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New Aspects for Non-Existence of Kneser Solutions of Neutral Differential Equations with Odd-Order

Osama Moaaz ^{1,†} , Dumitru Baleanu ^{2,3*,†}  and Ali Muhib ^{1,4,†}

¹ Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt; o_moaaz@mans.edu.eg (O.M.); muhib39@students.mans.edu.eg (A.M.)

² Department of Mathematics and Computer Science, Faculty of Arts and Sciences, Çankaya University Ankara, 06790 Etimesgut, Turkey

³ Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

⁴ Department of Mathematics, Faculty of Education, Al-Nadirah, Ibb University, PO Box 70270 Ibb, Yemen

* Correspondence: dumitru@cankaya.edu.tr or Baleanu@mail.cmuh.org.tw

† These authors contributed equally to this work.

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Abstract: Some new oscillatory and asymptotic properties of solutions of neutral differential equations with odd-order are established. Through the new results, we give sufficient conditions for the oscillation of all solutions of the studied equations, and this is an improvement of the relevant results. The efficiency of the obtained criteria is illustrated via example.

Keywords: odd-order differential equations; Kneser solutions; oscillatory solutions

1. Introduction

During this paper, we investigate the asymptotic properties of solutions to the odd-order neutral equation

$$(r(l)(z^{(n-1)}(l))^\alpha)' + f(l, u(\eta(l))) = 0, \quad l \geq l_0 > 0, \quad (1)$$

where $l \geq l_0$, $z(l) = u(l) + p(l)u(\theta(l))$, $0 \leq p(l) \leq p_0 < \infty$ and n is an odd natural number. Through the paper, we assume that

- (I) α is a ratio of odd positive integers, $r, \eta, \theta \in C^1(I_0, \mathbb{R}^+)$, $r'(l) \geq 0$, $\eta(l) < l$, $\eta'(l) > 0$, $(\eta^{-1}(l))' \geq \eta_0 > 0$, $\theta'(l) \geq \theta_0 > 0$, $\lim_{l \rightarrow \infty} \eta(l) = \infty$, $\lim_{l \rightarrow \infty} \theta(l) = \infty$, $I_\rho := [l_\rho, \infty)$, the function $f \in C(I_0 \times \mathbb{R}, \mathbb{R})$, and there exists a nonnegative function q such that $|f(l, u)| \geq q(l)|u|^\alpha$. Moreover, we study asymptotic behavior and oscillation of solutions of (1) in a canonical case, that is,

$$\int_{l_0}^{\infty} \frac{1}{r^{1/\alpha}(l)} dl = \infty. \quad (2)$$

- (II) $\theta(l) < l$ and $\theta \circ \eta = \eta \circ \theta$.

If there exists $l_u \geq l_0$ such that the real valued function u is continuous, $r(z^{(n-1)})^\alpha$ is continuously differentiable and satisfies (1), for all $l \in I_u$; then, u is said to be a solution of (1). We restrict our discussion to those solutions u of (1) which satisfy $\sup\{|u(l)| : l_1 \leq l\} > 0$ for every $l_1 \in I_u$.

Definition 1. A solution u of Equation (1) is called an *N-Kneser solution* if there exists a $l_* \in I_0$ such that $z(l)z'(l) < 0$ for all $l \in I_*$. The set of all eventually positive N-Kneser solutions of Equation (1) is denoted by \mathfrak{R} .

Definition 2. A solution u of (1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

There are many authors who studied the problem of oscillation of differential equations of a different order and presented many techniques in order to obtain criteria for oscillation of the studied equations, for example, [1–12].

For applications of odd-order equations in extrema, biology, and physics, we refer to the following examples. In 1701, James Bernoulli published the solution to the Isoperimetric Problem—a problem in which it is required to make one integral a maximum or minimum, while keeping constant the integral of a second given function, thus resulting in a differential equation of third-order (see [13]). In the early 1950s, Alan Lloyd Hodgkin and Andrew Huxley developed a mathematical model for the propagation of electrical pulses in the nerve of a squid. The Hodgkin–Huxley Model is a set of nonlinear ordinary differential equations. The model has played a seminal role in biophysics and neuronal modeling.

Recently, researchers have paid attention to neutral differential equations, as well as studying the oscillation behavior of their solutions. There is a practical side to study the problem of the oscillatory properties of solutions of neutral equations besides the theoretical side. For example, the neutral equations arise in applications to electric networks containing lossless transmission lines. Such networks appear in high-speed computers where lossless transmission lines are used to interconnect switching circuits. For more applications in science and technology, see [14–16].

