| &= \max_{t\in[0,1]}\left|\sum_{i=1}^m \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,z(\epsilon)\big)\,d\epsilon\right)ds\right| \\
&\leq \left\|\sum_{i=1}^m \lambda_i(t)\right\| + \int_0^1 \left|\mathcal{H}^\sigma(1,s)\right|\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,z(\epsilon)\big)\,d\epsilon\right)ds \\
&\leq \left\|\sum_{i=1}^m \lambda_i(t)\right\| + t^{\sigma-1}\int_0^1 \left|\mathcal{H}^\sigma(1,s)\right|\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\phi_{\min}^*(\epsilon,a)\,d\epsilon\right)ds \geq a \\
&= \|z\|.
\end{aligned} \tag{3.12}$$

Using Lemma 1.2, $z \in (\overline{\mathcal{A}^*(b)}) \setminus \mathcal{A}^*(a)$ is a fixed point of $\mathcal{G}^*$. By using Lemma 2.2 and Theorem 2.1, for $t \in (0,1)$ and $a \leq \|z\| \leq b$, we have $z(t) \geq t^{\sigma-1}\|z(t)\| \geq a t^{\sigma-1} > 0$. Therefore $z(t)$ is positive solution. It obeys

$$\begin{aligned}
\frac{\partial}{\partial t} z(t) &= \frac{\partial}{\partial t}\mathcal{G}^* z(t) \\
&= \frac{\partial}{\partial t}\sum_{i=1}^m \lambda_i(t) + \int_0^1 \frac{\partial}{\partial t}\mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,z(\epsilon)\big)\,d\epsilon\right)ds \\
&> 0.
\end{aligned} \tag{3.13}$$

□

3.1 Uniqueness result

Theorem 3.3 *Let us assume that assumptions* $(\mathcal{R}_1)$, $(\mathcal{R}_2)$ *and* $(\mathcal{R}_4)$ *are satisfied. Then there exists a unique solution for Eq.* (1.1) *on* $[0,1]$, *if*

$$\Delta^* = \mathcal{L}(q-1)\left[\frac{\overline{\mathsf{M}}}{\Gamma(\zeta+1)}\right]^{q-2}\left[\frac{1}{\Gamma(\sigma+1)}+\frac{1}{\Gamma(\sigma)}\right]\left[\frac{1}{\Gamma(\zeta+1)}\right]^{q-1} \le 1. \tag{3.14}$$

Proof We prove the uniqueness result for $p \ge 2$.

By (3.10) and, for all $t \in [0,1]$,

$$\begin{aligned} I^\zeta\left[\psi^*\left(t,\mathfrak{z}(t)\right)\right] &= \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\psi^*\left(s,\mathfrak{z}(s)\right)\,ds \\ &\le \frac{1}{\Gamma(\zeta)}\int_0^t (t-s)^{\zeta-1}\overline{\mathsf{M}}\,ds \le \frac{\overline{\mathsf{M}}}{\Gamma(\zeta+1)}. \end{aligned} \tag{3.15}$$

For each $\mathfrak{z} \in (\overline{\mathcal{A}^*(r)}) \setminus \mathcal{A}^*(r)$ and using (3.15) we have

$$\begin{aligned} &\left\|\mathcal{G}^*\mathfrak{z} - \mathcal{G}^*\upsilon\right\| \\ &\quad= \max_{t\in[0,1]}\left|\sum_{i=1}^m \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\left(\epsilon,\mathfrak{z}(\epsilon)\right)d\epsilon\right)ds\right. \\ &\qquad\left. - \sum_{i=1}^m \lambda_i(t) - \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\left(\epsilon,\upsilon(\epsilon)\right)d\epsilon\right)ds\right| \\ &\quad\le \int_0^1 \left|\mathcal{H}^\sigma(1,s)\right|\left|\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\left(\epsilon,\mathfrak{z}(\epsilon)\right)d\epsilon\right)ds\right. \\ &\qquad\left. - \phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\left(\epsilon,\upsilon(\epsilon)\right)d\epsilon\right)ds\right| \\ &\quad\le (q-1)\left[\frac{\overline{\mathsf{M}}}{\Gamma(\zeta+1)}\right]^{q-2}\int_0^1 \left|\mathcal{H}^\sigma(1,s)\right|\left|\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\left(\epsilon,\mathfrak{z}(\epsilon)\right)d\epsilon\right)ds\right. \\ &\qquad\left. - \left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*\left(\epsilon,\upsilon(\epsilon)\right)d\epsilon\right)ds\right| \\ &\quad\le \mathcal{L}(q-1)\left[\frac{\overline{\mathsf{M}}}{\Gamma(\zeta+1)}\right]^{q-2}\left[\frac{1}{\Gamma(\sigma+1)}+\frac{1}{\Gamma(\sigma)}\right]\left[\frac{1}{\Gamma(\zeta+1)}\right]^{q-1}\|\mathfrak{z}-\upsilon\| \\ &\quad= \Delta^* \quad \forall t \in [0,1], \end{aligned}$$

