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On a fractional hybrid multi-term integro-differential inclusion with four-point sum and integral boundary conditions

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Abstract

We investigate the existence of solutions for a fractional hybrid multi-term integro-differential inclusion with four-point sum and integral boundary value conditions. By using Dhage's fixed point results, we prove our main existence result. Finally, we give an example to illustrate our main result.

MSC: Primary 34A08; secondary 34A12

Keywords: Four-point sum boundary value condition; Fractional hybrid inclusion problem; Dhage's fixed point result; Fractional Caputo derivative

1 Introduction

One possible way that the mathematics can help the various fields of science is to become more powerful and flexible in modeling theory so that different types of phenomena with distinct parameters can be written in mathematical formulas. In this case, different types of software can be developed to allow for more cost-free testing and less material consumption. One of the basic methods in this way is working with fractional calculus and investigating different mathematical modelings based on fractional operators in fractional boundary value problems with applied boundary conditions. Nowadays, many researchers are studying different types of integro-differential equations [1–7] or inclusions [8, 9], q-differences [10–13], approximate solutions [14–20], the hybrid equations [21–28], and advanced fractional modelings [29–34].

The starting point for this field was a work of Dhage and Lakshmikantham in 2010 [35]. They introduced a new category of nonlinear differential equation called ordinary hybrid differential equation and studied the existence of extremal solutions for this boundary value problem by establishing some fundamental differential inequalities [35]. In 2012, Zhao et al. provided an extension for Dhage's work to fractional order and considered a boundary value problem of fractional hybrid differential equations [36]. Later, some papers on different properties of solutions for fractional hybrid boundary value problems were published. In 2015, Hilal and Kajouni discussed the existence of extremal solutions

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for the Caputo hybrid boundary value problem

$$\begin{cases} {}^{c}\mathcal{D}_0^{p}\left(\frac{k(t)}{h(t,k(t))}\right) = g(t,k(t)) = 0, \\ a\frac{k(0)}{h(0,k(0))} + b\frac{k(T)}{h(T,k(T))} = c, \end{cases}$$

where $t \in J = [0,T]$, $p \in (0,1)$, the functions $h : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ are continuous, and $a, b, c \in \mathbb{R}$ with $a+b \neq 0$ [37]. In 2016, Ahmad et al. studied the existence of solutions for the nonlocal boundary value problem of fractional hybrid inclusion problem

$$\begin{cases} {}^{c}\mathcal{D}_0^{\alpha}\left(\frac{k(t)-\sum_{i=1}^{m}\mathcal{I}_0^{\beta_i}h_i(t,k(t))}{g(t,k(t))}\right) \in \mathcal{G}(t,k(t)) = 0, \\ k(0) = \mu(x), \qquad k(1) = A \in \mathbb{R}, \end{cases}$$

where $t \in J = [0,1]$, ${}^{c}\mathcal{D}_0^{\alpha}$ denotes the Caputo fractional derivative of order $\alpha \in (1,2]$, and $\mathcal{I}_0^{\phi}$ is the Riemann–Liouville fractional integral of order $\phi > 0$ with $\phi \in \{\beta_1, \beta_2, \ldots, \beta_m\}$ [38]. In the same year, Baleanu et al. investigated some existence results and the dimension of the solution set for the fractional hybrid inclusion problem

$${}^{c}\mathcal{D}^{\nu}\left(\frac{k(t)}{\Lambda(t,k(t),\mathcal{I}^{\alpha_1}k(t),\ldots,\mathcal{I}^{\alpha_n}k(t))}\right) \in \Psi\left(t,k(t),\mathcal{I}^{\beta_1}k(t),\ldots,\mathcal{I}^{\beta_m}k(t)\right),$$

with boundary value conditions $k(0) = k_0^*$ and $k(1) = k_1^*$, where $t \in [0,1]$, $\nu \in (1,2]$, ${}^{c}\mathcal{D}^{\nu}$ and $\mathcal{I}^{\gamma}$ denote the Caputo derivative operator of the fractional order ν and the Riemann–Liouville integral operator of the fractional order $\gamma \in \{\alpha_i, \beta_j\} \subset (0,\infty)$ for $i = 1,\ldots n$ and $j = 1,\ldots,m$, respectively [8]. In 2019, Derbazi et al. studied the existence and uniqueness results for the fractional hybrid boundary value problem

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha}\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right) = \Theta(t,k(t)), \\ a_1\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right)|_{t=0} + b_1\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right)|_{t=T} = \lambda_1, \\ a_2{}^{c}\mathcal{D}^{\beta}\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right)|_{t=\eta} + b_2{}^{c}\mathcal{D}^{\beta}\left(\frac{k(t)-h(t,k(t))}{w(t,k(t))}\right)|_{t=T} = \lambda_2, \end{cases}$$

where $t \in [0,T]$, $\alpha \in (1,2]$, $\beta \in (0,1]$, $\eta \in (0,T)$, $a_1, a_2, b_1, b_2, \lambda_1, \lambda_2 \in \mathbb{R}$, and the fractional derivatives that appeared are Caputo-type ones [39].

By using the idea of these works, we investigate the fractional hybrid multi-term integro-differential inclusion of Caputo type

$${}^{C}\mathcal{D}_0^{\omega}\left(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\,\mathrm{d}s)}\right) \in \mathcal{S}\left(t,k(t),\phi_1(k(t)),\ldots,\phi_m(k(t))\right), \tag{1}$$

with four-point sum and integral hybrid boundary value conditions

$$\begin{cases} \left(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\,\mathrm{d}s)}\right)|_{t=0} + \sum_{j=1}^{r} b_j\mathcal{D}_0\left(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\,\mathrm{d}s)}\right)|_{t=\eta_1} = 0, \\ \lambda_1\left(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\,\mathrm{d}s)}\right)|_{t=\eta_2} + \lambda_2\sum_{j=1}^{r} b_j\mathcal{D}_0^2\left(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\,\mathrm{d}s)}\right)|_{t=1} = 0, \\ \lambda_3\int_0^1\left(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\,\mathrm{d}s)}\right)\mathrm{d}s = 0, \end{cases} \tag{2}$$

where $t \in [0,1]$, $0 < \eta_1 < \eta_2 < 1$, ${}^{C}\mathcal{D}_0^{\omega}$ denotes the Caputo fractional derivative of order $\omega \in (2,3]$, $\xi \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\mathcal{S} : [0,1] \times \mathbb{R}^{m+1} \to \mathcal{P}(\mathbb{R})$ is a set-valued map

via some properties, and $\lambda_1, \lambda_2, \lambda_3, b_j \in \mathbb{R}^+$ for $j = 1, \ldots, r$. Moreover, for each $i = 1, \ldots, m$, assume that $\phi_i(k(t)) = \int_0^t \frac{(t-s)^{\varpi_i - 1}}{\Gamma(\varpi_i)} k(s)\, \mathrm{d}s$ with $\varpi_i > 0$.

2 Preliminaries

Let $\omega > 0$. The Riemann–Liouville fractional integral of a function $k : [a, b] \to \mathbb{R}$ is defined by $\mathcal{I}_0^\omega k(t) = \int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} k(s)\, \mathrm{d}s$ provided that the right-hand side integral exists ([40, 41]). Now, let $n - 1 < \omega < n$ and $n = [\omega] + 1$. The Caputo fractional derivative of a function $k \in C^{(n)}([a, b], \mathbb{R})$ is defined by ${}^C\mathcal{D}_0^\omega k(t) = \int_0^t \frac{(t-s)^{n-\omega-1}}{\Gamma(n-\omega)} k^{(n)}(s)\, \mathrm{d}s$ provided that the right-hand side integral exists ([40, 41]). It has been proved that the general solution for the homogeneous fractional differential equation ${}^C\mathcal{D}_0^\omega k(t) = 0$ is in the form $k(t) = m_0^* + m_1^* t + m_2^* t^2 + \cdots + m_{n-1}^* t^{n-1}$, and we have

$$\mathcal{I}_0^\omega \left({}^C\mathcal{D}_0^\omega k(t) \right) = k(t) + \sum_{j=0}^{n-1} m_j^* t^j = k(t) + m_0^* + m_1^* t + m_2^* t^2 + \cdots + m_{n-1}^* t^{n-1},$$

where $m_0^*, \ldots, m_{n-1}^*$ are some real constants and $n = [\omega] + 1$ [42].

