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Abundant periodic wave solutions for fifth-order Sawada-Kotera equations

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ABSTRACT

In this manuscript, two nonlinear fifth-order partial differential equations, namely, the bidirectional and 2D-Sawada-Kotera equations are analytically treated using an extended form of homoclinic process. In the presence of a bilinear form, novel periodic waves with different categories including periodic soliton, solitary and kinky solitary wave solutions are constructed. In the meantime, The diverse features and mechanical qualities of these acquired solutions are elucidated by 3D figures and some contour plots.

1. Introduction

The attainment of analytical solutions for different models described by NLPDEs has a major part in various fields of applied physics. The quest for these solutions has now become a blistering subject in Neoteric nonlinear science disciplines. Over the last years, many effective methods availing from the headway of symbolic computation have been presented to solve these models like the *exp*-function method, adjusted simple technique, Hirota's bilinear, Wronskian, homogeneous balance, (G'/G)-expansion, F-expansion, Bäcklund transform, inverse scattering transform, and homoclinic test techniques and lots more [1–25].

In [26], Dai et al. investigated an extension of the homoclinic test algorithm for establishing solitary solutions of nonlinear systems with a higher dimension. While in [27–29], abundant solutions including double soliton, periodic kink-wave, and breather type soliton solutions are presented.

The Sawada-Kotera (SK) model is a remarkable unidirectional NLPDE that attracted considerable attention and is usually utilized to the gravitational force, conformal field theory, preserved current of Liouville sample. Furthermore, It has been studied comprehensively by many numbers of researchers. For instance, its multi-soliton solutions, traveling wave solutions, Darboux and Bäcklund transformations have been discussed in [30–33].

In many physical circumstances, It is desirable to have a model that permits us to describe waves that spread in opposite orientations. Here, we consider the bidirectional Sawada-Kotera (bSK) equation

$$\Xi_{5x} + 45\Xi^2\Xi_x + 15\Xi\Xi_t - 15\Xi_x\Xi_{xx} - 15\Xi\Xi_{3x} - 5\Xi_{xxt} - 5\int \Xi_{tt}dx + 15\,\Xi_x \int \Xi_t dx = 0. \tag{1}$$

This equation has been proposed in [34] via the Lax pair of the SK equation. Yunling and Xianguo [35] constructed the Darboux and Bäcklund transformations of the bSK equation to formulate rational and soliton solutions.

Next, we consider the general form of the integrable 2D-SK scheme [36]

$$\Xi_{5x} + 5\Xi^2\Xi_x + 5\Xi\Xi_y + 5\Xi\Xi_{3x} + 5\Xi_x\Xi_{2x} + 5\Xi_{xy} - \Xi_t - 5\int \Xi_{2y}dx + 5\,\Xi_x \int \Xi_y dx = 0, \tag{2}$$

relating to the totally integrable hierarchy of the KdV equations with higher order and has an infinite number of conservation laws. Bell-polynomial, truncated Painlevé expansion, multi-soliton solutions, Darboux transformations, lump solutions, and Lax pairs of this equation are discussed in [37–42]. However, the periodic wave solutions for Eqs. (1) and (2) via an extended form of homoclinic process have not been debated in the previous literature.

This essay aims to find exact periodic wave solutions with different types for the bSK and the 2D-SK equations by applying an extended

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form of homoclinic process and to demonstrate the structures of these wave solutions. The details are given in the upcoming sections.

2. Method description

To depict the fundamental steps of our proposed method, we are considering a general 2D NLPDE

$$G(\Xi, \Xi_t, \Xi_x, \Xi_y, \Xi_{xt}, \Xi_{yt}, \Xi_{xx}, \Xi_{yy}, \Xi_{tt}, \ldots) = 0, \tag{3}$$

where $\Xi = \Xi(x, y, t)$.

The main techniques of the extension form of homoclinic process (EFHP) [42] are:

Phase 1. Assuming

$$\Xi = \Lambda(\Im), \tag{4}$$

where $\Im$ is a novel function in its arguments. From Eq. (4), the Hirota's bilinear shape of Eq. (3) has the type

$$H(D_t, D_x, D_y; \Im \cdot \Im) = 0, \tag{5}$$

where D_t, D_x, and D_y are Hirota's bilinear operators [43].

Phase 2. The solitary solutions of (5) are created when we put $\Im$ in the form

$$\Im(x, y, t) = e^{-\vartheta_1} + \delta_1 \cos(\vartheta_2) + \delta_2 e^{\vartheta_1}, \tag{6}$$

where $\vartheta_j = a_j x + b_j y + c_j t$ and a_j, b_j, c_j, δ_j $(j = 1, 2)$ are free parameters.

Phase 3. Substituting (6) into (5) and setting the constant term and the coefficients of $e^{\pm\vartheta_1}\sin\vartheta_2$, $e^{\pm\vartheta_1}\cos\vartheta_2$ identical zero, we construct different algebraic equations encompassing a_j, b_j, c_j and δ_j $(j = 1, 2)$.

