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Some fractional calculus findings associated with the incomplete I-functions

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Abstract

In this article, several interesting properties of the incomplete I-functions associated with the Marichev–Saigo–Maeda (MSM) fractional operators are studied and investigated. It is presented that the order of the incomplete I-functions increases about the utilization of the above-mentioned operators toward the power multiple of the incomplete I-functions. Further, the Caputo-type MSM fractional order differentiation for the incomplete I-functions is studied and investigated. Saigo, Riemann–Liouville, and Erdélyi–Kober fractional operators are also discussed as specific cases.

MSC: 33B20; 33E20; 26A33

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1 Introduction and preliminaries

Recently, Jangid et al. [11] defined a new family of incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and ${}^{\Gamma}I_{p,q}^{m,n}(z)$. Incomplete I-functions are the natural generalization of the I-function defined by Rathie [27]. They are an expansion of a familiar Fox's H-function [6] and many other special functions. Fractional calculus for the variety of special functions is being widely used in mathematical modeling, statistical distribution, wireless communication, and engineering sciences (see [1, 3–5, 7–10, 12, 14–16, 18, 20–23, 28, 31–34]). The incomplete I-functions in the form of Mellin–Barnes type contour integrals are defined as

$$
\begin{aligned}
{}^{\gamma}I_{p,q}^{m,n}(z) &= {}^{\gamma}I_{p,q}^{m,n}\left[z \;\middle|\; \begin{matrix} (a_1,\sigma_1;A_1:x),(a_2,\sigma_2;A_2),\dots,(a_p,\sigma_p;A_p) \\ (b_1,\rho_1;B_1),\dots,(b_q,\rho_q;B_q) \end{matrix} \right] \\
&= {}^{\gamma}I_{p,q}^{m,n}\left[z \;\middle|\; \begin{matrix} (a_1,\sigma_1;A_1:x),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{matrix} \right] \\
&= \frac{1}{2\pi i}\int_{\$} \phi(s,x)z^s\,ds
\end{aligned}
\tag{1}
$$

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and

$$
\begin{aligned}
{}^{\Gamma}I_{p,q}^{m,n}(z) &= {}^{\Gamma}I_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:x),(a_2,\sigma_2;A_2),\ldots,(a_p,\sigma_p;A_p) \\ (b_1,\rho_1;B_1),\ldots,(b_q,\rho_q;B_q) \end{array}\right] \\
&= {}^{\Gamma}I_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:x),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{array}\right] \\
&= \frac{1}{2\pi i}\int_{\$}\Phi(s,x)z^s\,ds
\end{aligned}
\tag{2}
$$

for all $z \neq 0$, where

$$
\phi(s,x) = \frac{\{\gamma(1-a_1+\sigma_1 s,x)\}^{A_1}\prod_{j=1}^{m}\{\Gamma(b_j-\rho_j s)\}^{B_j}\prod_{j=2}^{n}\{\Gamma(1-a_j+\sigma_j s)\}^{A_j}}{\prod_{j=n+1}^{p}\{\Gamma(a_j-\sigma_j s)\}^{A_j}\prod_{j=m+1}^{q}\{\Gamma(1-b_j+\rho_j s)\}^{B_j}}
\tag{3}
$$

and

$$
\Phi(s,x) = \frac{\{\Gamma(1-a_1+\sigma_1 s,x)\}^{A_1}\prod_{j=1}^{m}\{\Gamma(b_j-\rho_j s)\}^{B_j}\prod_{j=2}^{n}\{\Gamma(1-a_j+\sigma_j s)\}^{A_j}}{\prod_{j=n+1}^{p}\{\Gamma(a_j-\sigma_j s)\}^{A_j}\prod_{j=m+1}^{q}\{\Gamma(1-b_j+\rho_j s)\}^{B_j}},
\tag{4}
$$

where $\gamma(\cdot,x)$ and $\Gamma(\cdot,x)$ are the lower and upper incomplete gamma functions defined in (6) and (7). The incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and ${}^{\Gamma}I_{p,q}^{m,n}(z)$ exist for all $x \geq 0$ under the same contour and conditions as stated in Rathie [27]. The incomplete I-functions fulfill the following relation (known as decomposition formula):

$$
{}^{\gamma}I_{p,q}^{m,n}(z) + {}^{\Gamma}I_{p,q}^{m,n}(z) = I_{p,q}^{m,n}(z)
\tag{5}
$$

for the familiar I-function given by Rathie [27]. Additionally, if we set $x = 0$ in (2), then we obtain the I-function [27].

In the sequence we shall use the following statements and descriptions.

The familiar lower and upper incomplete gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$, respectively, are laid out as follows:

$$
\gamma(s,x) = \int_0^x y^{s-1}e^{-y}\,dy \quad \left(\Re(s) > 0; x \geqq 0\right)
\tag{6}
$$

and

$$
\Gamma(s,x) = \int_x^{\infty} y^{s-1}e^{-y}\,dy \quad \left(x \geqq 0; \Re(s) > 0 \text{ when } x = 0\right).
\tag{7}
$$

These incomplete gamma functions fulfill the following relation (known as decomposition formula):

$$
\gamma(s,x) + \Gamma(s,x) = \Gamma(s) \quad \left(\Re(s) > 0\right).
\tag{8}
$$

In this article, several fractional calculus results associated with the incomplete I-functions are obtained. For $\sigma,\sigma',\rho,\rho',\eta \in \mathbb{C}$ and $x > 0$ with $\Re(\eta) > 0$, the left- and right-

hand sided MSM fractional integral operators (see [19]) are defined as

$$
\left(\mathcal{I}_{0+}^{\sigma,\sigma',\rho,\rho',\eta}f\right)(x) = \frac{x^{-\sigma}}{\Gamma(\eta)} \int_0^x (x-y)^{\eta-1} y^{-\sigma'}
$$
$$
\times F_3\left(\sigma,\sigma',\rho,\rho';\eta;1-\frac{y}{x},1-\frac{x}{y}\right)f(y)\,dy \tag{9}
$$

and

$$
\left(\mathcal{I}_{-}^{\sigma,\sigma',\rho,\rho',\eta}f\right)(x) = \frac{x^{-\sigma'}}{\Gamma(\eta)} \int_x^\infty (y-x)^{\eta-1} y^{-\sigma}
$$
$$
\times F_3\left(\sigma,\sigma',\rho,\rho';\eta;1-\frac{x}{y},1-\frac{y}{x}\right)f(y)\,dy, \tag{10}
$$

respectively. The left- and right-hand sided MSM fractional differential operators (see [30]) are defined as

$$
\left(\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta}f\right)(x) = \left(\frac{d}{dx}\right)^\alpha \left(\mathcal{I}_{0+}^{-\sigma',-\sigma,-\rho'+\alpha,-\rho,-\eta+\alpha}f\right)(x) \tag{11}
$$

and

$$
\left(\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta}f\right)(x) = \left(-\frac{d}{dx}\right)^\alpha \left(\mathcal{I}_{-}^{-\sigma',-\sigma,-\rho',-\rho+\alpha,-\eta+\alpha}f\right)(x), \tag{12}
$$

where $\alpha = [\Re(\eta)] + 1$ and $[\Re(\eta)]$ symbolizes the integer part in regard to $\Re(\eta)$. If $\max\{|x|,|y|\} < 1$, then the third Appell function F_3 is defined as

$$
F_3\left(\sigma,\sigma',\rho,\rho';\eta;x;y\right) = \sum_{i,j=0}^\infty \frac{(\sigma)_i(\sigma')_j(\rho)_i(\rho')_j}{(\eta)_{i+j}} \frac{x^i y^j}{i!j!}. \tag{13}
$$

Here, $(\sigma)_n$ is the well-known Pochhammer symbol. Recent papers [2, 17, 25] include a comprehensive demonstration related to the MSM operators together with their properties and applications. Saigo [29] introduced the fractional operators involving Gauss hypergeometric function $_2F_1()$. For $\sigma,\rho,\eta \in \mathbb{C}$, $x > 0$ along with $\Re(\sigma) > 0$, the left- and right-hand sided Saigo fractional integral operators are described as

$$
\left(\mathcal{I}_{0+}^{\sigma,\rho,\eta}f\right)(x) = \frac{x^{-\sigma-\rho}}{\Gamma(\sigma)} \int_0^x (x-y)^{\sigma-1} {}_2F_1\left(\sigma+\rho,-\eta;\sigma;1-\frac{y}{x}\right)f(y)\,dy \tag{14}
$$

and

$$
\left(\mathcal{I}_{-}^{\sigma,\rho,\eta}f\right)(x) = \frac{1}{\Gamma(\sigma)} \int_x^\infty (y-x)^{\sigma-1} y^{-\sigma-\rho} {}_2F_1\left(\sigma+\rho,-\eta;\sigma;1-\frac{x}{y}\right)f(y)\,dy, \tag{15}
$$

respectively. The left-hand and right-hand sided Saigo differential operators are defined as

$$
\left(\mathcal{D}_{0+}^{\sigma,\rho,\eta}f\right)(x) = \left(\frac{d}{dx}\right)^{[\Re(\sigma)]+1} \left(\mathcal{I}_{0+}^{-\sigma+[\Re(\sigma)]+1,-\rho-[\Re(\sigma)]-1,\sigma+\eta-[\Re(\sigma)]-1}f\right)(x) \tag{16}
$$

and

$$\left(\mathcal{D}_-^{\sigma,\rho,\eta}f\right)(x) = \left(-\frac{d}{dx}\right)^{[\Re(\sigma)]+1}\left(\mathcal{I}_-^{-\sigma+[\Re(\sigma)]+1,-\rho-[\Re(\sigma)]-1,\sigma+\eta}f\right)(x). \tag{17}$$