Assume that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a normed space. The set of all subsets of $\mathcal{X}$, the set of all closed subsets of $\mathcal{X}$, the set of all bounded subsets of $\mathcal{X}$, the set of all compact subsets of $\mathcal{X}$, and the set of all convex subsets of $\mathcal{X}$ are represented by $\mathcal{P}(\mathcal{X})$, $\mathcal{P}_{\mathrm{cl}}(\mathcal{X})$, $\mathcal{P}_b(\mathcal{X})$, $\mathcal{P}_{\mathrm{cp}}(\mathcal{X})$, and $\mathcal{P}_{\mathrm{cv}}(\mathcal{X})$, respectively. We say that $k^* \in \mathcal{X}$ is a fixed point for the set-valued map $\mathcal{S} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ if $k^* \in \mathcal{S}(k^*)$ [43]. The set of all fixed points of the set-valued map $\mathcal{S}$ is denoted by $\mathcal{FIX}(\mathcal{S})$ [43]. The Pompeiu–Hausdorff metric $\mathrm{PH}_d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\mathrm{PH}_{d_{\mathcal{X}}}(A_1, A_2) = \max\left\{ \sup_{a_1 \in A_1} d_{\mathcal{X}}(a_1, A_2), \sup_{a_2 \in A_2} d_{\mathcal{X}}(A_1, a_2) \right\},$$

where $d_{\mathcal{X}}(A_1, a_2) = \inf_{a_1 \in A_1} d_{\mathcal{X}}(a_1, a_2)$ and $d_{\mathcal{X}}(a_1, A_2) = \inf_{a_2 \in A_2} d_{\mathcal{X}}(a_1, a_2)$ [43]. A set-valued map $\mathcal{S} : \mathcal{X} \to \mathcal{P}_{\mathrm{cl}}(\mathcal{X})$ is said to be Lipschitz with constant $\lambda^* > 0$ whenever we have $\mathrm{PH}_{d_{\mathcal{X}}}(\mathcal{S}(k_1), \mathcal{S}(k_2)) \le \lambda^* d_{\mathcal{X}}(k_1, k_2)$ for all $k_1, k_2 \in \mathcal{X}$. A Lipschitz map $\mathcal{S}$ is called contraction whenever $\lambda^* \in (0, 1)$ [43]. We say that the set-valued map $\mathcal{S}$ is completely continuous whenever the set $\mathcal{S}(W)$ is relatively compact for every $W \in \mathcal{P}_b(\mathcal{X})$. A set-valued map $\mathcal{S} : [0, 1] \to \mathcal{P}_{\mathrm{cl}}(\mathbb{R})$ is said to be measurable if the function $t \longmapsto d_{\mathcal{X}}(\upsilon, \mathcal{S}(t))$ is measurable for all $\upsilon \in \mathbb{R}$ [43, 44]. We say that the set-valued map $\mathcal{S}$ is upper semi-continuous (u.s.c.) whenever, for each $k^* \in \mathcal{X}$, the set $\mathcal{S}(k^*)$ belongs to $\mathcal{P}_{\mathrm{cl}}(\mathcal{X})$, and for every open set $\mathcal{V}$ containing $\mathcal{S}(k^*)$, there exists an open neighborhood $\mathcal{U}_0^*$ of k^* such that $\mathcal{S}(\mathcal{U}_0^*) \subseteq \mathcal{V}$ [43]. The graph of the set-valued map $\mathcal{S} : \mathcal{X} \to \mathcal{P}_{\mathrm{cl}}(\mathcal{Y})$ is defined by $\mathrm{Graph}(\mathcal{S}) = \{(k, s) \in \mathcal{X} \times \mathcal{Y} : s \in \mathcal{S}(k)\}$. We say that graph of $\mathcal{S}$ is a closed set if, for each sequence $\{k_n\}_{n \ge 1}$ in $\mathcal{X}$ and $\{s_n\}_{n \ge 1}$ in $\mathcal{Y}$, $k_n \to k_0$, $s_n \to s_0$ and $s_n \in \mathcal{S}(k_n)$, we have $s_0 \in \mathcal{S}(k_0)$ [43, 44]. Suppose that the set-valued map $\mathcal{S} : \mathcal{X} \to \mathcal{P}_{\mathrm{cl}}(\mathcal{Y})$ is upper semi-continuous. Then $\mathrm{Graph}(\mathcal{S})$ is a subset of the product space $\mathcal{X} \times \mathcal{Y}$ which is a closed set. Conversely, if the set-valued map $\mathcal{S}$ is completely continuous and has a closed graph, then $\mathcal{S}$ is upper semi-continuous ([43], Proposition 2.1). A set-valued map $\mathcal{S}$ is convex-valued if $\mathcal{S}(k)$ is a convex set for each element $k \in \mathcal{X}$. A set of selections of set-valued map $\mathcal{S}$ at point $k \in C([0, 1], \mathbb{R})$ is defined by

$$(\mathcal{SEL})_{\mathcal{S},k} := \left\{ \vartheta \in \mathcal{L}^1([0, 1], \mathbb{R}) : \vartheta(t) \in \mathcal{S}(t, k(t)) \right\}$$

for almost all $t \in [0,1]$ [43, 44]. If $\mathcal{S}$ is an arbitrary set-valued map, then for each function $k \in C([0,1],\mathcal{X})$, we have $(\mathcal{SEL})_{\mathcal{S},k} \neq \emptyset$ whenever $\dim \mathcal{X} < \infty$ [43]. A set-valued map $\mathcal{S} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called Caratheodory whenever $t \mapsto \mathcal{S}(t,k)$ is a measurable mapping for each function $k \in \mathbb{R}$ and $k \mapsto \mathcal{S}(t,k)$ is an upper semi-continuous mapping for almost all $t \in [0,1]$ [43, 44]. Moreover, a Caratheodory set-valued map $\mathcal{S} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be $\mathcal{L}^1$-Caratheodory whenever, for each constant $\mu > 0$, there exists a function $\phi_\mu \in \mathcal{L}^1([0,1],\mathbb{R}^+)$ such that $\|\mathcal{S}(t,k)\| = \sup_{t\in[0,1]}\{|q| : q \in \mathcal{S}(t,k)\} \leq \phi_\mu(t)$ for all $|k| \leq \mu$ and for almost all $t \in [0,1]$ [43, 44]. We need the next results.

Theorem 1 ([45]) *Suppose that $\mathcal{X}$ is a separable Banach space, $\mathcal{S} : [0,1] \times \mathcal{X} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathcal{X})$ is an $\mathcal{L}^1$-Carathéodory set-valued map, and $\Xi : \mathcal{L}^1([0,1],\mathcal{X}) \to C([0,1],\mathcal{X})$ is a linear continuous mapping. Then the composition $\Xi \circ (\mathcal{SEL})_{\mathcal{S}} : C([0,1],\mathcal{X}) \to \mathcal{P}_{\mathrm{cp,cv}}(C([0,1],\mathcal{X}))$ is an operator in the product space $C([0,1],\mathcal{X}) \times C([0,1],\mathcal{X})$ with action $k \mapsto (\Xi \circ (\mathcal{SEL})_{\mathcal{S}})(k) = \Xi((\mathcal{SEL})_{\mathcal{S},k})$ having the closed graph property.*

Theorem 2 ([46]) *Let $\mathcal{X}$ be a Banach algebra. Assume that there exist a single-valued map $\Phi_1 : \mathcal{X} \to \mathcal{X}$ and a set-valued map $\Phi_2 : \mathcal{X} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathcal{X})$ such that*

(i) *Φ_1 is an operator including the Lipschitzian property with a Lipschitz constant l^*;*

(ii) *Φ_2 is an operator including upper semi-continuity and the compactness property;*

(iii) *$2l^*\hat{\Delta} < 1$ such that $\hat{\Delta} = \|\Phi_2(\mathcal{X})\|$.*

Then either the set $\mathcal{O}^* = \{v^* \in \mathcal{X} \mid \alpha_0 v^* \in \Phi_1 v^* \Phi_2 v^*, \alpha_0 > 1\}$ is unbounded or there is a solution in $\mathcal{X}$ for the operator inclusion $k \in \Phi_1 k \Phi_2 k$.

3 Main results

Now, we are ready to study the fractional hybrid multi-term inclusion problem (1)–(2). Consider the Banach space $\mathcal{X} = \{k(t) : k(t) \in C_{\mathbb{R}}([0,1])\}$ with the norm $\|k\|_{\mathcal{X}} = \sup_{t\in[0,1]} |k(t)|$. For convenience, consider the constants

$$
\begin{aligned}
&\Lambda_0 = \frac{2\lambda_1\eta_1\sum_{j=1}^{r} b_j - \lambda_1\eta_2^2 - 2\lambda_2\sum_{j=1}^{r} b_j}{\lambda_1}, \qquad \Lambda_5 = \frac{1}{\sum_{j=1}^{r} b_j - \eta_2},\\
&\Lambda_1 = \big(6\eta_2(\lambda_1\eta_1 + \lambda_2) + \lambda_1(2 - 3\eta_1)\big)\sum_{j=1}^{r} b_j, \qquad \Lambda_6 = \frac{\lambda_1\eta_2^2 + 2\lambda_2\sum_{j=1}^{r} b_j}{\lambda_1},\\
&\Lambda_2 = \left(3\lambda_1\eta_2^2 + 6\lambda_2\sum_{j=1}^{r} b_j\right)\left(1 - 2\sum_{j=1}^{r} b_j\right) - 2\lambda_1\eta_2, \qquad \Lambda_7 = \lambda_1(\Lambda_6 - \eta_2\Lambda_0\Lambda_5),\\
&\Lambda_3 = \frac{3(2\eta_2 - 1)}{\Lambda_1 + \Lambda_2}, \qquad \Lambda_8 = \big[\lambda_1(\Lambda_3 + \Lambda_4)[\eta_2\Lambda_0\Lambda_5 - \Lambda_6] - (\eta_2\Lambda_5 + 1)\big].\\
&\Lambda_4 = \frac{6(\sum_{j=1}^{r} b_j - \eta_2)}{\Lambda_1 + \Lambda_2}.
\end{aligned}
\tag{3}
$$

Here, we prove our first key result.

