DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **13**, Number **3**, March **2020**

doi:10.3934/dcdss.2020023

pp. **407–428**

NEW ASPECTS OF TIME FRACTIONAL OPTIMAL CONTROL PROBLEMS WITHIN OPERATORS WITH NONSINGULAR KERNEL

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Abstract. This paper deals with a new formulation of time fractional optimal control problems governed by Caputo-Fabrizio (CF) fractional derivative. The optimality system for this problem is derived, which contains the forward and backward fractional differential equations in the sense of CF. These equations are then expressed in terms of Volterra integrals and also solved by a new numerical scheme based on approximating the Volterra integrals. The linear rate of convergence for this method is also justified theoretically. We present three illustrative examples to show the performance of this method. These examples also test the contribution of using CF derivative for dynamical constraints and we observe the efficiency of this new approach compared to the classical version of fractional operators.

2010 *Mathematics Subject Classification.* Primary: 49K99, 34A08; Secondary: 34H05, 49M25.
Key words and phrases. Optimal control, nonsingular kernel, fractional calculus, error estimates, Volterra integrals.

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1. **Introduction.** Fractional calculus has been an active research field for decades and several promising ideas have been proposed in different fields of science and engineering. Fractional differentiation and integration operators, which are the generalization of classical integer-order counterparts, are capable of capturing memory effects due to their nonlocal nature [48, 51, 39]. Let's see the excellent paper of J. Liouville from 1832, namely: J. Liouville, M*é*moire: Sur quelques questions de g*é*om*é*trie et de m*é*canique, et sur un nouveau genre de calcul pour r*é*soudre ces questions, J l'Ecole Polyt*é*ch 13 (21) (1832) 1–66. In this very interesting work, we can find a simple fractional derivative which is not in the class that some colleagues tried to classified (see J. Liouville's paper [40, p. 3]). Thus, we think that any attempt to make a classification of fractional operators is just a nice discussion because the non-locality cannot be described by a single class of fractional operators. As J. Liouville said in his paper, we have to consider applications to validate our fractional calculus models. Only experimental tests, as it is very well known in science and engineering, will validate or not one of the fractional calculus operators for a given non-local model. Not always the more complicated fractional model will give a better result than a simple fractional one. In the literature, fractional operators have been incorporated to model different dynamics mathematically. In [14], it has been shown that physical properties of viscoelastic materials can be expressed through a model with fractional derivatives. In [61], Zhang et al. have reviewed the fractional calculus in terms of Earth sciences. Moreover, fractional calculus in continuous and discrete cases [33, 5, 4, 11, 7, 6, 1, 9, 2, 3] have been applied to physics [26], bioengineering [42], optics [24], fluid flow [21], energy systems [37] and biology [56, 54, 50, 47, 31, 30].

The Caputo fractional derivative is one of the mostly used fractional differentiation operators. Other than this operator, several different fractional derivatives have been defined and proposed in the literature. One of the newly defined operators is Caputo-Fabrizio (CF) fractional derivative, which is capable of eliminating singular kernel in the definition of classical Caputo derivative [18]. In the study [29], the underlying physical meaning of the nonsingular kernel was investigated and a collection of recent applications of fractional differentiation operators with non-singular kernels was presented. On the other hand, the effect of different kernels were compared for the fractional diffusion equation within the context of continuous time random walk, the efficiency and future contribution of CF derivative were underlined in [53]. A heat conduction equation with a relaxation term is obtained through the Cattaneo constitutive equation with Jeffrey's fading memory and the relaxation term corresponds to the CF time-fractional derivative in [28]. Moreover, the determination of the fractional order as a ratio of the relaxation time to the characteristic diffusion time of the process has been investigated in [27].

Several dynamical systems have been modeled through CF derivative. For example, material damping which is an incident where the actual stress depends on the entire strain history and the memory kernel which is used to define the stress cannot be chosen arbitrary. Indeed, singularity in the kernel must be eliminated. In addition, the fading memory is described by a monotonic and decreasing function of time. At that point, vital choice of exponential kernel must be underlined [23]. On the other hand, Casson fluid has been modeled in [52]. In [12], Bergman minimal blood glucose-insulin model has been generalized. The CF operator has also been used to investigate the motion of a charged field [44]. On the other hand, steady heat flow with CF derivative is solved in the study [58].

In addition to fractional differential equations (FDEs), several real-world problems can be modeled through optimal control theory. For instance, optimal treatment schedules for tuberculosis have been investigated in [25], while a SEIRS model is proposed with vaccination and treatment as control functions in [43]. In the study [19], a dynamic model for smoking–tuberculosis transmission has been studied based on the data in South Korea and optimal treatment strategy has been decided, while a model for fish harvesting has been investigated based on optimal control theory in [20]. On the other hand, a temperature control problem has been studied for the solid-liquid phase transition [46]. Moreover, it has been observed that the OCPs governed by FDEs have led to valuable results. For example, fractional optimal control problems (FOCPs) have been solved numerically using Legendre orthonormal polynomial basis [41]. In [32], the FOCP has been converted into a classical static optimization problem and Ritz's direct method has been used to obtain a new numerical scheme. The state and control functions have been approximated by the Legendre orthonormal basis in [45]. A pseudo-spectral method has been applied in [22]. Optimality conditions for fractional variational problems have been derived in [60], while a FOCP has been investigated for a model containing integer and fractional order derivatives in [34]. Different numerical algorithms for FOCPs governed by Caputo derivative have been investigated in [57]. Moreover, a constrained dynamic optimization problem defined by Riemann-Liouville derivative is solved with the use of Grünwald-Letnikov definition in [16]. As a different model, a free final time FOCP governed by a dynamical model involving integer and fractional order derivatives has been studied in [49]. On the other hand, the existence of optimal pairs of system governed by fractional evolution equations has been proven in [55]. Other than these mathematical studies, some applications of FOCPs can be mentioned as follows. Fractional descriptor continuous-time linear systems described by the CF derivative have been studied and necessary and sufficient conditions for the positivity and stability of the systems were established [36, 35]. The FOCP with Mittag-Leffler derivative has been analyzed in the recent paper [15].

Numerical methods for the FOCPs governed by Caputo derivative have been proposed in several studies as summarized above. However, the main drawback of this operator is its singular kernel, and hence, some nonlocal dynamics cannot be modeled properly through the use of Caputo derivative. Therefore, the new CF fractional derivative has been proposed in [17] by eliminating the singular kernel. Mathematical modeling of dynamical systems within the use of CF operator has been studied for a while; however, the FOCP formulation subject to a FDE with CF derivative is a new developing research field. Note that, determining the analytical solution of FOCPs in CF sense is not an easy task; hence, accurate and efficient numerical schemes are needed to obtain the numerical solution within this new calculus. Motivated by the aforementioned discussion, this paper proposes a new formulation of FOCPs governed by a FDE with CF derivative. The optimality system for this problem is derived, which contains forward and backward CF derivatives. We also provide a new numerical scheme to solve this problem, which exhibits linear rate of convergence. To the best of our knowledge, this is the first study investigating the performance of FOCPs in terms of a new first order numerical scheme designed for the CF derivative. We illustrate the performance of this method with three numerical examples. These examples also verify the efficiency of this new formulation over the use of Caputo derivative in terms of the asymptotic behaviour of controlled system like settling time.

The rest of this paper is organized as follows. In Sect. 2, we state the definition and properties of CF derivative. Section 3 introduces the FOCP with CF derivative and presents its necessary optimality conditions. In Sect. 4, a new scheme is developed for both of the state and adjoint equations. In Sect. 5, a convergence estimate is provided. Section 6 discusses three numerical examples to underline the efficiency of the new technique. Then, the paper ends with summary and conclusion.