Phase 4. Solving the previous system with the aid of Maple, we reach the required values a_j, b_j, c_j and δ_j $(j = 1, 2)$ and consequently we get a variety of analytical solutions of Eq. (3).

3. Implementations

Herein, we apply the EFHP to Eqs. (1) and (2) and construct their periodic solitary wave solutions.

3.1. Exact solutions of the bSK equation

With a view to construct the periodic solitary wave solutions of the bSK equation, we use the following bilinear transformation

$$\Xi(x, t) = -2(\ln \Im(x, t))_{xx}. \tag{7}$$

By using transformation (7), Eq. (1) is transformed to

$$(D_x^6 - 5D_t^2 - 5D_x^3 D_t)\Im \cdot \Im = 0. \tag{8}$$

Assuming the solution form of (8) as

$$\Im(x, t) = e^{-\vartheta_1} + \delta_1 \cos(\vartheta_2) + \delta_2 e^{\vartheta_1}, \tag{9}$$

where $\vartheta_j = a_j x + b_j t$ $(j = 1, 2)$.

Plugging (9) into (8) and applying phase 3 in the previous section, we reduce to

$$\begin{cases} a_1^6 - a_2^6 - 5a_1^3 b_1 - 5a_2^3 b_2 - 5b_1^2 + 5b_2^2 + 15a_1^2 a_2^4 - 15a_1^4 a_2^2 + 15a_1^2 a_2 b_2 + 15a_1 a_2^2 b_1 = 0, \\ 6a_1^5 a_2 + 6a_1 a_2^5 - 20a_1^3 a_2^3 + 5a_2^3 b_1 - 5a_1^3 b_2 - 10b_1 b_2 + 15a_1 a_2^2 b_2 - 15a_1^2 a_2 b_1 = 0, \\ 64a_1^6 \delta_2 - 80a_1^3 b_1 \delta_2 - 20a_2^3 b_2 \delta_1^2 + 5b_2^2 \delta_1^2 - 20b_1^2 \delta_2 - 16a_2^6 \delta_1^2 = 0. \end{cases}$$

Solving this system with the aid of Maple, we obtain the following results:

Case 1: $a_1 = \frac{[b_1(12\sqrt{5} - 20)]^{1/3}}{4}$, $a_2 = \frac{\sqrt{12}\,[b_1(12\sqrt{5} - 20)]^{1/3}}{8}$, $\delta_2 = \frac{9\delta_1^2(3\sqrt{5} - 7)}{4(3\sqrt{5} - 3)}$, $b_2 = 0$, where b_1 and δ_1 are real numbers.

Using the values in (7) along with (9), we obtain the analytical solution of (1) as

$$\Xi(x, t) = \frac{2\delta_1(a_2^2 - a_1^2)(\delta_2 e^{\vartheta_1} + e^{-\vartheta_1})\cos(\vartheta_2) - 8a_1^2\delta_2 + 2a_2^2\delta_1^2}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2 e^{\vartheta_1}]^2} - \frac{4a_1 a_2 \delta_1(\delta_2 e^{\vartheta_1} - e^{-\vartheta_1})\sin(\vartheta_2)}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2 e^{\vartheta_1}]^2}, \tag{10}$$

where $\vartheta_1 = \frac{[b_1(12\sqrt{5} - 20)]^{1/3}}{4}x + b_1 t$, $\vartheta_2 = \frac{\sqrt{12}\,[b_1(12\sqrt{5} - 20)]^{1/3}}{8}x$ and $\delta_2 = \frac{9\delta_1^2(3\sqrt{5} - 7)}{4(3\sqrt{5} - 3)}$.

When $\delta_2 > 0$, then the exact solution (10) can be reconstructed as

$$\Xi(x, t) = \frac{4\delta_1\sqrt{\delta_2}\,[(a_2^2 - a_1^2)\cos(\vartheta_2)\cosh(\vartheta_1 + \phi) - 2a_1 a_2 \sin(\vartheta_2)\sinh(\vartheta_1 + \phi)]}{[\delta_1\cos(\vartheta_2) + 2\sqrt{\delta_2}\cosh(\vartheta_1 + \phi)]^2} - \frac{8a_1^2\delta_2 - 2a_2^2\delta_1^2}{[\delta_1\cos(\vartheta_2) + 2\sqrt{\delta_2}\cosh(\vartheta_1 + \phi)]^2},$$

where $\phi = \ln\sqrt{\delta_2}$.

When $\delta_2 < 0$, then the exact solution (10) becomes

$$\Xi(x, t) = \frac{4\delta_1\sqrt{-\delta_2}\,[(a_1^2 - a_2^2)\cos(\vartheta_2)\sinh(\vartheta_1 + \phi) + 2a_1 a_2 \sin(\vartheta_2)\cosh(\vartheta_1 + \phi)]}{[\delta_1\cos(\vartheta_2) - 2\sqrt{-\delta_2}\sinh(\vartheta_1 + \phi)]^2} - \frac{8a_1^2\delta_2 - 2a_2^2\delta_1^2}{[\delta_1\cos(\vartheta_2) - 2\sqrt{-\delta_2}\sinh(\vartheta_1 + \phi)]^2},$$

where $\phi = \ln\sqrt{-\delta_2}$.