Karpuz et al. [17] studied the higher-order neutral differential equations of the following type:

$$(u(l) + p(l)u(\theta(l)))^{(n)} + q(l)u(\eta(l)) = 0, \text{ for } l \in [l_0, \infty) \tag{3}$$

where oscillatory and asymptotic behaviors of all solutions of higher-order neutral differential equations are compared with first-order delay differential equations, depending on two different ranges of the coefficient associated with the neutral part

Xing et al. [18] established some oscillation criteria for certain higher-order quasi-linear neutral differential equation

$$(r(l) \left((u(l) + p(l)u(\theta(l)))^{(n-1)} \right)^\alpha)' + q(l)u^\alpha(\eta(l)) = 0, \quad n \geq 2 \tag{4}$$

where $\alpha \leq 1$ is the quotient of odd positive integers.

Li and Rogovchenko [19] concerned with the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations

$$\left(r(t) \left((x(t) + p_0x(t - \omega_0))'' \right)^\alpha \right)' + q(t)x^\alpha(\tau(t)) = 0,$$

where $p_0 \geq 0$, $p_0 \neq 1$ and ω_0 are constants, $\omega_0 \geq 0$ (delayed argument) or $\omega_0 \leq 0$ (advanced argument).

Some results that are closely related to our work are presented as follows:

Theorem 1 ([17], Corollary 2, see [20], Theorem 3.1.1 and [21]). Assume that p satisfies the condition

$$p \in C([l_0, \infty), R^+) \text{ satisfies } \bar{l}_p := \limsup_{l \rightarrow \infty} p(l) < 1.$$

If

$$\limsup_{l \rightarrow \infty} \int_{\eta(l)}^l \frac{1}{(n-1)!} (\eta(\rho))^{n-1} q(\rho) d\rho > 1$$

or

$$\liminf_{l \rightarrow \infty} \int_{\eta(l)}^l \frac{1}{(n-1)!} (\eta(\rho))^{n-1} q(\rho) d\rho > \frac{1}{e}$$

holds, then (3) is almost oscillatory.

Theorem 2 ([18], Corollary 2.8). Let n be odd, $\alpha \leq 1$, $(\eta^{-1}(l))' \geq \eta_0 > 0$, $0 \leq p(l) \leq p_0 < \infty$, $\theta(l) \leq l$, and $\theta'(l) \geq \theta_0 > 0$, suppose that (2) holds. If $\theta^{-1}(\eta(l)) < l$ and

$$\liminf_{l \rightarrow \infty} \int_{\theta^{-1}(\eta(l))}^l \frac{\Theta(\rho) (\rho^{n-1})^\alpha}{r(\rho)} d\rho > \left(\frac{1}{\eta_0} + \frac{p_0^\alpha}{\eta_0 \theta_0} \right) \frac{((n-1)!)^\alpha}{e},$$

where $\Theta(l) = \min\{q(\eta^{-1}(l)), q(\eta^{-1}(\theta(l)))\}$; then, every solution of (4) is oscillatory or tends to zero as $l \rightarrow \infty$.

Lemma 1 ([18,22]). Assume that $u_1, u_2 \in [0, \infty)$. Then,

$$(u_1 + u_2)^\alpha \leq \mu (u_1^\alpha + u_2^\alpha),$$

and

$$\mu = \begin{cases} 1 & \text{for } 0 < \alpha \leq 1; \\ 2^{\alpha-1} & \text{for } \alpha > 1. \end{cases}$$

Lemma 2 ([23]). Let $u \in C^n([l_0, \infty), (0, \infty))$. Assume that $u^{(n)}(l)$ is of fixed sign and not identically zero on $[l_0, \infty)$ and that there exists a $l_1 \geq l_0$ such that $u^{(n-1)}(l) u^{(n)}(l) \leq 0$ for all $l \geq l_1$. If $\lim_{l \rightarrow \infty} u(l) \neq 0$, then, for every $\lambda \in (0, 1)$, there exists $l_\mu \geq l_1$ such that

$$u(l) \geq \frac{\lambda}{(n-1)!} l^{n-1} |u^{(n-1)}(l)| \text{ for } l \geq l_\mu.$$

2. Main Results

For the sake of convenience, we use the following notation:

$$R_0(\zeta, \varrho) = \int_\varrho^\zeta r^{-1/\alpha}(\rho) d\rho, R_k(\zeta, \varrho) = \int_\varrho^\zeta R_{k-1}(\zeta, \rho) d\rho, k = 1, 2, \dots, n-2$$

and

$$Q(l) = \min\{q(l), q(\theta(l))\}, Q_1(l) = \min\{q(\eta^{-1}(l)), q(\eta^{-1}(\theta(l)))\}.$$

The following lemma is a direct conclusion from Lemmas 2.1 and 2.4 in [18], so its proof was neglected.