2. **Preliminaries.** In the literature, there are several definitions of fractional differential operators including Caputo, Riemann-Liouville, Marchaud and Hadamard [39]; however, all of these operators suffer from singular kernel. In order to eliminate this drawback, some new fractional derivatives with nonsingular kernel have been formulated through the use of Mittag-Leffler function [13], Sinc function [59] and exponential function [17]. In this section, we briefly review the new fractional derivative with exponential kernel proposed by Caputo and Fabrizio in [17] and mention the required definitions and properties following the paper [8].

The left CF fractional derivative is defined for $0 \leq \alpha \leq 1$ in the Caputo sense as

$$ {}_0^{CF}\mathcal{D}_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(s) \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds, \tag{1} $$

and the corresponding right differentiation operator is modified by

$$ {}_t^{CF}\mathcal{D}_T^{\alpha} f(t) = -\frac{M(\alpha)}{1-\alpha} \int_t^T f'(s) \exp\left(-\frac{\alpha}{1-\alpha}(s-t)\right) ds, \tag{2} $$

where $M(\alpha)$ is a normalization function with $M(0) = 1 = M(1)$. In addition, the right Riemann-Liouville differentiation operator in the sense of CF is proposed in the form

$$ {}_t^{CFR}\mathcal{D}_T^{\alpha} f(t) = -\frac{M(\alpha)}{1-\alpha} \frac{d}{dt} \int_t^T f(s) \exp\left(-\frac{\alpha}{1-\alpha}(s-t)\right) \, ds. \tag{3} $$

The corresponding left and right fractional integrals are also given respectively as

$$ {}_0^{CF}\mathcal{I}_t^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} \int_0^t f(s) \, ds, \tag{4a} $$

$$ {}_t^{CF}\mathcal{I}_T^{\alpha} f(t) = \frac{1-\alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha)} \int_t^T f(s) \, ds. \tag{4b} $$

Before ending this section, we mention the following useful relations between the aforementioned fractional and integral operators

$$ {}_0^{CF}\mathcal{I}_t^{\alpha} \, {}_0^{CF}\mathcal{D}_t^{\alpha} f(t) = f(t) - f(0), \tag{5a} $$

$$ {}_t^{CF}\mathcal{I}_T^{\alpha} \, {}_t^{CF}\mathcal{D}_T^{\alpha} f(t) = f(t) - f(T). \tag{5b} $$

3. **Fractional optimal control problem formulation.** In this section, we consider the following formulation of FOCPs regarding the CF fractional derivative

$$ \min J(u) = \int_0^T F(t, x(t), u(t)) \, dt, \tag{6a} $$

$$ \text{subject to } {}_0^{CF}\mathcal{D}_t^{\alpha} x(t) = G(t, x(t), u(t)), \quad 0 < t \leq T, \tag{6b} $$

$$x(0) = x_0, \tag{6c}$$

where $x \in \mathbb{R}^n$ is the state vector while $u \in \mathbb{R}^m$ denotes the control variable, $F : \mathbb{R}^{n+m+1} \to \mathbb{R}$ and $G : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ are nonlinear scalar and vector functions, respectively. The symbol ${}_0^{CF}\mathcal{D}_t^{\alpha}x(t)$ denotes the left CF fractional derivative defined by Eq. (1), the parameter T is the final time, and x_0 is the specified initial state vector. The aim behind solving the FOCP (6) is to obtain the optimum control $u^*(t)$ for dynamical constraints (6b)-(6c) so that the cost functional $J(u)$ is minimized. In order to solve this problem, we give and prove a new theorem deriving the necessary optimality conditions associated to the FOCP (6).

Theorem 3.1. *The pair $(x(t), u(t))$ is a minimum of (6) if there exists an adjoint function $p(t)$, where the triple $(x(t), u(t), p(t))$ satisfies the following optimality system*

$$ {}_0^{CF}\mathcal{D}_t^{\alpha}x(t) = G(t, x(t), u(t)), \quad x(0) = x_0, \tag{7a}$$

$$ {}_t^{CF}\mathcal{D}_T^{\alpha}p(t) = \frac{\partial F(t, x(t), u(t))}{\partial x(t)} + p(t)\frac{\partial G(t, x(t), u(t))}{\partial x(t)}, \quad p(T) = 0, \tag{7b}$$

$$ \frac{\partial F(t, x(t), u(t))}{\partial u(t)} + p(t)\frac{\partial G(t, x(t), u(t))}{\partial u(t)} = 0. \tag{7c}$$

Proof. To construct the optimality system associated to the problem (6), we define the modified performance index as

$$\bar{J}(u) = \int_0^T \Big[F(t, x(t), u(t)) - p^T(t)\Big({}_0^{CF}\mathcal{D}_t^{\alpha}x(t) - G(t, x(t), u(t))\Big)\Big]\, dt, \tag{8}$$

where $p(t) \in \mathbb{R}^n$ is the co-state or adjoint function. We take the first variation of $\bar{J}(u)$ to find

$$\begin{aligned}\delta\bar{J}(u) = \int_0^T \Big[&\left(\frac{\partial F}{\partial x}\right)^T \delta x + \left(\frac{\partial F}{\partial u}\right)^T \delta u \\ &- (\delta p(t))^T \Big({}_0^{CF}\mathcal{D}_t^{\alpha}x(t) - G(t, x(t), u(t))\Big) \\ &- p^T(t)\Big({}_0^{CF}\mathcal{D}_t^{\alpha}\delta x(t) - \left(\frac{\partial G}{\partial x}\right)^T \delta x - \left(\frac{\partial G}{\partial u}\right)^T \delta u\Big)\Big]\, dt. \end{aligned} \tag{9}$$

For the time fractional term in (9), we apply the method of integration by parts as

$$\int_0^T p^T(t)\Big({}_0^{CF}\mathcal{D}_t^{\alpha}\delta x(t)\Big)\, dt = \int_0^T \Big({}_t^{CFR}\mathcal{D}_T^{\alpha}p(t)\Big)^T \delta x(t)\, dt, \tag{10}$$

with the condition $p(T) = 0$. Then, following the study [8, Prop. 2], we observe that

$$\begin{aligned} &\int_0^T \Big({}_t^{CFR}\mathcal{D}_T^{\alpha}p(t)\Big)^T \delta x(t)\, dt \\ &= \int_0^T \Big({}_t^{CF}\mathcal{D}_T^{\alpha}p(t)\Big)^T \delta x(t)\, dt + \frac{B(\alpha)}{1-\alpha}\underbrace{p(T)}_{=0}\exp\Big(-\frac{\alpha}{1-\alpha}(T-t)\Big) \\ &= \int_0^T \Big({}_t^{CF}\mathcal{D}_T^{\alpha}p(t)\Big)^T \delta x(t)\, dt. \end{aligned} \tag{11}$$

We substitute (11) into (9) to reach the equation

$$\delta\bar{J}(u) = \int_0^T \left(\frac{\partial F}{\partial x} + p(t)\frac{\partial G}{\partial x} - {}_t^{CF}\mathcal{D}_T^{\alpha} p(t)\right)^T \delta x \\ + \left(\frac{\partial F}{\partial u} + p(t)\frac{\partial G}{\partial u}\right)^T \delta u - (\delta p)^T \left({}_0^{CF}\mathcal{D}_t^{\alpha} x(t) - G(t, x(t), u(t))\right) dt. \quad (12)$$

The optimality system for Eq. (6) is obtained through the equation $\delta\bar{J}(u) = 0$, which must hold for all variations $\delta x(t), \delta u(t)$ and $\delta p(t)$. By taking the variations into consideration, we see that $\delta x(t) \neq 0$, $\delta u(t) \neq 0$ and $\delta p(t) \neq 0$ in general. Hence, for $\delta\bar{J}(u) = 0$ to be satisfied, the terms in the parenthesis in (12) must be equal to zero. Therefore, we obtain the adjoint equation (7b), the gradient equation (7c) and the state equation (7a), respectively. □

The optimality system serves as a tool to find the solution of FOCP (6). The state equation (7a), which can be linear or nonlinear, involves a left fractional derivative whereas the adjoint equation (7b) is modeled with a right fractional operator; hence, the analytical solution of these equations cannot be obtained for complicated FDEs. To overcome this difficulty, we need to use approximation techniques, which will be discussed in the next section.

