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A fractional derivative with two singular kernels and application to a heat conduction problem

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Abstract

In this article, we suggest a new notion of fractional derivative involving two singular kernels. Some properties related to this new operator are established and some examples are provided. We also present some applications to fractional differential equations and propose a numerical algorithm based on a Picard iteration for approximating the solutions. Finally, an application to a heat conduction problem is given.

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1 Introduction

In many applications in applied sciences, the use of fractional derivatives with singular kernels allows us to obtain more realistic models than those derived using the standard derivative (see e.g. [2–7, 10, 11, 13, 14]). The literature contains various notions of fractional derivatives with singular kernels. The best known are the Riemann–Liouville fractional derivative and the Caputo fractional derivative (see e.g. [12, 22]). For other definitions, see, for example [1, 8, 15–21] and the references therein.

In [1], Almeida introduced the notion of ψ -Caputo fractional derivative as a generalization of the Caputo derivative. Namely, given $\psi \in C^n([a, b], \mathbb{R})$ with $\psi' > 0$, and $f \in C^n([a, b], \mathbb{R})$, the left-sided fractional derivative order $\alpha \in (n - 1, n)$ of f with respect to ψ is defined by

$$({}^C D_a^{\alpha, \psi} f)(t) = \left(I_a^{n-\alpha, \psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n f \right)(t), \quad a < t \leq b,$$

where

$$(I_a^{\theta, \psi} h)(t) = \frac{1}{\Gamma(\theta)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\theta-1} h(s) ds, \quad \theta > 0.$$

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The right-sided fractional derivative of order α of f with respect to ψ is defined by

$$({}^C D_b^{\alpha, \psi} f)(t) = \left(I_b^{n-\alpha, \psi} \left(\frac{-1}{\psi'(t)} \frac{d}{dt} \right)^n f \right)(t), \quad a \leq t < b,$$

where

$$({}^I_b^{\theta, \psi} h)(t) = \frac{1}{\Gamma(\theta)} \int_t^b \psi'(s) (\psi(s) - \psi(t))^{\theta-1} h(s) ds, \quad \theta > 0.$$

In the particular case $\psi(t) = t$, ${}^C D_a^{\alpha, \psi}$ reduces to the left-sided Caputo fractional derivative, and ${}^C D_b^{\alpha, \psi}$ reduces to the right-sided Caputo fractional derivative. For other examples of ψ , one obtains other known fractional operators, as for example the fractional derivative of Caputo–Hadamard (see [7]) and the fractional derivative of Caputo–Erdélyi–Kober (see [9]). In all the above notions, the fractional derivatives involve only one singular kernel.

In this paper, a new concept of fractional derivative with two singular kernels $k_1(t, s) = \frac{1}{\Gamma(\theta+1)} \varphi'(s) (\varphi(t) - \varphi(s))^\theta$ and $k_2(s, \tau) = \frac{1}{\Gamma(\mu+1)} \psi'(\tau) (\psi(s) - \psi(\tau))^\mu$, where $-1 < \theta, \mu < 0$, is proposed. We establish some properties related to this introduced operator and present some applications to fractional differential equations. Namely, we investigate the existence and uniqueness of solutions of a nonlinear fractional boundary value problem of a higher order, and provide a numerical technique based on a Picard iteration for approximating solutions. An application to a heat conduction problem is also provided.

In Sect. 2, the fractional derivative operator with two singular kernels is introduced and some properties are established. The special case $\varphi = \psi$ is discussed in Sect. 3. In Sect. 4, we study a nonlinear fractional boundary value problem of a higher order. Namely, using Banach fixed point theorem, we establish the existence and uniqueness of solutions, and provide a numerical algorithm based on Picard iterations for approximating the solution. In Sect. 5, an application to a heat conduction problem is given.

2 Fractional derivative with two singular kernels

First, we fix some notations. We denote by \mathbb{N} the set of positive integers. Let $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $a < b$. Let

$$\Phi^{(n)} = \{ \varphi \in C^n([a, b], \mathbb{R}) : \varphi'(t) > 0, a \leq t \leq b \}.$$

For $\varphi \in \Phi^{(n)}$, let

$$L_\varphi^{(n)} = \left(\frac{1}{\varphi'(t)} \frac{d}{dt} \right)^n.$$

Definition 2.1 Let $\alpha, \beta \in (n-1, n)$, $\varphi \in \Phi^{(1)}$, $\psi \in \Phi^{(n)}$ and $f \in C^n([a, b], \mathbb{R})$. The left-sided (φ, ψ) -fractional derivative of f with parameters (α, β) is defined by

$$({}^{D_a^{(\alpha, \beta), (\varphi, \psi)}} f)(t) = I_a^{n-\alpha, \varphi} ({}^C D_a^{\beta, \psi} f)(t), \quad a < t \leq b. \tag{1}$$

The right-sided (φ, ψ) -fractional derivative of f with parameters (α, β) is defined by

$$({}^{D_b^{(\alpha, \beta), (\varphi, \psi)}} f)(t) = I_b^{n-\alpha, \varphi} ({}^C D_b^{\beta, \psi} f)(t), \quad a \leq t < b. \tag{2}$$

Remark 2.1 From (1), for all $a < t \leq b$, one has

$$\begin{aligned} (D_a^{(\alpha,\beta),(\varphi,\psi)}f)(t) &= \frac{1}{\Gamma(n-\alpha)\Gamma(n-\beta)} \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{n-\alpha-1} \\ &\quad \times \left(\int_a^s \psi'(\tau)(\psi(s)-\psi(\tau))^{n-\beta-1} (L_\psi^{(n)}f)(\tau) d\tau \right) ds. \end{aligned}$$

Similarly, from (2), for all $a \leq t < b$, one has

$$\begin{aligned} (D_b^{(\alpha,\beta),(\varphi,\psi)}f)(t) &= \frac{1}{\Gamma(n-\alpha)\Gamma(n-\beta)} \int_t^b \varphi'(s)(\varphi(s)-\varphi(t))^{n-\alpha-1} \\ &\quad \times \left(\int_s^b \psi'(\tau)(\psi(\tau)-\psi(s))^{n-\beta-1} (-1)^n (L_\psi^{(n)}f)(\tau) d\tau \right) ds. \end{aligned}$$

In $C([a, b], \mathbb{R})$ we consider the norm

$$\|f\|_\infty = \max\{|f(t)| : a \leq t \leq b\}, \quad f \in C([a, b], \mathbb{R}).$$

We endow $C^n([a, b], \mathbb{R})$ with the norm

$$\|f\| = \sum_{k=0}^n \|L_\psi^{(k)}f\|_\infty, \quad f \in C^n([a, b], \mathbb{R}),$$

where $\psi \in \Phi^{(n)}$.