Lemma 3. Assume that u is an eventually positive solution of (1). Then, there exists a sufficiently large $l_1 \geq l_0$ such that, for all $l \geq l_1$, either

$$\text{Case (1): } z(l) > 0, z'(l) > 0, z^{(n-1)}(l) > 0, (r(l)(z^{(n-1)}(l))^\alpha)' < 0$$

or

$$\text{Case (2): } (-1)^k z^{(k)}(l) > 0, \text{ for } k = 0, 1, 2, \dots, n.$$

Now, in the following theorem, we will provide a new criterion for non-existence of N-Kneser solutions of (1) by using the comparison theorem.

Theorem 3. Assume (I) and (II) holds. If there exists a function $\zeta(l) \in C([l_0, \infty), (0, \infty))$ satisfying $\eta(l) < \zeta(l)$ and $\theta^{-1}(\zeta(l)) < l$, such that the differential equation

$$G'(l) + \frac{1}{\mu} \frac{\theta_0}{\theta_0 + p_0^\alpha} R_{n-2}^\alpha(\zeta(l), \eta(l)) Q(l) G(\theta^{-1}(\zeta(l))) = 0 \tag{5}$$

is oscillatory, then \mathfrak{R} is an empty set.

Proof. Let u be a N-Kneser solution of (1), say $u(l) > 0$ and $u(\eta(l)) > 0$ for $l \geq l_1 \geq l_0$. This implies that

$$(-1)^k z^{(k)}(l) > 0, \text{ for } k = 0, 1, 2, \dots, n. \tag{6}$$

From (1), we see that

$$\begin{aligned} 0 &\geq \frac{p_0^\alpha}{\theta'(l)} \left(r(\theta(l)) \left(z^{(n-1)}(\theta(l)) \right)^\alpha \right)' + p_0^\alpha q(\theta(l)) u^\alpha(\eta(\theta(l))) \\ &\geq \frac{p_0^\alpha}{\theta_0} \left(r(\theta(l)) \left(z^{(n-1)}(\theta(l)) \right)^\alpha \right)' + p_0^\alpha q(\theta(l)) u^\alpha(\eta(\theta(l))) \\ &= \frac{p_0^\alpha}{\theta_0} \left(r(\theta(l)) \left(z^{(n-1)}(\theta(l)) \right)^\alpha \right)' + p_0^\alpha q(\theta(l)) u^\alpha(\theta(\eta(l))). \end{aligned} \tag{7}$$

Combining (1) and (7), we obtain

$$\begin{aligned} 0 &\geq (r(l) (z^{(n-1)}(l))^\alpha)' + \frac{p_0^\alpha}{\theta_0} \left(r(\theta(l)) \left(z^{(n-1)}(\theta(l)) \right)^\alpha \right)' + q(l) u^\alpha(\eta(l)) \\ &\quad + p_0^\alpha q(\theta(l)) u^\alpha(\theta(\eta(l))) \\ &\geq (r(l) (z^{(n-1)}(l))^\alpha)' + \frac{p_0^\alpha}{\theta_0} \left(r(\theta(l)) \left(z^{(n-1)}(\theta(l)) \right)^\alpha \right)' \\ &\quad + Q(l) (u^\alpha(\eta(l)) + p_0^\alpha u^\alpha(\theta(\eta(l))))). \end{aligned} \tag{8}$$

From definition of z and using (I), we have

$$z(\eta(l)) = u(\eta(l)) + p(\eta(l)) u(\theta(\eta(l))) \leq u(\eta(l)) + p_0 u(\theta(\eta(l))).$$

By using the latter inequality in (8), we get

$$\begin{aligned} 0 &\geq (r(l) (z^{(n-1)}(l))^\alpha)' + \frac{p_0^\alpha}{\theta_0} \left(r(\theta(l)) \left(z^{(n-1)}(\theta(l)) \right)^\alpha \right)' \\ &\quad + Q(l) (u(\eta(l)) + p_0 u(\theta(\eta(l))))^\alpha \\ &\geq (r(l) (z^{(n-1)}(l))^\alpha)' + \frac{p_0^\alpha}{\theta_0} \left(r(\theta(l)) \left(z^{(n-1)}(\theta(l)) \right)^\alpha \right)' + \frac{1}{\mu} Q(l) z^\alpha(\eta(l)), \end{aligned}$$

that is,

$$0 \geq \left(r(l) (z^{(n-1)}(l))^\alpha + \frac{p_0^\alpha}{\theta_0} r(\theta(l)) \left(z^{(n-1)}(\theta(l)) \right)^\alpha \right)' + \frac{1}{\mu} Q(l) z^\alpha(\eta(l)). \tag{9}$$