For $\rho = -\sigma$ and $\rho = 0$ in (14)–(17), the Riemann–Liouville and Erdélyi–Kober fractional operators are obtained respectively (for more explanation see [15]). $_2F_1$ is associated with F_3 as

$$F_3(\sigma, \gamma - \sigma, \rho, \gamma - \rho; \gamma; x; y) = {_2F_1}(\sigma, \rho; \gamma; x + y - xy).$$

The MSM fractional operators (9)–(12) are connected to Saigo operators (14)–(17) by

$$\left(\mathcal{I}_{0+}^{\sigma,0,\rho,\rho',\eta}f\right)(x) = \left(\mathcal{I}_{0+}^{\eta,\sigma-\eta,-\rho}f\right)(x),$$
$$\left(\mathcal{I}_-^{\sigma,0,\rho,\rho',\eta}f\right)(x) = \left(\mathcal{I}_-^{\eta,\sigma-\eta,-\rho}f\right)(x), \tag{18}$$

and

$$\left(\mathcal{D}_{0+}^{0,\sigma',\rho,\rho',\eta}f\right)(x) = \left(\mathcal{D}_{0+}^{\eta,\sigma'-\eta,\rho'-\eta}f\right)(x),$$
$$\left(\mathcal{D}_-^{0,\sigma',\rho,\rho',\eta}f\right)(x) = \left(\mathcal{D}_-^{\eta,\sigma'-\eta,\rho'-\eta}f\right)(x). \tag{19}$$

The following are well-known results (see [30]) and will be needed in proving the subsequent theorems.

Lemma 1.1 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda \in \mathbb{C}$ *and* $\Re(\eta) > 0$.
(a) *If* $\Re(\lambda) > \max\{0, \Re(\sigma' - \rho'), \Re(\sigma + \sigma' + \rho - \eta)\}$, *then*

$$\left(\mathcal{I}_{0+}^{\sigma,\sigma',\rho,\rho',\eta}t^{\lambda-1}\right)(x) = x^{-\sigma-\sigma'+\eta+\lambda-1}$$
$$\times \frac{\Gamma(\lambda)\Gamma(-\sigma'+\rho'+\lambda)\Gamma(-\sigma-\sigma'-\rho+\eta+\lambda)}{\Gamma(\rho'+\lambda)\Gamma(-\sigma-\sigma'+\eta+\lambda)\Gamma(-\sigma'-\rho+\eta+\lambda)}. \tag{20}$$

(b) *If* $\Re(\lambda) > \max\{\Re(\rho), \Re(-\sigma - \sigma' + \eta), \Re(-\sigma - \rho' + \eta)\}$, *then*

$$\left(\mathcal{I}_-^{\sigma,\sigma',\rho,\rho',\eta}t^{-\lambda}\right)(x) = x^{-\sigma-\sigma'+\eta-\lambda}$$
$$\times \frac{\Gamma(-\rho+\lambda)\Gamma(\sigma+\sigma'-\eta+\lambda)\Gamma(\sigma+\rho'-\eta+\lambda)}{\Gamma(\lambda)\Gamma(\sigma-\rho+\lambda)\Gamma(\sigma+\sigma'+\rho'-\eta+\lambda)}. \tag{21}$$

Motivated by the work of Srivastava et al. [32], we have derived the fractional calculus results associated with the incomplete I-functions. In Sect. 2, MSM fractional order integrals of left-hand and right-hand type are derived for the incomplete I-functions. In Sect. 3, MSM fractional order derivatives of left-hand and right-hand type are derived for the incomplete I-functions. In Sect. 4, Caputo-type MSM fractional order derivatives of left-hand and right-hand type are derived for the incomplete I-functions. In Sect. 5, we have derived the special cases of the incomplete I-functions.

2 Fractional integration of incomplete *I*-functions

Some fractional integrations pertaining to the incomplete *I*-functions are presented in this part. First, we shall investigate the MSM fractional order integrals of left-hand side type for the incomplete *I*-functions.

Theorem 2.1 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda, c \in \mathbb{C}$ *be so that* $\Re(\eta), \mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(\sigma' - \rho'),$ $\Re(\sigma + \sigma' + \rho - \eta)\}$. *Thereupon, for* $x > 0$,

$$
\left(\mathcal{I}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} z^{\lambda-1} {}^{\gamma} I_{p,q}^{m,n} \left[cz^{\mu} \; \middle| \; \begin{array}{c} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{-\sigma-\sigma'+\eta+\lambda-1}
$$

$$
\times {}^{\gamma} I_{p+3,q+3}^{m,n+3} \left[cx^{\mu} \; \middle| \; \begin{array}{c} (a_1,\sigma_1;A_1:y),(1-\lambda,\mu;1),(1+\sigma'-\rho'-\lambda,\mu;1), \\ (b_j,\rho_j;B_j)_{1,q},(1-\rho'-\lambda,\mu;1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1+\sigma+\sigma'+\rho-\eta-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (1+\sigma+\sigma'-\eta-\lambda,\mu;1),(1+\sigma+\sigma'+\rho-\eta-\lambda,\mu;1) \end{array} \right], \tag{22}
$$

provided every member in (22) *does exist*.

Proof The LHS of (22) is given by

$$
\left(\mathcal{I}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} \left(z^{\lambda-1} \frac{1}{2\pi i} \int_{\$} \phi(s,y) c^s z^{\mu s} \, ds \right) \right)(x), \tag{23}
$$

where $\phi(s,y)$ is given in (3). Interchanging the order of integration in the above equation yields

$$
\frac{1}{2\pi i} \int_{\$} \phi(s,y) c^s \left(\mathcal{I}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} z^{\lambda+\mu s-1} \right)(x) \, ds, \tag{24}
$$

using the results (20) and (3), we get the RHS of (22). □

   The properties given below are immediate consequences of definitions (1), (2), and (20), and consequently they are stated without proof here.

Theorem 2.2 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda, c \in \mathbb{C}$ *be such that* $\Re(\eta), \mu > 0$ *and* $\Re(\lambda) > \max\{0,$ $\Re(\sigma' - \rho'), \Re(\sigma + \sigma' + \rho - \eta)\}$. *Thereupon, for* $x > 0$,

$$
\left(\mathcal{I}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} z^{\lambda-1} {}^{\Gamma} I_{p,q}^{m,n} \left[cz^{\mu} \; \middle| \; \begin{array}{c} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{-\sigma-\sigma'+\eta+\lambda-1} \times
$$

$$
\times {}^{\Gamma} I_{p+3,q+3}^{m,n+3} \left[cx^{\mu} \; \middle| \; \begin{array}{c} (a_1,\sigma_1;A_1:y),(1-\lambda,\mu;1),(1+\sigma'-\rho'-\lambda,\mu;1), \\ (b_j,\rho_j;B_j)_{1,q},(1-\rho'-\lambda,\mu;1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1+\sigma+\sigma'+\rho-\eta-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (1+\sigma+\sigma'-\eta-\lambda,\mu;1),(1+\sigma+\sigma'+\rho-\eta-\lambda,\mu;1) \end{array} \right], \tag{25}
$$

provided every member in (25) *does exist*.

In accordance with (18) and rearranging the involved parameters, we have the subsequent results for Saigo operators.

Corollary 1 *Let* $\sigma, \rho, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\Re(\sigma), \mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(\rho - \eta)\}$. *Therefore, for* $x > 0$,

$$
\left(\mathcal{I}_{0+}^{\sigma,\rho,\eta} z^{\lambda-1} \gamma I_{p,q}^{m,n} \left[cz^\mu \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{-\rho+\lambda-1} \gamma I_{p+2,q+2}^{m,n+2} \left[cx^\mu \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1-\lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1+\rho-\lambda, \mu; 1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1+\rho-\eta-\lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1-\sigma-\eta-\lambda, \mu; 1) \end{array} \right], \tag{26}
$$

provided every member in (26) *does exist.*

Corollary 2 *Let* $\sigma, \rho, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\Re(\sigma), \mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(\rho - \eta)\}$. *Thereupon, for* $x > 0$,

$$
\left(\mathcal{I}_{0+}^{\sigma,\rho,\eta} z^{\lambda-1} \Gamma I_{p,q}^{m,n} \left[cz^\mu \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{-\rho+\lambda-1} \Gamma I_{p+2,q+2}^{m,n+2} \left[cx^\mu \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1-\lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1+\rho-\lambda, \mu; 1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1+\rho-\eta-\lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1-\sigma-\eta-\lambda, \mu; 1) \end{array} \right], \tag{27}
$$

provided every member in (27) *does exist.*

A similar type of image formulas associated with the Riemann–Liouville fractional integral are as follows.

Corollary 3 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\Re(\sigma), \mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\sigma - \eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{I}_{0+}^{\sigma,-\sigma,\eta} z^{\lambda-1} \gamma I_{p,q}^{m,n} \left[cz^\mu \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\sigma+\lambda-1} \gamma I_{p+1,q+1}^{m,n+1} \left[cx^\mu \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1-\lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1-\sigma-\lambda, \mu; 1) \end{array} \right], \tag{28}
$$

provided every member in (28) *does exist.*

Corollary 4 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\Re(\sigma), \mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\sigma - \eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{I}_{0+}^{\sigma,-\sigma,\eta} z^{\lambda-1} \Gamma I_{p,q}^{m,n} \left[cz^{\mu} \;\middle|\; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\sigma+\lambda-1} \Gamma I_{p+1,q+1}^{m,n+1} \left[cx^{\mu} \;\middle|\; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \sigma - \lambda, \mu; 1) \end{array} \right], \tag{29}
$$

provided every member in (29) *does exist.*

The corresponding corollary (1) introduces the Erdélyi–Kober fractional integral as follows.