Lemma 3 *Let $z \in \mathcal{X}$. Then k_0 is a solution for the fractional hybrid differential equation*

$$
{}^{C}\mathcal{D}_0^{\omega}\left(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\,\mathrm{d}s)}\right) = z(t), \quad \big(t \in [0,1], \omega \in (2,3]\big) \tag{4}
$$

with four-point hybrid integral boundary value conditions

$$\begin{cases} (\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\,\mathrm{d}s)})|_{t=0} + \sum_{j=1}^{r} b_j \mathcal{D}_0(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\mathrm{d}s)})|_{t=\eta_1} = 0, \\ \lambda_1(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\mathrm{d}s)})|_{t=\eta_2} + \lambda_2 \sum_{j=1}^{r} b_j \mathcal{D}_0^2(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\mathrm{d}s)})|_{t=1} = 0, \\ \lambda_3 \int_0^1 (\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\mathrm{d}s)})\mathrm{d}s = 0, \end{cases} \tag{5}$$

if and only if k_0 is a solution for the integral equation

$$\begin{aligned} k(t) = {} & \xi\left(t, k(t), \int_0^1 k(s)\,\mathrm{d}s\right)\Bigg[\frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1} z(s)\,\mathrm{d}s \\ & + \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3]\sum_{j=1}^{r} b_j}{\Gamma(\omega-1)} \\ & \times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} z(s)\,\mathrm{d}s \\ & + \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)}\int_0^{\eta_2} (\eta_2 - s)^{\omega-1} z(s)\,\mathrm{d}s \\ & + \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)]\sum_{j=1}^{r} b_j}{\lambda_1\Gamma(\omega-2)}\int_0^1 (1-s)^{\omega-3} z(s)\,\mathrm{d}s \\ & + \frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)}\int_0^1\int_0^s (s-\tau)^{\omega-1} z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg], \end{aligned} \tag{6}$$

where $\Lambda_0, \ldots, \Lambda_8$ are given in (3).

Proof Assume that k_0 is a solution for hybrid equation (4). Then there exist constants $m_0^*, m_1^*, m_2^* \in \mathbb{R}$ such that $\frac{k_0(t)}{\xi(t,k_0(t),\int_0^1 k_0(s)\,\mathrm{d}s)} = \mathcal{I}_0^{\omega} z(t) + m_0^* + m_1^* t + m_2^* t^2$. Hence,

$$k_0(t) = \xi\left(t, k_0(t), \int_0^1 k_0(s)\,\mathrm{d}s\right)\left[\int_0^t \frac{(t-s)^{\omega-1}}{\Gamma(\omega)} z(s)\,\mathrm{d}s + m_0^* + m_1^* t + m_2^* t^2\right], \tag{7}$$

and so

$$\begin{aligned} & \mathcal{D}_0\left(\frac{k_0(t)}{\xi(t,k_0(t),\int_0^1 k_0(s)\,\mathrm{d}s)}\right) = \int_0^t \frac{(t-s)^{\omega-2}}{\Gamma(\omega-1)} z(s)\,\mathrm{d}s + m_1^* + 2m_2^* t, \\ & \mathcal{D}_0^2\left(\frac{k_0(t)}{\xi(t,k_0(t),\int_0^1 k_0(s)\,\mathrm{d}s)}\right) = \int_0^t \frac{(t-s)^{\omega-3}}{\Gamma(\omega-2)} z(s)\,\mathrm{d}s + 2m_2^*, \\ & \int_0^1 \left(\frac{k_0(t)}{\xi(t,k_0(t),\int_0^1 k_0(s)\,\mathrm{d}s)}\right)\mathrm{d}s \\ & \quad = \int_0^1\int_0^s \frac{(s-\tau)^{\omega-1}}{\Gamma(\omega)} z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + m_0^* + \frac{1}{2}m_1^* + \frac{1}{3}m_2^*. \end{aligned}$$

By using the four-point hybrid boundary value conditions, we obtain

$$\begin{aligned}
m_0^* &= \frac{(\Lambda_3\Lambda_7 - \eta_2\Lambda_5)\sum_{j=1}^{r} b_j}{\Gamma(\omega-1)} \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} z(s)\,\mathrm{d}s \\
&\quad - \frac{\Lambda_7(\Lambda_3+\Lambda_4) + \eta_2\Lambda_5 + 1}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} z(s)\,\mathrm{d}s \\
&\quad + \frac{\lambda_2\Lambda_8 \sum_{j=1}^{r} b_j}{\lambda_1\Gamma(\omega-2)} \int_0^1 (1-s)^{\omega-3} z(s)\,\mathrm{d}s \\
&\quad - \frac{\lambda_1\Lambda_4(\eta_2\Lambda_0\Lambda_5 + \Lambda_6)}{\Gamma(\omega)} \int_0^1 \int_0^s (s-\tau)^{\omega-1} z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s, \\
m_1^* &= \frac{(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 \sum_{j=1}^{r} b_j}{\Gamma(\omega-1)} \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} z(s)\,\mathrm{d}s \\
&\quad + \frac{[1 - \lambda_1\Lambda_0(\Lambda_3+\Lambda_4)]\Lambda_5}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} z(s)\,\mathrm{d}s \\
&\quad + \frac{\lambda_2[1 - \lambda_1\Lambda_0(\Lambda_3+\Lambda_4)]\Lambda_5 \sum_{j=1}^{r} b_j}{\lambda_1\Gamma(\omega-2)} \int_0^1 (1-s)^{\omega-3} z(s)\,\mathrm{d}s \\
&\quad + \frac{\lambda_1\Lambda_0\Lambda_4\Lambda_5}{\Gamma(\omega)} \int_0^1 \int_0^s (s-\tau)^{\omega-1} z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
m_2^* &= -\frac{\lambda_1\Lambda_3 \sum_{j=1}^{r} b_j}{\Gamma(\omega-1)} \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} z(s)\,\mathrm{d}s \\
&\quad + \frac{\lambda_1(\Lambda_3+\Lambda_4)}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} z(s)\,\mathrm{d}s \\
&\quad + \frac{\lambda_2(\Lambda_3+\Lambda_4) \sum_{j=1}^{r} b_j}{\Gamma(\omega-2)} \int_0^1 (1-s)^{\omega-3} z(s)\,\mathrm{d}s \\
&\quad - \frac{\lambda_1\Lambda_4}{\Gamma(\omega)} \int_0^1 \int_0^s (s-\tau)^{\omega-1} z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.
\end{aligned}$$

By substituting the values m_0^*, m_1^*, and m_2^* in (7), we get

$$\begin{aligned}
k_0(t) &= \xi\left(t, k_0(t), \int_0^1 k_0(s)\,\mathrm{d}s\right)\Bigg[\frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} z(s)\,\mathrm{d}s \\
&\quad + \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3]\sum_{j=1}^{r} b_j}{\Gamma(\omega-1)} \\
&\quad \times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} z(s)\,\mathrm{d}s \\
&\quad + \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3+\Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} z(s)\,\mathrm{d}s \\
&\quad + \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3+\Lambda_4)]\sum_{j=1}^{r} b_j}{\lambda_1\Gamma(\omega-2)} \int_0^1 (1-s)^{\omega-3} z(s)\,\mathrm{d}s \\
&\quad + \frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)} \int_0^1 \int_0^s (s-\tau)^{\omega-1} z(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg].
\end{aligned}$$

This shows that the function k_0 is a solution for integral equation (6). Conversely, one can easily check that k_0 is a solution for problem (4)–(5) whenever k_0 is a solution function for integral equation (6). □

Definition 4 An absolutely continuous function $k : [0,1] \to \mathbb{R}$ is called a solution for the fractional hybrid inclusion problem (1)–(2) whenever there exists an integrable function $\vartheta \in \mathcal{L}^1([0,1],\mathbb{R})$ with $\vartheta(t) \in \mathcal{S}(t,k(t),\phi_1(k(t)),\dots,\phi_m(k(t)))$ for almost all $t \in [0,1]$ satisfying the four-point fractional hybrid sum and integral boundary value conditions

$$\begin{cases} (\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\,\mathrm{d}s)})|_{t=0} + \sum_{j=1}^{r} b_j \mathcal{D}_0(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\mathrm{d}s)})|_{t=\eta_1} = 0, \\ \lambda_1(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\mathrm{d}s)})|_{t=\eta_2} + \lambda_2 \sum_{j=1}^{r} b_j \mathcal{D}_0^2(\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\mathrm{d}s)})|_{t=1} = 0, \\ \lambda_3 \int_0^1 (\frac{k(t)}{\xi(t,k(t),\int_0^1 k(s)\mathrm{d}s)})\mathrm{d}s = 0, \end{cases}$$

and

$$\begin{aligned} k(t) = {} & \xi\left(t,k(t),\int_0^1 k(s)\,\mathrm{d}s\right)\Bigg[\frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ & + \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3]\sum_{j=1}^{r} b_j}{\Gamma(\omega-1)} \\ & \times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2}\vartheta(s)\,\mathrm{d}s \\ & + \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)}\int_0^{\eta_2} (\eta_2 - s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ & + \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)]\sum_{j=1}^{r} b_j}{\lambda_1\Gamma(\omega-2)}\int_0^1 (1-s)^{\omega-3}\vartheta(s)\,\mathrm{d}s \\ & + \frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)}\int_0^1\int_0^s (s-\tau)^{\omega-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg] \end{aligned}$$

for all $t \in [0,1]$.

