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A New Dynamic Scheme via Fractional Operators on Time Scale

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The present work investigates the applicability and effectiveness of the generalized Riemann-Liouville fractional integral operator integral method to obtain new Minkowski, Grüss type and several other associated dynamic variants on an arbitrary time scale, which are communicated as a combination of delta and fractional integrals. These inequalities extend some dynamic variants on time scales, and tie together and expand some integral inequalities. The present method is efficient, reliable, and it can be used as an alternative to establishing new solutions for different types of fractional differential equations applied in mathematical physics.

Keywords: Minkowski' inequlity, gruss inequality, fractional calculus, rimenn-liouville fractional integral operator, generalized riemann-liouville fractional integral operator, time sccale, holder inequality

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1. INTRODUCTION

Fractional calculus has also been comprehensively utilized in several instances, but the concept has been popularized and implemented in numerous disciplines of science, technology and engineering as a mathematical model (see [1, 2]). Numerous distinguished generalized fractional integral operators consist of the Hadamard operator, Erdlelyi-Kober operators, the Saigo operator, the Gaussian hypergeometric operator, the Marichev-Saigo-Maeda fractional integral operator, and so on.; out of the ones, the Riemann-Liouville fractional integral operator has been extensively utilized by researchers in theory as well as applications (see [1, 3–8]).

Stefan Hilger began the theories of time scales in his doctoral dissertation [9] and combined discrete and continuous analysis (see [10, 11]). From this moment, this hypothesis has received a lot of attention. In the book written by Bohner and Peterson [12] on the issues of time scale, a brief summary is given and several time calculations are performed. Over the past decade, many analysts working in specific applications have proved a reasonable number of dynamic inequalities on a time scale (see [13–15]). Several researchers have created various results relating to fractional calculus on time scales to obtain the corresponding dynamic inequalities (see [16–20]).

Recently, the idea of the fractional-order derivative has been expounded by Bastos et al. [16] via Riemann-Liouville fractional operators on scale versions by considering linear dynamic equations. Another approach on time scales shifts to the inverse Laplace transform [18]. Following such innovator work, the investigation of fractional calculus on time scales created in a mainstream look into research studies on time scales (see [18, 21–29] and references therein). Since the publications in 2015, several researchers made significant contributions to the history of time scales. Sun and Hou [30] employed the fractional q-symmetric systems on time scales. Yaslan and Liceli [29] obtained the three-point boundary value problem with delta Riemann-Liouville fractional derivative on time scales. Yan et al. [31] adopted the Caputo fractional techniques on differential equations on time scales. Zhu and Wu [32] employed Caputo nabla fractional derivatives in order to find the existence of solutions for Cauchy problems. As certifiable utilities, we refer to the study of calcium ion channels that are impeded with an infusion of calcium-chelator ethylene glycol tetraacetic acid [33]. Actually, physical utilization of initial value-fractional problems in diverse time scales proliferates [10, 34, 35]. For instance, the continuous time scale $\mathbb{T} = \mathbb{R}$, the fractional differential equations that oversee the practices of viscoelastic materials with memory and creep tendencies have been investigated in Chidouh et al. [36].

Integral Inequalities are an excellent way to investigate many scientific fields of research, including engineering, flow dynamics, biology, chaos, meteorology, vibration analysis, biochemistry, aerodynamics and many more. Since the productions of the above outcome in 1883, several works have been published in the literature of time calculus, with varied evidence, various speculations and improvements [37–51]. Recently, numerous analysts examined various inequalities, such as Hermite-Hadamard inequalities, Ostrowski inequalities and the expanded version of Hardy-type inequalities (see [13–15, 24, 52] and the references therein).

Here, we broaden accessible outcomes in the literature [53] by presenting increasingly broad ideas of fractional integral inequalities on time scales in the frame of generalized Riemann-Liouville fractional integral. At that point, we study the dynamic variants of corresponding generalized fractional-order on time scales. We obtain the inequalities Grüss, Minkowski and several others using the delta integrals in arbitrary time scales. For $\delta = 1$, the integral will become delta integral and for $\delta = 0$, it advances toward turning out to be nabla integral. An astounding audit about the time scale calculus can be found in the paper [54]. The proposed dynamical integral method is reliable and effective to obtain new solutions. This method has more advantages: it is direct and concise. Thus, the proposed method can be extended to solve many systems of non-linear fractional partial differential equations in mathematical and physical sciences. Also, the new exact analytical solutions can be obtained for the generalized ordinary differential equations to obtain new theorems related to stability and continuous dependence on parameters for dynamic equations on time scales.

The present work investigates the applicability and effectiveness of the several dynamic variants that are presented, which are based primarily on the generalized Riemann-Liouville fractional integral operators. We will show that the Grüss and Minkowski type, that we participated in are very specific to the current work. From an application point of view, the results ultimately relate to the study of Young's inequality, arithmetic, and geometry inequality. Our computed outcomes can be very useful as a starting point of comparison when some approximate methods are applied to this non-linear space-time fractional equation. Furthermore, there are likewise some occurrences that can be derived from our outcomes.

2. PRELIMINARIES

A non-empty closed subsets $\mathbb{R}$ of $\mathbb{T}$ is known as the time scale. The well-known examples of time scales theory are the set of real numbers $\mathbb{R}$ and the integers $\mathcal{Z}$. Throughout the paper, we refer $\mathbb{T}$ as time scale and a time-scaled interval is $\Upsilon_{\mathbb{T}} = [\upsilon_1, \upsilon_2]_{\mathbb{T}}$. We need the concept of jump operators. The forward jump operator is denoted by the symbol $\diamondsuit$ and the backward jump operator is denoted by ϑ, are said through the formulas:

$$\diamondsuit(t) = \inf\{\lambda \in \mathbb{T} : \rho > t\} \in \mathbb{T}, \;\; \vartheta(\omega) = \sup\{\rho \in \mathbb{T} : \rho < \omega\} \in \mathbb{T}.$$

We accumulate as:

$$\inf \emptyset := \sup \mathbb{T}, \quad \sup \emptyset := \inf \mathbb{T}.$$

If $\diamondsuit(t) > t$, then the term t is allude to be right-scattered and ω is allude to be left-scattered $\varrho(\omega) < \omega$. The elements that are most likely all the while appropriate-scattered and scattered are known as isolated. The term t is said to be right dense, if $\diamondsuit(t) = t$, and ω is said to be left dense, if $\varrho(\omega) = \omega$. In addition, the focuses t, ω are known to be dense if they are most likely right-dense and left-dense.
The mappings $\mu, \nu : \mathbb{T} \to [0, +\infty)$ defined by

$$\mu(t) := \diamondsuit(t) - t,$$

$$\nu(t) := t - \vartheta(t)$$

are called the forward and backward graininess functions, respectively.