Corollary 5 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\Re(\sigma), \mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{I}_{\eta,\sigma}^{+} z^{\lambda-1} \gamma I_{p,q}^{m,n} \left[cz^{\mu} \;\middle|\; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\lambda-1} \gamma I_{p+1,q+1}^{m,n+1} \left[cx^{\mu} \;\middle|\; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \sigma - \eta - \lambda, \mu; 1) \end{array} \right], \tag{30}
$$

provided every member in (30) *does exist.*

Corollary 6 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\Re(\sigma), \mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{I}_{\eta,\sigma}^{+} z^{\lambda-1} \Gamma I_{p,q}^{m,n} \left[cz^{\mu} \;\middle|\; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\lambda-1} \Gamma I_{p+1,q+1}^{m,n+1} \left[cx^{\mu} \;\middle|\; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \sigma - \eta - \lambda, \mu; 1) \end{array} \right], \tag{31}
$$

provided every member in (31) *does exist.*

The coming results lead to the right-hand sided MSM fractional order integrals of the incomplete I-functions.

Theorem 2.3 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda, c \in \mathbb{C}$ *be such that* $\Re(\eta), \mu > 0$ *and* $\Re(\lambda) > \max\{\Re(\rho), \Re(-\sigma - \sigma' + \eta), \Re(-\sigma - \rho' + \eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{I}_{-}^{\sigma,\sigma',\rho,\rho',\eta} z^{-\lambda} \gamma I_{p,q}^{m,n} \left[cz^{-\mu} \;\middle|\; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{-\sigma-\sigma'+\eta-\lambda}
$$

$$
\times {}^{\gamma} I_{p+3,q+3}^{m,n+3} \left[cx^{-\mu} \;\middle|\; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 + \rho - \lambda, \mu; 1), (1 - \sigma - \sigma' + \eta - \lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \lambda, \mu; 1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1 - \sigma - \rho' + \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1 - \sigma + \rho - \lambda, \mu; 1), (1 - \sigma - \sigma' - \rho' + \eta - \lambda, \mu; 1) \end{array} \right], \tag{32}
$$

provided every member in (32) *does exist.*

Proof The LHS of (32) is given by

$$
\left(\mathcal{I}_{-}^{\sigma,\sigma',\rho,\rho',\eta} \left(z^{-\lambda} \frac{1}{2\pi i} \int_{\math$} \phi(s,y) c^{s} z^{-\mu s} \, ds \right) \right)(x),
\tag{33}
$$

where $\phi(s,y)$ is given in (3). Interchanging the order of integration in the above equation yields

$$
\frac{1}{2\pi i} \int_{\math$} \phi(s,y) c^{s} \left(\mathcal{I}_{-}^{\sigma,\sigma',\rho,\rho',\eta} z^{-(\lambda+\mu s)} \right)(x) \, ds,
\tag{34}
$$

using the results (21) and (3), we get the RHS of (32). $\qquad\square$

Theorem 2.4 *Let* $\sigma,\sigma',\rho,\rho',\eta,\lambda,c \in \mathbb{C}$ *be such that* $\Re(\eta), \mu > 0$ *and* $\Re(\lambda) > \max\{\Re(\rho), \Re(-\sigma - \sigma' + \eta), \Re(-\sigma - \rho' + \eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{I}_{-}^{\sigma,\sigma',\rho,\rho',\eta} z^{-\lambda} \, {}^{\Gamma}I_{p,q}^{m,n} \left[cz^{-\mu} \;\middle|\; \begin{matrix} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{matrix} \right] \right)(x)
$$

$$
= x^{-\sigma-\sigma'+\eta-\lambda}
$$

$$
\times {}^{\Gamma}I_{p+3,q+3}^{m,n+3} \left[cx^{-\mu} \;\middle|\; \begin{matrix} (a_1,\sigma_1;A_1:y),(1+\rho-\lambda,\mu;1),(1-\sigma-\sigma'+\eta-\lambda,\mu;1), \\ (b_j,\rho_j;B_j)_{1,q},(1-\lambda,\mu;1), \end{matrix} \right.
$$

$$
\left. \begin{matrix} (1-\sigma-\rho'+\eta-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (1-\sigma+\rho-\lambda,\mu;1),(1-\sigma-\sigma'-\rho'+\eta-\lambda,\mu;1) \end{matrix} \right],
\tag{35}
$$

provided every member in (35) *does exist.*

The Saigo, Riemann–Liouville, and Erdélyi–Kober fractional order integrals of the incomplete I-functions are given below.

Corollary 7 *Let* $\sigma,\rho,\eta,\lambda,c \in \mathbb{C}$ *be such that* $\Re(\sigma), \mu > 0$ *and* $\Re(\lambda) > \max\{\Re(-\rho),\Re(-\eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{I}_{-}^{\sigma,\rho,\eta} z^{-\lambda\gamma} \, I_{p,q}^{m,n} \left[cz^{-\mu} \;\middle|\; \begin{matrix} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{matrix} \right] \right)(x)
$$

$$
= x^{-\rho-\lambda\gamma} I_{p+2,q+2}^{m,n+2} \left[cx^{-\mu} \;\middle|\; \begin{matrix} (a_1,\sigma_1;A_1:y),(1-\eta-\lambda,\mu;1), \\ (b_j,\rho_j;B_j)_{1,q},(1-\lambda,\mu;1), \end{matrix} \right.
$$

$$
\left. \begin{matrix} (1-\rho-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (1-\sigma-\rho-\eta-\lambda,\mu;1) \end{matrix} \right],
\tag{36}
$$

provided every member in (36) *does exist.*

Corollary 8 *Let $\sigma, \rho, \eta, \lambda, c \in \mathbb{C}$ be such that $\Re(\sigma), \mu > 0$ and $\Re(\lambda) > \max\{\Re(-\rho), \Re(-\eta)\}$. Then, for $x > 0$,*

$$
\left(\mathcal{I}_-^{\sigma,\rho,\eta} z^{-\lambda \, \Gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{-\rho-\lambda \, \Gamma} I_{p+2,q+2}^{m,n+2} \left[cx^{-\mu} \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:y),(1-\eta-\lambda,\mu;1), \\ (b_j,\rho_j;B_j)_{1,q},(1-\lambda,\mu;1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1-\rho-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (1-\sigma-\rho-\eta-\lambda,\mu;1) \end{array} \right], \tag{37}
$$

provided every member in (37) *does exist.*

Corollary 9 *Let $\sigma, \eta, \lambda, c \in \mathbb{C}$ be such that $\Re(\sigma), \mu > 0$ and $\Re(\lambda) > \max\{\Re(\sigma), \Re(-\eta)\}$. Then, for $x > 0$,*

$$
\left(\mathcal{I}_-^{\sigma,-\sigma,\eta} z^{-\lambda \gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\sigma-\lambda \gamma} I_{p+1,q+1}^{m,n+1} \left[cx^{-\mu} \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:y),(1+\sigma-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q},(1-\lambda,\mu;1) \end{array} \right], \tag{38}
$$

provided every member in (38) *does exist.*

Corollary 10 *Let $\sigma, \eta, \lambda, c \in \mathbb{C}$ be such that $\Re(\sigma), \mu > 0$ and $\Re(\lambda) > \max\{\Re(\sigma), \Re(-\eta)\}$. Then, for $x > 0$,*

$$
\left(\mathcal{I}_-^{\sigma,-\sigma,\eta} z^{-\lambda \, \Gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\sigma-\lambda \, \Gamma} I_{p+1,q+1}^{m,n+1} \left[cx^{-\mu} \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:y),(1+\sigma-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q},(1-\lambda,\mu;1) \end{array} \right], \tag{39}
$$

provided every member in (39) *does exist.*

Corollary 11 *Let $\sigma, \eta, \lambda, c \in \mathbb{C}$ be such that $\Re(\sigma), \mu > 0$ and $\Re(\lambda) > \max\{0, \Re(-\eta)\}$. Then the right-hand Erdélyi–Kober integration $\mathcal{K}_{\eta,\sigma}^-$ $(= \mathcal{I}_-^{\sigma,0,\eta})$ for $x > 0$,*

$$
\left(\mathcal{K}_{\eta,\sigma}^- z^{-\lambda \gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{-\lambda \gamma} I_{p+1,q+1}^{m,n+1} \left[cx^{-\mu} \,\middle|\, \begin{array}{c} (a_1,\sigma_1;A_1:y),(1-\eta-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q},(1-\sigma-\eta-\lambda,\mu;1) \end{array} \right], \tag{40}
$$

provided every member in (40) *does exist.*

Corollary 12 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\Re(\sigma), \mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\eta)\}$. *Then the right-hand Erdélyi–Kober integration* $\mathcal{K}_{\eta,\sigma}^{-}$ $(= \mathcal{I}_{-}^{\sigma,0,\eta})$ *for* $x > 0$ *is as follows:*

$$
\left(\mathcal{K}_{\eta,\sigma}^{-} z^{-\lambda} \, {}^{\Gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{-\lambda} \, {}^{\Gamma} I_{p+1,q+1}^{m,n+1} \left[cx^{-\mu} \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \sigma - \eta - \lambda, \mu; 1) \end{array} \right], \tag{41}
$$

provided every member in (41) *does exist.*

3 Fractional differentiation of incomplete *I*-functions

Right now, we study the MSM fractional order derivatives pertaining to the incomplete *I*-functions. The following are well-known results and will be used in subsequent theorems (see [30]).

Lemma 3.1 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda \in \mathbb{C}$.