On the other hand, it follows from the monotonicity of $r(l) (z^{(n-1)}(l))$ that

$$\begin{aligned} -z^{(n-2)}(\varrho) &\geq z^{(n-2)}(\varsigma) - z^{(n-2)}(\varrho) = \int_\varrho^\varsigma \frac{r^{1/\alpha}(\rho) z^{(n-1)}(\rho)}{r^{1/\alpha}(\rho)} d\rho \\ &\geq r^{1/\alpha}(\varsigma) z^{(n-1)}(\varsigma) R_0(\varsigma, \varrho). \end{aligned} \tag{10}$$

Integrating (10) from ϱ to ς , we have

$$-z^{(n-3)}(\varrho) \leq z^{(n-3)}(\varsigma) - z^{(n-3)}(\varrho) = r^{1/\alpha}(\varsigma) z^{(n-1)}(\varsigma) R_1(\varsigma, \varrho). \tag{11}$$

Integrating (11) $n - 3$ times from ϱ to ς and using (6), we get

$$z(\varrho) \geq r^{1/\alpha}(\varsigma) z^{(n-1)}(\varsigma) R_{n-2}(\varsigma, \varrho). \tag{12}$$

Thus, we have

$$z(\eta(l)) \geq r^{1/\alpha}(\zeta(l)) z^{(n-1)}(\zeta(l)) R_{n-2}(\zeta(l), \eta(l)),$$

which, by virtue of (9), yields that

$$0 \geq (r(l) (z^{(n-1)}(l))^\alpha + \frac{p_0^\alpha}{\theta_0} r(\theta(l)) (z^{(n-1)}(\theta(l)))^\alpha)' + \frac{1}{\mu} Q(l) r(\zeta(l)) (z^{(n-1)}(\zeta(l)) R_{n-2}(\zeta(l), \eta(l)))^\alpha. \tag{13}$$

Now, set

$$G(l) = r(l) (z^{(n-1)}(l))^\alpha + \frac{p_0^\alpha}{\theta_0} r(\theta(l)) (z^{(n-1)}(\theta(l)))^\alpha > 0.$$

From (I) and the fact that $r(l) (z^{(n-1)}(l))$ is non-increasing, we have

$$G(l) \leq r(\theta(l)) (z^{(n-1)}(\theta(l)))^\alpha \left(1 + \frac{p_0^\alpha}{\theta_0}\right)$$

or equivalently,

$$r(\zeta(l)) (z^{(n-1)}(\zeta(l)))^\alpha \geq \frac{\theta_0}{\theta_0 + p_0^\alpha} G(\theta^{-1}(\zeta(l))). \tag{14}$$

Using (14) in (13), we see that G is a positive solution of the differential inequality

$$G'(l) + \frac{1}{\mu} \frac{\theta_0}{\theta_0 + p_0^\alpha} R_{n-2}^\alpha(\zeta(l), \eta(l)) Q(l) G(\theta^{-1}(\zeta(l))) \leq 0.$$

In view of [24], Theorem 1, we have that (5) also has a positive solution, a contradiction. Thus, the proof is complete. \square

In the following theorem, we establish a hille and nehari type condition that confirms the non-existence of N-Kneser solutions of (1).

Theorem 4. Assume (I) and (II) hold. If there exists a function $\delta(l) \in C([l_0, \infty), (0, \infty))$ satisfying $\delta(l) < l$ and $\eta(l) < \theta(\delta(l))$ such that

$$\limsup_{l \rightarrow \infty} \frac{1}{\mu} \frac{R_{n-2}^\alpha(\theta(\delta(l)), \eta(l))}{r(\theta(\delta(l)))} \int_{\delta(l)}^l Q(\rho) d\rho > \frac{\theta_0 + p_0^\alpha}{\theta_0}, \tag{15}$$

then \mathfrak{R} is an empty set.