Definition 2.1. [12, 55] "Let $\hbar : \mathbb{T} \to \mathbb{R}$ be a real-valued function. Then $\hbar$ is said to be $\mathcal{RD}$-continuous on $\mathbb{R}$ if its left limit at any left dense point of $\mathbb{T}$ is finite and it is continuous on every right dense point of $\mathbb{T}$. All $\mathcal{RD}$-continuous functions are denoted by $\mathbb{C}_{\mathcal{RD}}$."

Definition 2.2. "A function $\mathcal{F} : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $\hbar : \mathbb{T} \to \mathbb{R}$ if $\mathcal{F}^{\Delta}(t) = \hbar(t)$, for all $t \in \mathbb{T}^k$. Then, one defines the delta integral by $\int_{\upsilon_1}^{t} \hbar(s)\Delta s = \mathcal{F}(t) - \mathcal{F}(\upsilon_1)$."

Theorem 2.1. *[55]. If $\hbar \in \mathbb{C}_{RD}$ and $t \in \mathbb{T}^k$, then*

$$\int_{t}^{\diamondsuit(t)} \hbar(s)\Delta s = \mu(t)\hbar(t).$$

Theorem 2.2. *[55]. Let $\upsilon_1, \upsilon_2, \upsilon_3 \in \mathbb{T}$, $\beta \in \mathbb{R}$ and $\hbar, \omega \in \mathbb{C}_{RD}$, then*

(i). $\int_{\upsilon_1}^{\upsilon_2} \big(\hbar_1(\rho) + \hbar_2(\rho)\big)\Delta\rho = \int_{\upsilon_1}^{\upsilon_2} \hbar_1(\rho)\Delta\rho + \int_{\upsilon_1}^{\upsilon_2} \hbar_2(\rho)\Delta\rho;$

(ii). $\int_{\upsilon_1}^{\upsilon_2} \beta\hbar(\rho)\Delta\rho = \beta \int_{\upsilon_1}^{\upsilon_2} \hbar(\rho)\Delta\rho;$

$(iii).\ \int_{\upsilon_1}^{\upsilon_2} \hbar(\rho)\Delta\rho = -\int_{\upsilon_2}^{\upsilon_1} \hbar(\rho)\Delta\rho;$

$(iv).\ \int_{\upsilon_1}^{\upsilon_2} \hbar(\rho)\Delta\rho = \int_{\upsilon_1}^{\varsigma_3} \hbar(\rho)\Delta\rho + \int_{\varsigma_3}^{\upsilon_2} \hbar(\rho)\Delta\rho;$

$(v).\ \int_{\upsilon_1}^{\upsilon_2} \hbar_1^{\diamond}(\rho)\hbar_2^{\Delta}\Delta\rho = \big(\hbar_1\hbar_2\big)(\upsilon_2) - \big(\hbar_1\hbar_2\big)(\upsilon_1) - \int_{\upsilon_1}^{\upsilon_2} \hbar_1^{\Delta}(\rho)\hbar_2(\rho)\Delta(\rho);$

$(vi).\ \int_{\upsilon_1}^{\upsilon_2} \hbar_1(\rho)\hbar_2^{\Delta}\Delta\rho = \big(\hbar_1\hbar_2\big)(\upsilon_2) - \big(\hbar_1\hbar_2\big)(\upsilon_1) - \int_{\upsilon_1}^{\upsilon_2} \hbar_1^{\Delta}(\rho)\hbar_2^{\diamond}(\rho)\Delta(\rho);$

$(vii).\ \int_{\upsilon_1}^{\upsilon_2} \hbar(\rho)\Delta(\rho) = 0;$

$(viii).$ *If* $\hbar(\rho) \geq 0$ *for all* ρ, *then* $\int_{\upsilon_1}^{\upsilon_2} \hbar(\rho)\Delta(\rho) \geq 0;$

$(ix).$ *If* $|\hbar_1(\rho)| \leq \hbar_2(\rho)$ *on* $[\upsilon_1, \upsilon_2]$, *then* $\Big|\int_{\upsilon_1}^{\upsilon_2} \hbar_1(\rho)\Delta\rho\Big| \leq \int_{\upsilon_1}^{\upsilon_2} \hbar_2(\rho)\Delta(\rho).$

From Theorem 2.2 (ix), for $\hbar_2(\rho) = |\hbar_1(\rho)|$ on $[\upsilon_1, \upsilon_2]$, we have

$$\Big|\int_{\upsilon_1}^{\upsilon_2} \hbar(\rho)\Delta\rho\Big| \leq \int_{\upsilon_1}^{\upsilon_2} \big|\hbar(\rho)\big|\Delta(\rho).$$

Proposition 2.1. *[56] Consider a time scale* $\mathbb{T}$ *and* $\hbar$ *is an increasing continuous function on* $\Upsilon_{\mathbb{T}}$. *An extension of* $\hbar$ *on* $\Upsilon_{\mathbb{T}}$ *is* $\mathcal{F}$ *given as*

$$\mathcal{F}(\theta) := \begin{cases} \hbar(\theta), & \text{if} \quad \theta \in \mathbb{T} \\ \hbar(\eta), & \text{if} \quad \theta \in (\eta, \sigma(\eta)) \not\subset \mathbb{T}, \end{cases}$$

then

$$\int_{\upsilon_1}^{\upsilon_2} \hbar(\eta)\Delta\hbar \leq \int_{\upsilon_1}^{\upsilon_2} \mathcal{F}(\hbar)d\hbar.$$

Next we demonstrate the idea of fractional integral on time scale, which is mainly due to [16].

Definition 2.3. [16] "For $0 < \delta < 1$, let $\Upsilon_{\mathbb{T}} \subset \mathbb{T}$ is a time scale and $\mathcal{F}$ be an integrable function on $\Upsilon_{\mathbb{T}}$. Then the (left) fractional integral of order δ of $\mathcal{F}$ is defined by

$$_{\upsilon_1}^{\mathbb{T}}\mathcal{J}_{\eta}^{\delta}(\eta) = \frac{1}{\Gamma(\delta)} \int_{\upsilon_1}^{\eta} (\eta - \theta)^{\delta-1}\mathcal{F}(\theta)\Delta\theta, \tag{1}$$

where Γ is the gamma function."