Theorem 2.1 Let $\alpha, \beta \in (n-1, n)$, $\varphi \in \Phi^{(1)}$, $\psi \in \Phi^{(n)}$ and $f \in C^n([a, b], \mathbb{R})$. Then

$$|(D_a^{(\alpha,\beta),(\varphi,\psi)}f)(t)| \leq \frac{(\varphi(t)-\varphi(a))^{n-\alpha}}{\Gamma(n-\alpha+1)} \frac{(\psi(t)-\psi(a))^{n-\beta}}{\Gamma(n-\beta+1)} \|f\|, \quad a < t \leq b, \tag{3}$$

and

$$|(D_b^{(\alpha,\beta),(\varphi,\psi)}f)(t)| \leq \frac{(\varphi(b)-\varphi(t))^{n-\alpha}}{\Gamma(n-\alpha+1)} \frac{(\psi(b)-\psi(t))^{n-\beta}}{\Gamma(n-\beta+1)} \|f\|, \quad a \leq t < b. \tag{4}$$

Proof Let $a < t \leq b$. Then

$$\begin{aligned} & |(D_a^{(\alpha,\beta),(\varphi,\psi)}f)(t)| \\ & \leq \frac{\|f\|}{\Gamma(n-\alpha)\Gamma(n-\beta)} \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{n-\alpha-1} \left(\int_a^s \psi'(\tau)(\psi(s)-\psi(\tau))^{n-\beta-1} d\tau \right) ds \\ & \leq \frac{\|f\|}{\Gamma(n-\alpha)\Gamma(n-\beta+1)} \int_a^t \varphi'(s)(\varphi(t)-\varphi(s))^{n-\alpha-1} (\psi(s)-\psi(a))^{n-\beta} ds \\ & \leq \frac{(\psi(t)-\psi(a))^{n-\beta} (\varphi(t)-\varphi(a))^{n-\alpha}}{\Gamma(n-\alpha+1)\Gamma(n-\beta+1)} \|f\|, \end{aligned}$$

which proves (3). Using similar estimates, one obtains (4). □

Corollary 2.1 *Let $\alpha, \beta \in (n - 1, n)$, $\varphi \in \Phi^{(1)}$, $\psi \in \Phi^{(n)}$ and $f \in C^n([a, b], \mathbb{R})$. Then*

$$\lim_{t \rightarrow a^+} (D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) = 0 \tag{5}$$

and

$$\lim_{t \rightarrow b^-} (D_b^{(\alpha, \beta), (\varphi, \psi)} f)(t) = 0. \tag{6}$$

Proof Taking the limit as $t \rightarrow a^+$ in (3), (5) follows. Similarly, taking the limit as $t \rightarrow b^-$ in (4), (6) follows. \square

Taking

$$(D_a^{(\alpha, \beta), (\varphi, \psi)} f)(a) = 0,$$

one deduces from (5) that $D_a^{(\alpha, \beta), (\varphi, \psi)} f \in C([a, b], \mathbb{R})$. Similarly, taking

$$(D_b^{(\alpha, \beta), (\varphi, \psi)} f)(b) = 0,$$

one deduces from (6) that $D_b^{(\alpha, \beta), (\varphi, \psi)} f \in C([a, b], \mathbb{R})$. Therefore, by Theorem 3.1, one obtains the following.

Corollary 2.2 *Let $\alpha, \beta \in (n - 1, n)$, $\varphi \in \Phi^{(1)}$ and $\psi \in \Phi^{(n)}$. Then, for any $g \in C^n([a, b], \mathbb{R})$, we have*

$$\|D_a^{(\alpha, \beta), (\varphi, \psi)} g\|_\infty \leq C \|g\| \quad \text{and} \quad \|D_b^{(\alpha, \beta), (\varphi, \psi)} g\|_\infty \leq C \|g\|,$$

where

$$C = \frac{(\varphi(b) - \varphi(a))^{n-\alpha} (\psi(b) - \psi(a))^{n-\beta}}{\Gamma(n - \alpha + 1) \Gamma(n - \beta + 1)}.$$

Lemma 2.1 *Let $\varphi \in \Phi^{(1)}$ and $f \in C^1([a, b], \mathbb{R})$. Then*

$$\lim_{\theta \rightarrow 0^+} (I_a^{\theta, \varphi} f)(t) = f(t), \quad a < t \leq b, \tag{7}$$

and

$$\lim_{\theta \rightarrow 0^+} (I_b^{\theta, \varphi} f)(t) = f(t), \quad a \leq t < b. \tag{8}$$

Proof Let $\theta > 0$. One has

$$(I_a^{\theta, \varphi} f)(t) = \frac{1}{\Gamma(\theta)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{\theta-1} f(s) ds.$$

Integrating by parts, one obtains

$$(I_a^{\theta, \varphi} f)(t) = \frac{1}{\Gamma(\theta + 1)} \left((\varphi(t) - \varphi(a))^\theta f(a) + \int_a^t (\varphi(t) - \varphi(s))^\theta f'(s) ds \right).$$

Passing to the limit as $\theta \rightarrow 0^+$ in the above equality, (7) follows. Similarly, one has

$$(I_b^{\theta, \varphi} f)(t) = \frac{1}{\Gamma(\theta)} \int_t^b \varphi'(s) (\varphi(s) - \varphi(t))^{\theta-1} f(s) ds.$$

Integrating by parts, one obtains

$$(I_b^{\theta, \varphi} f)(t) = \frac{1}{\Gamma(\theta + 1)} \left((\varphi(b) - \varphi(t))^\theta f(b) - \int_t^b (\varphi(s) - \varphi(t))^\theta f'(s) ds \right).$$

Passing to the limit as $\theta \rightarrow 0^+$ in the above equality, (8) follows. □

Theorem 2.2 *Let $n - 1 < \beta < n$, $\varphi \in \Phi^{(1)}$, $\psi \in \Phi^{(n)}$ and $f \in C^n([a, b], \mathbb{R})$.*

(I) *If ${}^C D_a^{\beta, \psi} f \in C^1([a, b], \mathbb{R})$, then*

$$\lim_{\alpha \rightarrow n^-} (D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) = ({}^C D_a^{\beta, \psi} f)(t), \quad a < t \leq b.$$

(II) *If ${}^C D_b^{\beta, \psi} f \in C^1([a, b], \mathbb{R})$, then*

$$\lim_{\alpha \rightarrow n^-} (D_b^{(\alpha, \beta), (\varphi, \psi)} f)(t) = ({}^C D_b^{\beta, \psi} f)(t), \quad a \leq t < b.$$

Proof Using (1) and (7), (I) follows. Similarly, using (2) and (8), (II) follows. □

Theorem 2.3 *Let $\alpha, \beta \in (n - 1, n)$, $\varphi \in \Phi^{(1)}$, $\psi \in \Phi^{(n+1)}$ and $f \in C^{n+1}([a, b], \mathbb{R})$. For all $a < t \leq b$,*

$$\begin{aligned} & (D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) \\ &= \frac{(L_\psi^{(n)} f)(a)}{\Gamma(n - \alpha)\Gamma(n + 1 - \beta)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{n-\alpha-1} (\psi(s) - \psi(a))^{n-\beta} ds \\ &+ \frac{1}{\Gamma(n - \alpha)\Gamma(n + 1 - \beta)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{n-\alpha-1} \\ &\times \left(\int_a^s (\psi(s) - \psi(\tau))^{n-\beta} \frac{d}{d\tau} (L_\psi^{(n)} f)(\tau) d\tau \right) ds. \end{aligned} \tag{9}$$