(a) *If* $\Re(\lambda) > \max\{0, \Re(-\sigma + \rho), \Re(-\sigma - \sigma' - \rho' + \eta)\}$, *then*

$$
\left(\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} t^{\lambda-1} \right)(x) = x^{\sigma+\sigma'-\eta+\lambda-1}
$$

$$
\times \frac{\Gamma(\lambda)\Gamma(\sigma - \rho + \lambda)\Gamma(\sigma + \sigma' + \rho' - \eta + \lambda)}{\Gamma(-\rho + \lambda)\Gamma(\sigma + \sigma' - \eta + \lambda)\Gamma(\sigma + \rho' - \eta + \lambda)}. \tag{42}
$$

(b) *If* $\Re(\lambda) > \max\{\Re(-\rho'), \Re(\sigma' + \rho - \eta), \Re(\sigma + \sigma' - \eta) + [\Re(\eta)] + 1\}$, *then*

$$
\left(\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta} t^{-\lambda} \right)(x) = x^{\sigma+\sigma'-\eta-\lambda}
$$

$$
\times \frac{\Gamma(\rho' + \lambda)\Gamma(-\sigma - \sigma' + \eta + \lambda)\Gamma(-\sigma' - \rho + \eta + \lambda)}{\Gamma(\lambda)\Gamma(-\sigma' + \rho' + \lambda)\Gamma(-\sigma - \sigma' - \rho + \eta + \lambda)}. \tag{43}
$$

Now, we represent the left-hand sided MSM fractional derivatives of the incomplete *I*-functions.

Theorem 3.2 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\sigma + \rho), \Re(-\sigma - \sigma' - \rho' + \eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} z^{\lambda-1} \, {}^{\gamma} I_{p,q}^{m,n} \left[cz^{\mu} \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\sigma+\sigma'-\eta+\lambda-1}
$$

$$
\times {}^{\gamma} I_{p+3,q+3}^{m,n+3} \left[cx^{\mu} \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \lambda, \mu; 1), (1 - \sigma + \rho - \lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1 + \rho - \lambda, \mu; 1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1 - \sigma - \sigma' - \rho' + \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1 - \sigma - \sigma' + \eta - \lambda, \mu; 1), (1 - \sigma - \rho' + \eta - \lambda, \mu; 1) \end{array} \right], \tag{44}
$$

provided every member in (44) *does exist.*

Proof The LHS of (44) is given by

$$\left(\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} \left(z^{\lambda-1} \frac{1}{2\pi i} \int_{\mathscr{S}} \phi(s,y) c^s z^{\mu s} \, ds \right) \right)(x), \tag{45}$$

where $\phi(s,y)$ is given in (3). Interchanging the order of integration in the above equation yields

$$\frac{1}{2\pi i} \int_{\mathscr{S}} \phi(s,y) c^s \left(\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} z^{\lambda+\mu s-1} \right)(x) \, ds, \tag{46}$$

using the results (42) and (3), we get the RHS of (44). □

The properties given below are immediate consequences of definitions (1), (2), and (42), and consequently they are disposed without demonstration as follows.

Theorem 3.3 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\sigma + \rho), \Re(-\sigma - \sigma' - \rho' + \eta)\}$. *Then, for* $x > 0$,

$$\left(\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} z^{\lambda-1} {}^{\Gamma}I_{p,q}^{m,n} \left[cz^{\mu} \,\middle|\, \begin{matrix} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{matrix} \right] \right)(x)$$

$$= x^{\sigma+\sigma'-\eta+\lambda-1}$$

$$\times {}^{\Gamma}I_{p+3,q+3}^{m,n+3} \left[cx^{\mu} \,\middle|\, \begin{matrix} (a_1,\sigma_1;A_1:y),(1-\lambda,\mu;1),(1-\sigma+\rho-\lambda,\mu;1), \\ (b_j,\rho_j;B_j)_{1,q},(1+\rho-\lambda,\mu;1), \end{matrix} \right.$$

$$\left. \begin{matrix} (1-\sigma-\sigma'-\rho'+\eta-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (1-\sigma-\sigma'+\eta-\lambda,\mu;1),(1-\sigma-\rho'+\eta-\lambda,\mu;1) \end{matrix} \right], \tag{47}$$

provided every member in (47) does exist.

The coming image formulas for incomplete I-functions involving the Saigo, Riemann–Liouville, and Erdélyi–Kober fractional derivatives are as follows.

Corollary 13 *Let* $\sigma, \rho, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\sigma - \rho - \eta)\}$. *Then, for* $x > 0$,

$$\left(\mathcal{D}_{0+}^{\sigma,\rho,\eta} z^{\lambda-1} {}^{\gamma}I_{p,q}^{m,n} \left[cz^{\mu} \,\middle|\, \begin{matrix} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{matrix} \right] \right)(x)$$

$$= x^{\rho+\lambda-1} {}^{\gamma}I_{p+2,q+2}^{m,n+2} \left[cx^{\mu} \,\middle|\, \begin{matrix} (a_1,\sigma_1;A_1:y),(1-\lambda,\mu;1), \\ (b_j,\rho_j;B_j)_{1,q},(1-\rho-\lambda,\mu;1), \end{matrix} \right.$$

$$\left. \begin{matrix} (1-\sigma-\rho-\eta-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (1-\eta-\lambda,\mu;1) \end{matrix} \right], \tag{48}$$

provided every member in (48) does exist.

Corollary 14 *Let* $\sigma, \rho, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\sigma - \rho - \eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{D}_{0+}^{\sigma,\rho,\eta} z^{\lambda-1} \Gamma I_{p,q}^{m,n} \left[cz^{\mu} \left|
\begin{array}{c}
(a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\
(b_j, \rho_j; B_j)_{1,q}
\end{array} \right. \right] \right)(x)
$$

$$
= x^{\rho+\lambda-1} \Gamma I_{p+2,q+2}^{m,n+2} \left[cx^{\mu} \left|
\begin{array}{c}
(a_1, \sigma_1; A_1 : y), (1 - \lambda, \mu; 1), \\
(b_j, \rho_j; B_j)_{1,q}, (1 - \rho - \lambda, \mu; 1),
\end{array} \right. \right.
$$

$$
\left.
\begin{array}{c}
(1 - \sigma - \rho - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\
(1 - \eta - \lambda, \mu; 1)
\end{array} \right], \tag{49}
$$

provided every member in (49) *does exist.*

Corollary 15 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{D}_{0+}^{\sigma,-\sigma,\eta} z^{\lambda-1} \gamma I_{p,q}^{m,n} \left[cz^{\mu} \left|
\begin{array}{c}
(a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\
(b_j, \rho_j; B_j)_{1,q}
\end{array} \right. \right] \right)(x)
$$

$$
= x^{-\sigma+\lambda-1} \gamma I_{p+1,q+1}^{m,n+1} \left[cx^{\mu} \left|
\begin{array}{c}
(a_1, \sigma_1; A_1 : y), (1 - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\
(b_j, \rho_j; B_j)_{1,q}, (1 - \rho - \lambda, \mu; 1)
\end{array} \right. \right], \tag{50}
$$

provided every member in (50) *does exist.*

Corollary 16 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\eta)\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{D}_{0+}^{\sigma,-\sigma,\eta} z^{\lambda-1} \Gamma I_{p,q}^{m,n} \left[cz^{\mu} \left|
\begin{array}{c}
(a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\
(b_j, \rho_j; B_j)_{1,q}
\end{array} \right. \right] \right)(x)
$$

$$
= x^{-\sigma+\lambda-1} \Gamma I_{p+1,q+1}^{m,n+1} \left[cx^{\mu} \left|
\begin{array}{c}
(a_1, \sigma_1; A_1 : y), (1 - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\
(b_j, \rho_j; B_j)_{1,q}, (1 - \rho - \lambda, \mu; 1)
\end{array} \right. \right], \tag{51}
$$

provided every member in (51) *does exist.*

Corollary 17 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{0, \Re(-\sigma - \eta)\}$. *Then the left-hand sided Erdélyi–Kober differential* $\mathcal{D}_{\eta,\sigma}^{+} (= \mathcal{D}_{0+}^{\sigma,0,\eta})$ *of the incomplete I-function is given for* $x > 0$ *by*

$$
\left(\mathcal{D}_{\eta,\sigma}^{+} z^{\lambda-1} \gamma I_{p,q}^{m,n} \left[cz^{\mu} \left|
\begin{array}{c}
(a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\
(b_j, \rho_j; B_j)_{1,q}
\end{array} \right. \right] \right)(x)
$$

$$
= x^{\lambda-1} \times
$$

$$
\times \gamma I_{p+1,q+1}^{m,n+1} \left[cx^{\mu} \left|
\begin{array}{c}
(a_1, \sigma_1; A_1 : y), (1 - \sigma - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\
(b_j, \rho_j; B_j)_{1,q}, (1 - \eta - \lambda, \mu; 1)
\end{array} \right. \right], \tag{52}
$$

provided every member in (52) *does exist.*

Corollary 18 *Let $\sigma, \eta, \lambda, c \in \mathbb{C}$ be such that $\mu > 0$ and $\Re(\lambda) > \max\{0, \Re(-\sigma - \eta)\}$. Then the left-hand sided Erdélyi–Kober differential $\mathcal{D}_{\eta,\sigma}^{+}$ $(= \mathcal{D}_{0+}^{\sigma,0,\eta})$ of the incomplete I-function is given for $x > 0$ by*

$$
\left(\mathcal{D}_{\eta,\sigma}^{+} z^{\lambda-1} {}^{\Gamma} I_{p,q}^{m,n} \left[cz^{\mu} \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right) (x)
$$

$$
= x^{\lambda-1}
$$

$$
\times {}^{\Gamma} I_{p+1,q+1}^{m,n+1} \left[cx^{\mu} \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \sigma - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \eta - \lambda, \mu; 1) \end{array} \right], \tag{53}
$$

provided every member in (53) *does exist.*

The next result leads to the right-hand MSM fractional derivative related to the incomplete I-functions.