Again, we demonstrate the concept of generalized Riemann-Liouville fractional integral operator which is proposed by [24].

Definition 2.4. [24] "For $0 < \delta < 1$, let $\mathbb{T}$ is a time scale and $[\upsilon_1, \upsilon_2]$ is an interval of $\mathbb{T}$. Suppose $\mathcal{F}$ be an integrable function on $[\upsilon_1, \upsilon_2]$ and Φ is monotone having a delta derivative Φ^{Δ} with $\Phi^{\Delta} \neq 0$ for any $\eta \in [\upsilon_1, \upsilon_2]$. Let $0 < \delta < 1$, then the (left) generalized fractional integral of order δ of $\mathcal{F}$ with respect to Φ is defined by

$$_{\upsilon_1;\Phi}^{\mathbb{T}}\mathcal{J}_{\eta}^{\delta}(\eta) = \frac{1}{\Gamma(\delta)} \int_{\upsilon_1}^{\eta} (\Phi(\eta) - \Phi(\theta))^{\delta-1}\Phi^{\Delta}(\theta)\mathcal{F}(\theta)\Delta\theta.'' \tag{2}$$

Remark 2.1. *If* $\mathbb{T} = \mathbb{R}$, *then Definitions 2.3 and 2.4 reduces to the well-known Riemann-Liouville and generalized Riemann-Liouville fractional integral, respectively (see [7]).*

3. MINKOWSKI TYPE INEQUALITIES FOR GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL ON TIME SCALE

This section is inaugurated to establishing generalizations of some reverse Minkowski inequality by introducing the generalized Riemann-Liouville fractional integral on time scale.

Theorem 3.1. *Let* $\delta, \gamma > 1$, *and* $\mathbb{T}$ *is a time scale. Suppose* $\mathcal{F}, \mathcal{G}$ *be two positive functions on* $[0, \infty)_{\mathbb{T}}$, *and* Φ *is monotone, delta differentiable* Φ^{Δ} *with* $\Phi^{\Delta} \neq 0$ *such that for all* $\eta > 0$, $_{0^+;\Phi}^{\mathbb{T}}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta) < \infty$, $_{0^+;\Phi}^{\mathbb{T}}\mathcal{J}_{\eta}^{\delta}\mathcal{G}(\eta) < \infty$. *If* $0 < \mathfrak{m} \leq \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \leq \mathcal{M}, \theta \in [0, \eta]$, *then*

$$\Big[{}_{0^+;\Phi}^{\mathbb{T}}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\Big]^{\frac{1}{\alpha}}\Big[{}_{0^+;\Phi}^{\mathbb{T}}\mathcal{J}_{\eta}^{\delta}\mathcal{G}(\eta)\Big]^{\frac{1}{\beta}} \leq \Big(\frac{\mathcal{M}}{\mathfrak{m}}\Big)^{\frac{1}{\alpha\beta}}\Big[{}_{0^+;\Phi}^{\mathbb{T}}\mathcal{J}_{\eta}^{\delta}\big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}}\big(\mathcal{G}(\eta)\big)^{\frac{1}{\beta}}\Big]. \tag{3}$$

Proof: Since $\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \leq \mathcal{M}, \theta \in [0, \eta]$, $\eta > 0$, we find that

$$\big(\mathcal{G}(\theta)\big)^{\frac{1}{\alpha}} \geq \mathcal{M}^{-\frac{1}{\beta}}\big(\mathcal{F}(\theta)\big)^{\frac{1}{\beta}} \tag{4}$$

and

$$\big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}}\big(\mathcal{G}(\theta)\big)^{\frac{1}{\beta}} \geq \mathcal{M}^{-\frac{1}{\beta}}\mathcal{F}(\theta). \tag{5}$$

Taking product on both sides of (5) $\frac{\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)}{\Gamma(\delta)}$, which is positive because $\theta \in (0, \eta)$, $\eta > 0$, we integrate the resulting identity with respect to θ from 0 to η we have

$$\frac{1}{\Gamma(\delta)} \int_0^{\eta} \big(\Phi(\eta) - \Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}}\big(\mathcal{G}(\theta)\big)^{\frac{1}{\beta}}\Delta\theta \geq \frac{\mathcal{M}^{-\frac{1}{\beta}}}{\Gamma(\delta)} \int_0^{\eta} \big(\Phi(\eta) - \Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\mathcal{F}(\theta)\Delta\theta, \tag{6}$$

which implies that

$$ {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}}\big(\mathcal{G}(\eta)\big)^{\frac{1}{\beta}} \geq \mathcal{M}^{-\frac{1}{\beta}}\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta). \tag{7} $$

It follows that

$$ \left({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}}\big(\mathcal{G}(\eta)\big)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} \geq \mathcal{M}^{-\frac{1}{\alpha\beta}}\left({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\right)^{\frac{1}{\alpha}}. \tag{8} $$

Accordingly, $\mathfrak{m}\mathcal{G}(\theta) \leq \mathcal{F}(\theta), \theta \in (0,\eta),\ \eta > 0$, therefore we have

$$ \big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}} \geq \mathfrak{m}^{\frac{1}{\alpha}}\big(\mathcal{G}(\theta)\big)^{\frac{1}{\alpha}}. \tag{9} $$

Taking product (9) by $(\mathcal{G}(\theta))^{\frac{1}{\beta}}$, we arrive at

$$ \big(\mathcal{G}(\theta)\big)^{\frac{1}{\beta}}\big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}} \geq \mathfrak{m}^{\frac{1}{\alpha}}\mathcal{G}(\theta). \tag{10} $$

Taking product on both sides of (11) $\frac{\left(\Phi(\eta)-\Phi(\theta)\right)^{\delta-1}\Phi^{\Delta}(\theta)}{\Gamma(\delta)}$, which is positive because $\theta \in (0,\eta),\ \eta > 0$, we integrate the resulting identity with respect to θ from 0 to η we have

$$ \frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\big(\mathcal{G}(\theta)\big)^{\frac{1}{\beta}}\big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}}\Delta\theta $$

$$ \geq \mathfrak{m}^{\frac{1}{\alpha}}\frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\mathcal{G}(\theta)\Delta\theta. \tag{11} $$

Hence, we can write

$$ \left({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}}\big(\mathcal{G}(\eta)\big)^{\frac{1}{\beta}}\right)^{\frac{1}{\beta}} \geq \mathfrak{m}^{\frac{1}{\alpha\beta}}\left({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}(\eta)\right)^{\frac{1}{\beta}}. \tag{12} $$