A periodic solitary solution with period $X = 2\pi = \vartheta_2$ and a solitary wave with $Y = \vartheta_1 + \phi$ are introduced by (10).

Fig. 1(a) illustrates the kinky-periodic solitary solution for specified values in case 1.

Case 2: $a_2 = a_1\sqrt{5 - 2\sqrt{5}}$, $b_1 = \frac{2a_1^3(3\sqrt{5} - 5)}{5}$, $b_2 = \frac{-8a_1^3(9\sqrt{5} - 20)}{(7\sqrt{5} - 15)}$, $\delta_1 = 0$, where a_1 and δ_2 are free real constants.

Substituting the resulting values along with (9) in (7), we gain

$$\Xi(x, t) = \frac{-8a_1^2\delta_2}{(e^{-\vartheta_1} + \delta_2 e^{\vartheta_1})^2}, \quad \text{where } \vartheta_1 = a_1 x + \frac{2a_1^3(3\sqrt{5} - 5)}{5}t. \tag{11}$$

When $\delta_2 > 0$, then the exact solution (11) can be written as

$$\Xi(x, t) = \frac{-8a_1^2\delta_2}{[2\sqrt{\delta_2}\cosh(\vartheta_1 + \phi)]^2}, \quad \text{where } \phi = \ln\sqrt{\delta_2}.$$

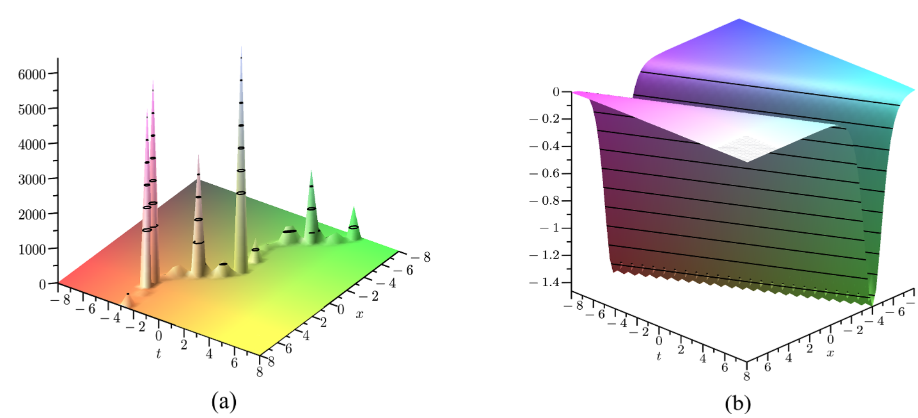

Fig. 1. (a) Solution (10) with $b_1 = 1$, $\delta_1 = 2.38$. (b) Solution (11) with $a_1 = 0.86$, $\delta_2 = 1$.

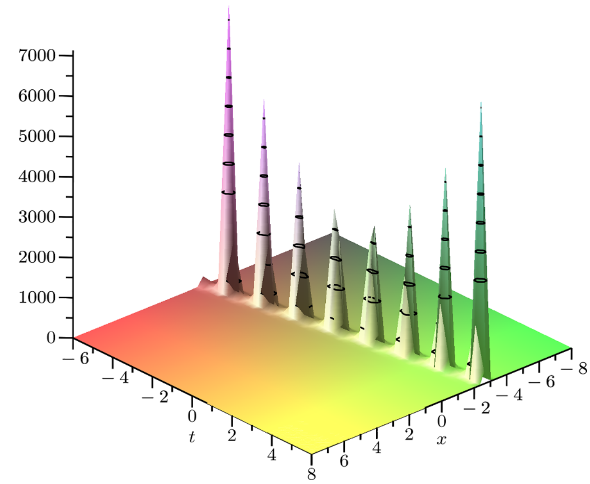

Fig. 2. Solution (12) with $b_1 = 0.25$, $\delta_1 = -5$.

When $\delta_2 < 0$, then the exact solution (11) can be written as

$$\Xi(x, t) = \frac{-8a_1^2\delta_2}{[-2\sqrt{-\delta_2}\sinh(\vartheta_1 + \phi)]^2}, \text{ where } \phi = \ln\sqrt{-\delta_2}.$$

The solution in (11) is a soliton solution. Fig. 1(b) reappears the soliton solution by selecting adequate measurable factors in case 2.

Case 3. $a_1 = \frac{[b_1(12\sqrt{5}+20)]^{1/3}}{2}$, $a_2 = 0$, $b_2 = 0$, $\delta_2 = 0$, where b_1 and δ_1 are free real parameters.

Inserting these values in (7) with the aid of (9), we introduce the solution of (1) as

$$\Xi(x, t) = \frac{-2\delta_1 a_1^2 e^{-\vartheta_1}}{[e^{-\vartheta_1} + \delta_1]^2}, \tag{12}$$

where $\vartheta_1 = \frac{[b_1(12\sqrt{5}+20)]^{1/3}}{2}x + b_1 t$ and $a_1 = \frac{[b_1(12\sqrt{5}+20)]^{1/3}}{2}$.