Proof. By using the same method in proof of Theorem 3, we obtain (9). Integrating (9) from $\delta(l)$ to l and using the fact that z is decreasing, we get

$$\begin{aligned} & r(\delta(l)) (z^{(n-1)}(\delta(l)))^\alpha + \frac{p_0^\alpha}{\theta_0} r(\theta(\delta(l))) (z^{(n-1)}(\theta(\delta(l))))^\alpha \\ & \geq r(l) (z^{(n-1)}(l))^\alpha + \frac{p_0^\alpha}{\theta_0} r(\theta(l)) (z^{(n-1)}(\theta(l)))^\alpha + \frac{1}{\mu} z^\alpha(\eta(l)) \int_{\delta(l)}^l Q(\rho) d\rho \\ & \geq \frac{1}{\mu} z^\alpha(\eta(l)) \int_{\delta(l)}^l Q(\rho) d\rho. \end{aligned}$$

Since $\theta(\delta(l)) < \theta(l)$ and $r(l)(z^{(n-1)}(l))$ is non-increasing, we have

$$r(\theta(\delta(l)))(z^{(n-1)}(\theta(\delta(l))))^\alpha \left(1 + \frac{p_0^\alpha}{\theta_0}\right) \geq \frac{1}{\mu} z^\alpha(\eta(l)) \int_{\delta(l)}^l Q(\rho) d\rho. \tag{16}$$

By using (12) with $\varsigma = \theta(\delta(l))$ and $\varrho = \eta(l)$ in (16), we obtain

$$\begin{aligned} & r(\theta(\delta(l)))(z^{(n-1)}(\theta(\delta(l))))^\alpha \left(1 + \frac{p_0^\alpha}{\theta_0}\right) \\ & \geq \frac{1}{\mu} \left(z^{(n-1)}(\theta(\delta(l)))\right)^\alpha R_{n-2}^\alpha(\theta(\delta(l)), \eta(l)) \int_{\delta(l)}^l Q(\rho) d\rho, \end{aligned}$$

that is,

$$\frac{\theta_0 + p_0^\alpha}{\theta_0} \geq \frac{1}{\mu} \frac{R_{n-2}^\alpha(\theta(\delta(l)), \eta(l))}{r(\theta(\delta(l)))} \int_{\delta(l)}^l Q(\rho) d\rho.$$

Now, we take the lim sup of both sides of the previous inequality, and we obtain a contradiction to (15). The proof is complete. \square

In the following theorem, we will provide another criterion for the non-existence of N-Kneser solutions of (1) using the comparison theorem.

Theorem 5. Assume (I), (II), and $\eta(\theta(l)) < l$ hold. If the differential equation

$$\Psi'(l) + Q_1(l) R_{n-2}^\alpha(\theta(l), l) \left(\frac{\eta_0 \theta_0}{\theta_0 + p_0^\alpha}\right) \Psi(\eta(l)) = 0 \tag{17}$$

is oscillatory, then \mathfrak{R} is an empty set.

Proof. Let u be a N-Kneser solution of (1), say $u(l) > 0$, $u(\theta(l)) > 0$ and $u(\eta(l)) > 0$ for $l \geq l_1 \geq l_0$. This implies that

$$(-1)^k z^{(k)}(l) > 0, \text{ for } k = 0, 1, 2, \dots, n.$$

By using (1) and (I), we see that

$$\begin{aligned} 0 & \geq \frac{1}{(\eta^{-1}(l))'} \left(r(\eta^{-1}(l)) \left(z^{(n-1)}(\eta^{-1}(l)) \right)^\alpha \right)' + q(\eta^{-1}(l)) u^\alpha(l) \\ & \geq \frac{1}{\eta_0} \left(r(\eta^{-1}(l)) \left(z^{(n-1)}(\eta^{-1}(l)) \right)^\alpha \right)' + q(\eta^{-1}(l)) u^\alpha(l), \end{aligned}$$

and, similarly,

$$\begin{aligned} 0 & \geq \frac{p_0^\alpha}{(\eta^{-1}(\theta(l)))'} \left(r(\eta^{-1}(\theta(l))) \left(z^{(n-1)}(\eta^{-1}(\theta(l))) \right)^\alpha \right)' \\ & \quad + p_0^\alpha q(\eta^{-1}(\theta(l))) u^\alpha(\theta(l)) \\ & \geq \frac{p_0^\alpha}{\eta_0 \theta_0} \left(r(\eta^{-1}(\theta(l))) \left(z^{(n-1)}(\eta^{-1}(\theta(l))) \right)^\alpha \right)' \\ & \quad + p_0^\alpha q(\eta^{-1}(\theta(l))) u^\alpha(\theta(l)). \end{aligned}$$