Theorem 3.4 *Let $\sigma, \sigma', \rho, \rho', \eta, \lambda, c \in \mathbb{C}$ be such that $\mu > 0$ and $\Re(\lambda) > \max\{\Re(-\rho'), \Re(\sigma' + \rho - \eta), \Re(\sigma + \sigma' - \eta) + [\Re(\eta)] + 1\}$. Then, for $x > 0$,*

$$
\left(\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta} z^{-\lambda\gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right) (x)
$$

$$
= x^{\sigma+\sigma'-\eta-\lambda}
$$

$$
\times {}^{\gamma} I_{p+3,q+3}^{m,n+3} \left[cx^{-\mu} \,\middle|\, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \rho' - \lambda, \mu; 1), (1 + \sigma + \sigma' - \eta - \lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \lambda, \mu; 1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1 + \sigma' + \rho - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1 + \sigma' - \rho' - \lambda, \mu; 1), (1 + \sigma + \sigma' + \rho - \eta - \lambda, \mu; 1) \end{array} \right], \tag{54}
$$

provided every member in (54) *does exist.*

Proof The LHS of (54) is given by

$$
\left(\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta} \left(z^{-\lambda} \frac{1}{2\pi i} \int_{\$} \phi(s,y) c^s z^{-\mu s} \, ds \right) \right)(x), \tag{55}
$$

where $\phi(s, y)$ is given in (3). Interchanging the order of integration in the above equation yields

$$
\frac{1}{2\pi i} \int_{\$} \phi(s,y) c^s \left(\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta} z^{-(\lambda+\mu s)} \right)(x) \, ds, \tag{56}
$$

using the results (43) and (3), we get the RHS of (54). □

Theorem 3.5 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{\Re(-\rho'),$ $\Re(\sigma' + \rho - \eta), \Re(\sigma + \sigma' - \eta) + [\Re(\eta)] + 1\}$. *Then, for* $x > 0$,

$$
\left(\mathcal{D}_-^{\sigma,\sigma',\rho,\rho',\eta} z^{-\lambda} \, {}^\Gamma I_{p,q}^{m,n} \left[cz^{-\mu} \, \middle| \, \begin{matrix} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{matrix} \right] \right)(x)
$$

$$
= x^{\sigma + \sigma' - \eta - \lambda}
$$

$$
\times {}^\Gamma I_{p+3,q+3}^{m,n+3} \left[cx^{-\mu} \, \middle| \, \begin{matrix} (a_1, \sigma_1; A_1 : y), (1 - \rho' - \lambda, \mu; 1), (1 + \sigma + \sigma' - \eta - \lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \lambda, \mu; 1), \end{matrix} \right.
$$

$$
\left. \begin{matrix} (1 + \sigma' + \rho - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1 + \sigma' - \rho' - \lambda, \mu; 1), (1 + \sigma + \sigma' + \rho - \eta - \lambda, \mu; 1) \end{matrix} \right], \tag{57}
$$

provided every member in (57) *does exist.*

The fractional order derivatives of Saigo, Riemann–Liouville, and Erdélyi–Kober type involving the incomplete I-functions are given as follows.

Corollary 19 *Let* $\sigma, \rho, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{\Re(-\sigma - \eta),$ $\Re(\rho) + [\Re(\sigma)] + 1\}$. *Then the right-hand Saigo derivative of the incomplete I-function is given for* $x > 0$ *by*

$$
\left(\mathcal{D}_-^{\sigma,\rho,\eta} z^{-\lambda} \gamma I_{p,q}^{m,n} \left[cz^{-\mu} \, \middle| \, \begin{matrix} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{matrix} \right] \right)(x)
$$

$$
= x^{\rho - \lambda \gamma} I_{p+2,q+2}^{m,n+2} \left[cx^{-\mu} \, \middle| \, \begin{matrix} (a_1, \sigma_1; A_1 : y), (1 - \sigma - \eta - \lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \lambda, \mu; 1), \end{matrix} \right.
$$

$$
\left. \begin{matrix} (1 + \rho - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1 + \rho - \eta - \lambda, \mu; 1) \end{matrix} \right], \tag{58}
$$

provided every member in (58) *does exist.*

Corollary 20 *Let* $\sigma, \rho, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{\Re(-\sigma - \eta),$ $\Re(\rho) + [\Re(\sigma)] + 1\}$. *Then the right-hand Saigo derivative of the incomplete I-function is given for* $x > 0$ *by*

$$
\left(\mathcal{D}_-^{\sigma,\rho,\eta} z^{-\lambda} {}^\Gamma I_{p,q}^{m,n} \left[cz^{-\mu} \, \middle| \, \begin{matrix} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{matrix} \right] \right)(x)
$$

$$
= x^{\rho - \lambda} {}^\Gamma I_{p+2,q+2}^{m,n+2} \left[cx^{-\mu} \, \middle| \, \begin{matrix} (a_1, \sigma_1; A_1 : y), (1 - \sigma - \eta - \lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \lambda, \mu; 1), \end{matrix} \right.
$$

$$
\left. \begin{matrix} (1 + \rho - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1 + \rho - \eta - \lambda, \mu; 1) \end{matrix} \right], \tag{59}
$$

provided every member in (59) *does exist.*

Corollary 21 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{\Re(-\sigma - \eta), \Re(-\sigma) + [\Re(\sigma)] + 1\}$. *Then, for* $x > 0$,

$$\left(\mathcal{D}_{-}^{\sigma,-\sigma,\eta} z^{-\lambda\gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \;\middle|\; \begin{matrix} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{matrix} \right]\right)(x)$$

$$= x^{-\sigma-\lambda\gamma} I_{p+1,q+1}^{m,n+1} \left[cx^{-\mu} \;\middle|\; \begin{matrix} (a_1, \sigma_1; A_1 : y), (1 + \rho - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \lambda, \mu; 1) \end{matrix} \right], \tag{60}$$

provided every member in (60) *does exist.*

Corollary 22 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{\Re(-\sigma - \eta), \Re(-\sigma) + [\Re(\sigma)] + 1\}$. *Then, for* $x > 0$,

$$\left(\mathcal{D}_{-}^{\sigma,-\sigma,\eta} z^{-\lambda\Gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \;\middle|\; \begin{matrix} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{matrix} \right]\right)(x)$$

$$= x^{-\sigma-\lambda\Gamma} I_{p+1,q+1}^{m,n+1} \left[cx^{-\mu} \;\middle|\; \begin{matrix} (a_1, \sigma_1; A_1 : y), (1 + \rho - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \lambda, \mu; 1) \end{matrix} \right], \tag{61}$$

provided every member in (61) *does exist.*

Corollary 23 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{\Re(-\sigma - \eta), [\Re(\sigma)] + 1\}$. *Then the right-hand Erdélyi–Kober fractional differentiation* $\mathcal{D}_{\eta,\sigma}^{-}$ $(= \mathcal{D}_{-}^{\sigma,0,\eta})$ *for* $x > 0$ *is*

$$\left(\mathcal{D}_{\eta,\sigma}^{-} z^{-\lambda\gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \;\middle|\; \begin{matrix} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{matrix} \right]\right)(x)$$

$$= x^{-\lambda\gamma} I_{p+1,q+1}^{m,n+1} \left[cx^{-\mu} \;\middle|\; \begin{matrix} (a_1, \sigma_1; A_1 : y), (1 - \sigma - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \eta - \lambda, \mu; 1) \end{matrix} \right], \tag{62}$$

provided every member in (62) *does exist.*

Corollary 24 *Let* $\sigma, \eta, \lambda, c \in \mathbb{C}$ *be such that* $\mu > 0$ *and* $\Re(\lambda) > \max\{\Re(-\sigma - \eta), [\Re(\sigma)] + 1\}$. *Then the right-hand Erdélyi–Kober fractional differentiation* $\mathcal{D}_{\eta,\sigma}^{-}$ $(= \mathcal{D}_{-}^{\sigma,0,\eta})$ *for* $x > 0$ *is*

$$\left(\mathcal{D}_{\eta,\sigma}^{-} z^{-\lambda\Gamma} I_{p,q}^{m,n} \left[cz^{-\mu} \;\middle|\; \begin{matrix} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{matrix} \right]\right)(x)$$

$$= x^{-\lambda\Gamma} I_{p+1,q+1}^{m,n+1} \left[cx^{-\mu} \;\middle|\; \begin{matrix} (a_1, \sigma_1; A_1 : y), (1 - \sigma - \eta - \lambda, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \eta - \lambda, \mu; 1) \end{matrix} \right], \tag{63}$$

provided every member in (63) *does exist.*

4 The Caputo-type fractional differentiation of incomplete *I*-functions

For $\sigma, \rho, \eta \in \mathbb{C}, x > 0$ along with $\Re(\sigma) > 0$, we characterize the respective Caputo fractional differential operators of left-hand and right-hand type related to the Gauss hypergeometric function as follows (see [26]):

$$\left({}^{c}\mathcal{D}_{0+}^{\sigma,\rho,\eta} f\right)(x) = \left(\mathcal{I}_{0+}^{-\sigma+[\Re(\sigma)]+1,-\rho-[\Re(\sigma)]-1,\sigma+\eta-[\Re(\sigma)]-1} f^{([\Re(\sigma)]+1)}\right)(x) \tag{64}$$

and

$$\left({}^{c}\mathcal{D}_{-}^{\sigma,\rho,\eta}f\right)(x) = (-1)^{[\Re(\sigma)]+1}\left(\mathcal{I}_{-}^{-\sigma+[\Re(\sigma)]+1,-\rho-[\Re(\sigma)]-1,\sigma+\eta}f^{([\Re(\sigma)]+1)}\right)(x), \tag{65}$$

where $f^{(n)}$ stands for the nth order derivative pertaining to f. The functional relationship within the MSM fractional order derivative of Caputo-type and MSM fractional derivative is similar to that between the Caputo fractional order derivative and fractional derivative of Riemann–Liouville type.