Fig. 2 depicts the kinky-periodic solitary solution (12) for specified choices in case 3.

3.2. Exact solutions of 2D SK model

To expand our review, we establish the exact solutions of 2D SK model given by Eq. (2) using EFHP. Under the following bilinear transformation

$$\Xi(x, y, t) = 6(\ln\hbar(x, y, t))_{xx}, \tag{13}$$

Eq. (2) is transformed into the bilinear form

$$(D_x^6 - D_t D_x + 5D_x^3 D_y - 5D_y^2)\hbar\cdot\hbar = 0. \tag{14}$$

Now, we assume (14) has the solution form

$$\hbar(x, y, t) = e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2 e^{\vartheta_1}, \tag{15}$$

where $\vartheta_j = a_j x + b_j y + c_j t$. Replacing (15) into (14) and repeating the above steps in the previous section we obtain the following equations:

$$\begin{cases} a_1^4 + a_2^4 - 6a_1^2a_2^2 + (a_1^2 - a_2^2)(b_1^2 - b_2^2 + c_1^2 - c_2^2) + a_1d_1 - a_2d_2 \\ \quad + 4a_1a_2(b_1b_2 + c_1c_2) = 0, \\ 2(a_1^2 - a_2^2)(b_1b_2 + c_1c_2 - 2a_1a_2) - 2a_1a_2(b_1^2 - b_2^2 + c_1^2 - c_2^2) - a_1d_2 - a_2d_1 = 0, \\ 16a_1^4\delta_2 + 4a_1d_1\delta_2 + 4a_2^2\delta_1^2 - a_2d_2\delta_1^2 = 0. \end{cases}$$

Solving this system with the aid of Maple, we have the following results:

Case 1: $b_1 = -a_1^3$, $b_2 = a_2^3$, $c_1 = -9a_1^5$, $c_2 = -9a_2^5$, where a_1, a_2, δ_1 and δ_2 are arbitrary real numbers.

Replacing these values in (13) along with (15), the solution of Eq. (2) can be written as

$$\Xi(x, y, t) = \frac{6\delta_1(a_1^2 - a_2^2)(\delta_2 e^{\vartheta_1} + e^{-\vartheta_1})\cos(\vartheta_2) + 12a_1a_2\delta_1(\delta_2 e^{\vartheta_1} - e^{-\vartheta_1})\sin(\vartheta_2)}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2 e^{\vartheta_1}]^2} + \frac{24a_1^2\delta_2 - 6a_2^2\delta_1^2}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2 e^{\vartheta_1}]^2}, \tag{16}$$

where $\vartheta_1 = a_1x - a_1^3y - 9a_1^5t$ and $\vartheta_2 = a_2x + a_2^3y - 9a_2^5t$.

When $\delta_2 > 0$, then the exact solution (16) can be reformulated to give

$$\Xi(x, y, t) = \frac{12\delta_1\sqrt{\delta_2}[(a_1^2 - a_2^2)\cos(\vartheta_2)\cosh(\vartheta_1 + \phi) + 2a_1a_2\sin(\vartheta_2)\sinh(\vartheta_1 + \phi)]}{[\delta_1\cos(\vartheta_2) + 2\sqrt{\delta_2}\cosh(\vartheta_1 + \phi)]^2} + \frac{24a_1^2\delta_2 - 6a_2^2\delta_1^2}{[\delta_1\cos(\vartheta_2) + 2\sqrt{\delta_2}\cosh(\vartheta_1 + \phi)]^2}, \tag{17}$$

where $\phi = \ln\sqrt{\delta_2}$.

When $\delta_2 < 0$, then the exact solution (16) becomes

$$\Xi(x, y, t) = \frac{12\delta_1\sqrt{-\delta_2}[(a_2^2 - a_1^2)\cos(\vartheta_2)\sinh(\vartheta_1 + \phi) - 2a_1a_2\sin(\vartheta_2)\cosh(\vartheta_1 + \phi)]}{[\delta_1\cos(\vartheta_2) - 2\sqrt{-\delta_2}\sinh(\vartheta_1 + \phi)]^2} + \frac{24a_1^2\delta_2 - 6a_2^2\delta_1^2}{[\delta_1\cos(\vartheta_2) - 2\sqrt{-\delta_2}\sinh(\vartheta_1 + \phi)]^2}, \tag{18}$$

where $\phi = \ln\sqrt{-\delta_2}$. A periodic solitary wave solution with period $2\pi = \vartheta_2 = X$ and is a solitary wave with $Y = \vartheta_1 + \phi$ are represented by (16).

Fig. 3 introduces the periodic solitary solution (16) along with its contour plot.

Case 2: $b_2 = \frac{k}{a_1}$, $c_1 = \frac{1}{a_1}(16a_1^6 + 20a_1^3b_1 - 5b_1^2)$, $c_2 = \frac{k}{a_1^2}(5a_1^3 - 15a_1a_2^2 - 10b_1) + \frac{1}{a_1^2}(6a_1^2a_2^5 - 5a_1^3a_2b_1 - 10a_1^6a_2 - 5a_1a_2^3b_1 - 20a_1^4a_2^3 + 5a_2b_1^2)$, $\delta_1 = 0$ where $k = a_1^3a_2 + a_1a_2^3 + a_2b_1 + \sqrt{6a_1^6a_2^2 + 3a_1^4a_2^4 + 6a_1^3a_2^2b_1 + 3a_1a_2^4b_1 + 3a_1^8 + 3a_1^5b_1}$.

