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Article

Planar System-Masses in an Equilateral Triangle: Numerical Study within Fractional Calculus

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Abstract: In this work, a system of three masses on the vertices of equilateral triangle is investigated. This system is known in the literature as a planar system. We first give a description to the system by constructing its classical Lagrangian. Secondly, the classical Euler-Lagrange equations (i.e., the classical equations of motion) are derived. Thirdly, we fractionalize the classical Lagrangian of the system, and as a result, we obtain the fractional Euler-Lagrange equations. As the final step, we give the numerical simulations of the fractional model, a new model which is based on Caputo fractional derivative.

Keywords: Planar system; masses in equilateral triangle; springs; Euler-Lagrange equations; fractional derivative

1 Introduction

In classical mechanics, the energy concept (used in building Lagrangian equation) finds a wide range of applications rather than the force concept (used in applying Newton's second law); this is because of the fact that dealing with scalars (energy) is more flexible than vectors (forces). In order to build the Lagrangian of a given system, one has to write and specify exactly both the kinetic and the potential energies of the system using suitable generalized coordinates. A wide range of systems including springs can be found in classical mechanics texts such as [1–3] where the Lagrangian of these systems have been constructed, and then the so-called classical Euler-Lagrange equations (or the equations of motion) have been derived. More to the point, a valuable study can be found in [4] discussing the linear stability of the well-known periodic orbits of the elliptic Lagrangian triangle solutions in a three-body problem.

A planar system is three masses in the vertices of an equilateral triangle attached together by same spirals having equal stiffness constant. This system is one of the interesting models described in the literature and solved using Lagrangian method [5]. The significance of this model arises from the fact that the three masses can vibrate in a plane not in a line as most systems; also, it can be considered as a good example of representation theory.

Fractional calculus is a branch of mathematics that extends derivatives and integrals to any order; this branch goes to more than 300 years back. It was initially considered that this calculus had no applications; nevertheless, the last 50 years showed that it plays an important role in nearly all branches of sciences [6–18]. Hence, some noticeable efforts have been done to enhance different aspects of the fractional calculus approach from both numerical and theoretical points of view. For instance, a spectral tau method was employed in [19] to analyze and implement new spectral solutions of the fractional Riccati differential equation. In [20], some numerical methods were proposed for the Riesz space fractional advection-dispersion equations with delay.

Many definitions for the fractional derivatives have been suggested; among these definitions, we have Riemann-Liouville and Caputo fractional operators. However, introducing new definitions in the fractional calculus allows us to become familiar with the newer aspects of some phenomena. This is the main reason for trying to come up with new ideas in this field of research. For more details about these definitions, the reader can refer to the following references [21–26].

Fractional calculus find a wide range of application in physics. As can be seen from the literature [27–34], one has to build the classical Lagrangian for an interesting system, and then generalize the classical Lagrangain by considering one of the fractional derivatives definitions. After that, one can get the fractional Euler-Lagrange equations (FELEs). Finally, the obtained equations of motion have to be solved either analytically or numerically; in most cases, we seek for the numerical solution.

In this work, we consider a system which has not been studied before numerically. The system is shown in Fig. 1. The importance of this system comes from the fact that each mass can oscillate in a plane and not on a single straight line, contrary to the most of physical systems [5]. Thus, we can here study the oscillatory of the system in the plane of triangle. Furthermore, this system is a good physical example on representation theory.

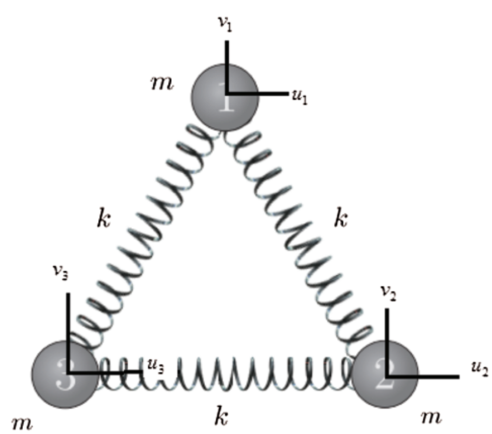

Figure 1: Identical masses on vertices of equilateral triangle

The structure of this work is prepared as follows. In Section 2, some preliminaries and definitions of the fractional calculus are listed briefly. In Section 3, the planar system considered is described classically and fractionally in details. In Section 4, numerical and simulation techniques used are illustrated, and lastly we end the paper with conclusion part.

2 Preliminaries

In the following, we give briefly the main meanings regarding the fractional derivatives (FDs) in the Caputo sense as well as their corresponding integrals. Let $x : [a, b] \to R$ be a time dependent function.

Then the left and right Caputo FDs of order α are, respectively, defined by

$$ {}_aD_t^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}x^{(n)}(\tau)d\tau, \tag{1} $$

$$ {}_tD_b^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}(-1)^n x^{(n)}(\tau)d\tau, \tag{2} $$

where $\Gamma(\cdot)$ is the Euler's gamma function, and n is an integer such that $n-1<\alpha<n$. The corresponding fractional integrals are also, respectively, described as

$$ {}_aI_t^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)d\tau, \tag{3} $$

$$ {}_tI_b^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1}x(\tau)d\tau. \tag{4} $$

Note that the Caputo derivative coincides with the ordinary differentiation when α is an integer, i.e.,

$$ {}_aD_t^{n}x(t) = x^{(n)}(t), $$
$$ {}_tD_b^{n}x(t) = (-1)^n x^{(n)}(t). \tag{5} $$

For more details, the interested reader can refer to [34].