Considering $\sigma, \sigma', \rho, \rho', \eta \in \mathbb{C}$, $x > 0$ together with $\Re(\eta) > 0$, the respective left-hand sided and right-hand sided Caputo-type MSM fractional differential operators, related to third Appell function, are defined as

$$\left({}^{c}\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta}f\right)(x) = \left(\mathcal{I}_{0+}^{-\sigma',-\sigma,-\rho'+\alpha,-\rho,-\eta+\alpha}f^{(\alpha)}\right)(x) \tag{66}$$

and

$$\left({}^{c}\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta}f\right)(x) = (-1)^{\alpha}\left(\mathcal{I}_{-}^{-\sigma',-\sigma,-\rho',-\rho+\alpha,-\eta+\alpha}f^{(\alpha)}\right)(x), \tag{67}$$

where $\alpha = [\Re(\eta)] + 1$.

The fractional operators (66) and (67) are connected to (64) and (65) as follows:

$$\left({}^{c}\mathcal{D}_{0+}^{0,\sigma',\rho,\rho',\eta}f\right)(x) = \left({}^{c}\mathcal{D}_{0+}^{\eta,\sigma'-\eta,\rho'-\eta}f\right)(x) \tag{68}$$

and

$$\left({}^{c}\mathcal{D}_{-}^{0,\sigma',\rho,\rho',\eta}f\right)(x) = \left({}^{c}\mathcal{D}_{-}^{\eta,\sigma'-\eta,\rho'-\eta}f\right)(x). \tag{69}$$

Now, we derive the Caputo-type MSM fractional order derivatives referring to the incomplete I-functions. The following are well-known results and will be used in subsequent theorems (see [13]).

Lemma 4.1 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda \in \mathbb{C}$ *and* $\kappa = [\Re(\eta)] + 1$.
(a) *If* $\Re(\lambda) - \kappa > \max\{0, \Re(-\sigma + \rho), \Re(-\sigma - \sigma' - \rho' + \eta)\}$, *then*

$$\left({}^{c}\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta}t^{\lambda-1}\right)(x) = x^{\sigma+\sigma'-\eta+\lambda-1}$$
$$\times \frac{\Gamma(\lambda)\Gamma(\sigma - \rho + \lambda - \kappa)\Gamma(\sigma + \sigma' + \rho' - \eta + \lambda - \kappa)}{\Gamma(-\rho + \lambda - \kappa)\Gamma(\sigma + \sigma' - \eta + \lambda)\Gamma(\sigma + \rho' - \eta + \lambda - \kappa)}. \tag{70}$$

(b) *If* $\Re(\lambda) + \kappa > \max\{\Re(-\rho'), \Re(\sigma' + \rho - \eta), \Re(\sigma + \sigma' - \eta) + [\Re(\eta)] + 1\}$, *then*

$$\left({}^{c}\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta}t^{-\lambda}\right)(x) = x^{\sigma+\sigma'-\eta-\lambda}$$
$$\times \frac{\Gamma(\rho' + \lambda + \kappa)\Gamma(-\sigma - \sigma' + \eta + \lambda)\Gamma(-\sigma' - \rho + \eta + \lambda + \kappa)}{\Gamma(\lambda)\Gamma(-\sigma' + \rho' + \lambda + \kappa)\Gamma(-\sigma - \sigma' - \rho + \eta + \lambda + \kappa)}. \tag{71}$$

First, we give the left-hand sided Caputo-type MSM fractional derivative of the incomplete *I*-functions.

Theorem 4.2 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda, a \in \mathbb{C}$, $\kappa = [\Re(\eta)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) - \kappa > \max\{0, \Re(-\sigma + \rho), \Re(-\sigma - \sigma' - \rho' + \eta)\}$. *Then, for* $x > 0$,

$$
\left({}^{c}\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} z^{\lambda-1} {}^{\gamma} I_{p,q}^{m,n} \left[az^{\mu} \left| \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right. \right] \right)(x)
$$

$$
= x^{\sigma + \sigma' - \eta + \lambda - 1}
$$

$$
\times {}^{\gamma} I_{p+3,q+3}^{m,n+3} \left[ax^{\mu} \left| \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \lambda, \mu; 1), (1 - \sigma + \rho - \lambda + \kappa, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1 + \rho - \lambda + \kappa, \mu; 1), \end{array} \right. \right.
$$

$$
\left. \begin{array}{c} (1 - \sigma - \sigma' - \rho' + \eta - \lambda + \kappa, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1 - \sigma - \sigma' + \eta - \lambda, \mu; 1), (1 - \sigma - \rho' + \eta - \lambda + \kappa, \mu; 1) \end{array} \right], \tag{72}
$$

provided every member in (72) *does exist.*

Proof The LHS of (72) is given by

$$
\left({}^{c}\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} \left(z^{\lambda-1} \frac{1}{2\pi i} \int_{\$} \phi(s, y) a^s z^{\mu s} \, ds \right) \right)(x), \tag{73}
$$

where $\phi(s, y)$ is given in (3). Interchanging the order of integration in the above equation yields

$$
\frac{1}{2\pi i} \int_{\$} \phi(s, y) a^s \left({}^{c}\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} z^{\lambda+\mu s-1} \right)(x) \, ds, \tag{74}
$$

using the results (70) and (3), we get the RHS of (72). □

The properties given below are immediate consequences of definitions (1), (2), and (70), and consequently they are provided without detailed proof as follows.

Theorem 4.3 *Let* $\sigma, \sigma', \rho, \rho', \eta, \lambda, a \in \mathbb{C}$, $\kappa = [\Re(\eta)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) - \kappa > \max\{0, \Re(-\sigma + \rho), \Re(-\sigma - \sigma' - \rho' + \eta)\}$. *Then, for* $x > 0$,

$$
\left({}^{c}\mathcal{D}_{0+}^{\sigma,\sigma',\rho,\rho',\eta} z^{\lambda-1} {}^{\Gamma} I_{p,q}^{m,n} \left[az^{\mu} \left| \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right. \right] \right)(x)
$$

$$
= x^{\sigma + \sigma' - \eta + \lambda - 1}
$$

$$
\times {}^{\Gamma} I_{p+3,q+3}^{m,n+3} \left[ax^{\mu} \left| \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \lambda, \mu; 1), (1 - \sigma + \rho - \lambda + \kappa, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1 + \rho - \lambda + \kappa, \mu; 1), \end{array} \right. \right.
$$

$$
\left. \begin{array}{c} (1 - \sigma - \sigma' - \rho' + \eta - \lambda + \kappa, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1 - \sigma - \sigma' + \eta - \lambda, \mu; 1), (1 - \sigma - \rho' + \eta - \lambda + \kappa, \mu; 1) \end{array} \right], \tag{75}
$$

provided every member in (75) *does exist.*

Corollary 25 *Let* $\sigma, \rho, \eta, \lambda, a \in \mathbb{C}$, $\kappa = [\Re(\sigma)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) - \kappa > \max\{0, \Re(-\sigma - \rho - \eta)\}$. *Then left-hand sided generalized Caputo fractional differentiation* $^{c}\mathcal{D}_{0+}^{\sigma,\rho,\eta}$ *of the incomplete I-function is given for* $x > 0$ *by*

$$
\left({}^{c}\mathcal{D}_{0+}^{\sigma,\rho,\eta} z^{\lambda-1} {}^{\gamma} I_{p,q}^{m,n} \left[az^{\mu} \; \middle| \; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\rho+\lambda-1} {}^{\gamma} I_{p+2,q+2}^{m,n+2} \left[ax^{\mu} \; \middle| \; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1-\lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1-\rho-\lambda, \mu; 1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1-\sigma-\rho-\eta-\lambda+\kappa, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1-\eta-\lambda+\kappa, \mu; 1) \end{array} \right], \tag{76}
$$

provided every member in (76) *does exist.*

Corollary 26 *Let* $\sigma, \rho, \eta, \lambda, a \in \mathbb{C}$, $\kappa = [\Re(\sigma)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) - \kappa > \max\{0, \Re(-\sigma - \rho - \eta)\}$. *Then left-hand sided generalized Caputo fractional differentiation* $^{c}\mathcal{D}_{0+}^{\sigma,\rho,\eta}$ *of the incomplete I-function is given for* $x > 0$ *by*

$$
\left({}^{c}\mathcal{D}_{0+}^{\sigma,\rho,\eta} z^{\lambda-1} {}^{\Gamma} I_{p,q}^{m,n} \left[az^{\mu} \; \middle| \; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\rho+\lambda-1} {}^{\Gamma} I_{p+2,q+2}^{m,n+2} \left[ax^{\mu} \; \middle| \; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1-\lambda, \mu; 1), \\ (b_j, \rho_j; B_j)_{1,q}, (1-\rho-\lambda, \mu; 1), \end{array} \right.
$$

$$
\left. \begin{array}{c} (1-\sigma-\rho-\eta-\lambda+\kappa, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (1-\eta-\lambda+\kappa, \mu; 1) \end{array} \right], \tag{77}
$$