4. **Proposed numerical model.** In this section, we propose an efficient discretization method for both of the state and adjoint equations (7a)-(7b). Here, we assume that the control variable $u(t)$ can be extracted explicitly from Eqn. (7c) as a function of the state and adjoint variables as $u(t) = K(t, x(t), p(t))$. We substitute $u(t) = K(t, x(t), p(t))$ into Eqs. (7a) and (7b) before discretizing the state and adjoint equations. Then, we solve the state equation forward starting with $x(0) = x_0$ and the adjoint equation backward through the use of $p(T) = 0$. Hence, we should construct two different discretization schemes. Before obtaining the approximate solution, we discretize the mesh $I = [0, T]$ as

$$0 = t_0 < t_1 < \ldots < t_M = T, \quad (13)$$

with constant time step $\Delta t = \frac{T}{M}$. We denote the numerical approximations of $x(t), p(t)$ at $t = t_i$ as x_i, p_i for $0 \leq i \leq M$, respectively. Now, we proceed with the derivation of discrete schemes for the state and adjoint equations, as the following subsections state.

4.1. **Discretization of state equation.** By using the properties of CF integral and derivative in (4a) and (5a), we obtain the Volterra integral equation corresponding to Eq. (7a) as

$$x(t) = x(0) + \frac{1-\alpha}{B(\alpha)} G_1(t, x(t), p(t)) + \frac{\alpha}{B(\alpha)} \int_0^t G_1(s, x(s), p(s))\, ds, \quad (14)$$

where $G_1(t, x(t), p(t)) := G(t, x(t), K(t, x(t), p(t))$. We discretize Eq. (14) at $t = t_k$ and apply the forward Euler method to approximate the integral on the right-hand side as

$$x(t_k) = x_0 + \frac{1-\alpha}{B(\alpha)} G_1(t_k, x_k, p_k) + \frac{\alpha}{B(\alpha)} \int_0^{t_k} G_1(s, x(s), p(s))\, ds$$

$$= x_0 + \frac{1-\alpha}{B(\alpha)} G_1(t_k, x_k, p_k) + \frac{\alpha}{B(\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} G_1(s, x(s), p(s))\, ds$$

$$\approx x_0 + \frac{1-\alpha}{B(\alpha)} G_1(t_k, x_k, p_k) + \Delta t \frac{\alpha}{B(\alpha)} \sum_{j=1}^{k} G_1(t_{j-1}, x_{j-1}, p_{j-1}). \tag{15}$$

Then, the discrete scheme for the state is expressed by

$$x_k = x_0 + \frac{1-\alpha}{B(\alpha)} G_1(t_k, x_k, p_k) + \Delta t \frac{\alpha}{B(\alpha)} \sum_{j=1}^{k} G_1(t_{j-1}, x_{j-1}, p_{j-1}). \tag{16}$$

4.2. **Discretization of adjoint equation.** We use the definition (4b) and property (5b) to obtain the Volterra integral form of adjoint equation (7b) as

$$p(t) = \underbrace{p(T)}_{=0} + \frac{1-\alpha}{B(\alpha)} H_1(t, x(t), p(t)) + \frac{\alpha}{B(\alpha)} \int_{t}^{T} H_1(s, x(s), p(s))\, ds, \tag{17}$$

where $H_1(t, x(t), p(t)) := \left(\frac{\partial F(t,x(t),u(t))}{\partial x(t)} + p(t) \frac{\partial G(t,x(t),u(t))}{\partial x(t)} \right) \Big|_{u(t)=K(t,x(t),p(t))}$. We fix $t = t_k$ in (17) and use the backward Euler method to approximate the integral in (17) to obtain

$$p(t_k) = \frac{1-\alpha}{B(\alpha)} H_1(t_k, x_k, p_k) + \frac{\alpha}{B(\alpha)} \int_{0}^{t_k} H_1(s, x(s), p(s))\, ds$$

$$= \frac{1-\alpha}{B(\alpha)} H_1(t_k, x_k, p_k) + \frac{\alpha}{B(\alpha)} \sum_{j=k+1}^{M} \int_{t_{j-1}}^{t_j} H_1(s, x(s), p(s))\, ds$$

$$\approx \frac{1-\alpha}{B(\alpha)} H_1(t_k, x_k, p_k) + \Delta t \frac{\alpha}{B(\alpha)} \sum_{j=k+1}^{M} H_1(t_j, x_j, p_j). \tag{18}$$

Then, we reach the following numerical scheme for the adjoint equation

$$p_k = \frac{1-\alpha}{B(\alpha)} H_1(t_k, x_k, p_k) + \Delta t \frac{\alpha}{B(\alpha)} \sum_{j=k+1}^{M} H_1(t_j, x_j, p_j). \tag{19}$$

As it is shown, Eqs. (16) and (19) form a coupled system of nonlinear algebraic equations. Solving these equations, the unknown variables x_k and p_k are determined for $k = 0, ..., M$; then, the discrete optimal control is obtained from $u_k = K(t_k, x_k, p_k)$. However, before applying the proposed numerical method, we must be sure that the proposed scheme has a good convergence property. In other words, we must be sure that the numerical error decays as we decrease the length of time step. Hence, we will present error estimates and determine the rate of convergence of the new technique in the next section.

5. **Error estimates and convergence analysis.** In this section, we derive the rate of convergence of the new approximation scheme to solve the following fractional initial value problem (IVP)

$$ {}_{0}^{CF}\mathcal{D}_t^{\alpha} x(t) = \mathcal{G}(t, x(t)), \quad 0 < t \leq 1, \tag{20a} $$

$$ x(0) = x_0. \tag{20b} $$

Here, we motivated by the study [38, Sec. 3] to derive error estimates. Before presenting the main theoretical results, we start with the justification of two lemmas.

Lemma 5.1. *Let $x(t)$ be the solution of Eq. (20) and $\mathcal{G}(t, x(t))$ be a bounded and Lipschitz continuous function with respect to both t and x with a Lipschitz constant L. Then, we have*

$$ |x(t) - x(t - \Delta t)| = \mathcal{O}(\Delta t), \tag{21} $$

under the condition $1 - L\frac{1-\alpha}{B(\alpha)} \geq 0$.