Combining the above inequalities yields that

$$\begin{aligned}
 0 \geq & \frac{1}{\eta_0} \left(r \left(\eta^{-1}(l) \right) \left(z^{(n-1)} \left(\eta^{-1}(l) \right) \right)^\alpha \right)' \\
 & + \frac{p_0^\alpha}{\eta_0 \theta_0} \left(r \left(\eta^{-1}(\theta(l)) \right) \left(z^{(n-1)} \left(\eta^{-1}(\theta(l)) \right) \right)^\alpha \right)' \\
 & + q \left(\eta^{-1}(l) \right) u^\alpha(l) + p_0^\alpha q \left(\eta^{-1}(\theta(l)) \right) u^\alpha(\theta(l)),
 \end{aligned}$$

that is,

$$\begin{aligned}
 0 \geq & \left(\frac{1}{\eta_0} r \left(\eta^{-1}(l) \right) \left(z^{(n-1)} \left(\eta^{-1}(l) \right) \right)^\alpha + \frac{p_0^\alpha}{\eta_0 \theta_0} r \left(\eta^{-1}(\theta(l)) \right) \left(z^{(n-1)} \left(\eta^{-1}(\theta(l)) \right) \right)^\alpha \right)' \\
 & + Q_1(l) z^\alpha(l).
 \end{aligned} \tag{18}$$

Now, we set

$$\Psi(l) = \frac{1}{\eta_0} r \left(\eta^{-1}(l) \right) \left(z^{(n-1)} \left(\eta^{-1}(l) \right) \right)^\alpha + \frac{p_0^\alpha}{\eta_0 \theta_0} r \left(\eta^{-1}(\theta(l)) \right) \left(z^{(n-1)} \left(\eta^{-1}(\theta(l)) \right) \right)^\alpha. \tag{19}$$

From (II) and the fact that $r(l) (z^{(n-1)}(l))$ is non-increasing, it is easy to see that

$$\Psi(l) \leq \frac{r \left(\eta^{-1}(\theta(l)) \right) \left(z^{(n-1)} \left(\eta^{-1}(\theta(l)) \right) \right)^\alpha}{\eta_0} \left(1 + \frac{p_0^\alpha}{\theta_0} \right). \tag{20}$$

By using (12) with $\varsigma = \theta(l)$ and $\varrho = l$ and (20), we have

$$z^\alpha(l) \geq r(\theta(l)) \left(z^{(n-1)}(\theta(l)) \right)^\alpha R_{n-2}^\alpha(\theta(l), l) \geq \Psi(\eta(l)) R_{n-2}^\alpha(\theta(l), l) \left(\frac{\eta_0 \theta_0}{\theta_0 + p_0^\alpha} \right).$$

From definition Ψ and using the above inequality in (18), we get

$$0 \geq \Psi'(l) + Q_1(l) R_{n-2}^\alpha(\theta(l), l) \left(\frac{\eta_0 \theta_0}{\theta_0 + p_0^\alpha} \right) \Psi(\eta(l)).$$

In view of [24], Theorem 1, we have that (17) also has a positive solution, a contradiction. Thus, the proof is complete. \square

3. New Oscillation Criteria

In the following lemma, we present criteria that ensure that non-existence of solutions satisfies case (1).

Lemma 4. Assume that u be an eventually positive solution of (1) and the differential equation

$$\Phi'(l) + \frac{Q(l)}{\left(1 + \frac{p_0^\alpha}{\theta_0}\right)} \left(\frac{\lambda_0}{(n-1)! r^{1/\alpha}(\eta(l))} (\eta(l))^{n-1} \right)^\alpha \Phi\left(\eta\left(\theta^{-1}(l)\right)\right) = 0 \tag{21}$$

or

$$\phi'(l) + \frac{Q_1(l)}{\left(\frac{1}{\eta_0} + \frac{p_0^\alpha}{\eta_0 \theta_0}\right)} \left(\frac{\lambda_1}{(n-1)! r^{1/\alpha}(l)} l^{n-1} \right)^\alpha \phi\left(\theta^{-1}(\eta(l))\right) = 0 \tag{22}$$

is oscillatory, then z does not satisfy the following case:

$$z(l) > 0, z'(l) > 0, z^{(n-1)}(l) > 0 \text{ and } z^{(n)}(l) \leq 0. \tag{23}$$

Proof. Assume on the contrary that u is an eventually positive solution of (1) and z satisfies (23). Proceeding as in the proof of Theorem 3, we obtain (9). By using Lemma 2, we get

$$z(l) \geq \frac{\lambda}{(n-1)!r^{1/\alpha}(l)} l^{n-1} r^{1/\alpha}(l) z^{(n-1)}(l). \tag{24}$$

