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Existence of Solutions for Nonlinear Fractional Differential Equations and Inclusions Depending on Lower-Order Fractional Derivatives

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Abstract: This article deals with the solutions of the existence and uniqueness for a new class of boundary value problems (BVPs) involving nonlinear fractional differential equations (FDEs), inclusions, and boundary conditions involving the generalized fractional integral. The nonlinearity relies on the unknown function and its fractional derivatives in the lower order. We use fixed-point theorems with single-valued and multi-valued maps to obtain the desired results, through the support of illustrations, the main results are well explained. We also address some variants of the problem.

Keywords: single-valued map; multi-valued map; Caputo derivative; generalized Riemann–Liouville integral

1. Introduction

The subject of the fractional boundary value problem (BVP) has been intensively discussed in recent years by several researchers and in the literature, for example [1–19] and the references cited therein, where a variety of findings relevant to both the theoretical and implementation aspects of the topic can be found. It has improved the classic modeling of many significant materials and processes with the use of fractional calculus tools as a fractional-order operator can take the history of the phenomena involved into account. The extensive applications of fractional calculus can easily be seen in many engineering and technical sciences such as biology, environmental problems, aerodynamics, electron-analytic chemistry, etc. We direct the viewer to the article [20–28] and the references listed in it for examples and information. Recently, some authors analyzed the problems of fractional differential equations and inclusions. Ahmad et al. discussed in [29] the fractional differential equations (FDEs) and inclusions with nonlocal Erdelyi–Kober integral conditions:

$$\begin{gathered} {}^{C}\mathcal{D}^{q}x(t) = f(t, x(t)), \ \ \tau \in [0, T] := \mathcal{K}, \\ x(0) = g(x), \ \ x(T) = \alpha\, \mathcal{J}_{\eta}^{\gamma,\delta} x(\xi). \end{gathered}$$

and:

$$\begin{gathered} {}^{C}\mathcal{D}^{q}x(t) \in f(t, x(t)), \ \ \tau \in [0, T] := \mathcal{K}, \\ x(0) = g(x), \ \ x(T) = \alpha\, \mathcal{J}_{\eta}^{\gamma,\delta} x(\xi). \end{gathered}$$

Ntouyas et al. [30] investigated the existence of solutions for fractional differential inclusion. Salem et al. [31] studied the FDEs and inclusions, under integral-multipoint conditions.

In this article, we examine a new BVP of FDEs and inclusions:

$$ {}^{C}\mathcal{D}^{\varsigma}y(\tau) = g(\tau, y(\tau), {}^{C}\mathcal{D}^{\varrho}y(\tau), {}^{C}\mathcal{D}^{\varrho+1}y(\tau)), \ \ \tau \in [0, T] := \mathcal{K}, \tag{1} $$

$$ {}^{C}\mathcal{D}^{\varsigma}y(\tau) \in \mathcal{G}(\tau, y(\tau), {}^{C}\mathcal{D}^{\varrho}y(\tau), {}^{C}\mathcal{D}^{\varrho+1}y(\tau)), \ \ \tau \in [0, T] := \mathcal{K}, \tag{2} $$

augmented with the boundary conditions given by:

$$ y(0) = 0, \ \ y^{'}(0) = 0, \ \ \int_0^T y(\sigma)d\sigma = \xi\, {}^{\rho}\mathcal{J}^{\omega}y(\zeta), \ \ y^{'}(T) = 0, \tag{3} $$

where ${}^{C}\mathcal{D}^{(\cdot)}$ denotes the Caputo fractional derivatives (CFDs) of order $(\cdot)$, $3 < \varsigma \leq 4$, $0 < \varrho \leq 1$, ${}^{\rho}\mathcal{J}^{\omega}$ denote the generalized Riemann–Liouville fractional integral (GRLFI) of order $0 < \omega < 1$, $\rho > 0$, $g : \mathcal{K} \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function, $\mathcal{G} : \mathcal{K} \times \mathbb{R}^3 \to \mathcal{T}(\mathbb{R})$ is a multivalued map, $\mathcal{T}(\mathbb{R})$ is all nonempty subsets of $\mathbb{R}$, and $0 < \zeta < T$, ξ is a real constant.

For $0 < \varrho \leq 1$, let $\mathcal{Y} = \{y : y, {}^{C}\mathcal{D}^{\varrho}y, {}^{C}\mathcal{D}^{\varrho+1}y \in C(\mathcal{K}, \mathbb{R})\}$ denote the $[0, T] \to \mathbb{R}$ continuous function space of Banach endowed with the $\|y\|_* = \|y\| + \|{}^{C}\mathcal{D}^{\varrho}y\| + \|{}^{C}\mathcal{D}^{\varrho+1}y\| = \sup_{\tau \in \mathcal{K}} \Big\{|y(\tau)| + |{}^{C}\mathcal{D}^{\varrho}y(\tau)| + |{}^{C}\mathcal{D}^{\varrho+1}y(\tau)|\Big\}$ norm. For a normed space $(\mathcal{Y}, \|\cdot\|)$, let $\mathcal{T}_{cld}(\mathcal{Y}) = \{\mathcal{Z} \in \mathcal{T}(\mathcal{Y}) : \mathcal{Y} \text{ is closed}\}$, $\mathcal{T}_{bd}(\mathcal{Y}) = \{\mathcal{Z} \in \mathcal{T}(\mathcal{Y}) : \mathcal{Y} \text{ is bounded}\}$, $\mathcal{T}_{cpt}(\mathcal{Y}) = \{\mathcal{Z} \in \mathcal{T}(\mathcal{Y}) : \mathcal{Y} \text{ is compact}\}$, and $\mathcal{T}_{cpt,cx}(\mathcal{Y}) = \{\mathcal{Z} \in \mathcal{T}(\mathcal{Y}) : \mathcal{Y} \text{ is compact and convex}\}$. Define the set of choices $\mathcal{G}$ by each $C(\mathcal{K}, \mathbb{R})$,

$$ \mathcal{W}_{\mathcal{G},y} = \{\phi : \mathcal{L}^1(\mathcal{K}, \mathbb{R}) : \phi(\tau) \in \mathcal{G}(\tau, y(\tau), {}^{C}\mathcal{D}^{\varrho}y(\tau), {}^{C}\mathcal{D}^{\varrho+1}y(\tau)) \text{ for a.e. } \tau \in \mathcal{K}\}. $$

The remaining part of the article is structured accordingly. We recall some definitions in Section 2 and establish a lemma regarding the linear problem variant (1)–(3). Sections 3 and 4 include the consequences of existence. We emphasize that the techniques used in these sections in fixed-point theory are the standard. Finally, we glance at a new problem similar to (1)–(3) and discuss the approach to solving them.

2. Preliminaries

Here, we are reminded of some basic concepts in the fractional calculus [23,32,33] and of the results that we need to accomplish during the upcoming analysis.

Definition 1. *A continuous function $g : (0, \infty) \to \mathbb{R}$ is defined by the RLFI of order $\varsigma > 0$:*

$$ \mathcal{I}^{\varsigma}g(\tau) = \frac{1}{\Gamma(\varsigma)} \int_0^{\tau} (\tau - \sigma)^{\varsigma-1} g(\sigma)d\sigma, $$

provided the right-hand side (RHS) is point-wise defined on $(0, \infty)$.

Definition 2. *The CFD of order ς for a function $g : [0, \infty) \to \mathbb{R}$ can be written as:*

$$ {}^{C}\mathcal{D}^{\varsigma}g(\tau) = \frac{1}{\Gamma(n - \varsigma)} (\tau - \sigma)^{n-\varsigma-1} g^{n}(\sigma)d\sigma, \ \ n - 1 < \varsigma < n, \ \ n = [\varsigma] + 1, $$

where $[\varsigma]$ denotes the integer part of the real number $[\varsigma]$.

Definition 3. *The GRLFI of order $\varsigma > 0$ and $\rho > 0$, of a function $g(\tau)$, $\forall\, 0 < \tau < \infty$, is defined as:*

$$ {}^{\rho}\mathcal{I}^{\varsigma}g(\tau) = \frac{\rho^{1-\varsigma}}{\Gamma(\varsigma)} \int_0^{\tau} \frac{\sigma^{\rho-1}}{(\tau^{\rho} - \sigma^{\rho})^{1-\varsigma}} d\sigma, $$

provided the RHS is point-wise defined on $(0, \infty)$.