but in (3.14) we assumed that $\Delta^* < 1$. This proves that $\mathcal{G}^*$ is a contraction map. Hence by the Banach contraction mapping principle there exists a unique fixed point for operator $\mathcal{G}^*$. Hence, there exists a unique solution of Eq. (1.1) on $[0,1]$. □

4 Hyers–Ulam stability

Here, we analyze the HU stability of (1.1). We define the HU stability as follows.

Definition 4.1 ([60]) The integral equation (3.1) is said to be HU stable if there exists a non-negative constant Δ^*, for some fixed non-negative constant $\gamma^* > 0$ satisfying the following:

If

$$\left| z(t) - \sum_{i=1}^{m} \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*(\epsilon, z(\epsilon))\,d\epsilon\right)ds\right| \le \gamma^*, \tag{4.1}$$

then there exists a function $\upsilon(t)$, which is continuous and satisfies

$$\upsilon(t) = \sum_{i=1}^{m} \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*(\epsilon, \upsilon(\epsilon))\,d\epsilon\right)ds, \tag{4.2}$$

with

$$\left| z(t) - \upsilon(t)\right| \le \Delta^*\gamma^*. \tag{4.3}$$

Theorem 4.1 *The FDE (1.1) with ϕ_p^* operator is HU stable for $p > 2$ provided that $(\mathcal{R}_1)$, $(\mathcal{R}_2)$ and $(\mathcal{R}_4)$ are satisfied.*

Proof For the HU stability of the problem (1.1), we prove that for the integral equation (3.1), with assumptions $(\mathcal{R}_1)$, $(\mathcal{R}_2)$ and $(\mathcal{R}_4)$. we have

$$\begin{aligned}
&\left| z(t) - \upsilon(t)\right| \\
&= \max_{t\in[0,1]}\left|\sum_{i=1}^{m} \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*(\epsilon, z(\epsilon))\,d\epsilon\right)ds\right. \\
&\quad \left. - \sum_{i=1}^{m} \lambda_i(t) - \int_0^1 \mathcal{H}^\sigma(t,s)\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*(\epsilon, \upsilon(\epsilon))\,d\epsilon\right)ds\right| \\
&\le \int_0^1 \left|\mathcal{H}^\sigma(1,s)\right|\left|\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*(\epsilon, z(\epsilon))\,d\epsilon\right)\right. \\
&\quad \left. - \phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*(\epsilon, \upsilon(\epsilon))\,d\epsilon\right)ds\right| \\
&\le (q-1)\left[\frac{\overline{\mathsf{M}}}{\Gamma(\zeta+1)}\right]^{q-2}\int_0^1 \left|\mathcal{H}^\sigma(1,s)\right|\left|\left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*(\epsilon, z(\epsilon))\,d\epsilon\right)ds\right. \\
&\quad \left. - \left(\frac{1}{\Gamma(\zeta)}\int_0^s (s-\epsilon)^{\zeta-1}\psi^*(\epsilon, \upsilon(\epsilon))\,d\epsilon\right)ds\right| \\
&\le \mathcal{L}(q-1)\left[\frac{\overline{\mathsf{M}}}{\Gamma(\zeta+1)}\right]^{q-2}\left[\frac{1}{\Gamma(\sigma+1)} + \frac{1}{\Gamma(\sigma)}\right]\left[\frac{1}{\Gamma(\zeta+1)}\right]^{q-1}\|z - \upsilon\| \\
&= \Delta^* \quad \forall t \in [0,1].
\end{aligned} \tag{4.4}$$

Hence using (4.4) Eq. (3.1) is HU stable. As a result, the FDE (1.1) is HU stable. □

5 Example

Here, we present some examples to illustrate our results.