Putting these values in (13) in the presence of (15), we get the solution of Eq. (2) as

$$\Xi(x, y, t) = \frac{24a_1^2\delta_2}{(e^{-\vartheta_1} + \delta_2 e^{\vartheta_1})^2}, \tag{19}$$

where $\vartheta_1 = a_1x + b_1y + \frac{1}{a_1}(16a_1^6 + 20a_1^3b_1 - 5b_1^2)t$.

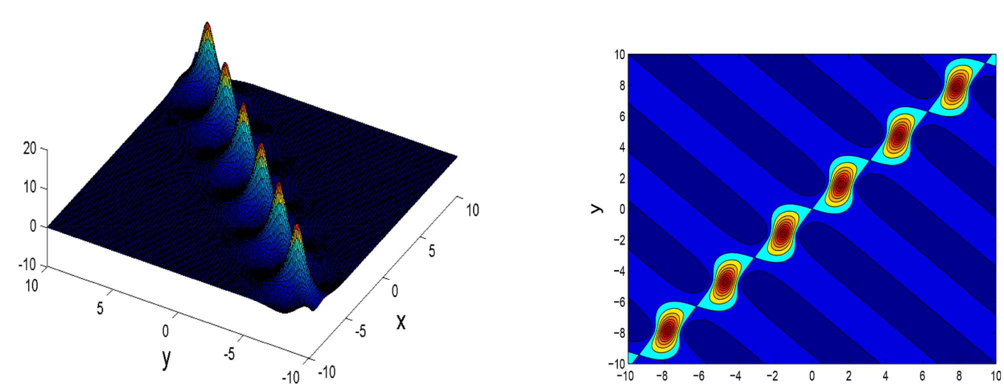

Fig. 3. Solution (16) with $a_1 = -1$, $a_2 = 1$, $b_1 = 0.5$, $\delta_1 = \delta_2 = 1$ and $t = 0$.

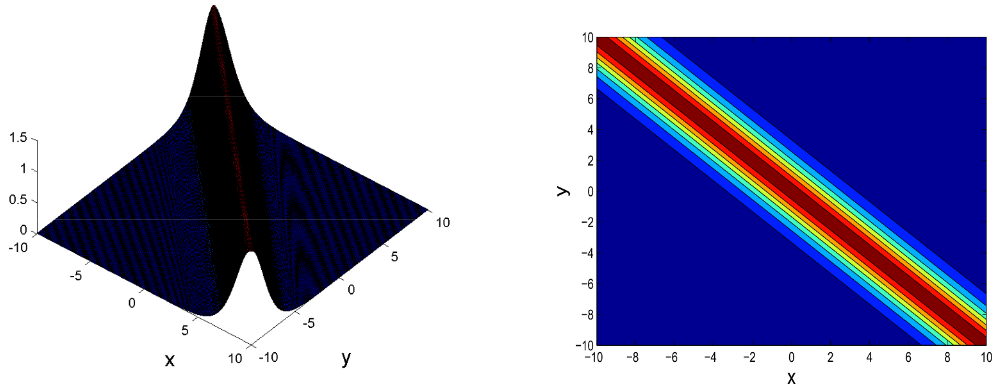

Fig. 4. Solution (19) with $a_1 = 0.5$, $b_1 = 0.5$, $\delta_2 = 1$ and $t = 0$.

When $\delta_2 > 0$, then the exact solution (19) can be written as

$$\Xi(x, y, t) = \frac{24a_1^2\delta_2}{[2\sqrt{\delta_2}\cosh(\vartheta_1 + \phi)]^2}, \quad \text{where } \phi = \ln\sqrt{\delta_2}.$$

When $\delta_2 < 0$, then we obtain

$$\Xi(x, y, t) = \frac{24a_1^2\delta_2}{[-2\sqrt{-\delta_2}\sinh(\vartheta_1 + \phi)]^2}, \quad \text{where } \phi = \ln\sqrt{-\delta_2}.$$

Clearly, Eq. (19) is a soliton solution. Fig. 4 represents this solution along with its contour plot.

Case 3: $b_1 = 3a_1a_2^2 - a_1^3$, $b_2 = a_2^3 - 3a_1^2a_2$, $c_1 = -9a_1(a_1^4 - 10a_1^2a_2^2 + 5a_2^4)$,
$c_2 = -9a_2(5a_1^4 - 10a_1^2a_2^2 + a_2^4)$, where a_1, a_2, δ_1 and δ_2 are arbitrary real numbers.