Definition 4. *A multi-valued map $\mathcal{G} : \mathcal{K} \times \mathbb{R}^3 \to \mathcal{T}(\mathbb{R})$ is Caratheodory if:*

- *(i) $\tau \longmapsto \mathcal{G}(\tau, y, z, w)$ is measurable for each $y, z, w \in \mathbb{R}$;*
- *(ii) $(y, z, w) \longmapsto \mathcal{G}(\tau, y, z, w)$ is upper semicontinuous (USC) $\forall\ \tau \in \mathcal{K}$; In addition, a $\mathcal{G}$ feature of Caratheodory is called $\mathcal{L}^1$-Caratheodory, if:*
- *(iii) for each $\alpha > 0$, there exists $\Lambda_\alpha \in \mathcal{L}^1(\mathcal{K}, \mathbb{R}^+)$ such that $\|\mathcal{G}(\tau, y, z, w)\| = \sup\{|\phi| : \phi \in \mathcal{G}(\tau, y, z, w) \leq \Lambda_\alpha(\tau)\}\ \forall\ \|y\|, \|z\|, \|w\| \leq \alpha$ and for almost everywhere $\tau \in \mathcal{K}$.*

Lemma 1. *Let $\hat{g} \in C[0, T]$. Then, the unique solution of the linear FDE:*

$$ {}^{C}\mathcal{D}^{\varsigma} y(\tau) = \hat{g}(\tau), \ \ \tau \in \mathcal{K}, \tag{4} $$

subject to the boundary condition (3) is given by:

$$ y(\tau) = \mathcal{I}^{\varsigma}\hat{g}(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\hat{g}(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\hat{g}(\sigma)d\sigma\Big] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\hat{g}(\tau), \tag{5} $$

where:

$$ \kappa_1(\tau) = \frac{\tau^2 3T^2 - 2T\tau^3}{\vartheta}, \ \ \kappa_2(\tau) = \frac{\tau^2\nu_2 - \nu_1\tau^3}{\vartheta}, \ \ \vartheta = 3T^2\nu_1 - 2T\nu_2, \tag{6} $$

$$ \nu_1 = \frac{T^3}{3} - \frac{\xi\zeta^{\rho\omega+2}}{\rho^{\omega}}\frac{\Gamma(\frac{2}{\rho}+1)}{\Gamma(\frac{2}{\rho}+\omega+1)}, \ \ \nu_2 = \frac{T^4}{4} - \frac{\xi\zeta^{\rho\omega+3}}{\rho^{\omega}}\frac{\Gamma(\frac{3}{\rho}+1)}{\Gamma(\frac{3}{\rho}+\omega+1)}. \tag{7} $$

Definition 5. *A multi-valued operator $\mathcal{U} : \mathcal{Y} \to \mathcal{T}_{cld}(\mathcal{Y})$ is called:*

(a) ι-Lipschitz iff there exists $\iota > 0$ such that $\mathcal{A}_d(\mathcal{U}(y), \mathcal{U}(z)) \leq \iota d(x, y)$ for each $y, z \in \mathcal{Y}$ and
(b) a contraction iff it is ι-Lipschitz with $\iota > 1$.

Definition 6. *A function $y \in C(\mathcal{K}, \mathbb{R})$ is said to be a solution of the BVP $y(0) = 0$, $y'(0) = 0$, $\int_0^T y(\sigma)d\sigma = \xi\ {}^{\rho}\mathcal{J}^{\omega}y(\zeta)$, $y'(T) = 0$, and there exists a function $\phi \in \mathcal{W}_{\mathcal{G},y}$ such that $\phi(\tau) \in \mathcal{G}(\tau, y(\tau), {}^{C}\mathcal{D}^{\varrho}y(\tau), {}^{C}\mathcal{D}^{\varrho+1}y(\tau))$, and:*

$$ y(\tau) = \mathcal{I}^{\varsigma}\phi(\sigma)(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi(\sigma)d\sigma\Big] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi(\sigma)(\tau). $$

3. Single-Valued Maps for the Problem (1) and (3)

With respect to Lemma 1, the problem (1) and (3) is turned into a fixed point problem equivalent to:

$$ y = \Upsilon y, \tag{8} $$

where $\Upsilon : \mathcal{Y} \to \mathcal{Y}$ is defined by:

$$ \begin{aligned} (\Upsilon y)(\tau) = {} & \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^{C}\mathcal{D}^{\varrho}y(\sigma), {}^{C}\mathcal{D}^{\varrho+1}y(\sigma))(\tau) \\ & + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^{C}\mathcal{D}^{\varrho}y(\sigma), {}^{C}\mathcal{D}^{\varrho+1}y(\sigma))(\zeta) \\ & - \int_0^T \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^{C}\mathcal{D}^{\varrho}y(\sigma), {}^{C}\mathcal{D}^{\varrho+1}y(\sigma))d\sigma\Big] \\ & + \kappa_2(\tau)\mathcal{I}^{\varsigma-1} g(\sigma, y(\sigma), {}^{C}\mathcal{D}^{\varrho}y(\sigma), {}^{C}\mathcal{D}^{\varrho+1}y(\sigma))(\tau). \end{aligned} \tag{9} $$

We represent it as suitable for computing:

$$\psi_1 = \frac{T^{\varsigma}}{\Gamma(\varsigma+1)} + \frac{\tilde{\kappa}_2 T^{\varsigma-1}}{\Gamma(\varsigma)} + \tilde{\kappa}_1\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}\Gamma(\varsigma+1)}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\Gamma(\varsigma+2)}\right), \tag{10}$$

$$\psi_2 = \frac{T^{\varsigma-\varrho}}{\Gamma(\varsigma-\varrho+1)} + \frac{\varphi_1}{\Gamma(\varsigma+1)}\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1}\right) + \frac{\varphi_2 T^{\varsigma-1}}{\Gamma(\varsigma)}, \tag{11}$$

$$\psi_3 = \frac{T^{\varsigma-\varrho-1}}{\Gamma(\varsigma-\varrho)} + \frac{\delta_1}{\Gamma(\varsigma+1)}\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1}\right) + \frac{\delta_2 T^{\varsigma-1}}{\Gamma(\varsigma)}, \tag{12}$$

$$\widehat{\psi}_1 = \psi_1 - \frac{T^{\varsigma}}{\Gamma(\varsigma+1)},\ \widehat{\psi}_2 = \psi_2 - \frac{T^{\varsigma-\varrho}}{\Gamma(\varsigma-\varrho+1)},\ \widehat{\psi}_3 = \psi_3 - \frac{T^{\varsigma-\varrho-1}}{\Gamma(\varsigma-\varrho)}. \tag{13}$$

Theorem 1. *Assume that there exists $\lambda \in C(\mathcal{K}, \mathbb{R}^+)$ such that $|g(\tau, y(\tau), {}^{C}\mathcal{D}^{\varrho}y(\tau), {}^{C}\mathcal{D}^{\varrho+1}y(\tau))| \leq \lambda(\tau)$ for $\tau \in \mathcal{K}$ with $\max_{\tau\in\mathcal{K}} |\lambda(\tau) = \|\lambda\|$. The problem (1) and (3) has at least one solution on $\mathcal{K}$.*

Proof. First, we demonstrate that operator Υ is completely continuous. Let $\mathcal{H} \subset \mathcal{Y}$ be a bounded set. Then, use the premise $|g(\tau, y(\tau), {}^{C}\mathcal{D}^{\varrho}y(\tau), {}^{C}\mathcal{D}^{\varrho+1}y(\tau))| \leq \lambda(\tau), \forall\, y \in \mathcal{H}$; we get:

$$|(\Upsilon y)(\tau)| \leq \mathcal{I}^{\varsigma}\lambda(\sigma)(\tau) + \kappa_1(\tau)\left[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\lambda(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\lambda(\sigma)d\sigma\right] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\lambda(\sigma)(\tau),$$

which yields when taking the norm for $\tau \in \mathcal{K}$,

$$\|\Upsilon y\| \leq \frac{\|\lambda\|}{\Gamma(\varsigma+1)}\left[(\tilde{\kappa}_2\varsigma T^{\varsigma-1} + T^{\varsigma}) + \tilde{\kappa}_1\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1}\right)\right] = \mathcal{P}_1,$$

where $\max_{\tau\in\mathcal{K}} |\kappa_i(\tau)| = \tilde{\kappa}_i,\ i = 1,2\ \kappa_i$'s are given by (6). Similarly, we can obtain:

$$\|{}^{C}\mathcal{D}^{\varrho}\Upsilon y\| \leq \|\lambda\|\left[\frac{T^{\varsigma-\varrho}}{\Gamma(\varsigma-\varrho+1)} + \frac{\varphi_1}{\Gamma(\varsigma+1)}\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1}\right) + \frac{\varphi_2 T^{\varsigma-1}}{\Gamma(\varsigma)}\right] = \mathcal{P}_2,$$

where $\varphi_i = \max_{\tau\in\mathcal{K}} |\varphi_i(\tau),\ i = 1,2$ and:

$$\varphi_1(\tau) = {}^{C}\mathcal{D}^{\varrho}\kappa_1(\tau) = \frac{6T^2\tau^{2-\varrho}}{\vartheta\Gamma(3-\varrho)} - \frac{12T\tau^{3-\varrho}}{\vartheta\Gamma(4-\varrho)},\quad \varphi_2(\tau) = {}^{C}\mathcal{D}^{\varrho}\kappa_2(\tau) = \frac{2\nu_2\tau^{2-\varrho}}{\vartheta\Gamma(3-\varrho)} - \frac{6\nu_1\tau^{3-\varrho}}{\vartheta\Gamma(4-\varrho)}.$$

Likewise, we can obtain:

$$\|{}^{C}\mathcal{D}^{\varrho+1}\Upsilon y\| \leq \|\lambda\|\left[\frac{T^{\varsigma-\varrho-1}}{\Gamma(\varsigma-\varrho)} + \frac{\delta_1}{\Gamma(\varsigma+1)}\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1}\right) + \frac{\delta_2 T^{\varsigma-1}}{\Gamma(\varsigma)}\right] = \mathcal{P}_3,$$

where $\delta_i = \max_{\tau\in\mathcal{K}} |\vartheta_i(\tau),\ i = 1,2$ and:

$$\delta_1(\tau) = {}^{C}\mathcal{D}^{\varrho+1}\kappa_1(\tau) = \frac{6T^2\tau^{1-\varrho}}{\vartheta\Gamma(2-\varrho)} - \frac{12T\tau^{2-\varrho}}{\vartheta\Gamma(3-\varrho)},\quad \delta_2(\tau) = {}^{C}\mathcal{D}^{\varrho+1}\kappa_2(\tau) = \frac{2\nu_2\tau^{1-\varrho}}{\vartheta\Gamma(2-\varrho)} - \frac{6\nu_1\tau^{2-\varrho}}{\vartheta\Gamma(3-\varrho)}.$$