Example 5.1 Let us take the following FDE:

$$\begin{cases} {}^c\mathcal{D}^\zeta \phi_p^*[{}^c\mathcal{D}^\sigma (\mathfrak{z}(t) - \sum_{i=1}^m \lambda_i(t))] = -\psi^*(t, \mathfrak{z}(t)), \quad t \in [0,1], \\ \phi_p^*[{}^c\mathcal{D}^\sigma \mathfrak{z}(t) - \sum_{i=1}^m \lambda_i(t)]|_{t=0} = 0, \qquad \mathfrak{z}(0) = \sum_{i=1}^m \lambda_i(0), \\ \mathfrak{z}'(1) = \sum_{i=1}^m \lambda_i'(1), \qquad \mathfrak{z}^j(0) = \sum_{i=1}^m \lambda_i^j(0) \quad \text{for } j = 2,3,4,\dots,n-1, \end{cases} \tag{5.1}$$

where $\zeta = 0.5$, $\sigma = 3.6$, $p = 5$, $q = 1.25$, $m = 3$. We have$\sum_{i=1}^m \lambda_i(t) = \frac{1}{t^2+100+i}$ $\forall t \in [0,1]$, $\psi^*(t,\mathfrak{z}(t)) = \frac{1}{t^2+20}[|\mathfrak{z}|^5 + \frac{1}{(1+7|\mathfrak{z}|^{\frac{5}{39}})}]$.

Let us consider

$$\begin{aligned} \overline{\mathsf{M}} = \phi_{\max}^*(t,r) &= \max_{t\in[0,1]}\left\{\frac{1}{t^2+20}\left[|\mathfrak{z}|^5 + \frac{1}{(1+7|\mathfrak{z}|^{\frac{5}{39}})}\right] : t^{\frac{13}{5}} r \le \mathfrak{z} \le r\right\} \\ &\le \max_{t\in[0,1]}\left(\frac{1}{t^2+20}\left[|r|^5 + \frac{1}{(1+7|t^{\frac{1}{3}} r^{\frac{5}{39}}|}\right]\right) \\ &\le 0.1 \quad \forall t \in [0,1], r = b = 1, \\ \phi_{\min}^*(t,r) &= \min_{t\in[0,1]}\left\{\frac{1}{t^2+20}\left[|\mathfrak{z}|^5 + \frac{1}{(1+7|\mathfrak{z}|^{\frac{5}{39}})}\right] : t^{\frac{13}{5}} r \le \mathfrak{z} \le r\right\} \\ &\ge \min_{t\in[0,1]}\left(\frac{1}{t^2+20}\left[\left|t^{13} r^5\right| + \frac{1}{(1+7|r^{\frac{5}{39}}|}\right]\right) \\ &\ge 0.0122503 \quad \forall t \in [0,1], r = a = \frac{1}{1000}. \end{aligned}$$

Then

$$\begin{aligned} &\max_{t\in[0,1]}\left(\sum_{i=1}^m \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,\mathfrak{z}(\epsilon)\big)\,d\epsilon\right)d\mathfrak{s}\right) \\ &\quad\le 0.115663 + \int_0^1 \mathcal{H}^\sigma(1,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\epsilon)^{\zeta-1}\phi_{\max}^*(t,\mathfrak{b})d\epsilon\right)d\mathfrak{s} \\ &\quad\le 0.115663 + \int_0^1 \mathcal{H}^\sigma(1,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\epsilon)^{\zeta-1}\phi_{\max}^*(t,1)d\epsilon\right)d\mathfrak{s} \\ &\quad\le 0.115663 + \int_0^1 \mathcal{H}^\sigma(1,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\epsilon)^{\zeta-1}\phi_{\max}^*(t,1)d\epsilon\right)d\mathfrak{s} \\ &\quad\le 0.115663 + \overline{\mathsf{M}}^{q-1}\left[\frac{1}{\Gamma(\sigma+1)} + \frac{1}{\Gamma(\sigma)}\right]\left[\frac{1}{\Gamma(\zeta+1)}\right]^{q-1} \\ &\quad\le 0.314901 < 1. \end{aligned} \tag{5.2}$$

Also we have

$$\begin{aligned} &\min_{t\in[0,1]}\left(\sum_{i=1}^m \lambda_i(t) + \int_0^1 \mathcal{H}^\sigma(t,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\epsilon)^{\zeta-1}\psi^*\big(\epsilon,\mathfrak{z}(\epsilon)\big)\,d\epsilon\right)d\mathfrak{s}\right) \\ &\quad\ge 0.029414 + \int_0^1 \mathcal{H}^\sigma(1,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}} (\mathfrak{s}-\epsilon)^{\zeta-1}\phi_{\min}^*(t,\mathfrak{a})d\epsilon\right)d\mathfrak{s} \end{aligned}$$

$$\begin{aligned} &\geq 0.029414 + \int_0^1 \mathcal{H}^\sigma(1,\mathfrak{s})\phi_q^*\left(\frac{1}{\Gamma(\zeta)}\int_0^{\mathfrak{s}}(\mathfrak{s}-\epsilon)^{\zeta-1}\phi_{\min}^*\left(t,\frac{1}{1000}\right)d\epsilon\right)d\mathfrak{s} \\ &\geq 0.029414 > \frac{1}{1000}. \end{aligned} \tag{5.3}$$

Using Theorem 3.2, Eq. (5.1) has a solution $\mathfrak{z}^*$ which satisfies $\frac{1}{1000} \leq \|\mathfrak{z}\| \leq 1$.