For all $a \leq t < b$,

$$\begin{aligned} & (D_b^{(\alpha, \beta), (\varphi, \psi)} f)(t) \\ &= \frac{(-1)^n (L_\psi^{(n)} f)(b)}{\Gamma(n - \alpha)\Gamma(n + 1 - \beta)} \int_t^b \varphi'(s) (\varphi(s) - \varphi(t))^{n-\alpha-1} (\psi(b) - \psi(s))^{n-\beta} ds \\ &- \frac{1}{\Gamma(n - \alpha)\Gamma(n + 1 - \beta)} \int_t^b \varphi'(s) (\varphi(s) - \varphi(t))^{n-\alpha-1} \\ &\times \left(\int_s^b (\psi(\tau) - \psi(s))^{n-\beta} (-1)^n \frac{d}{d\tau} (L_\psi^{(n)} f)(\tau) d\tau \right) ds. \end{aligned} \tag{10}$$

Proof Equation (9) follows from (1) and [1, Theorem 1]. (10) follows from (2) and [1, Theorem 1]. □

Corollary 2.3 *Let $\varphi \in \Phi^{(1)}$, $\psi \in \Phi^{(n+1)}$ and $f \in C^{n+1}([a, b], \mathbb{R})$. Then*

$$\lim_{\alpha \rightarrow n^-} \left(\lim_{\beta \rightarrow n^-} (D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) \right) = (L_\psi^{(n)} f)(t), \quad a < t \leq b, \tag{11}$$

and

$$\lim_{\alpha \rightarrow n^-} \left(\lim_{\beta \rightarrow n^-} (D_b^{(\alpha, \beta), (\varphi, \psi)} f)(t) \right) = (-1)^n (L_\psi^{(n)} f)(t), \quad a \leq t < b. \tag{12}$$

Proof Let $a < t \leq b$. From (9), for $n - 1 < \alpha < n$, one has

$$\begin{aligned} & \lim_{\beta \rightarrow n^-} (D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) \\ &= \frac{1}{\Gamma(n - \alpha)} (L_\psi^{(n)} f)(a) \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{n-\alpha-1} ds \\ & \quad + \frac{1}{\Gamma(n - \alpha)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{n-\alpha-1} \left(\int_a^s \frac{d}{d\tau} (L_\psi^{(n)} f)(\tau) d\tau \right) ds \\ &= \frac{(\varphi(t) - \varphi(a))^{n-\alpha}}{\Gamma(n - \alpha + 1)} (L_\psi^{(n)} f)(a) \\ & \quad + \frac{1}{\Gamma(n - \alpha)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{n-\alpha-1} ((L_\psi^{(n)} f)(s) - (L_\psi^{(n)} f)(a)) ds. \end{aligned}$$

Hence, taking the limit as $\alpha \rightarrow n^-$, and using (7), (11) follows. Similarly, for $a \leq t < b$, using (10) and (8), (12) follows. □

3 The case $\varphi = \psi$

Let $\alpha, \beta \in (n - 1, n)$, $\psi = \varphi \in \Phi^{(n)}$ and $f \in C^n([a, b], \mathbb{R})$. In this case, by (1), for all $a < t \leq b$, one obtains

$$\begin{aligned} (D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) &= (D_a^{(\alpha, \beta), (\varphi, \varphi)} f)(t) = I_a^{n-\alpha, \varphi} ({}^C D_a^{\beta, \varphi} f)(t) \\ &= I_a^{n-\alpha, \varphi} (I_a^{n-\beta, \varphi} L_\varphi^{(n)} f)(t). \end{aligned}$$

Using the semigroup property (see [1]), we have

$$(D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) = (I_a^{2n-(\alpha+\beta), \varphi} L_\varphi^{(n)} f)(t), \quad a < t \leq b. \tag{13}$$

Similarly, by (2), one obtains

$$(D_b^{(\alpha, \beta), (\varphi, \psi)} f)(t) = (I_b^{2n-(\alpha+\beta), \varphi} (-1)^n L_\varphi^{(n)} f)(t), \quad a \leq t < b. \tag{14}$$

3.1 The case $2n - 1 < \alpha + \beta < 2n$

In this case, using (13), one has

$$(D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) = (I_a^{n-(\alpha+\beta-n), \varphi} L_\varphi^{(n)} f)(t), \quad a < t \leq b,$$

i.e.

$$(D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) = ({}^C D_a^{\alpha+\beta-n, \varphi} f)(t), \quad a < t \leq b.$$

Similarly, using (14), one obtains

$$(D_b^{(\alpha,\beta),(\varphi,\psi)} f)(t) = ({}^C D_b^{\alpha+\beta-n,\varphi} f)(t), \quad a \leq t < b.$$

Hence, the following result holds.

Theorem 3.1 *Let $\alpha, \beta \in (n - 1, n)$, $\varphi \in \Phi^{(n)}$ and $f \in C^n([a, b], \mathbb{R})$. Suppose that $2n - 1 < \alpha + \beta < 2n$. Then*

$$(D_a^{(\alpha,\beta),(\varphi,\varphi)} f)(t) = ({}^C D_a^{\alpha+\beta-n,\varphi} f)(t) = (D_a^{(\beta,\alpha),(\varphi,\varphi)} f)(t), \quad a < t \leq b,$$

and

$$(D_b^{(\alpha,\beta),(\varphi,\varphi)} f)(t) = ({}^C D_b^{\alpha+\beta-n,\varphi} f)(t) = (D_b^{(\beta,\alpha),(\varphi,\varphi)} f)(t), \quad a \leq t < b.$$

3.2 The case $2n - 2 < \alpha + \beta < 2n - 1$

In this case, using (13), for $a < t \leq b$, one has

$$\begin{aligned} &(D_a^{(\alpha,\beta),(\varphi,\psi)} f)(t) \\ &= (I_a^{2n-(\alpha+\beta),\varphi} L_\varphi^{(n)} f)(t) \\ &= \frac{1}{\Gamma(2n - \alpha - \beta)} \int_a^t \varphi'(s) (\varphi(t) - \varphi(s))^{2n-\alpha-\beta-1} (L_\varphi^{(n)} f)(s) \, ds \\ &= \frac{1}{\Gamma(2n - \alpha - \beta)} \int_a^t (\varphi(t) - \varphi(s))^{2n-\alpha-\beta-1} \frac{d}{ds} (L_\varphi^{(n-1)} f)(s) \, ds. \end{aligned}$$

Integrating by parts, one obtains

$$\begin{aligned} &(D_a^{(\alpha,\beta),(\varphi,\psi)} f)(t) \\ &= \frac{1}{\Gamma(2n - \alpha - \beta)} [(\varphi(t) - \varphi(s))^{2n-\alpha-\beta-1} (L_\varphi^{(n-1)} f)(s)]_{s=a}^t \\ &\quad + \frac{(2n - \alpha - \beta - 1)}{\Gamma(2n - \alpha - \beta)} \int_a^t (\varphi(t) - \varphi(s))^{2n-\alpha-\beta-2} \varphi'(s) (L_\varphi^{(n-1)} f)(s) \, ds \\ &= -\frac{1}{\Gamma(2n - \alpha - \beta)} (\varphi(t) - \varphi(a))^{2n-\alpha-\beta-1} (L_\varphi^{(n-1)} f)(a) + g_n(t), \end{aligned} \tag{15}$$

where

$$\begin{aligned} g_n(t) &= \frac{1}{\Gamma((n - 1) - (\alpha + \beta - n))} \\ &\quad \times \int_a^t (\varphi(t) - \varphi(s))^{(n-1)-(\alpha+\beta-n)-1} \varphi'(s) (L_\varphi^{(n-1)} f)(s) \, ds. \end{aligned}$$

Now, we discuss two cases.