Put these values in (13) along with (15), we find that

$$\Xi(x, y, t) = \frac{6\delta_1(a_1^2 - a_2^2)(\delta_2 e^{\vartheta_1} + e^{-\vartheta_1})\cos(\vartheta_2) + 12a_1a_2\delta_1(\delta_2 e^{\vartheta_1} - e^{-\vartheta_1})\sin(\vartheta_2)}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2 e^{\vartheta_1}]^2} + \frac{24a_1^2\delta_2 - 6a_2^2\delta_1^2}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2 e^{\vartheta_1}]^2}, \tag{20}$$

where $\vartheta_1 = a_1x + (3a_1a_2^2 - a_1^3)y - 9a_1(a_1^4 - 10a_1^2a_2^2 + 5a_2^4)t$ and
$\vartheta_2 = a_2x + (a_2^3 - 3a_1^2a_2)y - 9a_2(5a_1^4 - 10a_1^2a_2^2 + a_2^4)t$.

When $\delta_2 > 0$ or $\delta_2 < 0$, then the exact solution (20) can be reconstructed as (17) or (18) respectively. Thus, Eq. (20) refers to a periodic solitary solution as shown in Fig. 5.

Case 4: $a_1 = 0$, $b_2 = \frac{3a_2^6\delta_1^2 + 4b_1^2\delta_2 - b_1^2\delta_1^2}{3a_2^3\delta_1^2}$, $c_1 = \frac{-5b_1(9a_2^6\delta_1^2 + 8b_1^2\delta_2 - 2b_1^2\delta_1^2)}{3a_2^4\delta_1^2}$,
$c_2 = \frac{90a_2^6b_1^2\delta_1^4 - 81a_2^{12}\delta_1^4 - 5b_1^4\delta_1^4 + 40b_1^4\delta_1^2\delta_2 - 180a_2^6b_1^2\delta_1^2\delta_2 - 80b_1^4\delta_2^2}{9a_2^7\delta_1^4} = l$, where a_2, b_1, δ_1 and δ_2 are free parameters.

Using these values in (13) along with (15), we get

$$\Xi(x, y, t) = \frac{-6a_2^2\delta_1(\delta_2 e^{\vartheta_1} + e^{-\vartheta_1})\cos(\vartheta_2) - 6a_2^2\delta_1^2}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2 e^{\vartheta_1}]^2}, \tag{21}$$

where $\vartheta_1 = b_1y - \left(\frac{5b_1(9a_2^6\delta_1^2 + 8b_1^2\delta_2 - 2b_1^2\delta_1^2)}{3a_2^4\delta_1^2}\right)t$ and $\vartheta_2 = a_2x + \left(\frac{3a_2^6\delta_1^2 + 4b_1^2\delta_2 - b_1^2\delta_1^2}{3a_2^3\delta_1^2}\right)y + lt$.

When $\delta_2 > 0$, then the exact solution (21) becomes

$$\Xi(x, y, t) = \frac{-12a_2^2\delta_1\sqrt{\delta_2}\cos(\vartheta_2)\cosh(\vartheta_1 + \phi) - 2a_2^2\delta_1^2}{[\delta_1\cos(\vartheta_2) + 2\sqrt{\delta_2}\cosh(\vartheta_1 + \phi)]^2},$$

where $\phi = \ln\sqrt{\delta_2}$.

When $\delta_2 < 0$, then

$$\Xi(x, y, t) = \frac{12a_2^2\delta_1\sqrt{-\delta_2}\cos(\vartheta_2)\sinh(\vartheta_1 + \phi) - 2a_2^2\delta_1^2}{[\delta_1\cos(\vartheta_2) - 2\sqrt{-\delta_2}\sinh(\vartheta_1 + \phi)]^2},$$

where $\phi = \ln\sqrt{-\delta_2}$.

Fig. 6 represents the periodic solitary wave solution (21) along with its contour plot.

Case 5: $a_2 = 0$, $b_1 = \frac{4b_2^2\delta_2 - 12a_1^6\delta_2 - b_2^2\delta_1^2}{12a_1^3\delta_2}$, $c_2 = \frac{5b_2(18a_1^6\delta_2 - 4b_2^2\delta_2 + b_2^2\delta_1^2)}{6a_1^4\delta_2}$,
$c_1 = \frac{1440a_1^6b_2^2\delta_2^2 - 1296a_1^{12}\delta_2^2 - 180a_1^6b_2^2\delta_1^2\delta_2 - 80b_2^4\delta_2^2 + 40b_2^4\delta_1^2\delta_2 - 5b_2^4\delta_1^4}{144a_1^7\delta_2^2} = m$,
where a_1, b_2, δ_1 and δ_2 are real parameters.

By using these values in (13) with (15), we reach to

$$\Xi(x, y, t) = \frac{6a_1^2\delta_1(\delta_2 e^{\vartheta_1} + e^{-\vartheta_1})\cos(\vartheta_2) + 24a_1^2\delta_2}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2 e^{\vartheta_1}]^2}, \tag{22}$$

where $\vartheta_1 = a_1x + \left(\frac{4b_2^2\delta_2 - 12a_1^6\delta_2 - b_2^2\delta_1^2}{12a_1^3\delta_2}\right)y + mt$ and $\vartheta_2 = b_2y + \left(\frac{5b_2(18a_1^6\delta_2 - 4b_2^2\delta_2 + b_2^2\delta_1^2)}{6a_1^4\delta_2}\right)t$.