Example 5.2 Let us take the following FDE:

$$\begin{cases} {}^c\mathcal{D}^\zeta\phi_p^*[{}^c\mathcal{D}^\sigma(\mathfrak{z}(t) - \sum_{i=1}^m \lambda_i(t))] = -\psi^*(t,\mathfrak{z}(t)), \quad t \in [0,1], \\ \phi_p^*[{}^c\mathcal{D}^\sigma\mathfrak{z}(t) - \sum_{i=1}^m \lambda_i(t)]|_{t=0} = 0, \qquad \mathfrak{z}(0) = \sum_{i=1}^m \lambda_i(0), \\ \mathfrak{z}'(1) = \sum_{i=1}^m \lambda_i'(1), \qquad \mathfrak{z}^j(0) = \sum_{i=1}^m \lambda_i^j(0) \quad \text{for } j = 2,3,4,\dots,n-1, \end{cases} \tag{5.4}$$

where $\zeta = 0.5$, $\sigma = 3.5$, $p = 5$, $q = 1.25$, $m = 3$. We have $\sum_{i=1}^m \lambda_i(t) = \frac{1}{t+100+i}$ $\forall t \in [0,1]$. $\psi^*(t,\mathfrak{z}(t)) = \frac{1}{t^2+20}[\frac{|\mathfrak{z}|}{(1+|\mathfrak{z}|)}]$, which satisfies the assumption $(\mathcal{R}_4)$ and where $\mathcal{L} = \frac{1}{20}$, that is,

$$\left|\psi^*\left(t,\mathfrak{z}(t)\right) - \psi^*\left(t,\upsilon(t)\right)\right| \leq \frac{1}{20}|\mathfrak{z} - \upsilon|.$$

Consider

$$\begin{aligned} \overline{\mathsf{M}} = \phi_{\max}^*(t,r) &= \max_{t\in[0,1]}\left\{\frac{1}{t^2+20}\left[\frac{|\mathfrak{z}|}{(1+|\mathfrak{z}|)}\right] : t^{\frac{5}{2}}r \leq \mathfrak{z} \leq r\right\} \\ &\leq \max_{t\in[0,1]}\left(\frac{1}{t^2+20}\left[\frac{|r|}{(1+|t^{\frac{5}{2}}r|)}\right]\right) \\ &\leq 0.05 \quad \forall t \in [0,1], r = b = 1. \end{aligned}$$

Then

$$\Delta^* = \mathcal{L}(q-1)\left[\frac{\overline{\mathsf{M}}}{\Gamma(\zeta+1)}\right]^{q-2}\left[\frac{1}{\Gamma(\sigma+1)} + \frac{1}{\Gamma(\sigma)}\right]\left[\frac{1}{\Gamma(\zeta+1)}\right]^{q-1} \approx 0.04306 \leq 1. \tag{5.5}$$

Hence there exists a unique solution of Eq. (5.4) on $[0,1]$ by Theorem 3.2. We can easily check all the conditions of Theorem 4.1 are also satisfied. As a result, Eq. (5.4) is HU stable.

6 Conclusion

In this investigation, the existence results for general FDEs (1.1) involving a ϕ_p^*-Laplacian operator is established by using Guo–Krasnoselskii's fixed point theorem [58]. The uniqueness results are proved by using the Banach contraction mapping principle and HU stability is also evaluated. The properties of the Green function also proved. The validity of our result is illustrated by examples. Also, one can study the multiple solutions, periodic solutions and controllability for the proposed general non-linear FDEs.

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Author details
[1]Department of Mathematics and Statistics, School of Basic and Applied Sciences, Central University of Punjab, Bathinda, India. [2]Department of Mathematics, Faculty of Art and Sciences, Cankaya University, Cankaya, Turkey. [3]Institute of Space Science, Magurele-Bucharest, Magurele-Bucharest, Romania. [4]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan. [5]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia.

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