For $0 < \tau_1 < \tau_2 < T$ and $\forall\, y \in \mathcal{H}$, we have:

$$|(\Upsilon y)(\tau_2) - (\Upsilon y)(\tau_1)| \leq \|\lambda\| \Bigg\{ \Bigg[\frac{|\tau_2^{\varsigma} - \tau_1^{\varsigma}| + 2(\tau_2 - \tau_1)^{\varsigma}}{\Gamma(\varsigma+1)} \Bigg] + \Bigg| \Bigg(\frac{(\tau_2^2 - \tau_1^2)\nu_2 - \nu_1(\tau_2^3 - \tau_1^3)}{\vartheta} \Bigg) \frac{T^{\varsigma-1}}{\Gamma(\varsigma)} \Bigg| + \Bigg| \frac{(\tau_2^2 - \tau_1^2)3T^2 - 2T(\tau_2^3 - \tau_1^3)}{\vartheta\Gamma(\varsigma+1)} \times \Bigg(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}} \frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1} \Bigg) \Bigg| \Bigg\}. \tag{14}$$

Similarly, we can accomplish:

$$|({}^{C}\mathcal{D}^{\varrho}\Upsilon y)(\tau_2) - ({}^{C}\mathcal{D}^{\varrho}\Upsilon y)(\tau_1)| \leq \|\lambda\| \Bigg\{ \Bigg[\frac{|\tau_2^{\varsigma-\varrho} - \tau_1^{\varsigma-\varrho}| + 2(\tau_2 - \tau_1)^{\varsigma-\varrho}}{\Gamma(\varsigma-\varrho+1)} \Bigg] + \Bigg| \Bigg(\frac{2\nu_2(\tau_2^{2-\varrho} - \tau_1^{2-\varrho})}{\vartheta\Gamma(3-\varrho)} - \frac{6\nu_1(\tau_2^{3-\varrho} - \tau_1^{3-\varrho})}{\vartheta\Gamma(4-\varrho)} \Bigg) \frac{T^{\varsigma-1}}{\Gamma(\varsigma)} \Bigg| + \Bigg| \frac{6T^2(\tau_2^{2-\varrho} - \tau_1^{2-\varrho})}{\vartheta\Gamma(3-\varrho)} - \frac{12T(\tau_2^{3-\varrho} - \tau_1^{3-\varrho})}{\vartheta\Gamma(4-\varrho)} \times \Bigg(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}\Gamma(\varsigma+1)} \frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\Gamma(\varsigma+2)} \Bigg) \Bigg| \Bigg\}. \tag{15}$$

Likewise, we obtain:

$$|({}^{C}\mathcal{D}^{\varrho+1}\Upsilon y)(\tau_2) - ({}^{C}\mathcal{D}^{\varrho+1}\Upsilon y)(\tau_1)| \leq \|\lambda\| \Bigg\{ \Bigg[\frac{|\tau_2^{\varsigma-\varrho-1} - \tau_1^{\varsigma-\varrho-1}| + 2(\tau_2 - \tau_1)^{\varsigma-\varrho-1}}{\Gamma(\varsigma-\varrho)} \Bigg] + \Bigg| \Bigg(\frac{2\nu_2(\tau_2^{1-\varrho} - \tau_1^{1-\varrho})}{\vartheta\Gamma(2-\varrho)} - \frac{6\nu_1(\tau_2^{2-\varrho} - \tau_1^{2-\varrho})}{\vartheta\Gamma(3-\varrho)} \Bigg) \frac{T^{\varsigma-1}}{\Gamma(\varsigma)} \Bigg| + \Bigg| \frac{6T^2(\tau_2^{1-\varrho} - \tau_1^{1-\varrho})}{\vartheta\Gamma(2-\varrho)} - \frac{12T(\tau_2^{2-\varrho} - \tau_1^{2-\varrho})}{\vartheta\Gamma(3-\varrho)} \times \Bigg(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}\Gamma(\varsigma+1)} \frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\Gamma(\varsigma+2)} \Bigg) \Bigg| \Bigg\}. \tag{16}$$

The RHS of the inequalities (14)–(16) tends to zero as $\tau_2 - \tau_1 \to 0$ independently of y. Thus, Υ is equicontinuous. Therefore, by the lemma (see Lemma 1.2 [21]), $\{\Upsilon y : y \in \mathcal{H}\}$, $\{{}^{C}\mathcal{D}^{\varrho}\Upsilon y : y \in \mathcal{H}\}$, and $\{{}^{C}\mathcal{D}^{\varrho+1}\Upsilon y : y \in \mathcal{H}\}$ are relatively compact in $C(\mathcal{K})$. Hence, $\Upsilon(\mathcal{H})$ is a relatively compact subset of $\mathcal{Y}$. Next, we take the set $\mathcal{V} = \{y \in \mathcal{Y} | y = \mu\Upsilon y,\ 0 < \mu < 1\}$, into consideration and prove it is bounded. Let $y \in \mathcal{V}$. Then, $y = \mu\Upsilon y$, $0 < \mu < 1$. For any $\tau \in \mathcal{K}$, it follows from $y(\tau) = \mu|\Upsilon y(\tau)|$ that:

$$\|\Upsilon y\| \leq \frac{\|\lambda\|}{\Gamma(\varsigma+1)} \Bigg[(\tilde{\kappa}_2\varsigma T^{\varsigma-1} + T^{\varsigma}) + \tilde{\kappa}_1 \Bigg(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}} \frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1} \Bigg) \Bigg]$$

This indicates that the set $\mathcal{V}$ is bounded. Thus, operator Υ has at least one fixed point by Theorem (see Theorem [20]) The problem (1) and (3) has at least one solution on $\mathcal{K}$. □

Theorem 2. *Let $g : \mathcal{K} \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function that holds the following conditions:*

($\mathcal{G}_1$) $|g(\tau, y_1, y_2, y_3) - g(\tau, z_1, z_2, z_3)| \leq \mathcal{P}(|y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3|)$, $\forall\ \tau \in \mathcal{K}$, $y_1, z_1, y_2, z_2, y_3, z_3 \in \mathbb{R}$, $\mathcal{P} > 0$.

$(\mathcal{G}_2)$ $|g(\tau, y(\tau), {}^C\mathcal{D}^{\varrho}y(\tau), {}^C\mathcal{D}^{\varrho+1}y(\tau))| \leq \Lambda(\tau)$ *for* $\tau \in \mathcal{K}$ *and* $\Lambda \in C(\mathcal{K}, \mathbb{R}^+)$ *with* $\max_{\tau\in\mathcal{K}} |\Lambda(\tau)| = \|\Lambda\|$.

$(\mathcal{G}_3)$ $\mathcal{P}\widehat{\psi} < 1$, *where* $\widehat{\psi} = \max\{\widehat{\psi}_1, \widehat{\psi}_2, \widehat{\psi}_3\}$ *and* $\widehat{\psi}_1, \widehat{\psi}_2$ *and* $\widehat{\psi}_3$ *are given by (13). The problem (1) and (3) has at least one solution on* $\mathcal{K}$.

Proof. Define $\mathcal{B}_\epsilon = \{y \in \mathcal{Y} : \|y\| \leq \epsilon\}$, where $\epsilon \geq \|\Lambda\|\psi$ with:

$$\psi = \max\{\psi_1, \psi_2, \psi_3\}, \tag{17}$$

where ψ_1, ψ_2, and ψ_3 are defined by (10)–(12), respectively. In order to demonstrate the premise of Theorem (see Theorem 4.4.1 [20]), the operator Υ provided by (9) is divided into $\Upsilon = \Upsilon_1 + \Upsilon_2$ by $\mathcal{B}_\epsilon$.

$$\begin{aligned}
(\Upsilon_1 y)(\tau) &= \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho}y(\sigma), {}^C\mathcal{D}^{\varrho+1}y(\sigma))(\tau) \\
(\Upsilon_2 y)(\tau) &= |\kappa_1(\tau)| \Big[\xi^{\rho} \mathcal{J}^{\omega} \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho}y(\sigma), {}^C\mathcal{D}^{\varrho+1}y(\sigma))(\zeta) \\
&\quad - \int_0^T \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho}y(\sigma), {}^C\mathcal{D}^{\varrho+1}y(\sigma)) d\sigma \Big] \\
&\quad + |\kappa_2(\tau)| \mathcal{I}^{\varsigma-1} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho}y(\sigma), {}^C\mathcal{D}^{\varrho+1}y(\sigma))(\tau).
\end{aligned}$$

It can be easily shown that for $y, z \in \mathcal{B}_\epsilon$, and using (17), $\|(\Upsilon_1 y) + (\Upsilon_2 z)\| \leq \|\Lambda\|\psi \leq \epsilon$, $\|({}^C\mathcal{D}^{\varrho}\Upsilon_1 y) + ({}^C\mathcal{D}^{\varrho}\Upsilon_2 z)\| \leq \|\Lambda\|\psi \leq \epsilon$ and $\|({}^C\mathcal{D}^{\varrho+1}\Upsilon_1 y) + ({}^C\mathcal{D}^{\varrho+1}\Upsilon_2 z)\| \leq \|\Lambda\|\psi \leq \epsilon$, that means $\Upsilon_1 y + \Upsilon_2 z \in \mathcal{B}_\epsilon$. Next, we are going to show that Υ_2 is a contraction. Let $y, z \in \mathbb{R}$, $\tau \in \mathcal{K}$. Then, we use the statement $(\mathcal{G}_1)$,

$$\begin{aligned}
\|\Upsilon_2 y - \Upsilon_2 z\| &\leq \mathcal{P}\|y - z\| \sup_{\tau\in\mathcal{K}} \Big[|\kappa_1(\tau)| \Big[\xi^{\rho} \mathcal{J}^{\omega} \mathcal{I}^{\varsigma}(1)(\zeta) + \int_0^T \mathcal{I}^{\varsigma}(1) d\sigma \Big] + |\kappa_2(\tau)| \mathcal{I}^{\varsigma-1}(1)(\tau) \Big] \\
&\leq \mathcal{P}\widehat{\psi}_1 \|y - z\| \leq \mathcal{P}\widehat{\psi}\|y - z\|.
\end{aligned}$$