When $\delta_2 > 0$, then the exact solution (22) can be written as

$$\Xi(x, y, t) = \frac{12a_1^2\delta_1\sqrt{\delta_2}\cos(\vartheta_2)\cosh(\vartheta_1 + \phi) + 24a_1^2\delta_2}{[\delta_1\cos(\vartheta_2) + 2\sqrt{\delta_2}\cosh(\vartheta_1 + \phi)]^2},$$

where $\phi = \ln\sqrt{\delta_2}$.

When $\delta_2 < 0$, then the exact solution (22) can be written as

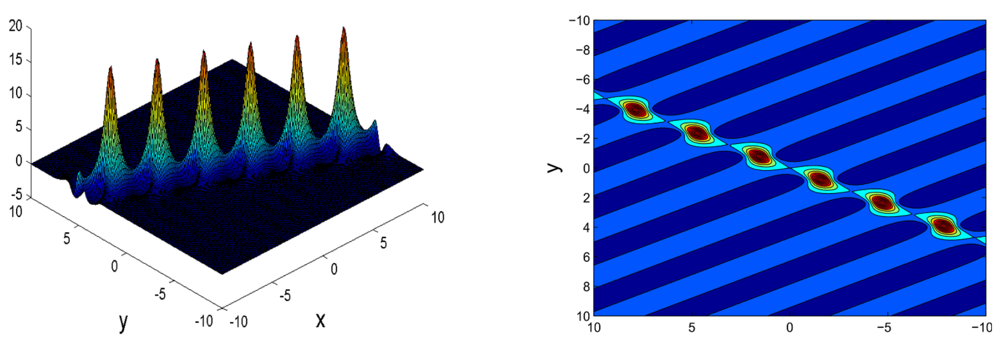

Fig. 5. Solution (20) with $a_1 = -1$, $a_2 = 1$, $b_1 = 0.5$, $\delta_1 = \delta_2 = 1$ and $t = 0$.

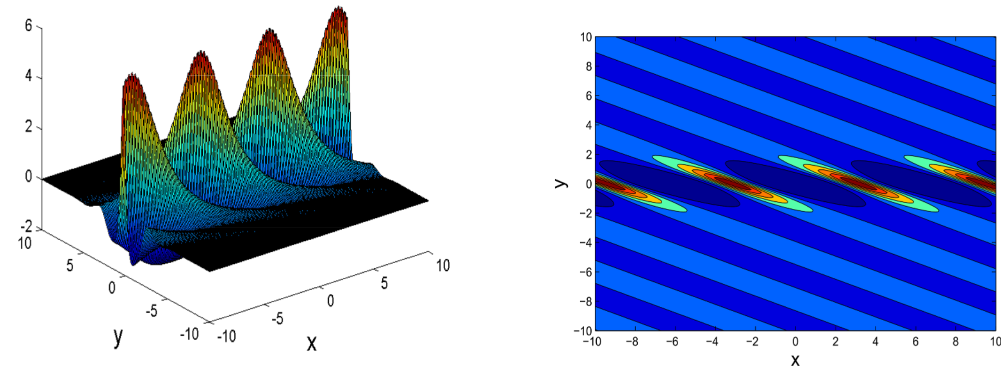

Fig. 6. Solution (21) with $a_2 = 1$, $b_1 = 1$, $\delta_1 = \delta_2 = 1$ and $t = 0$.

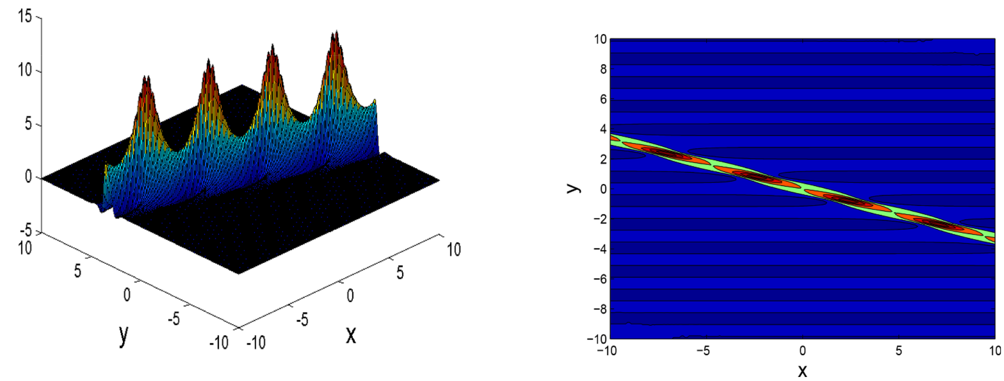

Fig. 7. Solution (22) with $a_1 = 1$, $b_2 = 4$, $\delta_1 = \delta_2 = 1$ and $t = 0$.