Now, we provide our main result.

Theorem 5 *Suppose that* $\xi : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ *is a continuous function and* $\mathcal{S} : [0,1] \times \mathbb{R}^{m+1} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R})$ *is a set-valued map. Assume that*

($\mathcal{C}1$) *there exists a bounded mapping* $\theta : [0,1] \to \mathbb{R}^+$ *such that*

$$\left|\xi\left(t,k_1(t),k_2(t)\right) - \xi\left(t,k_1'(t),k_2'(t)\right)\right| \le \theta(t)\sum_{i=1}^{2}\left|k_i(t) - k_i'(t)\right|$$

for all $k_1,k_2,k_1',k_2' \in \mathbb{R}$, *and* $t \in [0,1]$;

($\mathcal{C}2$) *the set-valued map* $\mathcal{S} : [0,1] \times \mathbb{R}^{m+1} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R})$ *has the* $\mathcal{L}^1$*-Caratheodory property*;

($\mathcal{C}3$) *there exists a positive mapping* $q(t) \in \mathcal{L}^1([0,1],\mathbb{R}^+)$ *such that*

$$\left\|\mathcal{S}(t,k_1,k_2,\dots,k_{m+1})\right\| = \sup\left\{|\vartheta| : \vartheta \in \mathcal{S}\left(t,k_1(t),k_2(t),\dots,k_{m+1}(t)\right)\right\} \le q(t)$$

for all $k_1,\dots,k_{m+1} \in \mathbb{R}$ *and for almost all* $t \in [0,1]$;

(C4) *there exists a positive real number* $\tilde{\rho} \in \mathbb{R}$ *such that*

$$\tilde{\rho} > \frac{\xi^* M \|q\|_{\mathcal{L}^1}}{1 - 2\theta^* M \|q\|_{\mathcal{L}^1}}, \tag{8}$$

where $\|q\|_{\mathcal{L}^1} = \int_0^1 |q(s)|\,\mathrm{d}s$, $\xi^* = \sup_{t\in[0,1]} |\xi(t,0,0)|$, $\theta^* = \sup_{t\in[0,1]} |\theta(t)|$ *and*

$$\begin{aligned} M = {} & \frac{1}{\Gamma(\omega+1)} \\ & + \frac{[|\Lambda_3\Lambda_7| + |\eta_2\Lambda_5| + (|\lambda_1\Lambda_0\Lambda_3| + 1)|\Lambda_5| + |\lambda_1\Lambda_3|]\eta_1^{\omega-1}\sum_{j=1}^r b_j}{\Gamma(\omega)} \\ & + \frac{[1 + |(\Lambda_8+\eta_2)\Lambda_5| + |(\Lambda_3+\Lambda_4)|(\lambda_1 + |\Lambda_7|)]\eta_2^{\omega}}{\Gamma(\omega+1)} \\ & + \frac{\lambda_2[|(1+\Lambda_5)\Lambda_8| + \lambda_1|\Lambda_3+\Lambda_4|]\sum_{j=1}^r b_j}{\lambda_1\Gamma(\omega-1)} \\ & + \frac{\lambda_1|\Lambda_4|[(1+\eta_2)|\Lambda_0\Lambda_5| + |\Lambda_6| + 1]}{\Gamma(\omega+2)}. \end{aligned} \tag{9}$$

Then the hybrid inclusion problem (1)–(2) *has a solution whenever* $4\theta^* M\|q\|_{\mathcal{L}^1} < 1$.

Proof For each $k \in \mathcal{X}$, define the set of selections of the operator $\mathcal{S}$ by

$$(\mathcal{SEL})_{\mathcal{S},k} = \big\{\vartheta \in \mathcal{L}^1\big([0,1]\big) : \vartheta(t) \in \mathcal{S}\big(t, k(t), \phi_1\big(k(t)\big), \ldots, \phi_m\big(k(t)\big)\big)\big\}$$

for almost all $t \in [0,1]$. Define $\mathcal{G} : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by

$$\mathcal{G}(k) = \big\{g \in \mathcal{X} : g(t) = a(t) \text{ for } t \in [0,1]\big\},$$

where

$$\begin{aligned} a(t) = {} & \xi\left(t, k(t), \int_0^1 k(s)\,\mathrm{d}s\right)\Bigg[\frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ & + \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3]\sum_{j=1}^r b_j}{\Gamma(\omega-1)}\int_0^{\eta_1}(\eta_1 - s)^{\omega-2}\vartheta(s)\,\mathrm{d}s \\ & + \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3+\Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)}\int_0^{\eta_2}(\eta_2 - s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\ & + \frac{\lambda_2[(1+t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3+\Lambda_4)]\sum_{j=1}^r b_j}{\lambda_1\Gamma(\omega-2)}\int_0^1 (1-s)^{\omega-3}\vartheta(s)\,\mathrm{d}s \\ & + \frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)}\int_0^1\int_0^s (s-\tau)^{\omega-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg], \end{aligned}$$

for some $\vartheta \in (\mathcal{SEL})_{\mathcal{S},k}$. One can easily check that g_0 is a solution for the hybrid inclusion problem (1)–(2) if and only if g_0 is a fixed point of the operator $\mathcal{G}$. Define the maps $\Phi_1 : \mathcal{X} \to \mathcal{X}$ by $(\Phi_1 k)(t) = \xi(t, k(t), \int_0^1 k(s)\,\mathrm{d}s)$ and $\Phi_2 : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by

$$(\Phi_2 k)(t) = \big\{\zeta \in \mathcal{X} : \zeta(t) = b(t) \text{ for } t \in [0,1]\big\},$$

where

$$
\begin{aligned}
b(t) = {} & \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \vartheta(s)\,\mathrm{d}s \\
& + \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3] \sum_{j=1}^r b_j}{\Gamma(\omega-1)} \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} \vartheta(s)\,\mathrm{d}s \\
& + \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} \vartheta(s)\,\mathrm{d}s \\
& + \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)] \sum_{j=1}^r b_j}{\lambda_1\Gamma(\omega-2)} \int_0^1 (1-s)^{\omega-3} \vartheta(s)\,\mathrm{d}s \\
& + \frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)} \int_0^1 \int_0^s (s-\tau)^{\omega-1} \vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}
$$

for some $\vartheta \in (\mathcal{SEL})_{\mathcal{S},k}$. Then we obtain $\mathcal{G}(k) = \Phi_1 k \Phi_2 k$. We prove that Φ_1 and Φ_2 satisfy the assumptions of Theorem 2. We first show that the operator Φ_1 is Lipschitz. Let $k_1, k_2 \in \mathcal{X}$. Assumption $(\mathcal{C}1)$ implies that

$$
\begin{aligned}
\big|(\Phi_1 k_1)(t) - (\Phi_1 k_2)(t)\big| &= \left|\xi\left(t, k_1(t), \int_0^1 k_1(s)\,\mathrm{d}s\right) - \xi\left(t, k_2(t), \int_0^1 k_2(s)\,\mathrm{d}s\right)\right| \\
&\le \theta(t)\big(\big|k_1(t) - k_2(t)\big| + \big|k_1(t) - k_2(t)\big|\big) \\
&= 2\theta(t)\big|k_1(t) - k_2(t)\big|
\end{aligned}
$$

for all $t \in [0,1]$. Hence, we get $\|\Phi_1 k_1 - \Phi_1 k_2\|_{\mathcal{X}} \le 2\theta^* \|k_1 - k_2\|_{\mathcal{X}}$ for all $k_1, k_2 \in \mathcal{X}$. This means that the operator Φ_1 is Lipschitz with constant $2\theta^*$. Now, we claim that the set-valued map Φ_2 has convex values. Let $k_1, k_2 \in \Phi_2 k$. Choose ϑ_1, ϑ_2 in $(\mathcal{SEL})_{\mathcal{S},k}$ such that

$$
\begin{aligned}
k_i(t) = {} & \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \vartheta_i(s)\,\mathrm{d}s \\
& + \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3] \sum_{j=1}^r b_j}{\Gamma(\omega-1)} \\
& \times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} \vartheta_i(s)\,\mathrm{d}s \\
& + \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} \vartheta_i(s)\,\mathrm{d}s \\
& + \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)] \sum_{j=1}^r b_j}{\lambda_1\Gamma(\omega-2)} \int_0^1 (1-s)^{\omega-3} \vartheta_i(s)\,\mathrm{d}s \\
& + \frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)} \int_0^1 \int_0^s (s-\tau)^{\omega-1} \vartheta_i(\tau)\,\mathrm{d}\tau\,\mathrm{d}s, \quad (i = 1,2)
\end{aligned}
$$