Similarly,

$$\|{}^C\mathcal{D}^{\varrho}\Upsilon_2 y - {}^C\mathcal{D}^{\varrho}\Upsilon_2 z\| \leq \mathcal{P}\widehat{\psi}_2 \|y - z\| \leq \mathcal{P}\widehat{\psi}\|y - z\|.$$

Likewise,

$$\|{}^C\mathcal{D}^{\varrho+1}\Upsilon_2 y - {}^C\mathcal{D}^{\varrho+1}\Upsilon_2 z\| \leq \mathcal{P}\widehat{\psi}_3 \|y - z\| \leq \mathcal{P}\widehat{\psi}\|y - z\|.$$

This follows from the observation $(\mathcal{G}_3)$ that the Υ_2 operator is a contraction. Next we are going to show the Υ_1 is compact and continuous. The continuity of g means operator Υ_1 is continuous. Υ_1 is uniformly bounded on $\mathcal{B}_\epsilon$ as $\|\Upsilon_1 y\| \leq \frac{\|\Lambda\| T^{\varsigma}}{\Gamma(\varsigma+1)}$, $\|{}^C\mathcal{D}^{\varrho}\Upsilon_1 y\| \leq \frac{\|\Lambda\| T^{\varsigma-\varrho}}{\Gamma(\varsigma-\varrho+1)}$ and $\|{}^C\mathcal{D}^{\varrho+1}\Upsilon_1 y\| \leq \frac{\|\Lambda\| T^{\varsigma-\varrho-1}}{\Gamma(\varsigma-\varrho)}$. Furthermore, with $\sup_{(\tau,y,z,w)\in\mathcal{K}\times\mathcal{B}_\epsilon} |g(\tau, y, z, w)| = \widehat{g} < \infty$ and $\tau_2 > \tau_1$, we have:

$$|(\Upsilon_1 y)(\tau_2) - (\Upsilon_1 y)(\tau_1)| \leq \left[\frac{\widehat{g}|\tau_2^{\varsigma} - \tau_1^{\varsigma}| + 2(\tau_2 - \tau_1)^{\varsigma}}{\Gamma(\varsigma+1)} \right]. \tag{18}$$

Similarly, we can obtain:

$$|({}^C\mathcal{D}^{\varrho}\Upsilon_1 y)(\tau_2) - ({}^C\mathcal{D}^{\varrho}\Upsilon_1 y)(\tau_1)| \leq \left[\frac{\widehat{g}|\tau_2^{\varsigma-\varrho} - \tau_1^{\varsigma-\varrho}| + 2(\tau_2 - \tau_1)^{\varsigma-\varrho}}{\Gamma(\varsigma-\varrho+1)} \right]. \tag{19}$$

Likewise, we obtain:

$$|({}^{C}\mathcal{D}^{\varrho+1}\Upsilon_1 y)(\tau_2) - ({}^{C}\mathcal{D}^{\varrho+1}\Upsilon_1 y)(\tau_1)| \le \left[\frac{\widehat{g}|\tau_2^{\varsigma-\varrho-1} - \tau_1^{\varsigma-\varrho-1}| + 2(\tau_2-\tau_1)^{\varsigma-\varrho-1}}{\Gamma(\varsigma-\varrho)}\right]. \tag{20}$$

The RHS of the inequalities (18)–(20) tend to zero as $\tau_2 - \tau_1 \to 0$ independently of y. Thus, Υ_1 is relatively compact on $\mathcal{B}_\epsilon$. Hence, by the lemma (see Lemma 1.2 [21]), Υ_1 is compact on $\mathcal{B}_\epsilon$. Therefore, all the claims of the theorem (see Theorem 4.4.1 [20]) are fulfilled. Therefore, at least one solution exists (1) and (3) for the problem on $\mathcal{K}$. □

Theorem 3. *Assume that $g : \mathcal{K} \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying condition $|g(\tau, y_1, y_2, y_3) - g(\tau, z_1, z_2, z_3)| \le \mathcal{Q}(|y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3|), \forall\, \tau \in \mathcal{K},\ y_1, z_1, y_2, z_2, y_3, z_3 \in \mathbb{R}$, with $\mathcal{Q} < \frac{1}{\psi}$, where $\psi = \max\{\psi_1, \psi_2, \psi_3\}$, and ψ_1, ψ_2 and ψ_3 are respectively given by (10)–(12). Therefore, a unique solution exists (1) and (3) for the problem on $\mathcal{K}$.*

Proof. We demonstrate that $\Upsilon\mathcal{B}_\epsilon \subset \mathcal{B}_\epsilon$, where Υ is described by (9), $\mathcal{B}_\epsilon = \{y \in \mathcal{Y} : \|y\| \le \epsilon\}$ with $\epsilon \ge \frac{\mathcal{M}\psi}{1-\mathcal{Q}\psi}$, $\mathcal{M} = \sup_{\tau\in\mathcal{K}} |g(\tau, 0, 0, 0)|$. For $y \in \mathcal{B}_\epsilon$, $\tau \in \mathcal{K}$, we have that:

$$\begin{aligned}|g(\tau, y, z, w)| &= |g(\tau, y, z, w) - g(\tau, 0, 0, 0) + g(\tau, 0, 0, 0)| \\ &\le |g(\tau, y, z, w) - g(\tau, 0, 0, 0)| + |g(\tau, 0, 0, 0)| \le \mathcal{Q}(|y| + |z| + |w|) + \mathcal{M} \\ &\le \mathcal{Q}\|y\| + \mathcal{M} \le \mathcal{Q}\epsilon + \mathcal{M},\end{aligned}$$

which yields along with the given conditions:

$$\begin{aligned}\|\Upsilon y\| &\le \left[\frac{T^\varsigma}{\Gamma(\varsigma+1)} + \frac{\tilde{\kappa}_2 T^{\varsigma-1}}{\Gamma(\varsigma)} + \frac{\tilde{\kappa}_1}{\Gamma(\varsigma+1)}\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^\omega}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1}\right)\right] \\ &\le (\mathcal{Q}\epsilon + \mathcal{M})\psi_1 \le (\mathcal{Q}\epsilon + \mathcal{M})\psi \le \epsilon.\end{aligned}$$

Similarly, we can obtain:

$$\|{}^{C}\mathcal{D}^{\varrho}\Upsilon y\| \le (\mathcal{Q}\epsilon + \mathcal{M})\psi_2 \le (\mathcal{Q}\epsilon + \mathcal{M})\psi \le \epsilon.$$

Likewise, we obtain:

$$\|{}^{C}\mathcal{D}^{\varrho+1}\Upsilon y\| \le (\mathcal{Q}\epsilon + \mathcal{M})\psi_3 \le (\mathcal{Q}\epsilon + \mathcal{M})\psi \le \epsilon.$$

Therefore, we get $\Upsilon y \in \mathcal{B}_\epsilon$, which means $\Upsilon\mathcal{B}_\epsilon \subset \mathcal{B}_\epsilon$. Then, for $y, z \in \mathcal{Y}$, for each $\tau \in \mathcal{K}$, we have:

$$\begin{aligned}\|\Upsilon y - \Upsilon z\| &\le \mathcal{Q}\|y - z\| \sup_{\tau\in\mathcal{K}}\left\{\mathcal{I}^\varsigma(1)(\tau) + \kappa_1(\tau)\left[\xi^\rho\mathcal{J}^\omega\mathcal{I}^\varsigma(1)(\zeta) - \int_0^T \mathcal{I}^\varsigma(1)d\sigma\right] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}(1)(\tau)\right\} \\ &\le \mathcal{Q}\psi_1\|y - z\| \le \mathcal{Q}\psi\|y - z\|.\end{aligned}$$

Similarly,

$$\|{}^{C}\mathcal{D}^{\varrho}\Upsilon_2 y - {}^{C}\mathcal{D}^{\varrho}\Upsilon_2 z\| \le \mathcal{Q}\psi_2\|y - z\| \le \mathcal{Q}\psi\|y - z\|.$$

Likewise,

$$\|{}^{C}\mathcal{D}^{\varrho+1}\Upsilon_2 y - {}^{C}\mathcal{D}^{\varrho+1}\Upsilon_2 z\| \le \mathcal{Q}\psi_3\|y - z\| \le \mathcal{Q}\psi\|y - z\|.$$

Therefore, the operator Υ is a contraction in the light of condition $\mathcal{Q} < \frac{1}{\psi}$. Thus, Theorem (see Theorem 1.2.2 [20]) follows that the problem (1) and (3) has a unique solution on $\mathcal{K}$. □

4. Multi-Valued Maps for the Problem (2) and (3)

Theorem 4. *Assume that:*

$(\mathcal{F}_1)$ *$\mathcal{G} : \mathcal{K} \times \mathbb{R}^3 \to \mathcal{T}(\mathbb{R})$ is $\mathcal{L}^1$-Caratheodory and has nonempty compact and convex values;*

$(\mathcal{F}_2)$ *there exists a function $\gamma \in C(\mathcal{K}, \mathbb{R})$, and a nondecreasing, sub-homogeneous function $\Delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that (i.e., $\Delta(\beta y) \leq \beta\Delta(y)$ $\forall$ $\beta \geq 1$ and $y \in \mathbb{R}^+$):*

$$\|\mathcal{G}(\tau, y)\|_{\mathcal{T}} := \sup\{|q| : q \in \mathcal{G}(\tau, y, z, w)\} \leq \gamma(\tau)\Delta(\|y\| + \|z\| + \|w\|)$$

for each $(\tau, y, z, w) \in \mathcal{K} \times \mathbb{R}^3$;

$(\mathcal{F}_3)$ *there exists a constant $\mathcal{E} > 0$ such that:*

$$\frac{\mathcal{E}}{\|\gamma\|\psi\Delta(\mathcal{E})} > 1,$$

Then, there is at least one solution for the BVP (2) and (3) on $\mathcal{K}$.