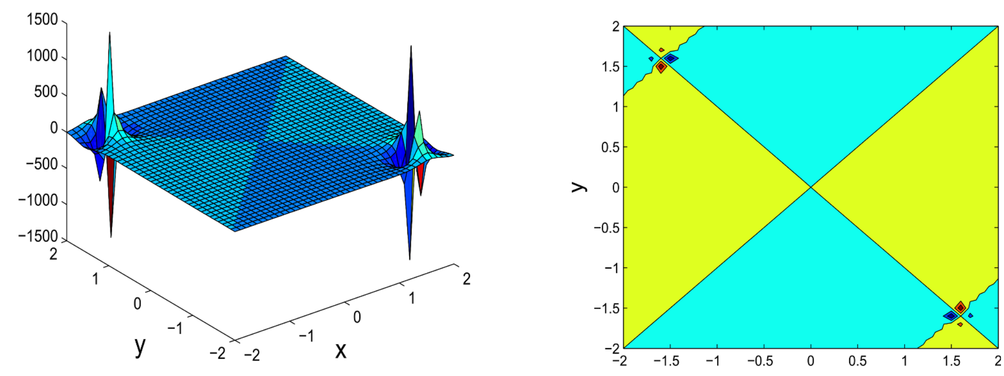

Fig. 8. Solution (22) with $a_1 = a_2 = 1$, $b_1 = 1$, $b_2 = -1$, $\delta_2 = 1$ and $t = 0$.

$$\Xi(x, y, t) = \frac{-12a_1^2\delta_1\sqrt{-\delta_2}\cos(\vartheta_2)\sinh(\vartheta_1 + \phi) + 24a_1^2\delta_2}{[\delta_1\cos(\vartheta_2) - 2\sqrt{-\delta_2}\sinh(\vartheta_1 + \phi)]^2},$$

where $\phi = \ln\sqrt{-\delta_2}$.

Fig. 7 displays the kinky periodic solitary wave solution (22) along with its contour plot.

Case 6: $c_1 = \frac{1}{a_1^2 + a_2^2}[5a_1(b_2^2 - b_1^2) + 5b_1(a_1^4 - a_2^4) - 10a_1a_2b_2(a_1^2 + a_2^2) + 5a_1a_2^4(a_2^2 - a_1^2) + a_1^7 - 9a_1^5a_2^2 - 10a_2b_1b_2] = l$, $c_2 = \frac{1}{a_1^2 + a_2^2}[5a_2(b_1^2 - b_2^2) + 5b_2(a_1^4 - a_2^4) + 10a_1a_2b_1(a_1^2 + a_2^2) + a_2^7 + 5a_1^4a_2(a_1^2 - a_2^2) - 9a_1^2a_2^5 - 10a_1b_1b_2] = m$,
$\delta_2 = \frac{-\delta_1^2}{4k_1}[a_1^4a_2^2(a_1^2 - a_2^2) + 2a_1a_2^2b_1(a_1^2 + a_2^2) + a_2b_2(a_1^4 + 3a_2^4) - a_2^6(5a_1^2 + 3a_2^2) - 2a_1a_2b_1b_2 + 4a_1^2a_2^3b_2 + a_1^2b_2^2 + a_2^2b_1^2]$, where a_1, a_2, b_1, b_2 and δ_1 are arbitrary real constants and

$$k_1 = [a_1^2a_2^4(a_1^2 - a_2^2) + 2a_1^2a_2b_2(a_1^2 + a_2^2) + 2a_1a_2b_1b_2 + a_1b_1(3a_1^4 + a_2^4) + a_1^6(3a_1^2 + 5a_2^2) + 4a_1^3a_2^2b_1 - a_1^2b_2^2 - a_2^2b_1^2].$$

Substituting these values in Eq. (13) with Eq. (15), we attain

$$\Xi(x, y, t) = \frac{6\delta_1(a_1^2 - a_2^2)(\delta_2e^{\vartheta_1} + e^{-\vartheta_1})\cos(\vartheta_2) + 12a_1a_2\delta_1(\delta_2e^{\vartheta_1} - e^{-\vartheta_1})\sin(\vartheta_2)}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2e^{\vartheta_1}]^2} + \frac{24a_1^2\delta_2 - 6a_2^2\delta_1^2}{[e^{-\vartheta_1} + \delta_1\cos(\vartheta_2) + \delta_2e^{\vartheta_1}]^2}, \tag{23}$$

where $\vartheta_1 = a_1x + b_1y + lt$ and $\vartheta_2 = a_2x + b_2y + mt$.

When $\delta_2 > 0$ or $\delta_2 < 0$, then the exact solution (23) can be reformulated as (17) or (18), respectively. We observe that Eq. (23) represents a periodic soliton solution. Fig. 8 represents the periodic soliton solution (23) along with its contour plot.

4. Conclusion

In conclusion, depending on the Hirota's bilinear form, we have generated novel different classes of exact periodic wave solutions for the bSK and the generalization of integrable 2D SK equations by employing the EFHP. Moreover, for the physical understanding, the derived solutions are represented by 3D graphs and by some contour plots. All the solutions gained in this essay are novel and deliver a worthwhile change in the existing literature.

Conflict of interest

The authors have declared no conflict of interest.

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