for almost all $t \in [0,1]$. Let $\lambda \in (0,1)$. Then we have

$$
\begin{aligned}
\lambda k_1(t) + (1-\lambda) k_2(t) = {} & \frac{1}{\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} \big[\lambda\vartheta_1(s) + (1-\lambda)\vartheta_2(s)\big]\,\mathrm{d}s \\
& + \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3] \sum_{j=1}^r b_j}{\Gamma(\omega-1)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2}\big[\lambda\vartheta_1(s) + (1-\lambda)\vartheta_2(s)\big]\,\mathrm{d}s \\
&+ \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)} \\
&\times \int_0^{\eta_2} (\eta_2 - s)^{\omega-1}\big[\lambda\vartheta_1(s) + (1-\lambda)\vartheta_2(s)\big]\,\mathrm{d}s \\
&+ \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)]\sum_{j=1}^{r} b_j}{\lambda_1\Gamma(\omega-2)} \\
&\times \int_0^{1} (1 - s)^{\omega-3}\big[\lambda\vartheta_1(s) + (1-\lambda)\vartheta_2(s)\big]\,\mathrm{d}s \\
&+ \frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)} \\
&\times \int_0^{1}\int_0^{s} (s-\tau)^{\omega-1}\big[\lambda\vartheta_1(\tau) + (1-\lambda)\vartheta_2(\tau)\big]\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}
$$

for almost all $t \in [0,1]$. Since $\mathcal{S}$ has convex values, $(\mathcal{SEL})_{\mathcal{S},k}$ is convex-valued. This gives that $\lambda\vartheta_1(t) + (1-\lambda)\vartheta_2(t) \in (\mathcal{SEL})_{\mathcal{S},k}$ for all $t \in [0,1]$, and so $\Phi_2 k$ is a convex set for all $k \in \mathcal{X}$.

Now, we prove that the operator Φ_2 is completely continuous. We have to prove the equi-continuity and uniform boundedness of the set $\Phi_2(\mathcal{X})$. First, we show that Φ_2 maps all bounded sets into bounded subsets of $\mathcal{X}$. For a positive number $\varepsilon^* \in \mathbb{R}$, consider the bounded ball $\mathcal{V}_{\varepsilon^*} = \{k \in \mathcal{X} : \|k\|_{\mathcal{X}} \le \varepsilon^*\}$. For every $k \in \mathcal{V}_{\varepsilon^*}$ and $\zeta \in \Phi_2 k$, there exists a function $\vartheta \in (\mathcal{SEL})_{\mathcal{S},k}$ so that

$$
\begin{aligned}
\zeta(t) = {}& \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\
&+ \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3]\sum_{j=1}^{r} b_j}{\Gamma(\omega-1)} \\
&\times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2}\vartheta(s)\,\mathrm{d}s \\
&+ \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)}\int_0^{\eta_2} (\eta_2 - s)^{\omega-1}\vartheta(s)\,\mathrm{d}s \\
&+ \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)]\sum_{j=1}^{r} b_j}{\lambda_1\Gamma(\omega-2)}\int_0^{1} (1 - s)^{\omega-3}\vartheta(s)\,\mathrm{d}s \\
&+ \frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)}\int_0^{1}\int_0^{s} (s-\tau)^{\omega-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}
$$

for all $t \in [0,1]$. Then we have

$$
\begin{aligned}
\big|\zeta(t)\big| \le {}& \frac{1}{\Gamma(\omega)}\int_0^t (t-s)^{\omega-1}\big|\vartheta(s)\big|\,\mathrm{d}s \\
&+ \frac{|(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3|\sum_{j=1}^{r} b_j}{\Gamma(\omega-1)} \\
&\times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2}\big|\vartheta(s)\big|\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|1+(t\Lambda_8+\eta_2)\Lambda_5+(\Lambda_3+\Lambda_4)(t^2\lambda_1-\Lambda_7)|}{\Gamma(\omega)}\int_0^{\eta_2}(\eta_2-s)^{\omega-1}\big|\vartheta(s)\big|\,\mathrm{d}s\\
&+\frac{\lambda_2|(1+t\Lambda_5)\Lambda_8+t^2\lambda_1(\Lambda_3+\Lambda_4)|\sum_{j=1}^{r}b_j}{\lambda_1\Gamma(\omega-2)}\\
&\times\int_0^1(1-s)^{\omega-3}\big|\vartheta(s)\big|\,\mathrm{d}s\\
&+\frac{\lambda_1|\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5-t^2-\Lambda_6]|}{\Gamma(\omega)}\\
&\times\int_0^1\int_0^s(s-\tau)^{\omega-1}\big|\vartheta(\tau)\big|\,\mathrm{d}\tau\,\mathrm{d}s\le\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}q(s)\,\mathrm{d}s\\
&+\frac{|(\Lambda_3\Lambda_7-\eta_2\Lambda_5)+t(\lambda_1\Lambda_0\Lambda_3-1)\Lambda_5-t^2\lambda_1\Lambda_3|\sum_{j=1}^{r}b_j}{\Gamma(\omega-1)}\\
&\times\int_0^{\eta_1}(\eta_1-s)^{\omega-2}q(s)\,\mathrm{d}s\\
&+\frac{|1+(t\Lambda_8+\eta_2)\Lambda_5+(\Lambda_3+\Lambda_4)(t^2\lambda_1-\Lambda_7)|}{\Gamma(\omega)}\int_0^{\eta_2}(\eta_2-s)^{\omega-1}q(s)\,\mathrm{d}s\\
&+\frac{\lambda_2|(1+t\Lambda_5)\Lambda_8+t^2\lambda_1(\Lambda_3+\Lambda_4)|\sum_{j=1}^{r}b_j}{\lambda_1\Gamma(\omega-2)}\int_0^1(1-s)^{\omega-3}q(s)\,\mathrm{d}s\\
&+\frac{\lambda_1|\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5-t^2-\Lambda_6]|}{\Gamma(\omega)}\int_0^1\int_0^s(s-\tau)^{\omega-1}q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\\
\le{}&\left[\frac{1}{\Gamma(\omega+1)}\right.\\
&+\frac{[|\Lambda_3\Lambda_7|+|\eta_2\Lambda_5|+(|\lambda_1\Lambda_0\Lambda_3|+1)|\Lambda_5|+|\lambda_1\Lambda_3|]\eta_1^{\omega-1}\sum_{j=1}^{r}b_j}{\Gamma(\omega)}\\
&+\frac{[1+|(\Lambda_8+\eta_2)\Lambda_5|+|(\Lambda_3+\Lambda_4)|(\lambda_1+|\Lambda_7|)]\eta_2^{\omega}}{\Gamma(\omega+1)}\\
&+\frac{\lambda_2[|(1+\Lambda_5)\Lambda_8|+\lambda_1|\Lambda_3+\Lambda_4|]\sum_{j=1}^{r}b_j}{\lambda_1\Gamma(\omega-1)}\\
&\left.+\frac{\lambda_1|\Lambda_4|[(1+\eta_2)|\Lambda_0\Lambda_5|+|\Lambda_6|+1]}{\Gamma(\omega+2)}\right]\|q\|_{\mathcal{L}^1}\\
={}&M\|q\|_{\mathcal{L}^1},
\end{aligned}
$$

where M is given in (9). Thus, $\|\zeta\|\le M\|q\|_{\mathcal{L}^1}$ and this shows that the set $\Phi_2(\mathcal{X})$ is uniformly bounded. Next, we prove that the operator Φ_2 maps bounded sets into equicontinuous sets. Let $k\in V_{\varepsilon^*}$ and $\zeta\in\Phi_2 k$. Choose $\vartheta\in(\mathcal{SEL})_{\mathcal{S},k}$ such that

$$
\begin{aligned}
\zeta(t)={}&\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s\\
&+\frac{[(\Lambda_3\Lambda_7-\eta_2\Lambda_5)+t(\lambda_1\Lambda_0\Lambda_3-1)\Lambda_5-t^2\lambda_1\Lambda_3]\sum_{j=1}^{r}b_j}{\Gamma(\omega-1)}\\
&\times\int_0^{\eta_1}(\eta_1-s)^{\omega-2}\vartheta(s)\,\mathrm{d}s
\end{aligned}
$$

$$+\frac{1+(t\Lambda_8+\eta_2)\Lambda_5+(\Lambda_3+\Lambda_4)(t^2\lambda_1-\Lambda_7)}{\Gamma(\omega)}\int_0^{\eta_2}(\eta_2-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s$$
$$+\frac{\lambda_2[(1+t\Lambda_5)\Lambda_8+t^2\lambda_1(\Lambda_3+\Lambda_4)]\sum_{j=1}^{r}b_j}{\lambda_1\Gamma(\omega-2)}\int_0^1(1-s)^{\omega-3}\vartheta(s)\,\mathrm{d}s$$
$$+\frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5-t^2-\Lambda_6]}{\Gamma(\omega)}\int_0^1\int_0^s(s-\tau)^{\omega-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

for all $t\in[0,1]$. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$. Then we have