Proof. Define an operator $\Delta_{\mathcal{G}} : C(\mathcal{K}, \mathbb{R}) \to \mathcal{T}(C(\mathcal{K}, \mathbb{R}))$ by $\Delta_{\mathcal{G}}(y) = \{u \in C(\mathcal{K}, \mathbb{R}) : u(\tau) = \mathcal{S}(y)(\tau)\}$ where:

$$\mathcal{S}(y)(\tau) = \left\{ \; \mathcal{I}^{\varsigma}\phi(\sigma)(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi(\sigma)d\sigma\Big] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi(\sigma)(\tau), \phi \in \mathcal{W}_{\mathcal{G},y}. \right.$$

We will prove that $\Delta_{\mathcal{G}}$ follows the theorem's assumptions (see 8. Theorem 8.5 [22]). The proof requires multiple measures. First, we demonstrate that for every $C(\mathcal{K}, \mathbb{R})$, $\Delta_{\mathcal{G}}$ is convex. This phase is evident as $\mathcal{W}_{\mathcal{G},y}$ is convex, so we skip the proof. Next, we show that $\Delta_{\mathcal{G}}$ maps in $C(\mathcal{K}, \mathbb{R})$ bounded sets to bound sets. Let $\mathcal{B}_{\epsilon} = \{y \in C(\mathcal{K}, \mathbb{R}) : \|y\| \leq \epsilon\}$ be a bounded ball in $C(\mathcal{K}, \mathbb{R})$ for a positive number ϵ. Then, for each $u \in \Delta_{\mathcal{G}}(y)$, $y \in \mathcal{B}_{\epsilon}$, there exists $\phi \in \mathcal{W}_{\mathcal{G},y}$ such that:

$$\mathcal{S}(y)(\tau) = \mathcal{I}^{\varsigma}\phi(\sigma)(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi(\sigma)d\sigma\Big] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi(\sigma)(\tau), \phi \in \mathcal{W}_{\mathcal{G},y}.$$

Then, we have for $\tau \in \mathcal{K}$:

$$\begin{aligned} |u(\tau)| &\leq \frac{1}{\Gamma(\varsigma+1)}\left[(\tilde{\kappa}_2\varsigma T^{\varsigma-1} + T^{\varsigma}) + \tilde{\kappa}_1\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1}\right)\right]\|\gamma\|\Delta(\|y\|_{\mathcal{Y}}) \\ &\leq \psi_1\|\gamma\|\Delta(\|y\|_{\mathcal{Y}}), \end{aligned}$$

which yields on the norm for $\tau \in \mathcal{K}$,

$$\|u\| \leq \psi_1\|\gamma\|\Delta(\|y\|_{\mathcal{Y}}) \leq \psi_1\|\gamma\|\Delta(\epsilon).$$

Similarly, we have:

$$\begin{aligned} \|^{C}\mathcal{D}^{\varrho}u(\tau)\| &\leq \left[\frac{T^{\varsigma-\varrho}}{\Gamma(\varsigma-\varrho+1)} + \frac{\varphi_1}{\Gamma(\varsigma+1)}\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1}\right) + \frac{\varphi_2 T^{\varsigma-1}}{\Gamma(\varsigma)}\right]\|\gamma\|\Delta(\|y\|_{\mathcal{Y}}) \\ &\leq \psi_2\|\gamma\|\Delta(\|y\|_{\mathcal{Y}}), \end{aligned}$$

Likewise, we have:

$$\begin{aligned}\|{}^{C}\mathcal{D}^{\varrho+1}u(\tau)\| &\leq \|\lambda\|\left[\frac{T^{\varsigma-\varrho-1}}{\Gamma(\varsigma-\varrho)}+\frac{\delta_1}{\Gamma(\varsigma+1)}\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)}+\frac{T^{\varsigma+1}}{\varsigma+1}\right)+\frac{\delta_2 T^{\varsigma-1}}{\Gamma(\varsigma)}\right]\|\gamma\|\Delta(\|y\|_{\mathcal{Y}})\\ &\leq \psi_3\|\gamma\|\Delta(\|y\|_{\mathcal{Y}}).\end{aligned}$$

As $u\in\Delta_{\mathcal{G}}(y)$, $y\in\mathcal{B}_\epsilon$ is arbitrary, so we have:

$$\begin{aligned}\|\Delta_{\mathcal{G}}(y)\|_{Y} &= \|\Delta_{\mathcal{G}}(y)\|+\|{}^{C}\mathcal{D}^{\varrho}\Delta_{\mathcal{G}}(y)\|+\|{}^{C}\mathcal{D}^{\varrho+1}\Delta_{\mathcal{G}}(y)\|\\ &\leq \|\gamma\|\Delta(\epsilon)(\psi).\end{aligned}$$

Next, we demonstrate that $\Delta_{\mathcal{G}}$ maps bounded into equicontinuous sets of $C(\mathcal{K},\mathbb{R})$. Let $\tau_1,\tau_2\in\mathcal{K}$ with $\tau_2>\tau_1$ and $y\in\mathcal{B}_\epsilon$. For each $u\in\Delta_{\mathcal{G}}(y)$, we obtain:

$$\begin{aligned}|u(\tau_2)-u(\tau_1)| \leq &\Bigg\{\left[\frac{|\tau_2^{\varsigma}-\tau_1^{\varsigma}|+2(\tau_2-\tau_1)^{\varsigma}}{\Gamma(\varsigma+1)}\right]+\left|\left(\frac{(\tau_2^2-\tau_1^2)\nu_2-\nu_1(\tau_2^3-\tau_1^3)}{\vartheta}\right)\frac{T^{\varsigma-1}}{\Gamma(\varsigma)}\right|\\ &+\Bigg|\frac{(\tau_2^2-\tau_1^2)3T^2-2T(\tau_2^3-\tau_1^3)}{\vartheta\Gamma(\varsigma+1)}\\ &\times\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)}+\frac{T^{\varsigma+1}}{\varsigma+1}\right)\Bigg|\Bigg\}\|\gamma\|\Delta(\epsilon).\end{aligned}$$

Similarly, we can obtain:

$$\begin{aligned}|{}^{C}\mathcal{D}^{\varrho}u(\tau_2)-{}^{C}\mathcal{D}^{\varrho}u(\tau_1)| \leq &\Bigg\{\left[\frac{|\tau_2^{\varsigma-\varrho}-\tau_1^{\varsigma-\varrho}|+2(\tau_2-\tau_1)^{\varsigma-\varrho}}{\Gamma(\varsigma-\varrho+1)}\right]\\ &+\left|\left(\frac{2\nu_2(\tau_2^{2-\varrho}-\tau_1^{2-\varrho})}{\vartheta\Gamma(3-\varrho)}-\frac{6\nu_1(\tau_2^{3-\varrho}-\tau_1^{3-\varrho})}{\vartheta\Gamma(4-\varrho)}\right)\frac{T^{\varsigma-1}}{\Gamma(\varsigma)}\right|\\ &+\Bigg|\frac{6T^2(\tau_2^{2-\varrho}-\tau_1^{2-\varrho})}{\vartheta\Gamma(3-\varrho)}-\frac{12T(\tau_2^{3-\varrho}-\tau_1^{3-\varrho})}{\vartheta\Gamma(4-\varrho)}\\ &\times\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}\Gamma(\varsigma+1)}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)}+\frac{T^{\varsigma+1}}{\Gamma(\varsigma+2)}\right)\Bigg|\Bigg\}\|\gamma\|\Delta(\epsilon).\end{aligned}$$

Likewise, we obtain:

$$\begin{aligned}|{}^{C}\mathcal{D}^{\varrho+1}u(\tau_2)-{}^{C}\mathcal{D}^{\varrho+1}u(\tau_1)| \leq &\Bigg\{\left[\frac{|\tau_2^{\varsigma-\varrho-1}-\tau_1^{\varsigma-\varrho-1}|+2(\tau_2-\tau_1)^{\varsigma-\varrho-1}}{\Gamma(\varsigma-\varrho)}\right]\\ &+\left|\left(\frac{2\nu_2(\tau_2^{1-\varrho}-\tau_1^{1-\varrho})}{\vartheta\Gamma(2-\varrho)}-\frac{6\nu_1(\tau_2^{2-\varrho}-\tau_1^{2-\varrho})}{\vartheta\Gamma(3-\varrho)}\right)\frac{T^{\varsigma-1}}{\Gamma(\varsigma)}\right|\\ &+\Bigg|\frac{6T^2(\tau_2^{1-\varrho}-\tau_1^{1-\varrho})}{\vartheta\Gamma(2-\varrho)}-\frac{12T(\tau_2^{2-\varrho}-\tau_1^{2-\varrho})}{\vartheta\Gamma(3-\varrho)}\\ &\times\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}\Gamma(\varsigma+1)}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)}+\frac{T^{\varsigma+1}}{\Gamma(\varsigma+2)}\right)\Bigg|\Bigg\}\|\gamma\|\Delta(\epsilon).\end{aligned}$$