- $n = 1$. In this case, one has

$$\begin{aligned} g_1(t) &= \frac{1}{\Gamma(1 - (\alpha + \beta))} \int_a^t (\varphi(t) - \varphi(s))^{-(\alpha+\beta)} \varphi'(s) f(s) \, ds \\ &= (I_a^{1-(\alpha+\beta),\varphi} f)(t). \end{aligned}$$

Hence, by (15), one deduces that

$$(D_a^{(\alpha,\beta),(\varphi,\psi)} f)(t) = \frac{-1}{\Gamma(2-\alpha-\beta)} (\varphi(t) - \varphi(a))^{1-\alpha-\beta} f(a) + (I_a^{1-(\alpha+\beta),\varphi} f)(t).$$

• $n \geq 2$. In this case, one has

$$g_n(t) = ({}^C D_a^{\alpha+\beta-n,\varphi} f)(t).$$

Hence, by (15), one deduces that

$$\begin{aligned} &(D_a^{(\alpha,\beta),(\varphi,\psi)} f)(t) \\ &= -\frac{1}{\Gamma(2n-\alpha-\beta)} (\varphi(t) - \varphi(a))^{2n-\alpha-\beta-1} (L_\varphi^{(n-1)} f)(a) + ({}^C D_a^{\alpha+\beta-n,\varphi} f)(t). \end{aligned}$$

Similarly, using (14), for $a \leq t < b$ and $n \geq 2$, one obtains

$$\begin{aligned} &(D_b^{(\alpha,\beta),(\varphi,\psi)} f)(t) \\ &= \frac{1}{\Gamma(2n-\alpha-\beta)} (\varphi(b) - \varphi(t))^{2n-\alpha-\beta-1} (-1)^n (L_\varphi^{(n-1)} f)(b) + ({}^C D_b^{\alpha+\beta-n,\varphi} f)(t) \end{aligned}$$

and for $n = 1$,

$$\begin{aligned} &(D_b^{(\alpha,\beta),(\varphi,\psi)} f)(t) \\ &= \frac{-1}{\Gamma(2-\alpha-\beta)} (\varphi(b) - \varphi(t))^{1-\alpha-\beta} f(b) + (I_b^{1-(\alpha+\beta),\varphi} f)(t). \end{aligned}$$

Hence, we have the following results.

Theorem 3.2 *Let $\alpha, \beta \in (n-1, n)$, $n \geq 2$, $\varphi \in \Phi^{(n)}$ and $f \in C^n([a, b], \mathbb{R})$. Suppose that $2n - 2 < \alpha + \beta < 2n - 1$. Then*

$$\begin{aligned} &(D_a^{(\alpha,\beta),(\varphi,\psi)} f)(t) \\ &= -\frac{(\varphi(t) - \varphi(a))^{2n-\alpha-\beta-1}}{\Gamma(2n-\alpha-\beta)} (L_\varphi^{(n-1)} f)(a) + ({}^C D_a^{\alpha+\beta-n,\varphi} f)(t) \\ &= (D_a^{(\beta,\alpha),(\varphi,\varphi)} f)(t), \quad a < t \leq b, \end{aligned}$$

and

$$\begin{aligned} &(D_b^{(\alpha,\beta),(\varphi,\psi)} f)(t) \\ &= \frac{(\varphi(b) - \varphi(t))^{2n-\alpha-\beta-1}}{\Gamma(2n-\alpha-\beta)} (-1)^n (L_\varphi^{(n-1)} f)(b) + ({}^C D_b^{\alpha+\beta-n,\varphi} f)(t) \\ &= (D_b^{(\beta,\alpha),(\varphi,\varphi)} f)(t), \quad a \leq t < b. \end{aligned}$$

Theorem 3.3 *Let $0 < \alpha, \beta < 1$, $\varphi \in \Phi^{(1)}$ and $f \in C^1([a, b], \mathbb{R})$. Suppose that $0 < \alpha + \beta < 1$. Then*

$$\begin{aligned} & (D_a^{(\alpha, \beta), (\varphi, \varphi)} f)(t) \\ &= \frac{-1}{\Gamma(2 - \alpha - \beta)} (\varphi(t) - \varphi(a))^{1 - \alpha - \beta} f(a) + (I_a^{1 - (\alpha + \beta), \varphi} f)(t) \\ &= (D_a^{(\beta, \alpha), (\varphi, \varphi)} f)(t), \quad a < t \leq b, \end{aligned}$$

and

$$\begin{aligned} & (D_b^{(\alpha, \beta), (\varphi, \varphi)} f)(t) \\ &= \frac{-1}{\Gamma(2 - \alpha - \beta)} (\varphi(b) - \varphi(t))^{1 - \alpha - \beta} f(b) + (I_b^{1 - (\alpha + \beta), \varphi} f)(t) \\ &= (D_b^{(\beta, \alpha), (\varphi, \varphi)} f)(t), \quad a \leq t < b. \end{aligned}$$

3.3 The case $\alpha + \beta = 2n - 1$

In this case, using (13), for $a < t \leq b$, one has

$$\begin{aligned} (D_a^{(\alpha, \beta), (\varphi, \psi)} f)(t) &= (I_a^{2n - (\alpha + \beta), \varphi} L_\varphi^{(n)} f)(t) \\ &= (I_a^{1, \varphi} L_\varphi^{(n)} f)(t) \\ &= \int_a^t \varphi'(s) (L_\varphi^{(n)} f)(s) ds \\ &= \int_a^t \frac{d}{ds} (L_\varphi^{(n-1)} f)(s) ds \\ &= (L_\varphi^{(n-1)} f)(t) - (L_\varphi^{(n-1)} f)(a). \end{aligned}$$

Similarly, using (14), for $a \leq t < b$, one obtains

$$(D_b^{(\alpha, \beta), (\varphi, \psi)} f)(t) = (-1)^n ((L_\varphi^{(n-1)} f)(b) - (L_\varphi^{(n-1)} f)(t)).$$

Hence, we obtain the following.

Theorem 3.4 *Let $\alpha, \beta \in (n - 1, n)$, $\varphi \in \Phi^{(n)}$ and $f \in C^n([a, b], \mathbb{R})$. Suppose that $\alpha + \beta = 2n - 1$. Then*