Proof. We will apply the technique in [38, Lemma 3.6]. Let $|\mathcal{G}(t, x(t))| < M$; then, we have

$$ \begin{aligned} & x(t) - x(t - \Delta t) \\ &= \frac{1-\alpha}{B(\alpha)} \left(\mathcal{G}(t, x(t)) - \mathcal{G}(t - \Delta t, x(t - \Delta t)) \right) \\ &\quad + \frac{\alpha}{B(\alpha)} \left(\int_0^t \mathcal{G}(s, x(s))\, ds - \int_0^{t-\Delta t} \mathcal{G}(s, x(s))\, ds \right) \\ &\leq \frac{1-\alpha}{B(\alpha)} \left(|\mathcal{G}(t, x(t)) - \mathcal{G}(t - \Delta t, x(t))| \right) \\ &\quad + \frac{1-\alpha}{B(\alpha)} \left(|\mathcal{G}(t - \Delta t, x(t)) - \mathcal{G}(t - \Delta t, x(t - \Delta t))| \right) \\ &\quad + \frac{\alpha}{B(\alpha)} \int_{t-\Delta t}^{t} |\mathcal{G}(s, x(s))|\, ds \\ &\leq \frac{1-\alpha}{B(\alpha)} \left(L\Delta t + L|x(t) - x(t - \Delta t)| \right) + \frac{\alpha}{B(\alpha)} M \Delta t. \end{aligned} \tag{22} $$

Hence, we obtain

$$ |x(t) - x(t - \Delta t)| \leq \Delta t \frac{(1-\alpha)(LB(\alpha) + M\alpha)}{B(\alpha)(B(\alpha) - (1-\alpha)L)}, \tag{23} $$

with $1 - L\frac{1-\alpha}{B(\alpha)} \geq 0$. Thus, the estimate (21) is achieved. □

Lemma 5.2. *Let $x(t)$ be the solution of Eq. (20) and $\mathcal{G}(t, x(t))$ be a bounded and Lipschitz continuous function with respect to both t and x with a Lipschitz constant L. Then, we have*

$$ \left| \int_0^{t_k} \mathcal{G}(t, x(t))\, dt - \Delta t \sum_{j=1}^{k} \mathcal{G}(t_{j-1}, x(t_{j-1})) \right| \leq C t_k\, \Delta t, \tag{24} $$

under the condition $1 - L\frac{1-\alpha}{B(\alpha)} \geq 0$.

Proof. The proof is motivated from the idea in [38, Lemma 3.7]. We have

$$\begin{aligned}
&\left|\int_0^{t_k} \mathcal{G}(t,x(t))\,dt - \Delta t \sum_{j=1}^{k} \mathcal{G}(t_{j-1},x(t_{j-1}))\right| \\
&\le \left|\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} [\mathcal{G}(t,x(t)) - \mathcal{G}(t_{j-1},x(t_{j-1}))]\,dt\right| \\
&\le \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} |[\mathcal{G}(t,x(t)) - \mathcal{G}(t,x(t_{j-1}))] + [\mathcal{G}(t,x(t_{j-1})) - \mathcal{G}(t_{j-1},x(t_{j-1}))]|\,dt \\
&\le L \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big[|x(t) - x(t_{j-1})| + |t - t_{j-1}|\Big]\,dt. \qquad (25)
\end{aligned}$$

Then, by applying Lemma 5.1 to the first term and bounding the second term, we obtain the final result. □

Now, we present the main theoretical result of this section. We prove that the error between the exact and numerical solution of Eq. (20) decays linearly as we halve the length of time step.

Theorem 5.3. *Let $x(t_k)$ and x_k be the exact and numerical solution of IVP (20) at $t = t_k$. Let $\mathcal{G}(t,x(t))$ be a bounded and Lipschitz continuous function with respect to both t and x with a Lipschitz constant L. Then, we have*

$$|x(t_k) - x_k| = \mathcal{O}(\Delta t), \qquad (26)$$

under the condition $1 - L\frac{1-\alpha}{B(\alpha)} \ge 0$.

Proof. We consider the error between the exact and numerical solution (obtained from Eq. (16)) at $t = t_k$. Using the Lipschitz continuity of $\mathcal{G}(t,x(t))$ and Lemma 5.2, we obtain

$$\begin{aligned}
&|x(t_k) - x_k| \\
&\le \frac{1-\alpha}{B(\alpha)} |\mathcal{G}(t_k,x(t_k)) - \mathcal{G}(t_k,x_k)| \\
&\quad + \frac{\alpha}{B(\alpha)} \left|\int_0^{t_k} \mathcal{G}(t,x(t))\,dt - \Delta t \sum_{j=1}^{k} \mathcal{G}(t_{j-1},x_{j-1})\right| \\
&\le \frac{1-\alpha}{B(\alpha)} |\mathcal{G}(t_k,x(t_k)) - \mathcal{G}(t_k,x_k)| \\
&\quad + \frac{\alpha}{B(\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} |\mathcal{G}(t,x(t)) - \mathcal{G}(t_{j-1},x_{j-1})|\,dt \\
&\le \frac{1-\alpha}{B(\alpha)} |\mathcal{G}(t_k,x(t_k)) - \mathcal{G}(t_k,x_k)|
\end{aligned}$$

$$+ \frac{\alpha}{B(\alpha)} \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} |\mathcal{G}(t, x(t)) - \mathcal{G}(t_{j-1}, x(t_{j-1}))| \, dt$$

$$+ \Delta t \frac{\alpha}{B(\alpha)} \sum_{j=1}^{k} |\mathcal{G}(t_{j-1}, x(t_{j-1})) - \mathcal{G}(t_{j-1}, x_{j-1})|$$

$$\leq L \frac{1-\alpha}{B(\alpha)} |x(t_k) - x_k| + \frac{\alpha}{B(\alpha)} L \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Big[|x(t) - x(t_{j-1})| + |t - t_{j-1}| \Big] \, dt$$

$$+ L \Delta t \frac{\alpha}{B(\alpha)} \sum_{j=1}^{k} |x(t) - x(t_{j-1})|. \tag{27}$$

Then, the Gronwall's Lemma is applied to Eq. (27) to reach the final estimate. □

6. **Illustrative examples.** In this section, we present three numerical examples to test the practical use of our proposed method. We depict the trajectory of controlled system to investigate the contribution of CF derivative over the classical Caputo fractional operator. In addition, we measure the computational time to observe the efficiency of the new scheme. Furthermore, we compute the decay rate of the cost functional J by halving the time step in case that exact solutions of the state and control equations are not available. We denote the value of J computed with M as J_M. Then, the rate ρ is calculated as

$$\rho = \log_2 \Big(\frac{J_M - J_{M/2}}{J_{M/2} - J_{M/4}} \Big).$$

If the exact solutions of the state and control are known, then we measure the absolute error in the state, control and cost functional as $e_x = \|x(t) - x^*(t)\|_\infty$, $e_u = \|u(t) - u^*(t)\|_\infty$ and $e_J = |J_M - J^*|$, respectively. Here, the term $\|g(t)\|_\infty$ denotes the maximum norm of the function $g(t)$ over $[0, T]$. Moreover, we compute the rate of convergence as

$$r_z = \log_2 \Big(\frac{(e_z)_{M/2}}{(e_z)_M} \Big),$$

by halving the time step where z stands for x, u or J.

Example 1. We consider the following linear time-invariant problem

$$\min \; J(u) = \frac{1}{2} \int_0^1 \big((x(t) - x_d(t))^2 + u^2(t) \big) \, dt, \tag{28}$$

subject to

$${}_0^{CF}D_t^{\alpha} x(t) = -x(t) + f(t) + u(t), \; 0 < t \leq 1, \quad x(0) = 2. \tag{29}$$

The necessary optimality conditions of the FOCP (28)-(29) are written in the form

$$\begin{cases} {}_0^{CF}D_t^{\alpha} x(t) = -x(t) - p(t) + f(t), \quad 0 < t \leq 1, \\ {}_t^{CF}D_1^{\alpha} p(t) = x(t) - x_d(t) - p(t), \quad 0 \leq t < 1, \\ x(0) = 2, \quad p(1) = 0. \end{cases} \tag{30}$$

We note that the corresponding gradient equation, which has been substituted into the state equation in (30), is written as

$$u^*(t) = -p(t), \quad 0 \leq t \leq 1. \tag{31}$$

TABLE 1. Example 1: The values of J, absolute error, order of convergence and computational time (CT) for $\alpha = \{0.6, 0.7\}$.