Clearly, the RHS of the above-mentioned inequalities tends to be zero as $\tau_2-\tau_1\to 0$. As $\Delta_{\mathcal{G}}$ fulfills the premises, the lemma (see Lemma 1.2 [21]) follows that $\Delta_{\mathcal{G}}: C(\mathcal{K},\mathbb{R})\to\mathcal{T}(C(\mathcal{K},\mathbb{R}))$ is completely continuous. We demonstrate that the $\Delta_{\mathcal{G}}$ is USC at the end. It is enough to prove that $\Delta_{\mathcal{G}}$ has a closed graph in the lemma (see Proposition 1.2 [32]). Let $y_n \to y_*$, $u_n \in \Delta_{\mathcal{G}}(y_n)$, and $u_n \to u_*$. Then, we have to prove that $u_* \in \Delta_{\mathcal{G}}(y_*)$; there exists $\phi_n \in \mathcal{W}_{\mathcal{G},y_n}$ such that for each $\tau \in \mathcal{K}$,

$$u_n(\tau) = \mathcal{I}^{\varsigma}\phi_n(\sigma)(\tau) + \kappa_1(\tau)\left[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi_n(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi_n(\sigma)d\sigma\right] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi_n(\sigma)(\tau).$$

Therefore, it is enough to prove that $\phi_* \in \mathcal{W}_{\mathcal{G},y_*}$ exists so that for each $\tau \in \mathcal{K}$,

$$u_*(\tau) = \mathcal{I}^{\varsigma}\phi_*(\sigma)(\tau) + \kappa_1(\tau)\left[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi_*(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi_*(\sigma)d\sigma\right] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi_*(\sigma)(\tau).$$

Consider the linear operator $\Psi : \mathcal{L}^1(\mathcal{K},\mathbb{R}) \to C(\mathcal{K},\mathbb{R})$ provided by:

$$\phi \mapsto \Psi(\phi])(\tau) = \mathcal{I}^{\varsigma}\phi(\sigma)(\tau) + \kappa_1(\tau)\left[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi(\sigma)d\sigma\right] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi(\sigma)(\tau).$$

Observe that:

$$\begin{aligned}\|u_n(\tau) - u_n(\tau)\| = \|&\mathcal{I}^{\varsigma}(\phi_n - \phi_*)(\sigma)(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}(\phi_n - \phi_*)(\sigma)(\zeta) \\ &- \int_0^T \mathcal{I}^{\varsigma}(\phi_n - \phi_*)(\sigma)d\sigma\Big] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}(\phi_n - \phi_*)(\sigma)(\tau)\| \to 0 \text{ as } n \to \infty.\end{aligned}$$

The lemma (see Lemma [34]) follows that $\Psi \circ \mathcal{W}_{\mathcal{G}}$ is a closed graph operator. We also have $u_n(\tau) \in \Psi(\mathcal{W}_{\mathcal{G},y_n})$. Therefore, since $y_n \to y_*$, we have:

$$\begin{aligned}u_*(\tau) = \mathcal{I}^{\varsigma}\phi_*(\sigma)(\tau) &+ \kappa_1(\tau)\left[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi_*(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi_*(\sigma)d\sigma\right] \\ &+ \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi_*(\sigma)(\tau), \text{ for some } \phi_* \in \mathcal{W}_{\mathcal{G},y_*}.\end{aligned}$$

Next, we demonstrate that an open set exists $\mathcal{V} \subseteq C(\mathcal{K},\mathbb{R})$ with $y \notin \Delta_{\mathcal{G}}(y)$ for any $\iota \in (0,T)$ and all $y \in \partial\mathcal{V}$. Let $\iota \in (0,T)$ and $y \in \iota\Delta_{\mathcal{G}}(y)$. Then, there exists $\phi \in \mathcal{L}^1(\mathcal{K},\mathbb{R})$ with $\phi \in \mathcal{W}_{\mathcal{G},y}$ such that for $\tau \in \mathcal{K}$, we can obtain:

$$\|y\|_Y = \|y\| + \|^C\mathcal{D}^{\varrho}y\| + \|^C\mathcal{D}^{\varrho+1}y\| \leq \|\gamma\|\Delta(\|y\|_Y)(\psi).$$

This ensures that $\frac{\|y\|_Y}{\|\gamma\|\Delta(\|y\|_Y)(\psi)} \leq 1$. With regard to (C3), there exists $\mathcal{E}$ such that $\|y\| \neq \mathcal{E}$. Let us set $\mathcal{V} = \{y \in C(\mathcal{K},\mathbb{R}) : \|y\| < \mathcal{E}\}$. Remember that operator $\Delta_{\mathcal{G}} : \overline{\mathcal{V}} \to \mathcal{T}(C(\mathcal{K},\mathbb{R}))$ is USC and completely continuous. There is no $y \in \partial\mathcal{V}$ of the choice of $\mathcal{V}$ such that $y \in \iota\Delta_{\mathcal{G}}(y)$ for some $\tau \in (0,T)$. Therefore, we deduce from Theorem (see 8.Theorem 8.5 [22]) that $\Delta_{\mathcal{G}}$ has a fixed point $\mathcal{V}$ to the problem (2) and (3). □

Theorem 5. *Assume that:*

($\mathcal{F}_4$) $\mathcal{G} : \mathcal{K} \times \mathbb{R}^3 \to \mathcal{T}_{cp}(\mathbb{R})$ *is such that* $\mathcal{G}(\cdot, y(\tau), {}^C\mathcal{D}^{\varrho}y(\tau), {}^C\mathcal{D}^{\varrho+1}y(\tau)) : \mathcal{K} \to \mathcal{T}_{cp}(\mathbb{R})$ *is measurable for each* $y \in \mathcal{R}$;

($\mathcal{F}_5$) $\mathcal{A}_d(\mathcal{G}(\tau,y,z,w), \mathcal{G}(\tau,\hat{y},\hat{z},\hat{w})) \leq s(\tau)[|y-\hat{y}| + |z-\hat{z}| + |w-\hat{w}|]\ \forall \in \mathcal{K}$ *and* $y,z,w,\hat{y},\hat{z},\hat{w} \in \mathbb{R}$ *with* $s \in C(\mathcal{K},\mathbb{R}^+)$ *and* $d(0,\mathcal{G}(\tau,0,0,0)) \leq s(\tau)\ \forall\ \tau \in \mathcal{K}$. *Then, the problem (2) and (3) has at least one solution for* $\mathcal{K}$ *if:*

$$\|s\|\psi < 1. \tag{21}$$

Proof. Consider operator $\Delta_{\mathcal{G}} : C(\mathcal{K},\mathbb{R}) \to \mathcal{T}(C(\mathcal{K},\mathbb{R}))$ defined in Theorem 4 at the beginning of the proof. Remember that for each $C(\mathcal{K},\mathbb{R})$, set $\mathcal{W}_{\mathcal{G},y}$ is not empty by Hypothesis ($\mathcal{F}_4$), so $\mathcal{G}$ has a measurable range (see Theorem III.6 [35]). Now, we prove that the operator $\Delta_{\mathcal{G}}$ fulfills the lemma's assumptions (see Lemma [36]). To show that $\Delta_{\mathcal{G}}(y) \in \mathcal{T}_{cl}(C(\mathcal{K},\mathbb{R}))$ for each $y \in C(\mathcal{K},\mathbb{R})$, let $p_{n_{n\geq 0}} \in \Delta_{\mathcal{G}}(y)$ be such that $p_n \to p$ $(n \to \infty)$ in $C(\mathcal{K},\mathbb{R})$. Then, $p \in C(\mathcal{K},\mathbb{R})$, and there exists $\phi_n \in \mathcal{W}_{\mathcal{G},y}$ such that, for each $\tau \in \mathcal{K}$,

$$p_n(\tau) = \mathcal{I}^{\varsigma}\phi_n(\sigma)(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi_n(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi_n(\sigma)d\sigma\Big] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi_n(\sigma)(\tau).$$

Since $\mathcal{G}$ has compact values, we move a subsequence (if required) to get p_n converging to p in $\mathcal{L}^1(\mathcal{K},\mathbb{R})$. Therefore, $\phi \in \mathcal{W}_{\mathcal{G},y}$, and for each $\tau \in \mathcal{K}$, we have:

$$p_n(\tau) \to p(\tau) = \mathcal{I}^{\varsigma}\phi(\sigma)(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi(\sigma)d\sigma\Big] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi(\sigma)(\tau).$$

Thus, $p \in \Delta_{\mathcal{G}}(y)$. Now, we demonstrate that there exists $\varepsilon := \|s\|\psi < 1$ such that $\mathcal{A}_d(\Delta_{\mathcal{G}}(y), \Delta_{\mathcal{G}}(\hat{y})) \leq \varepsilon\|y - \hat{y}\|_{\mathcal{Y}}$ for each $y, \hat{y} \in C(\mathcal{K},\mathbb{R})$. Let $y, \hat{y} \in C(\mathcal{K},\mathbb{R})$ and $u_1 \in \Delta_{\mathcal{G}}(y)$. Then, there exists $\phi_1(\tau) \in \mathcal{G}(\tau, y(\tau), {}^{C}\mathcal{D}^{\varrho}y(\tau), {}^{C}\mathcal{D}^{\varrho+1}y(\tau))$ such that, for each $\tau \in \mathcal{K}$,

$$u_1(\tau) = \mathcal{I}^{\varsigma}\phi_1(\sigma)(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi_1(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi_1(\sigma)d\sigma\Big] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi_1(\sigma)(\tau).$$