$$\begin{aligned}
\big|\zeta(t_2)-\zeta(t_1)\big| &\le \frac{1}{\Gamma(\omega)}\int_0^{t_1}\big[(t_2-s)^{\omega-1}-(t_1-s)^{\omega-1}\big]\big|\vartheta(s)\big|\,\mathrm{d}s\\
&\quad+\frac{1}{\Gamma(\omega)}\int_{t_1}^{t_2}(t_2-s)^{\omega-1}\big|\vartheta(s)\big|\,\mathrm{d}s\\
&\quad+\frac{[|(t_2-t_1)(\lambda_1\Lambda_0\Lambda_3-1)\Lambda_5|+|(t_2^2-t_1^2)\lambda_1\Lambda_3|]\sum_{j=1}^{r}b_j}{\Gamma(\omega-1)}\\
&\quad\times\int_0^{\eta_1}(\eta_1-s)^{\omega-2}\big|\vartheta(s)\big|\,\mathrm{d}s\\
&\quad+\frac{|(t_2-t_1)\Lambda_8\Lambda_5|+|\lambda_1(\Lambda_3+\Lambda_4)(t_2^2-t_1^2)|}{\Gamma(\omega)}\int_0^{\eta_2}(\eta_2-s)^{\omega-1}\big|\vartheta(s)\big|\,\mathrm{d}s\\
&\quad+\frac{\lambda_2[|(t_2-t_1)\Lambda_5\Lambda_8|+|(t_2^2-t_1^2)\lambda_1(\Lambda_3+\Lambda_4)|]\sum_{j=1}^{r}b_j}{\lambda_1\Gamma(\omega-2)}\\
&\quad\times\int_0^1(1-s)^{\omega-3}\big|\vartheta(s)\big|\,\mathrm{d}s\\
&\quad+\frac{\lambda_1|\Lambda_4|[|(t_2-t_1)\Lambda_0\Lambda_5|+|t_2^2-t_1^2|]}{\Gamma(\omega)}\int_0^1\int_0^s(s-\tau)^{\omega-1}\big|\vartheta(\tau)\big|\,\mathrm{d}\tau\,\mathrm{d}s\\
&\le \frac{1}{\Gamma(\omega)}\int_0^{t_1}\big[(t_2-s)^{\omega-1}-(t_1-s)^{\omega-1}\big]q(s)\,\mathrm{d}s+\frac{1}{\Gamma(\omega)}\\
&\quad\times\int_{t_1}^{t_2}(t_2-s)^{\omega-1}q(s)\,\mathrm{d}s\\
&\quad+\frac{[|(t_2-t_1)(\lambda_1\Lambda_0\Lambda_3-1)\Lambda_5|+|(t_2^2-t_1^2)\lambda_1\Lambda_3|]\sum_{j=1}^{r}b_j}{\Gamma(\omega-1)}\\
&\quad\times\int_0^{\eta_1}(\eta_1-s)^{\omega-2}q(s)\,\mathrm{d}s\\
&\quad+\frac{|(t_2-t_1)\Lambda_8\Lambda_5|+|\lambda_1(\Lambda_3+\Lambda_4)(t_2^2-t_1^2)|}{\Gamma(\omega)}\\
&\quad\times\int_0^{\eta_2}(\eta_2-s)^{\omega-1}q(s)\,\mathrm{d}s\\
&\quad+\frac{\lambda_2[|(t_2-t_1)\Lambda_5\Lambda_8|+|(t_2^2-t_1^2)\lambda_1(\Lambda_3+\Lambda_4)|]\sum_{j=1}^{r}b_j}{\lambda_1\Gamma(\omega-2)}\\
&\quad\times\int_0^1(1-s)^{\omega-3}q(s)\,\mathrm{d}s
\end{aligned}$$

$$+\frac{\lambda_1|\Lambda_4|[|(t_2-t_1)\Lambda_0\Lambda_5|+|t_2^2-t_1^2|]}{\Gamma(\omega)}\times\int_0^1\int_0^s(s-\tau)^{\omega-1}q(\tau)\,\mathrm{d}\tau\,\mathrm{d}s.$$

Note that the right-hand side tends to zero independently of $k\in\mathcal{V}_{\varepsilon^*}$ as $t_2\to t_1$. By using the Arzela–Ascoli theorem, the complete continuity of $\Phi_2: C([0,1],\mathbb{R})\to\mathcal{P}(C([0,1],\mathbb{R}))$ is deduced. Now, we show that Φ_2 has a closed graph, and this follows the upper semi-continuity of the operator Φ_2. Assume that $k_n\in\mathcal{V}_{\varepsilon^*}$ and $\zeta_n\in\Phi_2k_n$ with $k_n\to k^*$ and $\zeta_n\to\zeta^*$. We claim that $\zeta^*\in\Phi_2k^*$. For every $n\geq1$ and $\zeta_n\in\Phi_2k_n$, choose $\vartheta_n\in(\mathcal{SEL})_{\mathcal{S},k_n}$ such that

$$\begin{aligned}\zeta_n(t)&=\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\vartheta_n(s)\,\mathrm{d}s\\&\quad+\frac{[(\Lambda_3\Lambda_7-\eta_2\Lambda_5)+t(\lambda_1\Lambda_0\Lambda_3-1)\Lambda_5-t^2\lambda_1\Lambda_3]\sum_{j=1}^r b_j}{\Gamma(\omega-1)}\\&\quad\times\int_0^{\eta_1}(\eta_1-s)^{\omega-2}\vartheta_n(s)\,\mathrm{d}s\\&\quad+\frac{1+(t\Lambda_8+\eta_2)\Lambda_5+(\Lambda_3+\Lambda_4)(t^2\lambda_1-\Lambda_7)}{\Gamma(\omega)}\int_0^{\eta_2}(\eta_2-s)^{\omega-1}\vartheta_n(s)\,\mathrm{d}s\\&\quad+\frac{\lambda_2[(1+t\Lambda_5)\Lambda_8+t^2\lambda_1(\Lambda_3+\Lambda_4)]\sum_{j=1}^r b_j}{\lambda_1\Gamma(\omega-2)}\int_0^1(1-s)^{\omega-3}\vartheta_n(s)\,\mathrm{d}s\\&\quad+\frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5-t^2-\Lambda_6]}{\Gamma(\omega)}\int_0^1\int_0^s(s-\tau)^{\omega-1}\vartheta_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\end{aligned}$$

for all $t\in[0,1]$. It is sufficient to show that there exists a function $\vartheta^*\in(\mathcal{SEL})_{\mathcal{S},k^*}$ such that

$$\begin{aligned}\zeta^*(t)&=\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\vartheta^*(s)\,\mathrm{d}s\\&\quad+\frac{[(\Lambda_3\Lambda_7-\eta_2\Lambda_5)+t(\lambda_1\Lambda_0\Lambda_3-1)\Lambda_5-t^2\lambda_1\Lambda_3]\sum_{j=1}^r b_j}{\Gamma(\omega-1)}\\&\quad\times\int_0^{\eta_1}(\eta_1-s)^{\omega-2}\vartheta^*(s)\,\mathrm{d}s\\&\quad+\frac{1+(t\Lambda_8+\eta_2)\Lambda_5+(\Lambda_3+\Lambda_4)(t^2\lambda_1-\Lambda_7)}{\Gamma(\omega)}\int_0^{\eta_2}(\eta_2-s)^{\omega-1}\vartheta^*(s)\,\mathrm{d}s\\&\quad+\frac{\lambda_2[(1+t\Lambda_5)\Lambda_8+t^2\lambda_1(\Lambda_3+\Lambda_4)]\sum_{j=1}^r b_j}{\lambda_1\Gamma(\omega-2)}\int_0^1(1-s)^{\omega-3}\vartheta^*(s)\,\mathrm{d}s\\&\quad+\frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5-t^2-\Lambda_6]}{\Gamma(\omega)}\int_0^1\int_0^s(s-\tau)^{\omega-1}\vartheta^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\end{aligned}$$

for all $t\in[0,1]$. Define the continuous linear operator $\Xi:\mathcal{L}^1([0,1],\mathbb{R})\to\mathcal{X}$ by

$$\begin{aligned}\Xi(\vartheta)(t)=k(t)&=\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s\\&\quad+\frac{[(\Lambda_3\Lambda_7-\eta_2\Lambda_5)+t(\lambda_1\Lambda_0\Lambda_3-1)\Lambda_5-t^2\lambda_1\Lambda_3]\sum_{j=1}^r b_j}{\Gamma(\omega-1)}\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} \vartheta(s)\,\mathrm{d}s \\
&+ \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} \vartheta(s)\,\mathrm{d}s \\
&+ \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)] \sum_{j=1}^{r} b_j}{\lambda_1 \Gamma(\omega - 2)} \int_0^1 (1 - s)^{\omega-3} \vartheta(s)\,\mathrm{d}s \\
&+ \frac{\lambda_1\Lambda_4[(t - \eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)} \int_0^1 \int_0^s (s - \tau)^{\omega-1} \vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}
$$

for all $t \in [0,1]$, where $\mathcal{X} = C([0,1], \mathbb{R})$. Hence,

$$
\begin{aligned}
\left\| \zeta_n(t) - \zeta^*(t) \right\| = \Bigg\| & \frac{1}{\Gamma(\omega)} \int_0^t (t - s)^{\omega-1} \big(\vartheta_n(s) - \vartheta^*(s)\big)\,\mathrm{d}s \\
&+ \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3] \sum_{j=1}^{r} b_j}{\Gamma(\omega - 1)} \\
&\times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} \big(\vartheta_n(s) - \vartheta^*(s)\big)\,\mathrm{d}s \\
&+ \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)} \\
&\times \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} \big(\vartheta_n(s) - \vartheta^*(s)\big)\,\mathrm{d}s \\
&+ \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)] \sum_{j=1}^{r} b_j}{\lambda_1 \Gamma(\omega - 2)} \\
&\times \int_0^1 (1 - s)^{\omega-3} \big(\vartheta_n(s) - \vartheta^*(s)\big)\,\mathrm{d}s \\
&+ \frac{\lambda_1\Lambda_4[(t - \eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]}{\Gamma(\omega)} \\
&\times \int_0^1 \int_0^s (s - \tau)^{\omega-1} \big(\vartheta_n(\tau) - \vartheta^*(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s \Bigg\| \\
\to\ & 0.
\end{aligned}
$$