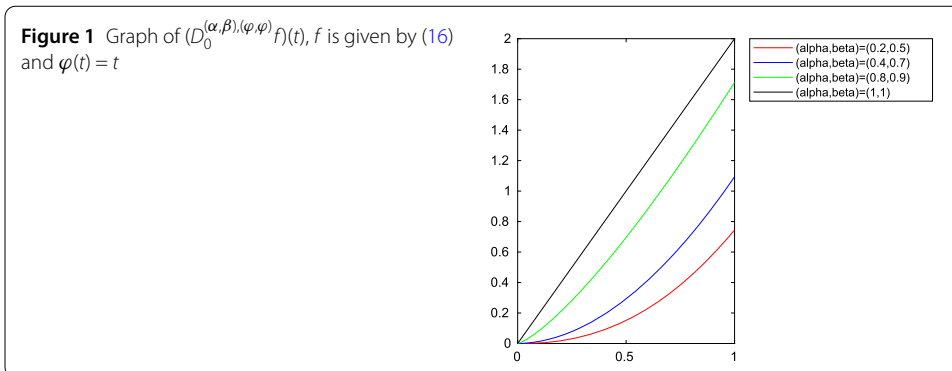
$$(D_a^{(\alpha, \beta), (\varphi, \varphi)} f)(t) = (L_\varphi^{(n-1)} f)(t) - (L_\varphi^{(n-1)} f)(a) = (D_a^{(\beta, \alpha), (\varphi, \varphi)} f)(t), \quad a < t \leq b,$$

and

$$\begin{aligned} (D_b^{(\alpha, \beta), (\varphi, \varphi)} f)(t) &= (-1)^n [(L_\varphi^{(n-1)} f)(b) - (L_\varphi^{(n-1)} f)(t)] \\ &= (D_b^{(\beta, \alpha), (\varphi, \varphi)} f)(t), \quad a \leq t < b. \end{aligned}$$

Example 3.1 Let $0 < \alpha, \beta < 1$. Consider the function

$$f(t) = (\varphi(t) - \varphi(0))^2, \quad 0 \leq t \leq 1, \tag{16}$$



where $\varphi \in \Phi^{(1)}$. By (13), one has

$$(D_0^{(\alpha,\beta),(\varphi,\varphi)} f)(t) = (I_0^{2-\alpha-\beta,\varphi} L_\varphi^{(1)} f)(t), \quad 0 < t \leq 1,$$

that is,

$$\begin{aligned} (D_0^{(\alpha,\beta),(\varphi,\varphi)} f)(t) &= \frac{1}{\Gamma(2-\alpha-\beta)} \int_0^t (\varphi(t) - \varphi(s))^{1-\alpha-\beta} f'(s) \, ds \\ &= \frac{2}{\Gamma(2-\alpha-\beta)} \int_0^t (\varphi(t) - \varphi(s))^{1-\alpha-\beta} (\varphi(s) - \varphi(0)) \varphi'(s) \, ds \\ &= \frac{2(\varphi(t) - \varphi(0))^{1-\alpha-\beta}}{\Gamma(2-\alpha-\beta)} \int_0^t \left[1 - \frac{\varphi(s) - \varphi(0)}{\varphi(t) - \varphi(0)} \right]^{1-\alpha-\beta} (\varphi(s) - \varphi(0)) \varphi'(s) \, ds. \end{aligned}$$

Using the change of variable $z = \frac{\varphi(s)-\varphi(0)}{\varphi(t)-\varphi(0)}$, one obtains

$$\begin{aligned} (D_0^{(\alpha,\beta),(\varphi,\varphi)} f)(t) &= \frac{2(\varphi(t) - \varphi(0))^{3-\alpha-\beta}}{\Gamma(2-\alpha-\beta)} \int_0^1 (1-z)^{(2-\alpha-\beta)-1} z^{2-1} \, dz \\ &= \frac{2(\varphi(t) - \varphi(0))^{3-\alpha-\beta}}{\Gamma(2-\alpha-\beta)} B(2-\alpha-\beta, 2) \\ &= \frac{2}{\Gamma(4-\alpha-\beta)} (\varphi(t) - \varphi(0))^{3-\alpha-\beta}, \end{aligned}$$

where B is the beta function. Observe that

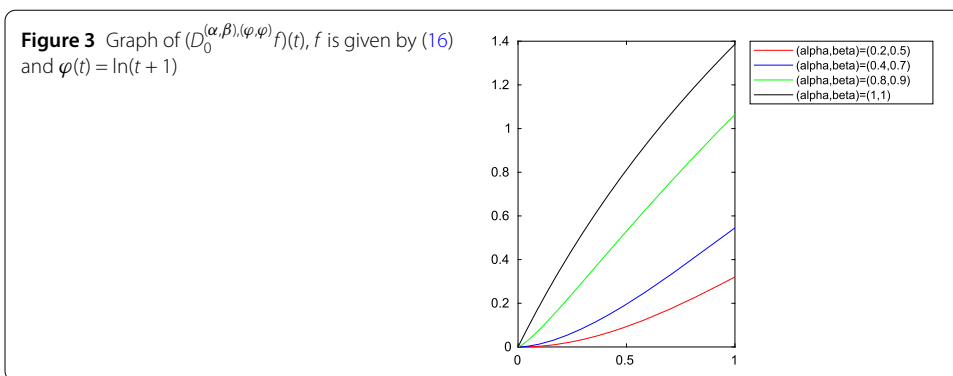
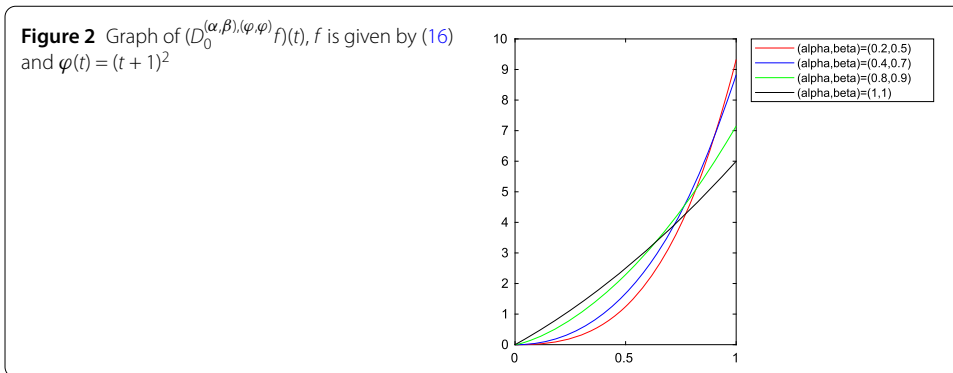
$$\lim_{(\alpha,\beta) \rightarrow (1^-,1^-)} (D_0^{(\alpha,\beta),(\varphi,\varphi)} f)(t) = 2(\varphi(t) - \varphi(0)) = \frac{f'(t)}{\varphi'(t)} = (L_\varphi^{(1)} f)(t),$$

which confirms (11). Figures 1–3 show some graphs of $(D_0^{(\alpha,\beta),(\varphi,\varphi)} f)(t)$ for different functions φ and different values of (α, β) .

Following a similar argument to above, one obtains a theorem.