	$\alpha = 0.6$				$\alpha = 0.7$			
M	J	e_J	r_J	CT	J	e_J	r_J	CT
50	4.2795	0.0116	-	0.33	6.1951	0.0309	-	0.29
100	4.2818	0.0058	1.00	0.43	6.2030	0.0154	1.00	0.41
200	4.2838	0.0029	1.00	0.74	6.2088	0.0077	1.00	0.77
400	4.2850	0.0015	0.95	2.97	6.2122	0.0039	0.98	2.88
800	4.2857	0.00073	1.03	18.91	6.2140	0.0019	1.03	19.74

TABLE 2. Example 1: The values of J, absolute error, order of convergence and computational time (CT) for $\alpha = \{0.8, 0.9\}$.

	$\alpha = 0.8$				$\alpha = 0.9$			
M	J	e_J	r_J	CT	J	e_J	r_J	CT
50	9.9900	0.0863	-	0.31	20.2108	0.2844	-	0.27
100	10.0091	0.0432	0.99	0.38	20.1666	0.1425	0.99	0.40
200	10.0247	0.0216	1.00	0.76	20.1912	0.0713	0.99	0.74
400	10.0339	0.0108	1.00	2.43	20.2152	0.0356	1.00	2.90
800	10.0390	0.0054	1.00	18.85	20.2301	0.0178	1.00	19.18

For this problem, we construct the exact solutions of the state and adjoint equations as

$$x^*(t) = 1 + \exp(-2t), \quad p^*(t) = \exp(t) - \exp(1). \tag{32}$$

We determine the source function $f(t)$ and desired state $x_d(t)$ by substituting (32) into (30) as

$$\begin{aligned} f(t) &= \frac{(2-\alpha)M(\alpha)}{2(2-3\alpha)} \left(2\exp(-2t) - \frac{\alpha}{1-\alpha} \exp\left(-\frac{\alpha}{1-\alpha} t \right) \right) \\ &\quad + \exp(-2t) + \exp(t) + 1 - \exp(1), \end{aligned} \tag{33}$$

$$\begin{aligned} x_d(t) &= -\exp(1) \frac{(2-\alpha)M(\alpha)}{2(2-3\alpha)} \left(\exp(t-1) - \frac{\alpha}{1-\alpha} \exp\left(-\frac{\alpha}{1-\alpha}(1-t) \right) \right) \\ &\quad + \exp(-2t) - \exp(t) + 1 + \exp(1). \end{aligned} \tag{34}$$

In Table 1-2, we present the values of the cost functional J, absolute error and decay rate for $\alpha = \{0.6, 0.7\}$ and $\alpha = \{0.8, 0.9\}$, respectively. The problem is solved in less then 20 seconds. As we increase α, the values of J increases. The absolute error does not follow the same pattern; but, we observe that the values of J decrease as we halve the step size and we reach the linear convergence rate which is compatible with Theorem 5.3.

In Table 3-4, absolute error and decay rates are shown for the state and control equations. For the state solution, the error is around 10^{-4}. Indeed, the error decays linearly as shown in Theorem. 5.3. On the other hand, the smallest error for the control is around 10^{-3}. We measure the first order decay rate for the control, too.

We present time evolution of the exact and numerical solutions of the state for $\alpha = \{0.6, 0.7, 0.8, 0.9\}$ in Fig. 1. We notice that the state is approximated very well with the use of this scheme.

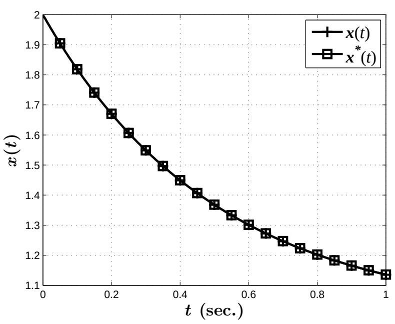
In Fig. 2, we show the exact and numerical solution of the control. We see that the control is solved very accurately over time.

TABLE 3. Example 1: The values of absolute error for $x(t)$ and the order of convergence for $\alpha = \{0.6, 0.7, 0.8, 0.9\}$.

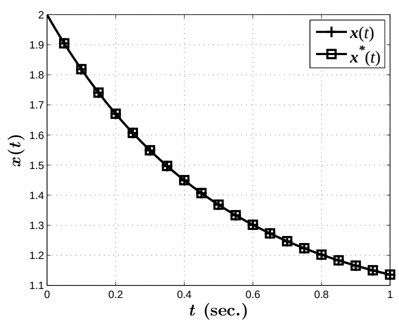
	$\alpha = 0.6$		$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
M	e_x	r_x	e_x	r_x	e_x	r_x	e_x	r_x
50	0.0090	-	0.0085	-	0.0146	-	0.0531	-
100	0.0045	1.00	0.0043	0.98	0.0072	1.01	0.0260	1.03
200	0.0022	1.03	0.0021	1.03	0.0036	1.00	0.0129	1.01
400	0.0011	1.00	0.0011	0.93	0.0018	1.00	0.0064	1.01
800	5.57e-04	0.98	5.34e-04	1.04	8.91e-04	1.01	0.0032	1.00

TABLE 4. Example 1: The values of absolute error for $u(t)$ and the order of convergence for $\alpha = \{0.6, 0.7, 0.8, 0.9\}$.

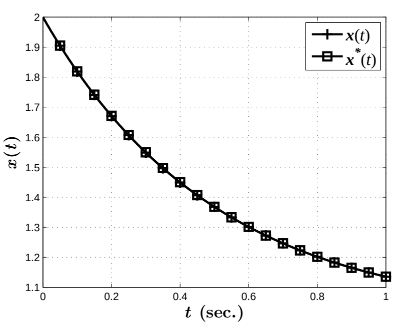
	$\alpha = 0.6$		$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$	
M	e_u	r_u	e_u	r_u	e_u	r_u	e_u	r_u
50	0.0240	-	0.0417	-	0.0770	-	0.1914	-
100	0.0120	1.00	0.0207	1.01	0.0381	1.01	0.0939	1.02
200	0.0060	1.00	0.0103	1.00	0.0190	1.00	0.0465	1.01
400	0.0030	1.00	0.0051	1.01	0.0095	1.00	0.0232	1.00
800	0.0015	1.00	0.0026	0.97	0.0047	1.01	0.0116	1.00



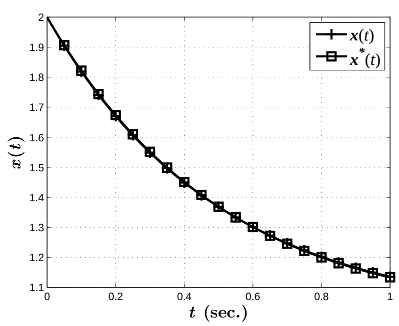
(a) $\alpha = 0.6$.



(b) $\alpha = 0.7$.