Conducting product between (8) and (12), we can draw the desired conclusion easily. □

Corollary 3.1. *Letting $\mathbb{T} = \mathbb{R}$, then under the assumption of Theorem 3.1, we have the following inequality in generalized Riemann-Liouville fractional integral:*

$$ \Big[\,\Phi\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\Big]^{\frac{1}{\alpha}}\Big[\,\Phi\mathcal{J}^{\delta}_{\eta}\mathcal{G}(\eta)\Big]^{\frac{1}{\beta}} \leq \Big(\frac{\mathcal{M}}{\mathfrak{m}}\Big)^{\frac{1}{\alpha\beta}}\Big[\,\Phi\mathcal{J}^{\delta}_{\eta}\big(\mathcal{F}(\theta)\big)^{\frac{1}{\alpha}}\big(\mathcal{G}(\eta)\big)^{\frac{1}{\beta}}\Big]. $$

Theorem 3.2. *Let $\delta,\gamma > 1$, and $\mathbb{T}$ is a time scale. Suppose $\mathcal{F},\mathcal{G}$ be two positive functions on $[0,\infty)_{\mathbb{T}}$, and Φ is monotone, delta differentiable Φ^{Δ} with $\Phi^{\Delta} \neq 0$ such that for all $\eta > 0$, ${}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha}(\eta) < \infty$, ${}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta}(\eta) < \infty$. If $0 < \mathfrak{m} \leq \frac{\mathcal{F}^{\alpha}(\theta)}{\mathcal{G}^{\beta}(\theta)} \leq \mathcal{M}, \theta \in [0,\eta]$, then*

$$ \Big[{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha}(\eta)\Big]^{\frac{1}{\alpha}}\Big[{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta}(\eta)\Big]^{\frac{1}{\beta}} \leq \Big(\frac{\mathcal{M}}{\mathfrak{m}}\Big)^{\frac{1}{\alpha\beta}}\Big[{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\big(\mathcal{F}(\theta)\mathcal{G}(\eta)\big)\Big], \tag{13} $$

where $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta} = 1$.

Proof: Replacing $\mathcal{F}(\theta)$ and $\mathcal{G}(\theta)$ by $\mathcal{F}^{\alpha}(\theta)$ and $\mathcal{G}^{\beta}(\theta)$, $\theta \in [0,\eta]$, $\eta > 0$ in Theorem 3.1, we acquire the desired result. This completes the proof. □

4. GRÜSS TYPE INEQUALITIES VIA GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL ON TIME SCALE

Our coming result is the generalization of Grüss type inequality via generalized Reimann-Liouville fractional integral operator on time scale.

Theorem 4.1. *Let $\delta,\gamma > 1$, and $\mathbb{T}$ is a time scale. Suppose there is a positive function $\mathcal{F}$ on $[0,\infty)_{\mathbb{T}}$, and Φ is monotone, delta differentiable Φ^{Δ} with $\Phi^{\Delta} \neq 0$ such that for all $\eta > 0$. Assume that the subsequent.*
(I)There exist two integrable functions φ_1,φ_2 on $[0,\infty)_{\mathbb{T}}$ such that

$$ \varphi_1(\eta) \leq \mathcal{F}(\eta) \leq \varphi_2(\eta), \quad \forall\eta \in [0,\infty)_{\mathbb{T}}. \tag{14} $$

Then, for $\eta > 0, \delta,\gamma > 1$, one has

$$ {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}(\eta) + {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\lambda}_{\eta}\varphi_1(\eta) $$
$$ \geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\lambda}_{\eta}\varphi_1(\eta) + {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\lambda}_{\eta}\mathcal{F}(\eta), \tag{15} $$

Proof: From (I), for all $\theta \geq 0, \lambda \geq 0$, we have

$$ \big(\varphi_2(\theta)-\mathcal{F}(\theta)\big)\big(\mathcal{F}(\lambda)-\varphi_1(\lambda)\big) \geq 0. \tag{16} $$

Therefore,

$$ \varphi_2(\theta)\mathcal{F}(\lambda)+\varphi_1(\lambda)\mathcal{F}(\theta) \geq \varphi_1(\lambda)\varphi_2(\theta)+\mathcal{F}(\theta)\mathcal{F}(\lambda). \tag{17} $$

Taking product on both sides of (17) $\frac{\left(\Phi(\eta)-\Phi(\theta)\right)^{\delta-1}\Phi^{\Delta}(\theta)}{\Gamma(\delta)}$, which is positive because $\theta \in (0,\eta),\ \eta > 0$, we integrate the resulting identity with respect to θ from 0 to η we have

$$ \mathcal{F}(\lambda)\frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\varphi_2(\theta)\Delta\theta $$
$$ +\varphi_1(\lambda)\frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\mathcal{F}(\theta)\Delta\theta $$
$$ \geq \varphi_1(\lambda)\frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\varphi_2(\theta)\Delta\theta $$
$$ +\mathcal{F}(\lambda)\frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\mathcal{F}(\theta)\Delta\theta, \tag{18} $$

arrives at

$$ \mathcal{F}(\lambda)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta)+\varphi_1(\lambda)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta) \geq \varphi_1(\lambda)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta) $$
$$ +\mathcal{F}(\lambda)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta). \tag{19} $$

Taking product on both sides of (19) $\frac{\left(\Phi(\eta)-\Phi(\lambda)\right)^{\gamma-1}\Phi^{\Delta}(\lambda)}{\Gamma(\gamma)}$, which is positive because $\lambda \in (0,\eta),\ \eta > 0$, we integrate the resulting identity with respect to λ from 0 to η we have

$$
\begin{aligned}
& {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\mathcal{F}(\lambda)\Delta\lambda \\
& +{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\varphi_1(\lambda)\Delta\lambda \\
\geq\ & {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\varphi_1(\lambda)\Delta\lambda \\
& +{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\mathcal{F}(\lambda)\Delta\lambda.
\end{aligned}
\tag{20}
$$

Hence, we conclude the desired inequality. This completes the proof. □

Special cases of Theorem 4.1, we attain the subsequent results.