Therefore, by setting $w(l) = r(l) (z^{(n-1)}(l))^\alpha$ in (9) and utilizing (24), we see that w is a positive solution of the equation

$$\left(w(l) + \frac{p_0^\alpha}{\theta_0} w(\theta(l)) \right)' + Q(l) \left(\frac{\lambda}{(n-1)!r^{1/\alpha}(\eta(l))} (\eta(l))^{n-1} \right)^\alpha w(\eta(l)) = 0. \tag{25}$$

Since $w(l) = r(l) (z^{(n-1)}(l))^\alpha$ is non-increasing and it satisfies (25), let us denote

$$\Phi(l) = w(l) + \frac{p_0^\alpha}{\theta_0} w(\theta(l)).$$

It follows from $\theta(l) < l$

$$\Phi(l) \leq w(\theta(l)) \left(1 + \frac{p_0^\alpha}{\theta_0} \right).$$

Substituting these terms into (25), we get that Φ is a positive solution of

$$\Phi'(l) + \frac{Q(l)}{\left(1 + \frac{p_0^\alpha}{\theta_0} \right)} \left(\frac{\lambda}{(n-1)!r^{1/\alpha}(\eta(l))} (\eta(l))^{n-1} \right)^\alpha \Phi(\eta(\theta^{-1}(l))) \leq 0.$$

In view of [24], Theorem 1, we have that (21) also has a positive solution, which is a contradiction (21).

Now, proceeding as in the proof of Theorem 5, we obtain (18). In the same style as the first part, we have

$$0 \geq \left(\frac{1}{\eta_0} r(\eta^{-1}(l)) (z^{(n-1)}(\eta^{-1}(l)))^\alpha + \frac{p_0^\alpha}{\eta_0 \theta_0} r(\eta^{-1}(\theta(l))) (z^{(n-1)}(\eta^{-1}(\theta(l))))^\alpha \right)' + Q_1(l) z^\alpha(l).$$

By using Lemma 2, we get

$$z(l) \geq \frac{\lambda}{(n-1)!r^{1/\alpha}(l)} l^{n-1} r^{1/\alpha}(l) z^{(n-1)}(l).$$

Therefore, by setting $U(l) = r(l) (z^{(n-1)}(l))^\alpha$ in (18) and utilizing (24), we see that U is a positive solution of the equation

$$\left(\frac{1}{\eta_0} U(\eta^{-1}(l)) + \frac{p_0^\alpha}{\eta_0 \theta_0} U(\eta^{-1}(\theta(l))) \right)' + Q_1(l) \left(\frac{\lambda}{(n-1)!r^{1/\alpha}(l)} l^{n-1} \right)^\alpha U(l) = 0. \tag{26}$$

Since $U(l) = r(l) (z^{(n-1)}(l))^\alpha$ is non-increasing and it satisfies (26), let us denote

$$\phi(l) = \frac{1}{\eta_0} U(\eta^{-1}(l)) + \frac{p_0^\alpha}{\eta_0 \theta_0} U(\eta^{-1}(\theta(l))).$$

It follows from $\theta(l) < l$

$$\phi(l) \leq U(\eta^{-1}(\theta(l))) \left(\frac{1}{\eta_0} + \frac{p_0^\alpha}{\eta_0 \theta_0} \right).$$

Substituting these terms into (26), we get that ϕ is a positive solution of

$$\phi'(l) + \frac{Q_1(l)}{\left(\frac{1}{\eta_0} + \frac{p_0^\alpha}{\eta_0 \theta_0}\right)} \left(\frac{\lambda}{(n-1)!r^{1/\alpha}(l)} l^{n-1}\right)^\alpha \phi(\theta^{-1}(\eta(l))) \leq 0.$$

In view of [24], Theorem 1, we have that (22) also has a positive solution, which is a contradiction (22). Thus, the proof is complete. \square

The following theorems give the criteria for oscillation for all solutions of Equation (1).

Theorem 6. *If (5) and (21) are oscillatory, then (1) is oscillatory.*

Proof. Assume on the contrary that u is an eventually positive solution of (1). Then, from Lemma 3, we conclude that there are two possible cases for the behavior of z and its derivatives. By using Theorem 3 and Lemma 4, conditions (5) and (21) ensure that there are no solutions for Equation (1) satisfy case (1) and case (2) respectively. Thus, the proof is complete. \square

Theorem 7. *If (17) and (21) are oscillatory, then (1) is oscillatory.*

Proof. Assume on the contrary that u is an eventually positive solution of (1). Then, from Lemma 3, we conclude that there are two possible cases for the behavior of z and its derivatives. By using Theorem 5 and Lemma 4, conditions (17) and (21) ensure that there are no solutions for Equation (1) satisfying case (1) and case (2), respectively. Thus, the proof is complete. \square

The following corollaries provided criteria for the oscillation of the first-order equations that were used in the comparison.