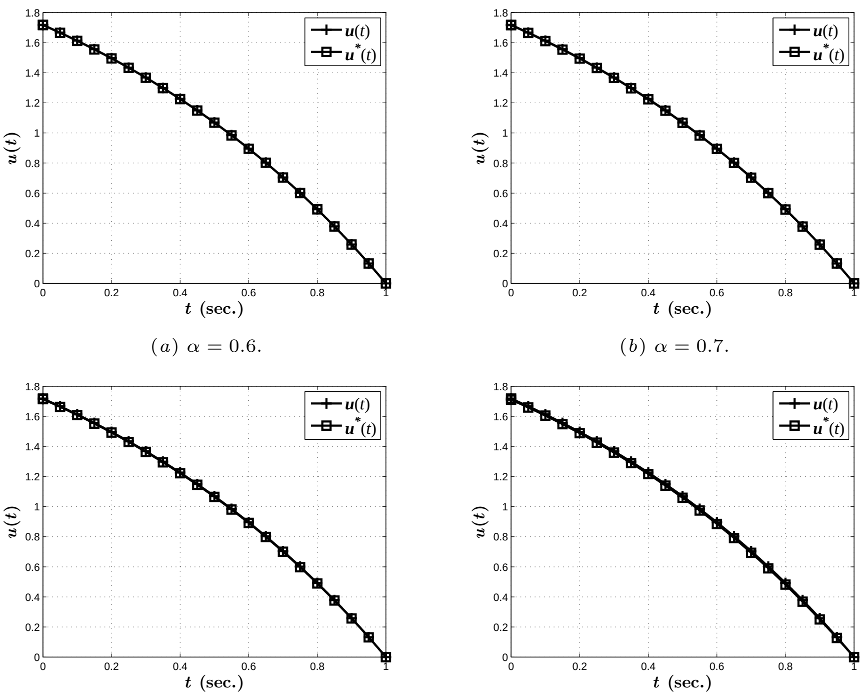


(c) $\alpha = 0.8$.



(d) $\alpha = 0.9$.

FIGURE 1. Example 1: Comparative results of $x(t)$ and $x^*(t)$ for $M = 800$ and $\alpha = \{0.6, 0.7, 0.8, 0.9\}$.



(a) $\alpha = 0.6$. (b) $\alpha = 0.7$.

(c) $\alpha = 0.8$. (d) $\alpha = 0.9$.

FIGURE 2. Example 1: Comparative results of $u(t)$ and $u^*(t)$ for $M = 800$ and $\alpha = \{0.6, 0.7, 0.8, 0.9\}$.

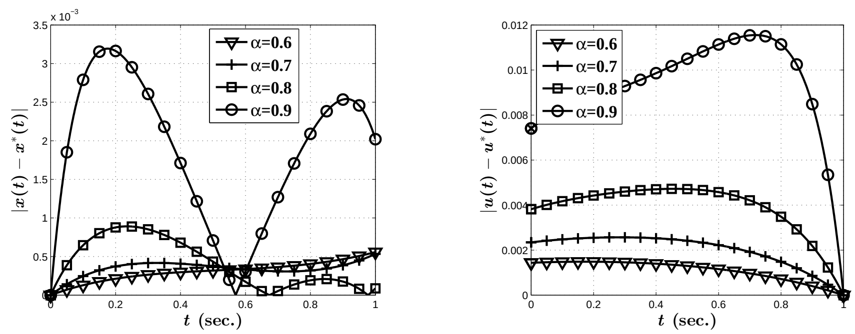


FIGURE 3. Example 1: The absolute error plots for $x(t)$ (*left*) and $u(t)$ (*right*) with $M = 800$ and $\alpha = \{0.6, 0.7, 0.8, 0.9\}$.

To examine the error distribution, we depict the absolute error in the state and control in Fig. 3. Although the error is not distributed equally over time, it is less than 10^{-3} and 10^{-2} for both of the state and control, respectively.

Example 2. We consider the following linear time-invariant problem [10]

$$\min \; J(u) = \frac{1}{2}\int_0^{20} \left(x^2(t) + u^2(t)\right) dt, \tag{35}$$

TABLE 5. Example 2: The values of J and computational time (CT).

	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$		$\alpha = 1$	
M	J	CT	J	CT	J	CT	J	CT
50	0.2048	0.08	0.1912	0.08	0.1817	0.08	0.1754	0.08
100	0.2008	0.15	0.1909	0.15	0.1856	0.14	0.1842	0.14
200	0.2019	0.41	0.1945	0.40	0.1920	0.39	0.1940	0.41
400	0.2031	1.70	0.1971	1.67	0.1963	1.71	0.2002	1.61
800	0.2039	17.17	0.1986	17.55	0.1986	16.96	0.2035	17.18

subject to

$$ {}_0^{CF}D_t^{\alpha} x(t) = -x(t) + u(t),\ 0 < t \leq 20, \quad x(0) = 1. \tag{36}$$

The necessary optimality conditions of the FOCP (35)-(36) are written in the form

$$\begin{cases} {}_0^{CF}D_t^{\alpha} x(t) = -x(t) - p(t),\ 0 < t \leq 20, \\ {}_t^{CF}D_{20}^{\alpha} p(t) = x(t) - p(t),\ 0 \leq t < 20, \\ x(0) = 1, \quad p(20) = 0. \end{cases} \tag{37}$$

We note that the corresponding gradient equation is obtained as

$$u^*(t) = -p(t), \quad 0 \leq t \leq 20. \tag{38}$$

For $\alpha = 1$, the exact solution of this problem is given by

$$x^*(t) = \gamma \sinh(\sqrt{2}t) + \cosh(\sqrt{2}t), \tag{39}$$

$$u^*(t) = (\gamma + \sqrt{2}) \sinh(\sqrt{2}t) + (\gamma\sqrt{2} + 1) \cosh(\sqrt{2}t), \tag{40}$$

where

$$\gamma = -\frac{\sqrt{2}\sinh(20\sqrt{2}) + \cosh(20\sqrt{2})}{\sinh(20\sqrt{2}) + \sqrt{2}\cosh(20\sqrt{2})}. \tag{41}$$

We present the values of J and computational time measured by implementing the proposed method for some values of M and α in Table 5. These results underline the convergence and speed of the algorithm, which is less than 18 seconds for $M = 800$. The approximate solutions of $x(t)$ and $u(t)$ obtained with $M = 800$ for $\alpha = \{0.7, 0.8, 0.9, 1\}$ are shown in Fig. 4. For comparison purposes, we also plotted the exact solution for $\alpha = 1$ given by Eqs. (39)-(40) in Fig. 4. We observe that the fractional numerical solution gets closer to the integer-order exact solution as α approaches 1. Moreover, as the fractional order α is decreased, the settling time of the response decreases. It means that the behavior of controlled system strongly depends on the real number α. Hence, the order of differentiation can be regarded as a control parameter to reach a desired performance in terms of system's specifications like settling time. In Table 6, we compare the values of J computed through the Caputo and CF derivatives by fixing $N = 800$. We observe that the values of J computed using the CF derivative are less than those of the classical Caputo for different values of α. In Fig. 5, we present the numerical solution of state variable for different values of α within the Caputo and CF fractional operators. We obtain a better settling time and asymptotic behavior of the response through the CF derivative than the classical Caputo for all values of $0 < \alpha < 1$. We note that the results for $\alpha = 1$ obtained within the Caputo and CF coincide with each other, as expected. To summarize, we can underline the superiority of CF to the classical Caputo for the FOCPs based on the results shown in Table 6 and Figure 5.