Corollary 4.1. *Letting $\Phi(\eta)=\eta$, then Theorem 4.1 will lead to the Riemann-Liouville fractional integral on time scales:*

$$
\begin{aligned}
& {}^{\mathbb{T}}_{0^+}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}_{\eta}^{\gamma}\mathcal{F}(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}_{\eta}^{\lambda}\varphi_1(\eta) \\
\geq\ & {}^{\mathbb{T}}_{0^+}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}_{\eta}^{\lambda}\varphi_1(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}_{\eta}^{\lambda}\mathcal{F}(\eta).
\end{aligned}
$$

Remark 4.1. *If $\mathbb{T}=\mathbb{R}$, then Theorem 4.1 will lead to Theorem 2.11 in [57] and corollary 4.1 will lead to Corollary 3 in [57]. Also, if we choose $\mathbb{T}=\mathbb{R}$ along with $\Phi(\eta)=\eta$, then Theorem 4.1 will lead to Theorem 2 in [58].*

Theorem 4.2. *Let $\delta,\gamma>1$, and $\mathbb{T}$ is a time scale. Suppose there are two positive functions $\mathcal{F},\mathcal{G}$ on $[0,\infty)_{\mathbb{T}}$, and Φ is monotone, delta differentiable Φ^{Δ} with $\Phi^{\Delta}\neq 0$ such that for all $\eta>0$. Suppose that (I) holds and moreover one assumes the following.*
(II) There exist ω_1 and ω_2 integrable functions on $[0,\infty)_{\mathbb{T}}$ such that

$$
\omega_1(\eta)\leq\mathcal{G}(\eta)\leq\omega_2(\eta)\quad\forall\eta\in[0,\infty)_{\mathbb{T}}. \tag{21}
$$

Then, for $\eta>0,\delta,\gamma>1$,the following inequalities hold:

(A_1)
$$
\begin{aligned}
& {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\mathcal{G}(\eta)+{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\omega_1(\eta) \\
\geq\ & {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\omega_1(\eta) \\
& +{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\mathcal{G}(\eta),
\end{aligned}
$$

(B_1)
$$
\begin{aligned}
& {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\varphi_1(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{G}(\eta)+{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\omega_2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\mathcal{F}(\eta) \\
\geq\ & {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\varphi_1(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\omega_2(\eta) \\
& +{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{G}(\eta),
\end{aligned}
$$

(C_1)
$$
\begin{aligned}
& {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\omega_2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)+{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\mathcal{G}(\eta) \\
\geq\ & {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\mathcal{G}(\eta) \\
& +{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\omega_2(\eta),
\end{aligned}
$$

(D_1)
$$
\begin{aligned}
& {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_1(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\omega_1(\eta) \\
& +{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\mathcal{G}(\eta) \\
\geq\ & {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_1(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\mathcal{G}(\eta) \\
& +{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\gamma}\omega_1(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta).
\end{aligned}
\tag{22}
$$

Proof: To prove (A_1), from (I) and (II), we have for $x\in[0,\infty)_{\mathbb{T}}$ that

$$
\big(\varphi_2(\theta)-\mathcal{F}(\theta)\big)\big(\mathcal{G}(\lambda)-\omega_1(\lambda)\big)\geq 0. \tag{23}
$$

Therefore,

$$
\varphi_2(\theta)\mathcal{G}(\lambda)+\omega_1(\lambda)\mathcal{F}(\theta)\geq\omega_1(\lambda)\varphi_2(\theta)+\mathcal{G}(\lambda)\mathcal{F}(\theta). \tag{24}
$$

Taking product on both sides of (24) $\frac{\left(\Phi(\eta)-\Phi(\theta)\right)^{\delta-1}\Phi^{\Delta}(\theta)}{\Gamma(\delta)}$, which is positive because $\theta\in(0,\eta)$, $\eta>0$, we integrate the resulting identity with respect to θ from 0 to η we have

$$
\begin{aligned}
& \mathcal{G}(\lambda)\frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\varphi_2(\theta)\Delta\theta \\
& +\omega_1(\lambda)\frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\mathcal{F}(\theta)\Delta\theta \\
\geq\ & \omega_1(\lambda)\frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\varphi_2(\theta)\Delta\theta \\
& +\mathcal{G}(\lambda)\frac{1}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\mathcal{F}(\theta)\Delta\theta.
\end{aligned}
\tag{25}
$$

Then we have

$$
\begin{aligned}
& \mathcal{G}(\lambda)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)+\omega_1(\lambda)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta) \\
\geq\ & \omega_1(\lambda)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)+\mathcal{G}(\lambda)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta).
\end{aligned}
\tag{26}
$$

Again, multiplying both sides of (26) by $\frac{\left(\Phi(\eta)-\Phi(\lambda)\right)^{\gamma-1}\Phi^{\Delta}(\lambda)}{\Gamma(\gamma)}$, which is positive because $\lambda\in(0,\eta)$, $\eta>0$, we integrate the resulting identity with respect to λ from 0 to η we have

$$
\begin{aligned}
& {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\mathcal{G}(\lambda)\Delta\lambda \\
& +{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\omega_1(\lambda)\Delta\lambda \\
\geq\ & {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\varphi_2(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\omega_1(\lambda)\Delta\lambda \\
& +{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}_{\eta}^{\delta}\mathcal{F}(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\mathcal{G}(\lambda)\Delta\lambda.
\end{aligned}
$$

This follows that

$$
\begin{aligned}
&{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta)+{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\omega_1(\eta)\\
&\geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\omega_1(\eta)+{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta),
\end{aligned}
$$

we acquire the desired inequality (A_1).

To prove $(B_1)-(D_1)$, we utilizes the subsequent variants:

$$
\begin{aligned}
(B_1)\quad &\big(\omega_2(\theta)-\mathcal{G}(\theta)\big)\big(\mathcal{F}(\lambda)-\varphi_1(\lambda)\big)\geq 0,\\
(C_1)\quad &\big(\varphi_2(\theta)-\mathcal{F}(\theta)\big)\big(\mathcal{G}(\lambda)-\omega_2(\lambda)\big)\leq 0,\\
(D_1)\quad &\big(\varphi_1(\theta)-\mathcal{F}(\theta)\big)\big(\mathcal{G}(\lambda)-\omega_1(\lambda)\big)\leq 0.
\end{aligned}
$$

□

Special case of Theorem 4.2, we have the subsequent corollaries.