Corollary 1. *If there exists a function $\zeta(l) \in C([l_0, \infty), (0, \infty))$ satisfying $\eta(l) < \zeta(l)$ and $\theta^{-1}(\zeta(l)) < l$, such that*

$$\liminf_{l \rightarrow \infty} \int_{\theta^{-1}(\zeta(l))}^l R_{n-2}^\alpha(\zeta(\rho), \eta(\rho)) \frac{Q(\rho)}{\mu} d\rho \geq \frac{\theta_0 + p_0^\alpha}{\theta_0 e} \tag{27}$$

and

$$\liminf_{l \rightarrow \infty} \int_{\eta(\theta^{-1}(l))}^l Q(l) \left(\frac{\lambda_0}{(n-1)!r^{1/\alpha}(\eta(l))} (\eta(l))^{n-1}\right)^\alpha d\rho \geq \frac{\theta_0 + p_0^\alpha}{\theta_0 e} \tag{28}$$

hold, then (1) is oscillatory.

Corollary 2. *Let $\delta(l) = \theta(l)$ in Theorem 4. If $\eta(l) < \theta(\theta(l))$, such that (28) and*

$$\limsup_{l \rightarrow \infty} \frac{1}{\mu} \frac{R_{n-2}^\alpha(\theta(\theta(l)), \eta(l))}{r(\theta(\theta(l)))} \int_{\theta(l)}^l Q(\rho) d\rho > \frac{\theta_0 + p_0^\alpha}{\theta_0} \tag{29}$$

hold, then (1) is oscillatory.

Corollary 3. *If $\eta(\theta(l)) < l$, such that (28) and*

$$\liminf_{l \rightarrow \infty} \int_{\eta(\theta(l))}^l Q_1(\rho) R_{n-2}^\alpha(\theta(\rho), \rho) > \frac{\theta_0 + p_0^\alpha}{\eta_0 \theta_0 e} \tag{30}$$

hold, then (1) is oscillatory.

Example 1. Consider the differential equation

$$\left((u(l) + pu(\delta l))^{(n-1)\alpha} \right)' + \frac{q_0}{l^{\alpha(n-1)+1}} u^\alpha(\lambda l) = 0, \quad l \geq 1 \tag{31}$$

From (31), we have $r(l) = 1, p(l) = p, \theta(l) = \delta l, \eta(l) = \lambda l$ and $q(l) = q_0/l^{\alpha(n-1)+1}$. Using some mathematical operations. By using Corollary 1, we find that (31) is oscillatory if

$$q_0 \ln \left(\frac{2\delta}{\delta + \lambda} \right) > \frac{\mu(\theta_0 + p_0^\alpha)}{\theta_0 e} \left((n-1)! \left(\frac{2}{\delta - \lambda} \right)^{n-1} \right)^\alpha$$

and

$$q_0 \ln \left(\frac{\lambda}{\delta} \right) > \frac{(\theta_0 + p_0^\alpha) ((n-1)!)^\alpha}{\theta_0 e \lambda_0^\alpha \lambda^{\alpha(n-1)}}.$$

By using Corollary 3, we find that (31) is oscillatory if

$$q_0 \ln \left(\frac{\lambda}{\delta} \right) > \frac{(\theta_0 + p_0^\alpha) ((n-1)!)^\alpha}{\theta_0 e \lambda_0^\alpha \lambda^{\alpha(n-1)}}$$

and

$$q_0 \ln \left(\frac{1}{\delta \lambda} \right) > \frac{(\theta_0 + p_0^\alpha)}{\eta_0 \theta_0 e} \frac{((n-1)!)^\alpha}{\lambda^{\alpha(n-1)+1} (\delta - 1)^{\alpha(n-1)}}.$$

4. Conclusions

This article is concerned with oscillatory properties of solutions for the odd-order neutral equation. Many works have studied the oscillatory properties of solutions of an odd-order equation; see [17,18]. However, in these works, we find sufficient conditions to ensure that every non-oscillatory solution tends to zero, that is, conditions that guarantee that all solutions are oscillatory or tend to zero. Unusually, in this paper, we presented new criteria ensuring that all solutions of (1) are oscillatory, which in turn is an improvement and extension of the results in [17,18]. For this purpose, we used the comparison technique with first-order equations. For ease of application in the examples, Corollaries 1–3 provided criteria for the oscillation of the first-order equations that were used in the comparison.

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