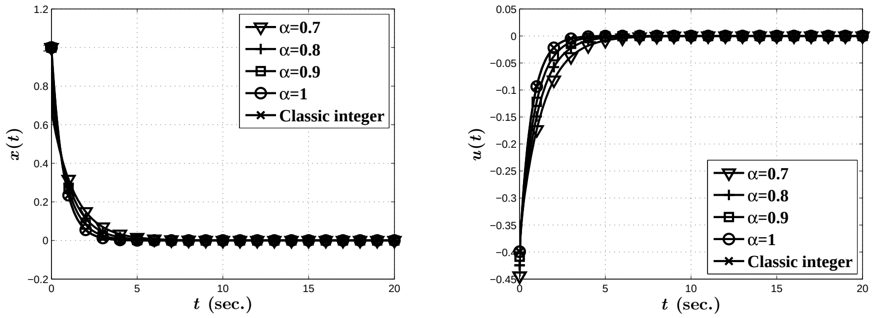
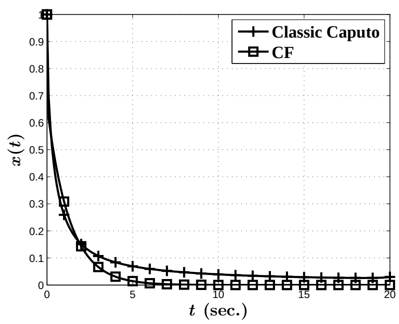


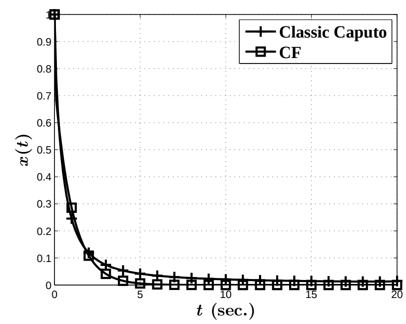
FIGURE 4. Example 2: Numerical results of $x(t)$ (*left*) and $u(t)$ (*right*).

TABLE 6. Example 2: The comparative values of J with $M = 800$.

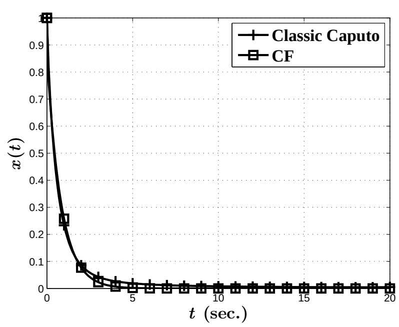
FD	J			
	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
Caputo	0.2301	0.2073	0.2002	0.2035
CF	0.2039	0.1986	0.1986	0.2035



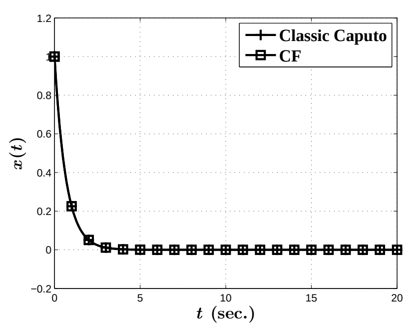
(a) $\alpha = 0.7$.



(b) $\alpha = 0.8$.



(c) $\alpha = 0.9$.



(d) $\alpha = 1$.

FIGURE 5. Example 2: Numerical results of $x(t)$ for Caputo and CF derivatives.

Example 3. In this example, we consider the fractional Duffing equation

$$\begin{cases} {}_0^C D_t^{\alpha} x_1(t) = x_2(t),\ 0 < t \leq 20, \\ {}_0^C D_t^{\alpha} x_2(t) = -x_1(t) - 0.02 x_2(t) - 5 x_1^3(t) + u(t),\ 0 \leq t < 20, \\ x_1(0) = 1, \quad x_2(0) = 0. \end{cases} \tag{42}$$

Table 7. Example 3: The values of J, rate of convergence and computational time (CT) for $\alpha = \{0.7, 0.8\}$.

	$\alpha = 0.7$				$\alpha = 0.8$			
N	J_N	$J_N - J_{N/2}$	ρ	CT	J_N	$J_N - J_{N/2}$	ρ	CT
50	12.1411	-	-	0.63	13.8799	-	-	0.64
100	7.9860	4.1551	-	1.32	8.6678	5.2121	-	1.25
200	6.6614	1.3246	1.65	4.31	6.9545	1.7133	1.61	4.33
400	6.1602	0.5012	1.40	17.18	6.3194	0.6351	1.43	16.97
800	5.9455	0.2147	1.22	146.48	6.0524	0.2670	1.25	144.69

Table 8. Example 3: The values of J, rate of convergence and computational time (CT) for $\alpha = \{0.9, 1\}$.

	$\alpha = 0.9$				$\alpha = 1$			
N	J_N	$J_N - J_{N/2}$	ρ	CT	J_N	$J_N - J_{N/2}$	ρ	CT
50	15.6616	-	-	0.73	18.3533	-	-	0.69
100	9.9329	5.7287	-	1.39	12.3690	5.9843	-	1.73
200	7.7685	2.1644	1.40	4.30	9.7979	2.5711	1.22	3.81
400	6.9493	0.8192	1.40	18.26	8.7483	1.0496	1.29	18.53
800	6.6071	0.3422	1.26	148.13	8.3018	0.4465	1.23	167.40

The goal is to find the optimal control $u^*(t)$ so that the quadratic cost functional

$$J = \frac{1}{2}\int_0^{20} \left(10(x_1^2(t) + x_2^2(t)) + u^2(t)\right) dt, \tag{43}$$

subject to the constraint (42) is minimized.

The necessary optimality conditions of the FOCP (42)-(43) are formulated as

$$\begin{cases} {}_0^C D_t^\alpha x_1(t) = x_2(t), \\ {}_0^C D_t^\alpha x_2(t) = -x_1(t) - 0.02x_2(t) - 5x_1^3(t) - p_2(t), \\ {}_t^R D_{20}^\alpha p_1(t) = 10x_1(t) - p_2(t) - 15x_1^2(t)p_2(t), \\ {}_t^R D_{20}^\alpha p_2(t) = 10x_2(t) + p_1(t) - 0.02p_2(t), \\ x_1(0) = 1,\ x_2(0) = 0, \quad p_1(20) = p_2(20) = 0, \end{cases} \tag{44}$$

where the connection between the optimal control and adjoint variable is realized through the equation

$$u^*(t) = -p_2(t), \quad 0 \le t \le 20. \tag{45}$$

In Tables 7-8, the values of J are listed for some values of M and α. For different values of α, we measure a rate around 1.2 for fractional case, which is better than the linear rate of convergence proved in Theorem 5.3. Numerical results of $x_1(t)$, $x_2(t)$ and $u(t)$ with $\alpha = \{0.7, 0.8, 0.9, 1\}$ and $M = 800$ are shown in Fig. 6. We observe that the difference between the approximate solution of Eq. (44) and the integer-order solution vanishes as α approaches 1. Moreover, there is a positive correlation between the order of differentiation α and the settling time of the response. Thus, some performance requirements like settling time can be controlled through the parameter α. In Table 9, the values of J measured for the model with Caputo and CF derivatives are listed. The smaller values of J are computed through the model