provided every member in (77) *does exist.*

Corollary 27 *Let* $\sigma, \eta, \lambda, a \in \mathbb{C}$, $\kappa = [\Re(\sigma)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) - \kappa > \max\{0, \Re(-\sigma - \eta)\}$. *Then left-hand sided Caputo-type Erdélyi–Kober fractional differentiation* $^{c}\mathcal{D}_{\eta,\sigma}^{+}$ $(= \mathcal{D}_{0+}^{\sigma,0,\eta})$ *of the incomplete I-function is given for* $x > 0$ *by*

$$
\left({}^{c}\mathcal{D}_{\eta,\sigma}^{+} z^{\lambda-1} {}^{\gamma} I_{p,q}^{m,n} \left[az^{\mu} \; \middle| \; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{\lambda-1}
$$

$$
\times {}^{\gamma} I_{p+1,q+1}^{m,n+1} \left[ax^{\mu} \; \middle| \; \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1-\sigma-\eta-\lambda+\kappa, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1-\eta-\lambda+\kappa, \mu; 1) \end{array} \right], \tag{78}
$$

provided every member in (78) *does exist.*

Corollary 28 *Let* $\sigma, \eta, \lambda, a \in \mathbb{C}$, $\kappa = [\Re(\sigma)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) - \kappa > \max\{0, \Re(-\sigma - \eta)\}$. *Then left-hand sided Caputo-type Erdélyi–Kober fractional differen-*

tiation $^c\mathcal{D}_{\eta,\sigma}^+$ $(= \mathcal{D}_{0+}^{\sigma,0,\eta})$ *of the incomplete I-function is given for $x > 0$ by*

$$
\left({}^c\mathcal{D}_{\eta,\sigma}^+ z^{\lambda-1} {}^\Gamma I_{p,q}^{m,n} \left[az^\mu \,\middle|\, \begin{matrix} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{matrix} \right] \right)(x)
$$

$$
= x^{\lambda-1}
$$

$$
\times {}^\Gamma I_{p+1,q+1}^{m,n+1} \left[ax^\mu \,\middle|\, \begin{matrix} (a_1,\sigma_1;A_1:y),(1-\sigma-\eta-\lambda+\kappa,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q},(1-\eta-\lambda+\kappa,\mu;1) \end{matrix} \right], \quad (79)
$$

provided every member in (79) *does exist.*

Lastly, we represent the right-hand sided Caputo-type MSM fractional derivative of the incomplete I-functions.

Theorem 4.4 *Let $\sigma,\sigma',\rho,\rho',\eta,\lambda,a \in \mathbb{C}$, $\kappa = [\Re(\eta)] + 1$ be such that $\mu > 0$ and $\Re(\lambda) + \kappa > \max\{\Re(-\rho'),\Re(\sigma'+\rho-\eta),\Re(\sigma+\sigma'-\eta)\} + \kappa\}$. Then, for $x > 0$,*

$$
\left({}^c\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta} z^{-\lambda\gamma} I_{p,q}^{m,n} \left[az^{-\mu} \,\middle|\, \begin{matrix} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{matrix} \right] \right)(x)
$$

$$
= x^{\sigma+\sigma'-\eta-\lambda}
$$

$$
\times {}^\gamma I_{p+3,q+3}^{m,n+3} \left[ax^{-\mu} \,\middle|\, \begin{matrix} (a_1,\sigma_1;A_1:y),(1-\rho'-\lambda-\kappa,\mu;1),(1+\sigma+\sigma'-\eta-\lambda,\mu;1), \\ (b_j,\rho_j;B_j)_{1,q},(1-\lambda,\mu;1), \end{matrix} \right.
$$

$$
\left. \begin{matrix} (1+\sigma'+\rho-\eta-\lambda-\kappa,\mu;1),(a_j,\sigma_j;A_j)_{2,p} \\ (1+\sigma'-\rho'-\lambda-\kappa,\mu;1),(1+\sigma+\sigma'+\rho-\eta-\lambda-\kappa,\mu;1) \end{matrix} \right], \quad (80)
$$

provided every member in (80) *does exist.*

Proof The LHS of (80) is given by

$$
\left({}^c\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta} \left(z^{-\lambda} \frac{1}{2\pi i} \int_\$ \phi(s,y)a^s z^{-\mu s}\,ds \right) \right)(x), \quad (81)
$$

where $\phi(s,y)$ is given in (3). Interchanging the order of integration in the above equation yields

$$
\frac{1}{2\pi i} \int_\$ \phi(s,y)a^s \left({}^c\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta} z^{-(\lambda+\mu s)} \right)(x)\,ds, \quad (82)
$$

using the results (71) and (3), we get the RHS of (80).   □

Theorem 4.5 *Let $\sigma,\sigma',\rho,\rho',\eta,\lambda,a \in \mathbb{C}$, $\kappa = [\Re(\eta)] + 1$ be such that $\mu > 0$ and $\Re(\lambda) + \kappa > \max\{\Re(-\rho'),\Re(\sigma'+\rho-\eta),\Re(\sigma+\sigma'-\eta)\} + \kappa\}$. Then, for $x > 0$,*

$$
\left({}^c\mathcal{D}_{-}^{\sigma,\sigma',\rho,\rho',\eta} z^{-\lambda} {}^\Gamma I_{p,q}^{m,n} \left[az^{-\mu} \,\middle|\, \begin{matrix} (a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p} \\ (b_j,\rho_j;B_j)_{1,q} \end{matrix} \right] \right)(x)
$$

$$
= x^{\sigma+\sigma'-\eta-\lambda}
$$

$$\times \, ^{\Gamma}I_{p+3,q+3}^{m,n+3}\left[ax^{-\mu}\,\middle|\,\begin{array}{l}(a_1,\sigma_1;A_1:y),(1-\rho'-\lambda-\kappa,\mu;1),(1+\sigma+\sigma'-\eta-\lambda,\mu;1),\\ \qquad\qquad (b_j,\rho_j;B_j)_{1,q},(1-\lambda,\mu;1),\end{array}\right.$$

$$\left.\begin{array}{l}(1+\sigma'+\rho-\eta-\lambda-\kappa,\mu;1),(a_j,\sigma_j;A_j)_{2,p}\\ (1+\sigma'-\rho'-\lambda-\kappa,\mu;1),(1+\sigma+\sigma'+\rho-\eta-\lambda-\kappa,\mu;1)\end{array}\right], \tag{83}$$

provided every member in (83) *does exist.*

Corollary 29 *Let* $\sigma,\rho,\eta,\lambda,a \in \mathbb{C}$, $\kappa = [\Re(\sigma)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) + \kappa > \max\{\Re(-\sigma-\eta),\Re(\rho)+\kappa\}$. *Then the right-hand sided generalized Caputo fractional derivative* $^c\mathcal{D}_-^{\sigma,\rho,\eta}$ *of the incomplete I-function is given for* $x > 0$ *by*

$$\left(^c\mathcal{D}_-^{\sigma,\rho,\eta}z^{-\lambda\gamma}I_{p,q}^{m,n}\left[az^{-\mu}\,\middle|\,\begin{array}{l}(a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p}\\ \qquad (b_j,\rho_j;B_j)_{1,q}\end{array}\right]\right)(x)$$

$$= x^{\rho-\lambda\gamma}I_{p+2,q+2}^{m,n+2}\left[ax^{-\mu}\,\middle|\,\begin{array}{l}(a_1,\sigma_1;A_1:y),(1-\sigma-\eta-\lambda-\kappa,\mu;1),\\ \qquad (b_j,\rho_j;B_j)_{1,q},(1-\lambda,\mu;1),\end{array}\right.$$

$$\left.\begin{array}{l}(1+\rho-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p}\\ (1+\rho-\eta-\lambda-\kappa,\mu;1)\end{array}\right], \tag{84}$$

provided every member in (84) *does exist.*

Corollary 30 *Let* $\sigma,\rho,\eta,\lambda,a \in \mathbb{C}$, $\kappa = [\Re(\sigma)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) + \kappa > \max\{\Re(-\sigma-\eta),\Re(\rho)+\kappa\}$. *Then the right-hand sided generalized Caputo fractional derivative* $^c\mathcal{D}_-^{\sigma,\rho,\eta}$ *of the incomplete I-function is given for* $x > 0$ *by*

$$\left(^c\mathcal{D}_-^{\sigma,\rho,\eta}z^{-\lambda\Gamma}I_{p,q}^{m,n}\left[az^{-\mu}\,\middle|\,\begin{array}{l}(a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p}\\ \qquad (b_j,\rho_j;B_j)_{1,q}\end{array}\right]\right)(x)$$

$$= x^{\rho-\lambda\Gamma}I_{p+2,q+2}^{m,n+2}\left[ax^{-\mu}\,\middle|\,\begin{array}{l}(a_1,\sigma_1;A_1:y),(1-\sigma-\eta-\lambda-\kappa,\mu;1),\\ \qquad (b_j,\rho_j;B_j)_{1,q},(1-\lambda,\mu;1),\end{array}\right.$$

$$\left.\begin{array}{l}(1+\rho-\lambda,\mu;1),(a_j,\sigma_j;A_j)_{2,p}\\ (1+\rho-\eta-\lambda-\kappa,\mu;1)\end{array}\right], \tag{85}$$

provided every member in (85) *does exist.*

Corollary 31 *Let* $\sigma,\eta,\lambda,a \in \mathbb{C}$, $\kappa = [\Re(\sigma)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) + \kappa > \max\{\Re(-\sigma-\eta),\kappa\}$. *Then the right-hand sided Caputo-type Erdélyi–Kober fractional derivative* $^c\mathcal{D}_{\eta,\sigma}^-$ $(= {}^c\mathcal{D}_-^{\sigma,\rho,\eta})$ *of the incomplete I-function is given for* $x > 0$ *by*