Hence, Theorem 1 implies that the operator $\Xi \circ (\mathcal{SEL})_{\mathcal{S}}$ has a closed graph. Since $\zeta_n \in \Xi((\mathcal{SEL})_{\mathcal{S},k_n})$ and $k_n \to k^*$, there exists $\vartheta^* \in (\mathcal{SEL})_{\mathcal{S},k^*}$ such that

$$
\begin{aligned}
\zeta^*(t) = & \frac{1}{\Gamma(\omega)} \int_0^t (t - s)^{\omega-1} \vartheta^*(s)\,\mathrm{d}s \\
&+ \frac{[(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3] \sum_{j=1}^{r} b_j}{\Gamma(\omega - 1)} \\
&\times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} \vartheta^*(s)\,\mathrm{d}s \\
&+ \frac{1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} \vartheta^*(s)\,\mathrm{d}s \\
&+ \frac{\lambda_2[(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)] \sum_{j=1}^{r} b_j}{\lambda_1 \Gamma(\omega - 2)} \int_0^1 (1 - s)^{\omega-3} \vartheta^*(s)\,\mathrm{d}s
\end{aligned}
$$

$$+\frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5-t^2-\Lambda_6]}{\Gamma(\omega)}\int_0^1\int_0^s(s-\tau)^{\omega-1}\vartheta^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

for all $t\in[0,1]$. Hence, $\zeta^*\in\Phi_2k^*$ and so Φ_2 has a closed graph. From this it follows that the operator Φ_2 is upper semi-continuous. Since the operator Φ_2 has compact values, Φ_2 is a compact and upper semi-continuous operator. By using assumption $(\mathcal{C}3)$, we have

$$\begin{aligned}\hat{\Delta}&=\|\Phi_2(\mathcal{X})\|=\sup_{t\in[0,1]}\{|\Phi_2k|:k\in\mathcal{X}\}\\&=\Bigg[\frac{1}{\Gamma(\omega+1)}\\&\quad+\frac{[|\Lambda_3\Lambda_7|+|\eta_2\Lambda_5|+(|\lambda_1\Lambda_0\Lambda_3|+1)|\Lambda_5|+|\lambda_1\Lambda_3|]\eta_1^{\omega-1}\sum_{j=1}^r b_j}{\Gamma(\omega)}\\&\quad+\frac{[1+|(\Lambda_8+\eta_2)\Lambda_5|+|(\Lambda_3+\Lambda_4)|(\lambda_1+|\Lambda_7|)]\eta_2^{\omega}}{\Gamma(\omega+1)}\\&\quad+\frac{\lambda_2[|(1+\Lambda_5)\Lambda_8|+\lambda_1|\Lambda_3+\Lambda_4|]\sum_{j=1}^r b_j}{\lambda_1\Gamma(\omega-1)}\\&\quad+\frac{\lambda_1|\Lambda_4|[(1+\eta_2)|\Lambda_0\Lambda_5|+|\Lambda_6|+1]}{\Gamma(\omega+2)}\Bigg]\|q\|_{\mathcal{L}^1}\\&=M\|q\|_{\mathcal{L}^1}.\end{aligned}$$

Put $l^*=2\theta^*$. Then $\hat{\Delta}l^*<\frac{1}{2}$. Now, by using Theorem 2 for Φ_2, we get that one of the conditions, (a) or (b), holds. We claim that condition (b) is impossible. By considering Theorem 2 and assumption $(\mathcal{C}4)$, assume that k is an arbitrary element of $\mathcal{O}^*$ with $\|k\|=\tilde{\rho}$. Then $\alpha_0k(t)\in\Phi_1k(t)\Phi_2k(t)$ for all $\alpha_0>1$. Choose the related function $\vartheta\in(\mathcal{SEL})_{S,k}$. Then, for each $\alpha_0>1$, we have

$$\begin{aligned}k(t)&=\frac{1}{\alpha_0}\xi\Big(t,k(t),\int_0^1k(s)\,\mathrm{d}s\Big)\Bigg[\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s\\&\quad+\frac{[(\Lambda_3\Lambda_7-\eta_2\Lambda_5)+t(\lambda_1\Lambda_0\Lambda_3-1)\Lambda_5-t^2\lambda_1\Lambda_3]\sum_{j=1}^r b_j}{\Gamma(\omega-1)}\\&\quad\times\int_0^{\eta_1}(\eta_1-s)^{\omega-2}\vartheta(s)\,\mathrm{d}s\\&\quad+\frac{1+(t\Lambda_8+\eta_2)\Lambda_5+(\Lambda_3+\Lambda_4)(t^2\lambda_1-\Lambda_7)}{\Gamma(\omega)}\int_0^{\eta_2}(\eta_2-s)^{\omega-1}\vartheta(s)\,\mathrm{d}s\\&\quad+\frac{\lambda_2[(1+t\Lambda_5)\Lambda_8+t^2\lambda_1(\Lambda_3+\Lambda_4)]\sum_{j=1}^r b_j}{\lambda_1\Gamma(\omega-2)}\int_0^1(1-s)^{\omega-3}\vartheta(s)\,\mathrm{d}s\\&\quad+\frac{\lambda_1\Lambda_4[(t-\eta_2)\Lambda_0\Lambda_5-t^2-\Lambda_6]}{\Gamma(\omega)}\int_0^1\int_0^s(s-\tau)^{\omega-1}\vartheta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\Bigg]\end{aligned}$$

for all $t\in[0,1]$. Thus, one can write

$$\begin{aligned}|k(t)|&=\frac{1}{\alpha_0}\Big|\xi\Big(t,k(t),\int_0^1k(s)\,\mathrm{d}s\Big)\Big|\Bigg[\frac{1}{\Gamma(\omega)}\int_0^t(t-s)^{\omega-1}|\vartheta(s)|\,\mathrm{d}s\\&\quad+\frac{|(\Lambda_3\Lambda_7-\eta_2\Lambda_5)+t(\lambda_1\Lambda_0\Lambda_3-1)\Lambda_5-t^2\lambda_1\Lambda_3|\sum_{j=1}^r b_j}{\Gamma(\omega-1)}\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} \big|\vartheta(s)\big| \, \mathrm{d}s \\
&+ \frac{|1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)|}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} \big|\vartheta(s)\big| \, \mathrm{d}s \\
&+ \frac{\lambda_2 |(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)| \sum_{j=1}^{r} b_j}{\lambda_1 \Gamma(\omega - 2)} \int_0^1 (1 - s)^{\omega-3} \big|\vartheta(s)\big| \, \mathrm{d}s \\
&+ \frac{\lambda_1 |\Lambda_4[(t - \eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]|}{\Gamma(\omega)} \int_0^1 \int_0^s (s - \tau)^{\omega-1} \big|\vartheta(\tau)\big| \, \mathrm{d}\tau \, \mathrm{d}s \Bigg] \\
= &\left[\left| \xi\left(t, k(t), \int_0^1 k(s) \, \mathrm{d}s\right) - \xi(t, 0, 0) \right| + \big|\xi(t, 0, 0)\big| \right] \\
&\times \left[\frac{1}{\Gamma(\omega)} \int_0^t (t - s)^{\omega-1} \big|\vartheta(s)\big| \, \mathrm{d}s \right. \\
&+ \frac{|(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3| \sum_{j=1}^{r} b_j}{\Gamma(\omega - 1)} \\
&\times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} \big|\vartheta(s)\big| \, \mathrm{d}s \\
&+ \frac{|1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)|}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} \big|\vartheta(s)\big| \, \mathrm{d}s \\
&+ \frac{\lambda_2 |(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)| \sum_{j=1}^{r} b_j}{\lambda_1 \Gamma(\omega - 2)} \int_0^1 (1 - s)^{\omega-3} \big|\vartheta(s)\big| \, \mathrm{d}s \\
&+ \frac{\lambda_1 |\Lambda_4[(t - \eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]|}{\Gamma(\omega)} \int_0^1 \int_0^s (s - \tau)^{\omega-1} \big|\vartheta(\tau)\big| \, \mathrm{d}\tau \, \mathrm{d}s \Bigg] \\
\le &\left[2\theta^* \|k\| + \xi^* \right] \Bigg[\frac{1}{\Gamma(\omega)} \int_0^t (t - s)^{\omega-1} q(s) \, \mathrm{d}s \\
&+ \frac{|(\Lambda_3\Lambda_7 - \eta_2\Lambda_5) + t(\lambda_1\Lambda_0\Lambda_3 - 1)\Lambda_5 - t^2\lambda_1\Lambda_3| \sum_{j=1}^{r} b_j}{\Gamma(\omega - 1)} \\
&\times \int_0^{\eta_1} (\eta_1 - s)^{\omega-2} q(s) \, \mathrm{d}s \\
&+ \frac{|1 + (t\Lambda_8 + \eta_2)\Lambda_5 + (\Lambda_3 + \Lambda_4)(t^2\lambda_1 - \Lambda_7)|}{\Gamma(\omega)} \int_0^{\eta_2} (\eta_2 - s)^{\omega-1} q(s) \, \mathrm{d}s \\
&+ \frac{\lambda_2 |(1 + t\Lambda_5)\Lambda_8 + t^2\lambda_1(\Lambda_3 + \Lambda_4)| \sum_{j=1}^{r} b_j}{\lambda_1 \Gamma(\omega - 2)} \int_0^1 (1 - s)^{\omega-3} q(s) \, \mathrm{d}s \\
&+ \frac{\lambda_1 |\Lambda_4[(t - \eta_2)\Lambda_0\Lambda_5 - t^2 - \Lambda_6]|}{\Gamma(\omega)} \int_0^1 \int_0^s (s - \tau)^{\omega-1} q(\tau) \, \mathrm{d}\tau \, \mathrm{d}s \Bigg] \\
\le &\left[2\theta^* \tilde{\rho} + \xi^* \right] M \|q\|_{\mathcal{L}^1}
\end{aligned}
$$