Theorem 3.5 *Let $\alpha, \beta \in (0, 1)$ and $\theta > 0$. Let*

$$f(t) = (\varphi(t) - \varphi(a))^\theta, \quad a \leq t \leq b,$$



where $\varphi \in \Phi^{(1)}$. Then

$$(D_a^{(\alpha,\beta),(\varphi,\varphi)} f)(t) = \frac{\Gamma(\theta + 1)}{\Gamma(2 - \alpha - \beta + \theta)} (\varphi(t) - \varphi(a))^{\theta+1-\alpha-\beta}, \quad a < t \leq b.$$

The Mittag-Leffler function E_θ , $\theta > 0$, is defined by

$$E_\theta(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\theta k + 1)}, \quad t \geq 0.$$

Theorem 3.6 Let $\rho > 0$ and $0 < \alpha, \beta < 1$ with $1 < \alpha + \beta < 2$. Let

$$f(t) = E_{\alpha+\beta-1}(\rho(\varphi(t) - \varphi(a))^{\alpha+\beta-1}), \quad a \leq t \leq b,$$

where $\varphi \in \Phi^{(1)}$. Then

$$(D_a^{(\alpha,\beta),(\varphi,\varphi)} f)(t) = \rho f(t), \quad a < t \leq b.$$

Proof By Theorem 3.1, one has

$$(D_a^{(\alpha,\beta),(\varphi,\varphi)} f)(t) = ({}^C D_a^{\alpha+\beta-1, \varphi} f)(t).$$

Next, using [1, Lemma 2], the desired result follows. □

Theorem 3.7 Let $\rho > 0$ and $0 < \alpha, \beta < 1$ with $0 < \alpha + \beta < 1$. Let

$$f(t) = E_{1-\alpha-\beta}(\rho(\varphi(t) - \varphi(a))^{1-\alpha-\beta}), \quad a \leq t \leq b,$$

where $\varphi \in \Phi^{(1)}$. Then

$$(D_a^{(\alpha,\beta),(\varphi,\varphi)}f)(t) = \frac{f(t)}{\rho} - \left[\frac{(\varphi(t) - \varphi(a))^{1-\alpha-\beta}}{\Gamma(2-\alpha-\beta)} + \frac{1}{\rho} \right], \quad a < t \leq b.$$

Proof By Theorem 3.3, one has

$$(D_a^{(\alpha,\beta),(\varphi,\varphi)}f)(t) = \frac{-1}{\Gamma(2-\alpha-\beta)}(\varphi(t) - \varphi(a))^{1-\alpha-\beta} + (I_a^{1-(\alpha+\beta),\varphi}f)(t),$$

for all $a < t \leq b$. On the other hand, an elementary calculation gives us

$$(I_a^{1-(\alpha+\beta),\varphi}f)(t) = \frac{f(t) - 1}{\rho},$$

for all $a < t \leq b$. Hence, combining the above equalities, we obtain the desired result. \square

Remark 3.1 By Theorems 3.6 and 3.7, one observes that, if $0 < \rho < 1$, then

$$\lim_{(\alpha+\beta) \rightarrow 1^-} (D_a^{(\alpha,\beta),(\varphi,\varphi)}f)(t) = \lim_{(\alpha+\beta) \rightarrow 1^+} (D_a^{(\alpha,\beta),(\varphi,\varphi)}f)(t) = \frac{\rho}{1-\rho}.$$

4 Applications to fractional differential equations

Let $\alpha, \beta \in (n-1, n)$, $\varphi \in \Phi^{(1)}$ and $\psi \in \Phi^{(n)}$. We first consider the problem

$$\begin{cases} (D_a^{(\alpha,\beta),(\varphi,\psi)}y)(t) = \sigma(t), & a < t < b, \\ (L_\psi^{(k)}y)(a) = \mu_k, & k = 0, 1, \dots, n-1, \end{cases} \tag{17}$$

where $\sigma \in C^1([a, b], \mathbb{R})$ and $\sigma(a) = 0$.

Proposition 4.1 Problem (17) has a unique solution $y \in C^n([a, b], \mathbb{R})$, which is given by

$$y(t) = \sum_{k=0}^{n-1} \frac{\mu_k}{k!} (\psi(t) - \psi(a))^k + I_a^{\beta,\psi} (I_a^{1-(n-\alpha),\varphi} L_\varphi^{(1)}\sigma)(t), \quad a \leq t \leq b. \tag{18}$$

Proof Let y be the function given by (18). One observes easily that

$$(D_a^{(\alpha,\beta),(\varphi,\psi)}(\psi(\cdot) - \psi(a))^k)(t) = 0, \quad k = 0, 1, \dots, n-1.$$

Hence, using (1), one has

$$\begin{aligned} (D_a^{(\alpha,\beta),(\varphi,\psi)}y)(t) &= D_a^{(\alpha,\beta),(\varphi,\psi)} I_a^{\beta,\psi} (I_a^{1-(n-\alpha),\varphi} L_\varphi^{(1)}\sigma)(t) \\ &= I_a^{n-\alpha,\varphi} {}^C D_a^{\beta,\psi} I_a^{\beta,\psi} (I_a^{1-(n-\alpha),\varphi} L_\varphi^{(1)}\sigma)(t). \end{aligned}$$

Using the property (see [1]) ${}^C D_a^{\beta, \psi} I_a^{\beta, \psi} f = f$, one obtains

$$(D_a^{(\alpha, \beta), (\varphi, \psi)} y)(t) = I_a^{n-\alpha, \varphi} (I_a^{1-(n-\alpha), \varphi} L_\varphi^{(1)} \sigma)(t).$$

Next, using the semigroup property, we have

$$\begin{aligned} (D_a^{(\alpha, \beta), (\varphi, \psi)} y)(t) &= (I_a^{1, \varphi} L_\varphi^{(1)} \sigma)(t) \\ &= \int_a^t \varphi'(s) \frac{1}{\varphi'(s)} \sigma'(s) ds \\ &= \sigma(t) - \sigma(a). \end{aligned}$$

Since $\sigma(a) = 0$, one deduces that

$$(D_a^{(\alpha, \beta), (\varphi, \psi)} y)(t) = \sigma(t).$$

On the other hand, one can check easily that

$$(L_\psi^{(k)} y)(a) = \mu_k$$

for all $k = 0, 1, \dots, n - 1$. Therefore, the function y given by (18) solves (17).

Now, suppose that $y \in C^n([a, b], \mathbb{R})$ is a solution of (18). By (1), one has

$$I_a^{n-\alpha, \varphi} ({}^C D_a^{\beta, \psi} y)(t) = \sigma(t),$$

which yields

$${}^C D_a^{n-\alpha, \varphi} I_a^{n-\alpha, \varphi} ({}^C D_a^{\beta, \psi} y)(t) = ({}^C D_a^{n-\alpha, \varphi} \sigma)(t),$$

i.e.

$$({}^C D_a^{\beta, \psi} y)(t) = ({}^C D_a^{n-\alpha, \varphi} \sigma)(t).$$

Then we have

$$I_a^{\beta, \psi} ({}^C D_a^{\beta, \psi} y)(t) = I_a^{\beta, \psi} ({}^C D_a^{n-\alpha, \varphi} \sigma)(t) = I_a^{\beta, \psi} (I_a^{1-(n-\alpha), \varphi} L_\varphi^{(1)} \sigma)(t). \tag{19}$$

On the other hand, one has (see [1])

$$I_a^{\beta, \psi} ({}^C D_a^{\beta, \psi} y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{(L_\psi^{(k)} y)(a)}{k!} (\psi(t) - \psi(a))^k.$$

Using the initial conditions, one obtains

$$I_a^{\beta, \psi} ({}^C D_a^{\beta, \psi} y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{\mu_k}{k!} (\psi(t) - \psi(a))^k. \tag{20}$$

Further, combining (19) with (20), one deduces that

$$y(t) = \sum_{k=0}^{n-1} \frac{\mu_k}{k!} (\psi(t) - \psi(a))^k + I_a^{\beta, \psi} (I_a^{1-(n-\alpha), \varphi} L_\varphi^{(1)} \sigma)(t). \quad \square$$