Corollary 4.2. *Letting $\Phi(\eta)=\eta$, then Theorem 4.2 will lead to a new result for Riemann-Liouville fractional integral on time scales:*

$$
\begin{aligned}
(A_2)\quad &{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\omega_1(\eta)\\
&\geq {}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\omega_1(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta),\\
(B_2)\quad &{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\varphi_1(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\mathcal{G}(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\omega_2(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}(\eta)\\
&\geq {}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\varphi_1(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\omega_2(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\mathcal{G}(\eta),\\
(C_2)\quad &{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\omega_2(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta)\\
&\geq {}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\varphi_2(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\omega_2(\eta),\\
(D_2)\quad &{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\varphi_1(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\omega_1(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta)\\
&\geq {}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\varphi_1(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta)+{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\gamma}_{\eta}\omega_1(\eta)\,{}^{\mathbb{T}}_{0^+}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta).
\end{aligned}
$$

Remark 4.2. *If $\mathbb{T}=\mathbb{R}$, then Theorem 4.2 will lead to Theorem 2.15 in [57] and corollary 4.2 will lead to Corollary 2.16 in [57]. Also, If we choose $\mathbb{T}=\mathbb{R}$ along with $\Phi(\eta)=\eta$, then Theorem 4.2 will lead to Theorem 5 in [58].*

5. SOME OTHER BOUNDS VIA GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL ON TIME SCALE

Theorem 5.1. *Let $\delta,\gamma>1$, and $\mathbb{T}$ is a time scale. Suppose there are two positive functions $\mathcal{F},\mathcal{G}$ on $[0,\infty)_{\mathbb{T}}$, and Φ is monotone, delta differentiable Φ^{Δ} with $\Phi^{\Delta}\neq 0$ such that for all $\eta>0$, $\alpha,\beta>1$ satisfying $\frac{1}{\alpha}+\frac{1}{\beta}=1$. Then, for $\eta>0$, one has*

$$
\begin{aligned}
(A_3)\quad &\frac{1}{\alpha}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{\alpha}(\eta)\\
&+\frac{1}{\beta}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{\beta}(\eta)\\
&\geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta)\mathcal{F}(\eta),\\
(B_3)\quad &\frac{1}{\alpha}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{\beta}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha}(\eta)\\
&+\frac{1}{\beta}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{\alpha}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta}(\eta)\\
&\geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{\beta-1}(\eta)\mathcal{F}^{\alpha-1}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta),\\
(C_3)\quad &\frac{1}{\alpha}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{2}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha}(\eta)\\
&+\frac{1}{\beta}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{2}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta}(\eta)\\
&\geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{\frac{2}{\beta}}(\eta)\mathcal{G}^{\frac{2}{\alpha}}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta),\\
(D_3)\quad &\frac{1}{\alpha}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{\beta}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{2}(\eta)\\
&+\frac{1}{\beta}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{\alpha}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{2}(\eta)\\
&\geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{\alpha-1}(\eta)\mathcal{G}^{\beta-1}(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\frac{2}{\alpha}}(\eta)\mathcal{G}^{\frac{2}{\beta}}(\eta).
\end{aligned}
\tag{27}
$$

Proof: Taking into account the Young's inequality [59]:

$$
\frac{1}{\alpha}a^{\alpha}+\frac{1}{\beta}b^{\beta}\geq ab,\quad \forall a,b\geq 0,\ \alpha,\beta>0,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \tag{28}
$$

setting $a=\mathcal{F}(\theta)\mathcal{G}(\lambda)$ and $b=\mathcal{F}(\lambda)\mathcal{G}(\theta)$, $\theta,\lambda>0$, we have

$$
\frac{1}{\alpha}\big(\mathcal{F}(\theta)\mathcal{G}(\lambda)\big)^{\alpha}+\frac{1}{\beta}\big(\mathcal{F}(\lambda)\mathcal{G}(\theta)\big)^{\beta}\geq(\mathcal{F}(\theta)\mathcal{G}(\lambda))(\mathcal{F}(\lambda)\mathcal{G}(\theta)). \tag{29}
$$

Taking product on both sides of (29) $\frac{\left(\Phi(\eta)-\Phi(\theta)\right)^{\delta-1}\Phi^{\Delta}(\theta)}{\Gamma(\delta)}$, which is positive because $\theta\in(0,\eta)$, $\eta>0$, we integrate the resulting identity with respect to θ from 0 to η we have

$$
\begin{aligned}
&\frac{\mathcal{G}^{\alpha}(\lambda)}{\alpha\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\Gamma(\delta)\mathcal{F}^{\alpha}(\theta)\Delta\theta\\
&+\frac{\mathcal{F}^{\beta}(\lambda)}{\beta\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\Gamma(\delta)\mathcal{G}^{\beta}(\theta)\Delta\theta\\
&\geq\frac{\mathcal{G}(\lambda)\mathcal{F}(\lambda)}{\Gamma(\delta)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\theta)\big)^{\delta-1}\Phi^{\Delta}(\theta)\Gamma(\delta)\mathcal{F}(\theta)\mathcal{G}(\theta)\Delta\theta,
\end{aligned}
\tag{30}
$$

we get

$$
\begin{aligned}
&\frac{\mathcal{G}^{\alpha}(\lambda)}{\alpha}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha}(\eta)+\frac{\mathcal{F}^{\beta}(\lambda)}{\beta}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta}(\eta)\\
&\geq\mathcal{G}(\lambda)\mathcal{F}(\lambda)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta).
\end{aligned}
\tag{31}
$$

Again, multiplying both sides of (31) by $\frac{\left(\Phi(\eta)-\Phi(\lambda)\right)^{\gamma-1}\Phi^{\Delta}(\lambda)}{\Gamma(\gamma)}$, which is positive because $\lambda\in(0,\eta)$, $\eta>0$, we integrate the resulting identity with respect to λ from 0 to η we have

$$
\begin{aligned}
&\frac{1}{\alpha}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha}(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\mathcal{G}^{\alpha}(\lambda)\Delta\lambda\\
&+\frac{1}{\beta}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta}(\eta)\frac{1}{\Gamma(\gamma)}\int_0^{\eta}\big(\Phi(\eta)-\Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\mathcal{F}^{\beta}(\lambda)\Delta\lambda
\end{aligned}
$$

$$\geq \frac{{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta)}{\Gamma(\gamma)} \int_0^{\eta} \big(\Phi(\eta) - \Phi(\lambda)\big)^{\gamma-1}\Phi^{\Delta}(\lambda)\mathcal{G}(\lambda)\mathcal{F}(\lambda)\Delta\lambda, \tag{32}$$

consequently, we get

$$\frac{1}{\alpha}\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{\alpha}(\eta) + \frac{1}{\beta}\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{\beta}(\eta)$$
$$\geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta)\mathcal{F}(\eta), \tag{33}$$

which implies (A_3). The remaining variants can be proved by adopting the same technique as we did in (A_3).