for all $t \in [0, 1]$. Hence, we get $\tilde{\rho} \le \frac{\xi^* M \|q\|_{\mathcal{L}^1}}{1 - 2\theta^* M \|q\|_{\mathcal{L}^1}}$. Now, by using (8), we conclude that condition (b) of Theorem 2 is impossible. Thus, $k \in \Phi_1 k \Phi_2 k$. Hence, the operator $\mathcal{G}$ has a fixed point, and so the hybrid inclusion problem (1)–(2) has a solution. □

Here, we provide an example to illustrate our main results.

Example 1 Consider the fractional hybrid multi-term integro-differential inclusion

$$\begin{aligned} &{}^{C}\mathcal{D}_0^{2.71}\left(\frac{k(t)}{\frac{t|\cos k(t)|}{(1+|\cos k(t)|)10{,}000}+0.0001t\int_0^1 k(s)\,\mathrm{d}s+0.0009}\right)\\ &\quad\in\left[-2,(t+1)\cos k(t)+2\sin\left(\mathcal{I}_0^{0.03}k(t)\right)+\frac{7}{10}\sin^2\left(\mathcal{I}_0^{0.05}k(t)\right)+\frac{8}{10}\right], \end{aligned} \tag{10}$$

with four-point sum and integral boundary value conditions

$$\begin{cases} \left(\frac{k(t)}{\frac{t|\cos k(t)|}{(1+|\cos k(t)|)10{,}000}+0.0001t\int_0^1 k(s)\,\mathrm{d}s+0.0009}\right)|_{t=0}+\sum_{j=1}^{3}b_j\\ \quad\times\mathcal{D}_0\left(\frac{k(t)}{\frac{t|\cos k(t)|}{(1+|\cos k(t)|)10{,}000}+0.0001t\int_0^1 k(s)\,\mathrm{d}s+0.0009}\right)|_{t=0.1}=0,\\ 1.27\left(\frac{k(t)}{\frac{t|\cos k(t)|}{(1+|\cos k(t)|)10{,}000}+0.0001t\int_0^1 k(s)\,\mathrm{d}s+0.0009}\right)|_{t=0.22}+0.9\sum_{j=1}^{3}b_j\\ \quad\times\mathcal{D}_0^2\left(\frac{k(t)}{\frac{t|\cos k(t)|}{(1+|\cos k(t)|)10{,}000}+0.0001t\int_0^1 k(s)\,\mathrm{d}s+0.0009}\right)|_{t=1}=0,\\ 1.4\int_0^1\left(\frac{k(t)}{\frac{t|\cos k(t)|}{(1+|\cos k(t)|)10{,}000}+0.0001t\int_0^1 k(s)\,\mathrm{d}s+0.0009}\right)\mathrm{d}s=0, \end{cases} \tag{11}$$

where $t\in[0,1]$, $\omega=2.71$, $\eta_1=0.1$, $\eta_2=0.22$, $\lambda_1=1.27$, $\lambda_2=0.9$, $\lambda_3=1.4$, and $r=3$. Then we have $\sum_{j=1}^{3}b_j=0.24$ with $b_1=0.09$, $b_2=0.08$, and $b_3=0.07$. For $m=2$, put $\phi_1(k(t))=\mathcal{I}_0^{0.03}k(t)$ and $\phi_2(k(t))=\mathcal{I}_0^{0.05}k(t)$, where $\varpi_1=0.03$ and $\varpi_2=0.05$. Consider the continuous map $\xi:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ defined by

$$\xi\left(t,k_1(t),k_2(t)\right)=\frac{t|\cos k_1(t)|}{(1+|\cos k_1(t)|)10{,}000}+0.0001t\int_0^1 k_2(s)\,\mathrm{d}s+0.0009$$

with $\xi^*=\sup_{t\in[0,1]}|\xi(t,0,0)|=\frac{1}{20{,}000}+0.0009=0.00095$. On the other hand, it is clear that the function ξ is Lipschitzian. Indeed, for each $k_1,k_2\in\mathbb{R}$, we have

$$\begin{aligned} &\left|\xi\left(t,k_1(t),\int_0^1 k_1(s)\,\mathrm{d}s\right)-\xi\left(t,k_2(t),\int_0^1 k_2(s)\,\mathrm{d}s\right)\right|\\ &\quad\le\frac{t}{10{,}000}\left[\left|k_1(t)-k_2(t)\right|+\left|k_1(t)-k_2(t)\right|\right]\\ &\quad=\frac{2t}{10{,}000}\left|k_1(t)-k_2(t)\right|. \end{aligned}$$

If we set $\theta(t)=\frac{t}{10{,}000}$, then $\theta^*=\sup_{t\in[0,1]}|\theta(t)|=0.0001$. In this position, we define the set-valued map $\mathcal{S}:[0,1]\times\mathbb{R}^{2+1}\to\mathcal{P}(\mathbb{R})$ by

$$\begin{aligned} &\mathcal{S}\left(t,k(t),\phi_1\left(k(t)\right),\phi_2\left(k(t)\right)\right)\\ &\quad=\left[-2,(t+1)\cos k(t)+2\sin\left(\mathcal{I}_0^{0.03}k(t)\right)+\frac{7}{10}\sin^2\left(\mathcal{I}_0^{0.05}k(t)\right)+\frac{8}{10}\right]. \end{aligned}$$

Since

$$|\zeta|\le\max\left[-2,(t+1)\cos k(t)+2\sin\left(\mathcal{I}_0^{0.03}k(t)\right)+\frac{7}{10}\sin^2\left(\mathcal{I}_0^{0.05}k(t)\right)+\frac{8}{10}\right]\le t+\frac{9}{2}$$

for all $\zeta \in \mathcal{S}(t, k(t), \phi_1(k(t)), \phi_2(k(t)))$, we obtain

$$\begin{aligned}\left\|\mathcal{S}\left(t, k(t), \phi_1\left(k(t)\right), \phi_2\left(k(t)\right)\right)\right\| &= \sup\left\{|\vartheta| : \vartheta \in \mathcal{S}\left(t, k(t), \phi_1\left(k(t)\right), \phi_2\left(k(t)\right)\right)\right\} \\ &\leq t + 4.5.\end{aligned}$$

Put $q(t) = t + 4.5$ for all $t \in [0, 1]$. Then $\|q\|_{\mathcal{L}^1} = \int_0^1 |q(s)|\,\mathrm{d}s = \int_0^1 (s + 4.5)\,\mathrm{d}s = 5$ and $M \simeq 42.2585$. Now, we choose $\tilde{\rho} > 0$ so that

$$\tilde{\rho} > \frac{\xi^* M \|q\|_{\mathcal{L}^1}}{1 - 2\theta^* M \|q\|_{\mathcal{L}^1}} = \frac{0.00095 \times 42.2585 \times 5}{1 - 2(0.0001 \times 42.2585 \times 5)} \simeq 0.209574.$$

Thus $\tilde{\rho} > 0.209574$. Then $4\theta^* M \|q\|_{\mathcal{L}^1} \simeq 0.0845 < 1$. Now, by using Theorem 5, the hybrid multi-term inclusion problem (10)–(11) has a solution.

4 Conclusion

It is known that most natural phenomena are modeled by different types of fractional differential equations and inclusions. This diversity in investigating complicate fractional differential equations and inclusions increases our ability for exact modelings of more phenomena. This is useful in designing modern software which helps us to allow for more cost-free testing and less material consumption. In this work, we study the existence of solutions for a fractional hybrid multi-term integro-differential inclusion problem with four-point sum and integral boundary value conditions. By using Dhage's fixed point results, we prove our main existence result. Finally, we give an example to illustrate our main result.

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