Consider now the nonlinear problem

$$\begin{cases} (D_a^{(\alpha, \beta), (\varphi, \psi)} y)(t) = \int_a^t f(s, y(s)) ds, & a < t < b, \\ (L_\psi^{(k)} y)(a) = \mu_k, & k = 0, 1, \dots, n-1, \end{cases} \quad (21)$$

where $f : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. We suppose that

$$|f(t, \lambda) - f(t, \eta)| \leq C_f |\lambda - \eta| \quad (22)$$

for all $t \in [a, b]$ and $\lambda, \eta \in \mathbb{R}$, where

$$0 < C_f < \frac{\Gamma(\beta + 1)\Gamma(2 - n + \alpha)}{(\varphi(b) - \varphi(a))^{1-n+\alpha} (\psi(b) - \psi(a))^\beta}. \quad (23)$$

Theorem 4.1 *Problem (21) admits one and only one solution $y^* \in C^n([a, b], \mathbb{R})$. Moreover, for any $y_0 \in C([a, b], \mathbb{R})$, the Picard sequence $\{y_n\} \subset C([a, b], \mathbb{R})$ defined by*

$$y_{n+1}(t) = \sum_{k=0}^{n-1} \frac{\mu_k}{k!} (\psi(t) - \psi(a))^k + I_a^{\beta, \psi} \left(I_a^{1-(n-\alpha), \varphi} \frac{1}{\varphi} f(\cdot, y_n(\cdot)) \right)(t), \quad a \leq t \leq b,$$

converges uniformly to y^* .

Proof Let A be the self-mapping defined in $C([a, b], \mathbb{R})$ by

$$(Az)(t) = \sum_{k=0}^{n-1} \frac{\mu_k}{k!} (\psi(t) - \psi(a))^k + I_a^{\beta, \psi} \left(I_a^{1-(n-\alpha), \varphi} \frac{1}{\varphi} f(\cdot, z(\cdot)) \right)(t), \quad a \leq t \leq b,$$

i.e.

$$\begin{aligned} (Az)(t) &= \sum_{k=0}^{n-1} \frac{\mu_k}{k!} (\psi(t) - \psi(a))^k \\ &\quad + \frac{1}{\Gamma(\beta)\Gamma(1 - n + \alpha)} \\ &\quad \times \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \left(\int_a^s (\varphi(s) - \varphi(\tau))^{-(n-\alpha)} f(\tau, z(\tau)) d\tau \right) ds. \end{aligned}$$

By Proposition 4.1, $y \in C^n([a, b], \mathbb{R})$ is a solution of (21) if and only if $y \in C([a, b], \mathbb{R})$ is a fixed point of A . We shall show that A is a contraction in $(C([a, b], \mathbb{R}), \|\cdot\|_\infty)$, and then, by the fixed point theorem of Banach, we obtain the desired result. For any $y, z \in C([a, b], \mathbb{R})$,

one has

$$\begin{aligned} & |(Az)(t) - (Ay)(t)| \\ & \leq \frac{1}{\Gamma(\beta)\Gamma(1-n+\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \\ & \quad \times \left(\int_a^s (\varphi(s) - \varphi(\tau))^{-(n-\alpha)} |f(\tau, z(\tau)) - f(\tau, y(\tau))| d\tau \right) ds. \end{aligned}$$

Using (22), we have

$$\begin{aligned} & |(Az)(t) - (Ay)(t)| \\ & \leq \frac{C_f}{\Gamma(\beta)\Gamma(1-n+\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \\ & \quad \times \left(\int_a^s (\varphi(s) - \varphi(\tau))^{-(n-\alpha)} |z(\tau) - y(\tau)| d\tau \right) ds \\ & \leq \frac{C_f \|y - z\|_\infty}{\Gamma(\beta)\Gamma(1-n+\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \left(\int_a^s (\varphi(s) - \varphi(\tau))^{-(n-\alpha)} d\tau \right) ds \\ & = \frac{C_f \|y - z\|_\infty}{\Gamma(\beta)\Gamma(2-n+\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} (\varphi(s) - \varphi(a))^{1-n+\alpha} ds \\ & \leq \frac{C_f (\varphi(b) - \varphi(a))^{1-n+\alpha} \|y - z\|_\infty}{\Gamma(\beta)\Gamma(2-n+\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} ds \\ & = \frac{C_f (\varphi(b) - \varphi(a))^{1-n+\alpha} (\psi(t) - \psi(a))^\beta}{\Gamma(\beta + 1)\Gamma(2-n+\alpha)} \|y - z\|_\infty \\ & \leq L \|y - z\|_\infty, \end{aligned}$$

where

$$L = \frac{(\varphi(b) - \varphi(a))^{1-n+\alpha} (\psi(b) - \psi(a))^\beta}{\Gamma(\beta + 1)\Gamma(2-n+\alpha)} C_f.$$

Hence,

$$\|Ay - Az\|_\infty \leq L \|y - z\|_\infty$$

for all $y, z \in C([a, b], \mathbb{R})$. On the other hand, by (23), one has $0 < L < 1$. Therefore, A is a contraction. □

Example 4.1 Consider the fractional boundary value problem

$$\begin{cases} (D_0^{(\alpha,\beta),(\varphi,\psi)} y)(t) = \int_0^t \frac{\cos(y(s))}{s+\rho} ds, & 0 < t < 1, \\ y(0) = 0, \end{cases} \tag{24}$$

where $0 < \alpha, \beta < 1$, $\varphi(t) = t$, $\psi(t) = \ln(t + 1)$ and $\Gamma(\alpha + 1)\Gamma(\beta + 1)\rho > (\ln 2)^\beta$. Problem (24) is a particular case of problem (21) with $(a, b) = (0, 1)$, $n = 1$, $\mu_0 = 0$ and

$$f(t, \lambda) = \frac{\cos \lambda}{t + \rho}, \quad (t, \lambda) \in [0, 1] \times \mathbb{R}.$$

For all $t \in [0, 1]$ and $\lambda, \eta \in \mathbb{R}$, one has

$$\begin{aligned} |f(t, \lambda) - f(t, \eta)| &= \left| \frac{\cos \lambda}{t + \rho} - \frac{\cos \eta}{t + \rho} \right| \\ &\leq \frac{|\lambda - \eta|}{t + \rho} \\ &\leq C_f |\lambda - \rho|, \end{aligned}$$

where $C_f = \rho^{-1}$. On the other hand, one has

$$C_f < \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{(\ln 2)^\beta} = \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{(\varphi(1) - \varphi(0))^\alpha (\psi(1) - \psi(0))^\beta}.$$

Hence, by Theorem 4.1, problem (24) admits a unique solution $y^* \in C^1([0, 1], \mathbb{R})$. Moreover, for any $y_0 \in C([0, 1], \mathbb{R})$, the Picard sequence

$$y_{n+1}(t) = \frac{1}{\Gamma(\beta)\Gamma(\alpha)} \int_0^t \frac{1}{s+1} \left[\ln \left(\frac{t+1}{s+1} \right) \right]^{\beta-1} \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\tau + \rho} \cos(y_n(\tau)) d\tau \right) ds,$$

for all $t \leq 1$, converges uniformly to y^* .