(B_3) $a = \frac{\mathcal{F}(\theta)}{\mathcal{F}(\lambda)}$, $b = \frac{\mathcal{G}(\theta)}{\mathcal{G}(\lambda)}$, $\mathcal{F}(\lambda), \mathcal{G}(\lambda) \neq 0$,

(C_3) $a = \mathcal{F}(\theta)\mathcal{G}^{\frac{2}{\alpha}}(\lambda)$, $b = \mathcal{F}^{\frac{2}{\beta}}(\lambda)\mathcal{G}(\theta)$,

(D_3) $a = \mathcal{F}^{\frac{2}{\alpha}}(\theta)\mathcal{F}(\lambda)$, $b = \mathcal{G}^{\frac{2}{\beta}}(\theta)\mathcal{G}(\lambda)$, $\mathcal{F}(\lambda), \mathcal{G}(\lambda) \neq 0$.

Repeating the foregoing argument, we obtain $(B_3) - (D_3)$. □

Theorem 5.2. *Let $\delta, \gamma > 1$, and $\mathbb{T}$ is a time scale. Suppose $\mathcal{F}, \mathcal{G}$ be two positive functions on $[0, \infty)_{\mathbb{T}}$, and Φ is monotone, delta differentiable Φ^{Δ} with $\Phi^{\Delta} \neq 0$ such that for all $\eta > 0$, and $\alpha, \beta > 0$ satisfying $\alpha + \beta = 1$. Then, for $\eta > 0$, one has*

(A_4)
$$p\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta) + q\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}(\beta) \geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\big(\mathcal{F}^{\alpha}(\eta)\mathcal{G}^{\beta}(\eta)\big)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\big(\mathcal{F}^{\beta}(\eta)\mathcal{G}^{\alpha}(\eta)\big),$$

(B_4)
$$p\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha-1}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\big(\mathcal{F}(\eta)\mathcal{G}^{\beta}(\eta)\big) + q\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{\beta-1}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\big(\mathcal{F}^{\beta}(\eta)\mathcal{G}(\eta)\big) \geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{\alpha}(\eta),$$

(C_4)
$$p\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{\frac{2}{\alpha}}(\eta) + q\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{\frac{2}{\beta}}(\eta) \geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\alpha}(\eta)\mathcal{G}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{\beta}(\eta)\mathcal{F}^{2}(\eta),$$

(D_4)
$$p\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{\frac{2}{\alpha}}(\eta)\mathcal{G}^{\beta}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{\alpha-1}(\eta) + q\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{\beta-1}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}^{\frac{2}{\beta}}(\eta)\mathcal{G}^{\alpha}(\eta) \geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{2}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}^{2}(\eta). \tag{34}$$

Proof: Taking into account the weighted $AM - GM$ inequality

$$\alpha a + \beta b \geq a^{\alpha}b^{\beta}, \quad \forall a, b \geq 0,\ \alpha, \beta > 0,\ \alpha + \beta = 1, \tag{35}$$

by setting $a = \mathcal{F}(\theta)\mathcal{G}(\lambda)$ and $b = \mathcal{F}(\lambda)\mathcal{G}(\theta)$, $\lambda, \theta > 0$, we have

$$\alpha\mathcal{F}(\theta)\mathcal{G}(\lambda) + \beta\mathcal{F}(\lambda)\mathcal{G}(\theta) \geq \big(\mathcal{F}(\theta)\mathcal{G}(\lambda)\big)^{\alpha}\big(\mathcal{F}(\lambda)\mathcal{G}(\theta)\big)^{\beta}. \tag{36}$$

Multiplying both sides of (36) by $\frac{1}{\Gamma(\delta)\Gamma(\gamma)}(\Phi(\eta) - \Phi(\theta))^{\delta-1}\Phi^{\Delta}(\theta)(\Phi(\eta) - \Phi(\lambda))^{\gamma-1}\Phi^{\Delta}(\lambda)$, which is positive because $\theta, \lambda \in (0, \eta)$, $\eta > 0$ and integrating the resulting identity from 0 to η we have

$$\frac{\alpha}{\Gamma(\delta)\Gamma(\gamma)}\int_0^{\eta}\int_0^{\eta}(\Phi(\eta) - \Phi(\theta))^{\delta-1}(\Phi(\eta) - \Phi(\lambda))^{\gamma-1}\Phi^{\Delta}(\theta)\Phi^{\Delta}(\lambda)\mathcal{F}(\theta)\mathcal{G}(\lambda)\Delta\theta\Delta\lambda$$
$$+ \frac{\beta}{\Gamma(\delta)\Gamma(\gamma)}\int_0^{\eta}\int_0^{\eta}(\Phi(\eta) - \Phi(\theta))^{\delta-1}(\Phi(\eta) - \Phi(\lambda))^{\gamma-1}\Phi^{\Delta}(\theta)\Phi^{\Delta}(\lambda)\mathcal{F}(\lambda)\mathcal{G}(\theta)\Delta\theta\Delta\lambda$$
$$\geq \frac{1}{\Gamma(\delta)\Gamma(\gamma)}\int_0^{\eta}\int_0^{\eta}(\Phi(\eta) - \Phi(\theta))^{\delta-1}(\Phi(\eta) - \Phi(\lambda))^{\gamma-1}\Phi^{\Delta}(\theta)\Phi^{\Delta}(\lambda) \times \big(\mathcal{F}(\theta)\mathcal{G}(\lambda)\big)^{\alpha}\big(\mathcal{F}(\lambda)\mathcal{G}(\theta)\big)^{\beta}\Delta\lambda\Delta\theta, \tag{37}$$

we conclude that

$$p\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{G}(\eta) + q\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\mathcal{F}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}(\eta)$$
$$\geq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\big(\mathcal{F}^{\alpha}(\eta)\mathcal{G}^{\beta}(\eta)\big)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\gamma}_{\eta}\big(\mathcal{F}^{\beta}(\eta)\mathcal{G}^{\alpha}(\eta)\big), \tag{38}$$

which implies (A_4). The rest of inequalities can be shown in similar way by the following choice of parameters in $AM - GM$ inequality.

(B_4) $a = \frac{\mathcal{F}(\lambda)}{\mathcal{F}(\theta)}$, $b = \frac{\mathcal{G}(\theta)}{\mathcal{G}(\lambda)}$, $\mathcal{F}(\theta), \mathcal{G}(\lambda) \neq 0$.