By $\mathcal{F}_2$, we have:

$$\mathcal{A}_d(\mathcal{G}(\tau, y, z, w), \mathcal{G}(\tau, \hat{y}, \hat{z}, \hat{w})) \leq s(\tau)[|y - \hat{y}| + |z - \hat{z}| + |w - \hat{w}|].$$

Therefore, there exists $l \in \mathcal{G}(\tau, \hat{y}, \hat{z}, \hat{w})$ such that:

$$|\phi_1(\tau) - l| \leq s(\tau)[|y(\tau) - \hat{y}(\tau)| + |z(\tau) - \hat{z}(\tau)| + |w(\tau) - \hat{w}(\tau)|], \ \ \tau \in \mathcal{K}.$$

Define $\mathcal{V} : \mathcal{K} \to \mathcal{T}(\mathbb{R})$ by:

$$\mathcal{V}(\tau) = \{l \in \mathbb{R} : |\phi_1(\tau) - l| \leq s(\tau)[|y(\tau) - \hat{y}(\tau)| + |z(\tau) - \hat{z}(\tau)| + |w(\tau) - \hat{w}(\tau)|]\}.$$

As the $\mathcal{V}(\tau) \cap \mathcal{G}(\tau, \hat{y}, \hat{z}, \hat{w})$ operator can be measurable (Proposition III.4 [35]), a $\phi_2(\tau)$ function exists, which is a selection measurable for $\mathcal{V}(\tau) \cap \mathcal{G}(\tau, \hat{y}, \hat{z}, \hat{w})$. Therefore, $\phi_2(\tau) \in \mathcal{G}(\tau, \hat{y}, \hat{z}, \hat{w})$, and for each $\tau \in \mathcal{K}$, we have $|\phi_1(\tau) - \phi_2(\tau)| \leq s(\tau)[|y(\tau) - \hat{y}(\tau)| + |z(\tau) - \hat{z}(\tau)| + |w(\tau) - \hat{w}(\tau)|]$. For each $\tau \in \mathcal{K}$, define:

$$u_2(\tau) = \mathcal{I}^{\varsigma}\phi_2(\sigma)(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}\phi_2(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}\phi_2(\sigma)d\sigma\Big] + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}\phi_2(\sigma)(\tau).$$

Thus,

$$\begin{aligned} |u_1(\tau) - u_2(\tau)| &= \mathcal{I}^{\varsigma}|\phi_1 - \phi_2|(\sigma)(\tau) + \kappa_1(\tau)\Big[\xi^{\rho}\mathcal{J}^{\omega}\mathcal{I}^{\varsigma}|\phi_1 - \phi_2|(\sigma)(\zeta) - \int_0^T \mathcal{I}^{\varsigma}|\phi_1 - \phi_2|(\sigma)d\sigma\Big] \\ &\quad + \kappa_2(\tau)\mathcal{I}^{\varsigma-1}|\phi_1 - \phi_2|(\sigma)(\tau) \\ &\leq \frac{\|s\|}{\Gamma(\varsigma+1)}\left[(\tilde{\kappa}_2\varsigma T^{\varsigma-1} + T^{\varsigma}) + \tilde{\kappa}_1\left(\frac{\xi\zeta^{\varsigma+\rho\omega}}{\rho^{\omega}}\frac{\Gamma(\frac{\varsigma}{\rho}+1)}{\Gamma(\frac{\varsigma}{\rho}+\omega+1)} + \frac{T^{\varsigma+1}}{\varsigma+1}\right)\right]\|y - \hat{y}\|_{\mathcal{Y}}, \end{aligned}$$

which yields $\|u_1 - u_2\| \leq \|s\|\psi_1\|y - \hat{y}\|_{\mathcal{Y}}$. Furthermore, we have:

$$\|{}^{C}\mathcal{D}^{\varrho}u_1(\tau) - {}^{C}\mathcal{D}^{\varrho}u_2(\tau)\| \leq \|s\|\psi_2\|y - \hat{y}\|_{\mathcal{Y}}.$$

In a similar manner, we have:

$$\|{}^{C}\mathcal{D}^{\varrho+1}u_1(\tau)-{}^{C}\mathcal{D}^{\varrho+1}u_2(\tau)\|\leq\|s\|\psi_3\|y-\hat{y}\|_{\mathcal{Y}}.$$

As a result, we get $\|u_1-u_2\|\leq\|s\|\psi\|y-\hat{y}\|_{\mathcal{Y}}$. Likewise, swap the positions of y and $\hat{y}$; we can get $\mathcal{A}_d(\Delta_{\mathcal{G}}(y),\Delta_{\mathcal{G}}(\hat{y}))\leq\|s\|\psi\|y-\hat{y}\|_{\mathcal{Y}}$. Since $\Delta_{\mathcal{G}}$ is a contraction by (21), it follows that it has a fixed point y by the lemma (see Lemma [36]), which is a solution of the problem (2) and (3). □

5. Examples

Example 1. *Consider a fractional BVP given by:*

$$ {}^{C}\mathcal{D}^{\frac{94}{25}}y(\tau)=g(\tau,y(\tau),{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau),{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)),\ \ \tau\in[0,1], \tag{22}$$

$$ y(0)=0,\ \ y'(0)=0,\ \ \int_0^1 y(\sigma)d\sigma=\frac{11}{20}\,{}^{\frac{9}{25}}\mathcal{J}^{\frac{23}{50}}y\Big(\frac{31}{50}\Big),\ \ y'(1)=0. \tag{23}$$

Here, $\varsigma=\frac{94}{25}$, $\varrho=\frac{17}{50}$, $\xi=\frac{11}{20}$, $\zeta=\frac{31}{50}$, $T=1$, $\rho=\frac{9}{25}$, $\omega=\frac{23}{50}$. *We have that with the data* $\kappa_1=20.988457646948103$, $\kappa_2=1.5906878444726738$, $\nu_1=0.1992226279453819$, $\nu_2=0.17972085404600993$, $\vartheta=0.2382261757441258$. *Now, we demonstrate the outcomes by selecting different* $g(\tau,y(\tau),{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau),{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau))$ *values.*

- *(i) Consider:*

$$g(\tau,y(\tau),{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau),{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau))=\frac{1}{\sqrt{\tau^2+64}}\left(\frac{|y(\tau)|}{|y(\tau)|+1}+cos^2({}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau))+sin({}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau))\right)$$

Clearly, $g(\tau,y(\tau),{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau),{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau))=\frac{3}{\sqrt{\tau^2+64}}=\lambda(\tau)$. *Therefore, the assumption of Theorem 1 holds. Hence, in Theorem 1, at least one solution has been found for the problem (22)–(23) on* $[0,1]$.

- *(ii) To prove that Theorem 2 is valid, nonlinear function* g *is taken from the form:* $g(\tau,y(\tau),{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau),{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau))=\frac{1}{7(3+\tau^2)}\left(y(\tau)+\frac{|{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau)|}{|{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau)|+1}+cos({}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau))\right)$. *With the data given, we get* $\kappa_1=20.988457646948103$, $\kappa_2=1.5906878444726738$, $\nu_1=0.1992226279453819$, $\nu_2=0.17972085404600993$, $\vartheta=0.2382261757441258$, $\widehat{\psi}_1=0.6727808331171499$, $\widehat{\psi}_2=0.9525720580682466$, $\widehat{\psi}_3=2.054165331280194$, *and* $\widehat{\psi}=\max\{\widehat{\psi}_1,\widehat{\psi}_2,\widehat{\psi}_3\}=2.054165331280194$. *By the resulting inequalities* $|g(\tau,y(\tau),{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau),{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau))-g(\tau,z(\tau),{}^{C}\mathcal{D}^{\frac{17}{50}}z(\tau),{}^{C}\mathcal{D}^{\frac{67}{50}}z(\tau))|=\frac{1}{7(3+\tau^2)}\Big(|y-z|+|{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau)-{}^{C}\mathcal{D}^{\frac{17}{50}}z(\tau)|+|{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)-{}^{C}\mathcal{D}^{\frac{67}{50}}z(\tau)|\Big)\leq\frac{1}{21}\|y-z\|$, *we have* $\mathcal{P}=\frac{1}{21}$. *Therefore, the hypothesis of Theorem 2 is fulfilled. Consequently, Theorem 2's assumption applies, and the problem (22)–(23) has at least one solution on* $[0,1]$.