5 Fractional model of a heat conduction problem

The standard Fourier law of thermal conduction in one dimension is given by

$$-\rho \frac{dy}{dx} = z(x), \quad x > 0, \tag{25}$$

where ρ is the material's thermal conductivity, z is the density of the heat flux and y is the temperature. Replacing $\frac{d}{dx}$ by $D_0^{(\alpha, \beta), (\varphi, \psi)}$, where $\alpha, \beta \in (0, 1)$, we obtain the fractional version of (25)

$$-\rho (D_0^{(\alpha, \beta), (\varphi, \psi)} y)(x) = z(x), \quad x > 0. \tag{26}$$

If $z(0) = 0$ and $y(0) = y_0$, by Proposition 4.1, the unique solution of (26) is given by

$$y(x) = y_0 - I_0^{\beta, \psi} \left(I_0^{\alpha, \varphi} L_\varphi^{(1)} \frac{z}{\rho} \right)(x), \quad x \geq 0,$$

i.e.

$$\begin{aligned} y(x) &= y_0 - \frac{1}{\rho \Gamma(\beta)\Gamma(\alpha)} \int_0^x \psi'(\eta) (\psi(x) - \psi(\eta))^{\beta-1} \\ &\quad \times \left(\int_0^\eta (\varphi(\eta) - \varphi(\lambda))^{\alpha-1} z'(\lambda) d\lambda \right) d\eta. \end{aligned} \tag{27}$$

Example 5.1 Consider (26) with $\varphi = \psi$ and $z(x) = \varphi(x) - \varphi(0)$. In this case, by (27), one has

$$\begin{aligned} y(x) &= y_0 - \frac{1}{\rho \Gamma(\beta) \Gamma(\alpha + 1)} \int_0^x \varphi'(\eta) (\varphi(x) - \varphi(\eta))^{\beta-1} (\varphi(\eta) - \varphi(0))^\alpha d\eta \\ &= y_0 - \frac{(\varphi(x) - \varphi(0))^{\beta-1}}{\rho \Gamma(\beta) \Gamma(\alpha + 1)} \int_0^x \varphi'(\eta) \left[1 - \frac{(\varphi(\eta) - \varphi(0))}{(\varphi(x) - \varphi(0))} \right]^{\beta-1} (\varphi(\eta) - \varphi(0))^\alpha d\eta \\ &= y_0 - \frac{(\varphi(x) - \varphi(0))^{\alpha+\beta}}{\rho \Gamma(\beta) \Gamma(\alpha + 1)} \int_0^1 (1-w)^{\beta-1} w^\alpha dw, \end{aligned}$$

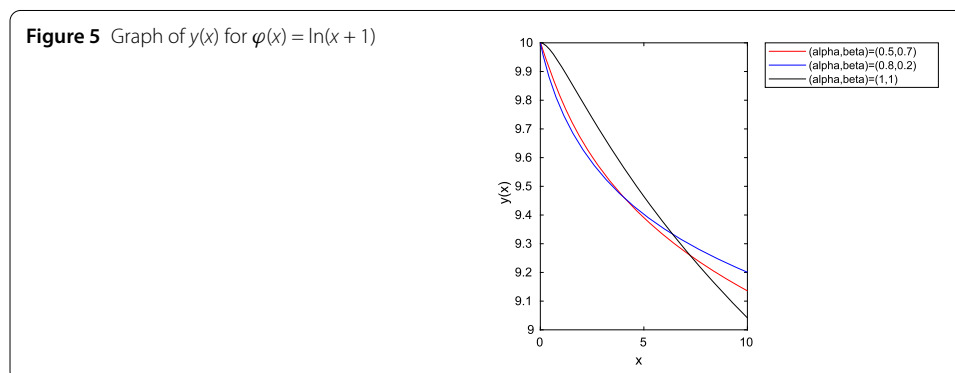
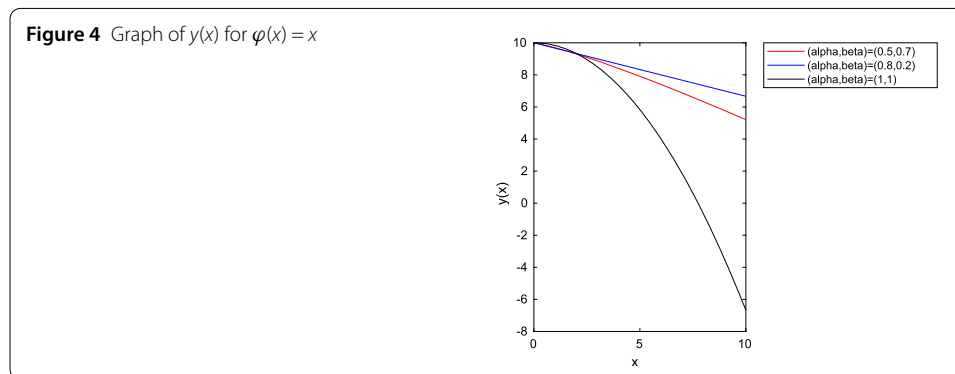
which yields

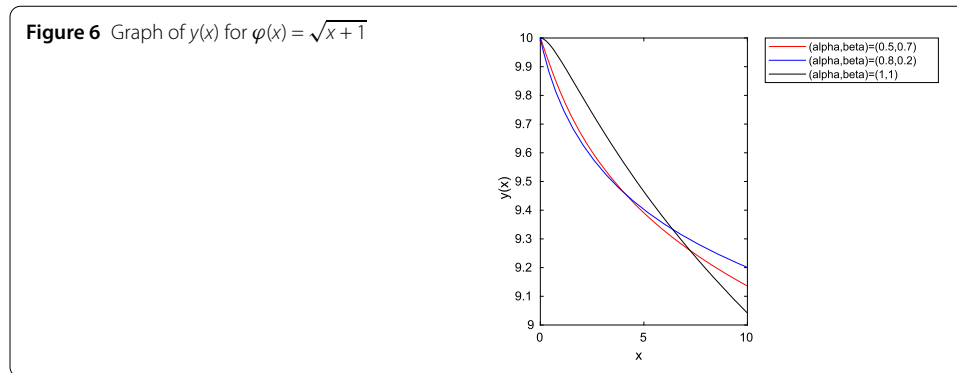
$$y(x) = y_0 - \frac{1}{\rho \Gamma(\alpha + \beta + 1)} (\varphi(x) - \varphi(0))^{\alpha+\beta}, \quad x \geq 0.$$

Observe that in the case $\varphi(x) = x$ one has

$$\lim_{(\alpha, \beta) \rightarrow (1^-, 1^-)} y(x) = y_0 - \frac{1}{2\rho} x^2,$$

which is the unique solution of (25) with $z(x) = x$ and $y(0) = y_0$. Figures 4–6 show some graphs of y for different functions φ and different values of (α, β) .





6 Conclusion

The goal of this article was to propose a new notion of fractional derivative involving two singular kernels. Some properties of this introduced operator were proved and some examples were provided. We also presented some applications to fractional differential equations. Namely, an existence and uniqueness result was established for a nonlinear fractional boundary value problem with a higher order, and a numerical algorithm based on Picard iteration was provided for approximating the unique solution. Moreover, an application to a heat conduction problem was presented. It will be interesting to develop new numerical methods for solving fractional differential equations (or partial differential equations that are fractional in time) involving this new concept, in particular in the case $\varphi \neq \psi$.

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Authors' contributions

All authors carried out the proofs and conceived of the study. All authors read and approved the final manuscript.

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