(C_4) $a = \mathcal{F}(\theta)\mathcal{G}^{\frac{2}{\alpha}}(\lambda)$, $b = \mathcal{F}^{\frac{2}{\beta}}(\lambda)\mathcal{G}(\theta)$,

(D_4) $a = \frac{\mathcal{F}^{\frac{2}{\alpha}}(\theta)}{\mathcal{G}(\lambda)}$, $b = \frac{\mathcal{F}^{\frac{2}{\beta}}(\lambda)}{\mathcal{G}(\theta)}$, $\mathcal{G}(\theta), \mathcal{G}(\theta) \neq 0$.

□

Example 5.1. Let $\delta, \gamma > 1$, and $\mathbb{T}$ is a time scale. Suppose $\mathcal{F}, \mathcal{G}$ be two positive functions on $[0, \infty)_{\mathbb{T}}$, and Φ is monotone, delta differentiable Φ^{Δ} with $\Phi^{\Delta} \neq 0$ such that for all $\eta > 0$, and $\alpha, \beta > 0$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Let

$$\mathfrak{m} = \min_{0\leq\theta\leq\eta}\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} \quad \textit{and} \quad \mathcal{M} = \max_{0\leq\theta\leq\eta}\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}. \tag{39}$$

Then, for $\eta > 0$, $\delta, \gamma > 1$, one has the following inequalities:

(1)
$$0 \leq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{2}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{2}(\eta) \leq \frac{\mathfrak{m} + \mathcal{M}}{4\mathfrak{m}\mathcal{M}}\big({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta)\big)^{2},$$

(2)
$$0 \leq \sqrt{{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{2}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{2}(\eta)} - \big({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta)\big) \leq \frac{\sqrt{\mathcal{M}} - \sqrt{\mathfrak{m}}}{2\sqrt{\mathfrak{m}\mathcal{M}}}\big({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta)\big),$$

(3)
$$0 \leq {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^{2}(\eta)\, {}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^{2}(\eta) - \big({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta)\big)^{2} \leq \frac{\mathcal{M} - \mathfrak{m}}{4\mathfrak{m}\mathcal{M}}\big({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta)\big)^{2}.$$

Proof: From Equation (39) and the inequality

$$\Big(\frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)} - \mathfrak{m}\Big)\Big(\mathcal{M} - \frac{\mathcal{F}(\theta)}{\mathcal{G}(\theta)}\Big)\mathcal{G}^{2}(\theta) \geq 0, \ \ 0 \leq \theta \leq \eta, \tag{40}$$

then we can write as,

$$\mathcal{F}^2(\theta) + \mathfrak{m}\mathcal{M}\mathcal{G}^2(\theta) \leq (\mathfrak{m} + \mathcal{M})\mathcal{F}(\theta)\mathcal{G}(\theta). \tag{41}$$

Multiplying both sides of (41) by $\frac{1}{\Gamma(\delta)}(\Phi(\eta)-\Phi(\theta))\Phi^{\Delta}(\theta)$, which is positive because $\theta \in (0, \eta)$, $\eta > 0$ and integrating the resulting identity from 0 to η, we have

$$\frac{1}{\Gamma(\delta)}\int_0^{\eta}(\Phi(\eta) - \Phi(\theta))\Phi^{\Delta}(\theta)\mathcal{F}^2(\theta)\Delta\theta + \mathfrak{m}\mathcal{M}\frac{1}{\Gamma(\delta)}\int_0^{\eta}(\Phi(\eta) - \Phi(\theta))\Phi^{\Delta}(\theta)\mathcal{G}^2(\theta)\Delta\theta \leq (\mathfrak{m} + \mathcal{M})\frac{1}{\Gamma(\delta)}\int_0^{\eta}(\Phi(\eta) - \Phi(\theta))\Phi^{\Delta}(\theta)\mathcal{F}(\theta)\mathcal{G}(\theta)\Delta\theta, \tag{42}$$

implies that

$${}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^2(\eta) + \mathfrak{m}\mathcal{M}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^2(\eta) \leq (\mathfrak{m} + \mathcal{M})\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta), \tag{43}$$

on the other hand, it follows from $\mathfrak{m}\mathcal{M} > 0$ and

$$\left(\sqrt{{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^2(\eta)} - \sqrt{\mathfrak{m}\mathcal{M}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^2(\eta)}\right)^2 \geq 0, \tag{44}$$

that

$$2\sqrt{{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^2(\eta)}\sqrt{\mathfrak{m}\mathcal{M}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^2(\eta)} \leq \sqrt{{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^2(\eta)} + \sqrt{\mathfrak{m}\mathcal{M}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^2(\eta)} \tag{45}$$

then from equation (43) and (45), we obtain,

$$4\mathfrak{m}\mathcal{M}\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}^2(\eta)\,{}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{G}^2(\eta) \leq (\mathfrak{m} + \mathcal{M})^2({}^{\mathbb{T}}_{0^+;\Phi}\mathcal{J}^{\delta}_{\eta}\mathcal{F}(\eta)\mathcal{G}(\eta)). \tag{46}$$

Which implies (1). By some transformation of (1), similarly, we obtain (2) and (3). □

6. CONCLUSION

The succinct view of this paper to establish numerous inequalities on an arbitrary time scale for generalized Riemann-Liouville fractional integrals. For the suitable selection of Φ on time scale, one can discover numerous novel and existing outcomes as specific cases. This shows the idea of generalized Riemann-Liouville fractional integral is wide and unifying one, yet additionally, improve few consequences in the study on the time scale hypothesis. Numerous variants are explored, when $\mathbb{T} = \mathbb{R}$. Finally, we introduced various dynamic variants by employing generalized Riemann-Liouville fractional integral as an example. Our consequences have potential applications in calcium ion channels, fractional calculus of variations on time scales, involving fractional fundamentalism in mechanics and physics, quantization, control theory, and description of conservative, nonconservative, and constrained systems. The performance of the fractional dynamical integral method is reliable and effective to obtain new solutions. This method has more advantages: it is direct and concise. Thus, the proposed method can be extended to solve many systems of nonlinear fractional partial differential equations in mathematical and physical sciences. Also, the new exact analytical solutions, can be obtained for the generalized ordinary differential equations to obtain new theorems related to stability and continuous dependence on parameters for dynamic equations on time scales. Our computed outcomes can be very useful as a starting point of comparison when some approximate methods are applied to this nonlinear space-time fractional equation.

AUTHOR CONTRIBUTIONS

SR and MA: Conceptualization; SR, MA, and KN: Writing original draft preparation; DB and GR: Formal Analysis; SR and DB: Methodology; KN, DB, and GR: Writing review and Editing.

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