- *(iii) Let us consider:*

$$g(\tau, y(\tau), {}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau), {}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)) = \frac{1}{4\sqrt{\tau^2+16}}\left(y(\tau) + ({}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau)) + \frac{{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)|}{|{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)|+1} + 1\right).$$

By the resulting inequalities, $|g(\tau, y(\tau), {}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau), {}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)) - g(\tau, z(\tau), {}^{C}\mathcal{D}^{\frac{17}{50}}z(\tau), {}^{C}\mathcal{D}^{\frac{67}{50}}z(\tau))| = \frac{1}{4\sqrt{\tau^2+16}}\Big(|y-z| + |{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau) - {}^{C}\mathcal{D}^{\frac{17}{50}}z(\tau)| + |{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau) - {}^{C}\mathcal{D}^{\frac{67}{50}}z(\tau)|\Big) \leq \frac{1}{16}\|y-z\|$*, we have* $\mathcal{Q} = \frac{1}{16}$*. With the data given, we get* $\kappa_1 = 20.988457646948103$, $\kappa_2 = 1.5906878444726738$, $\nu_1 = 0.1992226279453819$, $\nu_2 = 0.17972085404600993$, $\vartheta = 0.2382261757441258$, $\widehat{\psi}_1 = 0.7322037594741672$, $\widehat{\psi}_2 = 1.0485700387520187$, $\widehat{\psi}_3 = 2.3824784252186952$*, and* $\widehat{\psi} = \max\{\widehat{\psi}_1, \widehat{\psi}_2, \widehat{\psi}_3\} = 2.3824784252186952$*. Therefore, the hypothesis of Theorem 3 is fulfilled. Consequently, Theorem 3's assumption applies, and the problem (22)–(23) has a unique solution on* $[0,1]$.

Example 2. *Consider the following problem of inclusions:*

$${}^{C}\mathcal{D}^{\frac{94}{25}}y(\tau) \in \mathcal{G}(\tau, y(\tau), {}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau), {}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)), \quad \tau \in [0,1], \tag{24}$$

$$y(0) = 0, \quad y'(0) = 0, \quad \int_0^1 y(\sigma)d\sigma = \frac{11}{20}\,\frac{9}{25}\mathcal{J}^{\frac{23}{50}}\,y\left(\frac{31}{50}\right), \quad y'(1) = 0. \tag{25}$$

- *(i) To show the illustration of Theorem 4, we take* $\mathcal{G}$ *under consideration.*

$$\begin{aligned}\mathcal{G}(\tau, y(\tau), {}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau), {}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)) = \frac{1}{3\sqrt{625+\tau^2}}\Bigg(&y(\tau) + sin({}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau)) \\ &+ \Big(\frac{|{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)|}{|{}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)|+1}\Big) + 1\Bigg).\end{aligned} \tag{26}$$

Apparently, $\gamma = \frac{1}{75}$, $\Delta(\|y\|_{\mathcal{Y}}) = 1 + \|y\|_{\mathcal{Y}}$*, and state* $(\mathcal{F}_3)$ *with* $\mathcal{E} > \mathcal{E}_1$ *is satisfied. Therefore, all criteria of Theorem 4 have been fulfilled, and at least one solution exists to the problem (24)–(25) with* $\mathcal{G}$ *given in (26) on* $[0,1]$.

- *(ii) Let us choose* $\mathcal{G}$ *for the example of Theorem 5.*

$$\mathcal{G}(\tau, y(\tau)) = \left[0, \frac{1}{8(4+\tau^2)}\left(y(\tau) + \frac{|{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau)|}{|{}^{C}\mathcal{D}^{\frac{17}{50}}y(\tau)|+1} + cos({}^{C}\mathcal{D}^{\frac{67}{50}}y(\tau)) + \frac{1}{\tau+1}\right)\right]. \tag{27}$$

It is clear that:

$$\mathcal{A}_d(\mathcal{G}(\tau, y), \mathcal{G}(\tau, \hat{y})) \leq \frac{1}{8(\tau^2+4)}\|y - \hat{y}\|_{\mathcal{Y}}.$$

Allowing $s(\tau) = \frac{1}{8(\tau^2+4)}$, *we can check easily that* $d(0, \mathcal{G}(\tau, 0)) \leq s(\tau)$ *holds* $\forall\ \tau \in [0,1]$ *and that* $\|s\|\psi < 1$. *Since we are satisfied with the assumptions of Theorem 5, we conclude that the problem (24)–(25) with* $\mathcal{G}$ *as indicated by (27) has at least one solution for* $[0,1]$.

6. Discussion

We discussed the solutions of the existence and uniqueness for FDEs and inclusions supplemented by GRLFI boundary conditions. We used fixed point theorems for single-valued and multi-valued maps to evaluate the desired results. When we fixed the parameters involved in the problem (ρ, ω, ξ) (1)–(3), our results corresponded to certain specific problems. Suppose that taking $\rho = 1$ in the results provided, we are given the problems (1) with the form:

$$y(0) = y^{'}(0) = 0, \quad \int_0^T y(\sigma)d\sigma = \xi\ \mathcal{J}^{\omega} y(\zeta), \quad y^{'}(T) = 0, \tag{28}$$

while the results are:

$$y(0) = y^{'}(0) = 0, \quad \int_0^T y(\sigma)d\sigma = 0, \quad y^{'}(T) = 0, \tag{29}$$

followed by $\xi = 0$. When $\rho = \omega = 1$, we can obtain:

$$y(0) = y^{'}(0) = 0, \quad \int_0^T y(\sigma)d\sigma = \xi \int_0^{\zeta} y(\sigma)d\sigma, \quad y^{'}(T) = 0. \tag{30}$$

Concerning the problem (1) with (28) instead of (3), we obtained the operator $\widehat{\Upsilon} : \mathcal{Y} \to \mathcal{Y}$ defined by:

$$\begin{aligned}(\widehat{\Upsilon} y)(\tau) = \kappa_1(\tau)\Big[&\xi \mathcal{I}^{\varsigma+\omega} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho} y(\sigma), {}^C\mathcal{D}^{\varrho+1} y(\sigma))(\zeta) \\ &- \int_0^T \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho} y(\sigma), {}^C\mathcal{D}^{\varrho+1} y(\sigma))d\sigma\Big] \\ &+ \kappa_2(\tau)\mathcal{I}^{\varsigma-1} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho} y(\sigma), {}^C\mathcal{D}^{\varrho+1} y(\sigma))(\tau) \\ &+ \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho} y(\sigma), {}^C\mathcal{D}^{\varrho+1} y(\sigma))(\tau),\end{aligned}$$

where:

$$\begin{aligned}\kappa_1(\tau) &= \frac{\tau^2 3T^2 - 2T\tau^3}{\vartheta}, \quad \kappa_2(\tau) = \frac{\tau^2\nu_2 - \nu_1\tau^3}{\vartheta}, \quad \vartheta = 3T^2\nu_1 - 2T\nu_2, \\ \nu_1 &= \frac{T^3}{3} - \frac{2\xi\zeta^{\omega+2}}{\Gamma(3+\omega)}, \quad \nu_2 = \frac{T^4}{4} - \frac{6\xi\zeta^{\omega+3}}{\Gamma(4+\omega)}.\end{aligned}$$

Similarly, the problem (1) related to operator $\overline{\Upsilon} : \mathcal{Y} \to \mathcal{Y}$ with conditions (29) rather than (3) is:

$$\begin{aligned}(\widehat{\Upsilon} y)(\tau) = \kappa_2(\tau)&\mathcal{I}^{\varsigma-1} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho} y(\sigma), {}^C\mathcal{D}^{\varrho+1} y(\sigma))(\tau) \\ &- \kappa_1(\tau)\Big[\int_0^T \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho} y(\sigma), {}^C\mathcal{D}^{\varrho+1} y(\sigma))d\sigma\Big] \\ &+ \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^C\mathcal{D}^{\varrho} y(\sigma), {}^C\mathcal{D}^{\varrho+1} y(\sigma))(\tau),\end{aligned}$$

where:

$$\kappa_1(\tau) = \frac{\tau^2 3T^2 - 2T\tau^3}{\vartheta}, \quad \kappa_2(\tau) = \frac{\tau^2\nu_2 - \nu_1\tau^3}{\vartheta}, \quad \vartheta = 3T^2\nu_1 - 2T\nu_2, \quad \nu_1 = \frac{T^3}{3} \quad \nu_2 = \frac{T^4}{4}.$$

Likewise, the problem (1) related to operator $\tilde{\Upsilon} : \mathcal{Y} \to \mathcal{Y}$ with conditions (30) rather than (3) is:

$$(\tilde{\Upsilon}y)(\tau) = \kappa_1(\tau)\Big[\xi \int_0^{\zeta} \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^{C}\mathcal{D}^{\varrho} y(\sigma), {}^{C}\mathcal{D}^{\varrho+1} y(\sigma))(\tau) d\sigma$$
$$- \int_0^{T} \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^{C}\mathcal{D}^{\varrho} y(\sigma), {}^{C}\mathcal{D}^{\varrho+1} y(\sigma)) d\sigma\Big]$$
$$+ \kappa_2(\tau)\mathcal{I}^{\varsigma-1} g(\sigma, y(\sigma), {}^{C}\mathcal{D}^{\varrho} y(\sigma), {}^{C}\mathcal{D}^{\varrho+1} y(\sigma))(\tau)$$
$$+ \mathcal{I}^{\varsigma} g(\sigma, y(\sigma), {}^{C}\mathcal{D}^{\varrho} y(\sigma), {}^{C}\mathcal{D}^{\varrho+1} y(\sigma))(\tau),$$

where:

$$\kappa_1(\tau) = \frac{\tau^2 3T^2 - 2T\tau^3}{\vartheta}, \ \kappa_2(\tau) = \frac{\tau^2 \nu_2 - \nu_1 \tau^3}{\vartheta}, \ \vartheta = 3T^2\nu_1 - 2T\nu_2,$$
$$\nu_1 = \frac{T^3 - \xi\zeta^3}{3}, \ \nu_2 = \frac{T^4 - \xi\zeta^4}{4}.$$

The existence and uniqueness of solutions for the new problems can be defined by the $\widehat{\Upsilon}$, $\overline{\Upsilon}$, and $\tilde{\Upsilon}$ operators similar to those obtained for (1)–(3).

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