$$\left(^c\mathcal{D}_{\eta,\sigma}^-z^{-\lambda\gamma}I_{p,q}^{m,n}\left[az^{-\mu}\,\middle|\,\begin{array}{l}(a_1,\sigma_1;A_1:y),(a_j,\sigma_j;A_j)_{2,p}\\ \qquad (b_j,\rho_j;B_j)_{1,q}\end{array}\right]\right)(x)$$

$$= x^{-\lambda}$$

$$\times \, ^{\gamma}I_{p+1,q+1}^{m,n+1}\left[ax^{-\mu}\,\middle|\,\begin{array}{l}(a_1,\sigma_1;A_1:y),(1-\sigma-\eta-\lambda-\kappa,\mu;1),(a_j,\sigma_j;A_j)_{2,p}\\ \qquad (b_j,\rho_j;B_j)_{1,q},(1-\eta-\lambda-\kappa,\mu;1)\end{array}\right], \tag{86}$$

provided every member in (86) *does exist.*

Corollary 32 *Let* $\sigma, \eta, \lambda, a \in \mathbb{C}$, $\kappa = [\Re(\sigma)] + 1$ *be such that* $\mu > 0$ *and* $\Re(\lambda) + \kappa >$ $\max\{\Re(-\sigma - \eta), \kappa\}$. *Then the right-hand sided Caputo-type Erdélyi–Kober fractional derivative* ${}^{c}\mathcal{D}_{\eta,\sigma}^{-}$ $(= {}^{c}\mathcal{D}_{-}^{\sigma,\rho,\eta})$ *of the incomplete I-function is given for* $x > 0$ *by*

$$
\left({}^{c}\mathcal{D}_{\eta,\sigma}^{-} z^{-\lambda} \, {}^{\Gamma}I_{p,q}^{m,n} \left[az^{-\mu} \, \middle| \, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q} \end{array} \right] \right)(x)
$$

$$
= x^{-\lambda}
$$

$$
\times {}^{\Gamma}I_{p+1,q+1}^{m,n+1} \left[ax^{-\mu} \, \middle| \, \begin{array}{c} (a_1, \sigma_1; A_1 : y), (1 - \sigma - \eta - \lambda - \kappa, \mu; 1), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; B_j)_{1,q}, (1 - \eta - \lambda - \kappa, \mu; 1) \end{array} \right], \quad (87)
$$

provided every member in (87) *does exist.*

5 Special cases and concluding remarks

By specializing the parameters in definition (2), we obtain the following functions as special cases of the incomplete *I*-function ${}^{\Gamma}I_{p,q}^{m,n}(z)$:

(i) *Incomplete* $\overline{I}$*-function* ${}^{\Gamma}\overline{I}_{p,q}^{m,n}(z)$: if we set B_j $(j = 1, \ldots, m) = 1$ in (2), then we obtain

$$
{}^{\Gamma}\overline{I}_{p,q}^{m,n}(z) = {}^{\Gamma}\overline{I}_{p,q}^{m,n} \left[z \, \middle| \, \begin{array}{c} (a_1, \sigma_1; A_1 : x), (a_j, \sigma_j; A_j)_{2,p} \\ (b_j, \rho_j; 1)_{1,m}, (b_j, \rho_j; B_j)_{m+1,q} \end{array} \right]
$$

$$
= \frac{1}{2\pi i} \int_{\$} \overline{\phi}(s,x) z^s \, ds, \quad (88)
$$

where

$$
\overline{\phi}(s,x) = \frac{\{\gamma(1 - a_1 + \sigma_1 s, x)\}^{A_1} \prod_{j=1}^{m} \Gamma(b_j - \rho_j s) \prod_{j=2}^{n} \{\Gamma(1 - a_j + \sigma_j s)\}^{A_j}}{\prod_{j=n+1}^{p} \{\Gamma(a_j - \sigma_j s)\}^{A_j} \prod_{j=m+1}^{q} \{\Gamma(1 - b_j + \rho_j s)\}^{B_j}}. \quad (89)
$$

(ii) *Incomplete* $\overline{H}$*-function* $\overline{\Gamma}_{p,q}^{m,n}(z)$: if we set B_j $(j = 1, \ldots, m) = 1$ and A_j $(j = n + 1, \ldots, p) = 1$ in (2), then we obtain (see [32])

$$
\overline{\Gamma}_{p,q}^{m,n}(z) = {}^{\Gamma}\overline{I}_{p,q}^{m,n} \left[z \, \middle| \, \begin{array}{c} (a_1, \sigma_1; A_1 : x), (a_j, \sigma_j; A_j)_{2,n}, (a_j, \sigma_j; 1)_{n+1,p} \\ (b_j, \rho_j; 1)_{1,m}, (b_j, \rho_j; B_j)_{m+1,q} \end{array} \right]
$$

$$
= \overline{\Gamma}_{p,q}^{m,n} \left[z \, \middle| \, \begin{array}{c} (a_1, \sigma_1; A_1 : x), (a_j, \sigma_j; A_j)_{2,n}, (a_j, \sigma_j)_{n+1,p} \\ (b_j, \rho_j)_{1,m}, (b_j, \rho_j; B_j)_{m+1,q} \end{array} \right]
$$

$$
= \frac{1}{2\pi i} \int_{\$} \overline{\psi}(s,x) z^s \, ds, \quad (90)
$$

where

$$
\overline{\psi}(s,x) = \frac{\{\gamma(1 - a_1 + \sigma_1 s, x)\}^{A_1} \prod_{j=1}^{m} \Gamma(b_j - \rho_j s) \prod_{j=2}^{n} \{\Gamma(1 - a_j + \sigma_j s)\}^{A_j}}{\prod_{j=n+1}^{p} \Gamma(a_j - \sigma_j s) \prod_{j=m+1}^{q} \{\Gamma(1 - b_j + \rho_j s)\}^{B_j}}. \quad (91)
$$

(iii) *Incomplete H-function $\Gamma_{p,q}^{m,n}(z)$:* if we set B_j $(j = 1,\ldots,q) = 1$ and A_j $(j = 1,\ldots,p) = 1$ in (2), then we obtain (see [32])

$$
\begin{aligned}
\Gamma_{p,q}^{m,n}(z) &= {}^{\Gamma}I_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1,\sigma_1;1:x),(a_j,\sigma_j;1)_{2,p} \\ (b_j,\rho_j;1)_{1,q} \end{matrix} \right] \\
&= \Gamma_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_1,\sigma_1:x),(a_j,\sigma_j)_{2,p} \\ (b_j,\rho_j)_{1,q} \end{matrix} \right] \\
&= \frac{1}{2\pi i}\int_{\$} \psi(s,x)z^s \, ds,
\end{aligned}
\tag{92}
$$

where

$$
\psi(s,x) = \frac{\gamma(1-a_1+\sigma_1 s,x)\prod_{j=1}^{m}\Gamma(b_j-\rho_j s)\prod_{j=2}^{n}\Gamma(1-a_j+\sigma_j s)}{\prod_{j=n+1}^{p}\Gamma(a_j-\sigma_j s)\prod_{j=m+1}^{q}\Gamma(1-b_j+\rho_j s)}.
\tag{93}
$$

(iv) *Incomplete Fox–Wright function ${}_p\Psi_q^{(\Gamma)}(z)$:* if we take the substitutions $z = -z$, A_j $(j = 1,\ldots,p) = B_j$ $(j = 1,\ldots,q) = 1$, $a_j \to (1-a_j)$ $(j = 1,\ldots,p)$ and $b_j \to (1-b_j)$ $(j = 1,\ldots,q)$ in (2), then we obtain (see [24])

$$
\begin{aligned}
{}_p\Psi_q^{(\Gamma)}(z) &= {}^{\Gamma}I_{p,q+1}^{1,p}\left[-z \,\middle|\, \begin{matrix} (1-a_1,\sigma_1;1:x),(1-a_j,\sigma_j;1)_{2,p} \\ (0,1),(1-b_j,\rho_j;1)_{1,q} \end{matrix} \right] \\
&= {}_p\Psi_q^{(\Gamma)}\left[\begin{matrix} (a_1,\sigma_1,x),(a_j,\sigma_j)_{2,p}; \\ (b_j,\rho_j)_{1,q}; \end{matrix} z \right].
\end{aligned}
\tag{94}
$$

Remark 1 Similarly, one can easily obtain another class of incomplete functions as special cases of the incomplete *I*-function ${}^{\gamma}\overline{I}_{p,q}^{m,n}(z)$.

It is to note that if we use the relations (90) and (92), then one can obtain the fractional calculus results associated with the incomplete *H*-functions and incomplete $\overline{H}$-functions (see [32]) as special cases of our results. Moreover, all the results investigated in this paper, taking into account the decomposition formula (8) (or $y = 0$ in the results involving ${}^{\Gamma}I_{p,q}^{m,n}(z)$), lead to the known results provided earlier by Kataria and Vellaisamy [13].

In the present article, we investigated a number of fractional calculus image formulas involving incomplete *I*-functions associated with the MSM operators. The incomplete *I*-functions generalize *I*-function, $\overline{H}$-function, *H*-function, Meijer *G*-function, hypergeometric function, and many other functions. Additionally, the MSM fractional operators generalize Saigo, Riemann–Liouville, Erdélyi–Kober fractional calculus operators. In consideration of the indicated fact, one can obtain numerous image formulas comprising a class of special functions (see [14, 15, 27, 30, 32]) as limiting cases of the main outcomes.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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