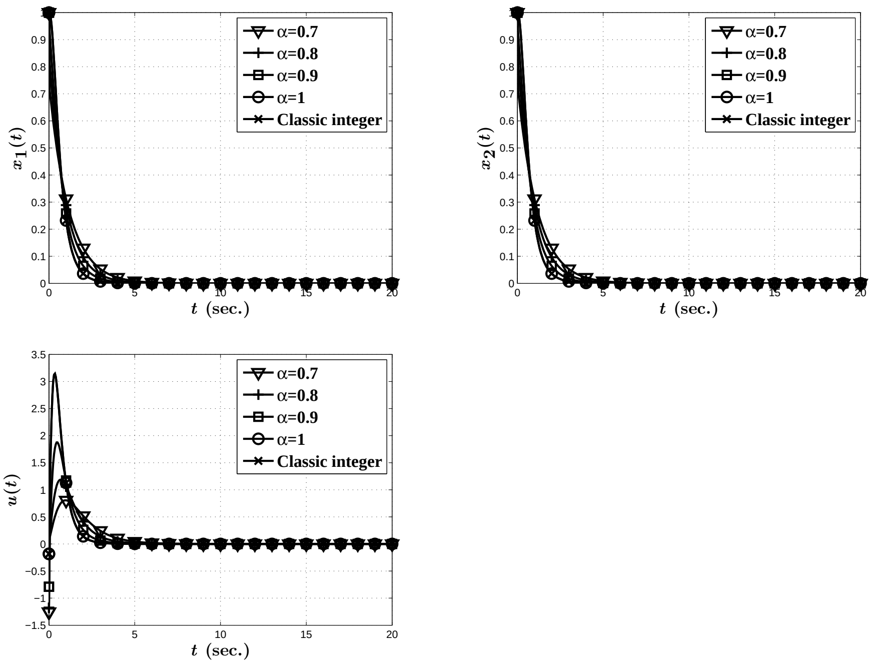


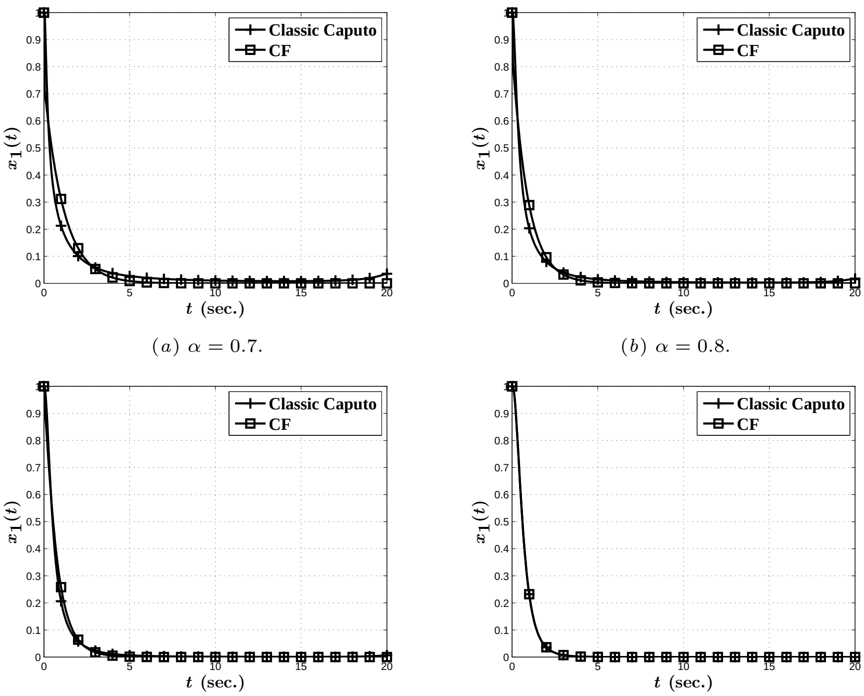
FIGURE 6. Example 3: Numerical results of $x_1(t)$, $x_2(t)$ and $u(t)$.

TABLE 9. Example 3: The comparative values of J with $M = 800$.

FD	J			
	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
Caputo	6.6976	6.6329	7.1959	8.3018
CF	5.9455	6.0524	6.6071	8.3018

with CF derivative when compared to the one that contains the classical Caputo for some values of α. In Figs. 7-8, the solution associated with Caputo and CF fractional operators are compared. We observe that better results for settling time and asymptotic behavior of the response are achieved by applying CF derivative instead of Caputo for different values of α. Moreover, there is not a visible difference between the results obtained with both operators for $\alpha = 1$. According to the results reported in Table 9 and Figs. 7-8, we deduce that the FOCP governed by a FDE with CF derivative leads to better results (compared to the classical Caputo) for settling time and asymptotic behavior of the response as well as cost value J.

7. **Summary and conclusion.** The FDEs have been a promising research topic for a long time since they are capable of modeling real-world problems arising in different fields such as physics, biology, optics and control systems. In addition, the fractional derivatives have eliminated the drawback of integer-order counterparts on account of their nonlocal characteristics. Furthermore, due to elimination of singular kernel in CF derivative, we can obtain more accurate models within this new fractional differentiation. However, analytical solution of FOCPs in the CF



(a) $\alpha = 0.7$. (b) $\alpha = 0.8$.

(c) $\alpha = 0.9$. (d) $\alpha = 1$.

FIGURE 7. Example 3: Numerical results of $x_1(t)$ for Caputo and CF derivatives.

sense cannot be derived explicitly and numerical tools are required at this point. In this paper, we proposed a new formulation of FOCPs involving CF fractional derivative. We derived the optimality system for this problem in terms of Volterra integrals and solved them by an efficient numerical method. We justified that the new scheme exhibits linear rate of convergence. In addition, we enriched the study with some numerical examples to test the efficiency of the proposed scheme. According to the results, the convergence to the optimal solution was achieved very quickly. In addition, a smaller value of the cost functional as well as better settling time and asymptotic behavior of the response were measured when the constraint is modeled through the CF derivative rather than classical Caputo. To sum up, the FDEs with CF derivative is a promising constraint for FOCPs over the dynamical models with classical Caputo derivative.

Acknowledgments. The authors are grateful to the anonymous reviewers for their careful reading and valuable suggestions which helped in significant improvement of the manuscript.

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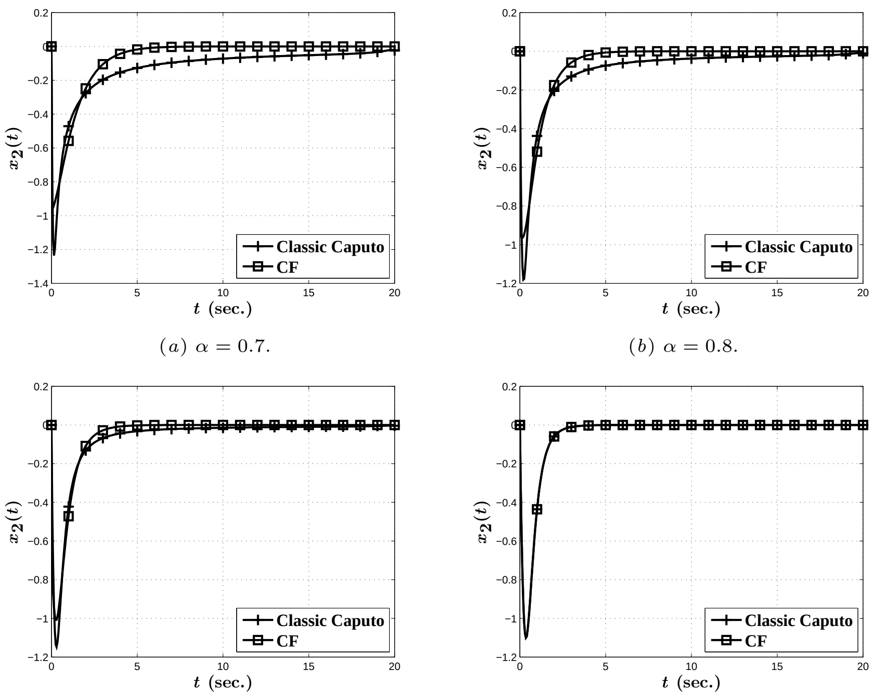

(a) $\alpha = 0.7$. (b) $\alpha = 0.8$.

(c) $\alpha = 0.9$. (d) $\alpha = 1$.

FIGURE 8. Example 3: Numerical results of $x_2(t)$ for Caputo and CF derivatives.

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Received June 2018; revised September 2018.

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