3 Planar System: Masses in an Equilateral Triangle

The arrangement shown in Fig. 1 is composed of three equal masses (m) located at the corners of an equilateral triangle and connected by three spiral springs with stiffness (k). As usual, we begin our description by constructing the classical Lagrangian L_c, which is the subtract of kinetic and potential energies, respectively. To do this, we label the three masses by 1, 2, and 3 as shown in Fig. 1. Let us describe the displacements from the equilibrium positions by $u_1, v_1, u_2, v_2, u_3, v_3$. The kinetic energy T of the system is then obtained from

$$ T = \frac{1}{2}m\left[\left(\dot{u}_1^2+\dot{v}_1^2\right)+\left(\dot{u}_2^2+\dot{v}_2^2\right)+\left(\dot{u}_3^2+\dot{v}_3^2\right)\right]. \tag{6} $$

The potential energy stored in the spring attached between Nos. 1 and 2 is also given as

$$ U_{12} = \frac{1}{2}k\left[(u_1-u_2)^2+(v_1-v_2)^2\right]. \tag{7} $$

Thus, the total potential energy $U$of the system reads

$$ U = U_{12}+U_{13}+U_{23} $$

$$= \frac{1}{2}k\left[(u_1-u_2)^2+(v_1-v_2)^2+(u_2-u_3)^2+(v_2-v_3)^2+(u_3-u_1)^2+(v_3-v_1)^2\right]. \tag{8}$$

According to definition of the classical Lagragian L_c and Eqs. (6) and (8), we can write:

$$L_c = T - U$$

$$= \frac{1}{2}m\left[\left(\dot{u}_1^2+\dot{v}_1^2\right)+\left(\dot{u}_2^2+\dot{v}_2^2\right)+\left(\dot{u}_3^2+\dot{v}_3^2\right)\right]-\frac{1}{2}k\begin{bmatrix}(u_1-u_2)^2+(v_1-v_2)^2+(u_2-u_3)^2\\+(v_2-v_3)^2+(u_3-u_1)^2+(v_3-v_1)^2\end{bmatrix}. \tag{9}$$

Applying $\frac{\partial L_c}{\partial q}-\frac{d}{dt}\frac{\partial L_c}{\partial \dot{q}}=0$ to Eq. (9) with $q = u_1,\ u_2,\ u_3,\ v_1,\ v_2$, and v_3, respectively, the classical equations of motion for our system read:

$$m\ddot{u}_1 + k[(u_1-u_2)-(u_3-u_1)] = 0. \tag{10}$$

$$m\ddot{u}_2 + k[(u_2-u_3)-(u_1-u_2)] = 0. \tag{11}$$

$$m\ddot{u}_3 + k[(u_3-u_1)-(u_2-u_3)] = 0. \tag{12}$$

$$m\ddot{v}_1 + k[(v_1-v_2)-(v_3-v_1)] = 0 \tag{13}$$

$$m\ddot{v}_2 + k[(v_2-v_3)-(v_1-v_2)] = 0. \tag{14}$$

$$m\ddot{v}_3 + k[(v_3-v_1)-(v_2-v_3)] = 0 \tag{15}$$

Our next step is to generalize the Eq. (9). Thus, the fractional Lagrangian reads:

$$\begin{aligned}L_f = {} & \frac{1}{2}m\left[\left(\left(\mathcal{D}_t^\alpha u_1\right)^2+\left(\mathcal{D}_t^\alpha v_1\right)^2\right)+\left(\left(\mathcal{D}_t^\alpha u_2\right)^2+\left(\mathcal{D}_t^\alpha v_2\right)^2\right)+\left(\left(\mathcal{D}_t^\alpha u_3\right)^2+\left(\mathcal{D}_t^\alpha v_3\right)^2\right)\right]\\ & -\frac{1}{2}k\begin{bmatrix}(u_1-u_2)^2+(v_1-v_2)^2+(u_2-u_3)^2\\+(v_2-v_3)^2+(u_3-u_1)^2+(v_3-v_1)^2\end{bmatrix}\end{aligned}. \tag{16}$$

Now the fractional equations of motion can be obtained by applying Eq. (16) to $\frac{\partial L^F}{\partial q}+\mathcal{D}_t^\alpha\frac{\partial L^F}{\partial_a D_t^\alpha q}+\mathcal{D}_t^\beta\frac{\partial L^F}{\partial_t D_b^\alpha q}=0$ with $q = u_i$, $q = v_i$ for $i = 1, 2, 3$. As a result, the following equations are obtained

$$m\mathcal{D}_t^{2\alpha}u_1 + k[(u_3-u_1)-(u_1-u_2)] = 0 \tag{17}$$

$$m\mathcal{D}_t^{2\alpha}u_2 + k[(u_1-u_2)-(u_2-u_3)] = 0 \tag{18}$$

$$m\mathcal{D}_t^{2\alpha}u_3 + k[(u_2-u_3)-(u_3-u_1)] = 0 \tag{19}$$

$$m\mathcal{D}_t^{2\alpha}v_1 + k[(v_3-v_1)-(v_1-v_2)] = 0 \tag{20}$$

$$m\mathcal{D}_t^{2\alpha}v_2 + k[(v_3-v_1)-(v_1-v_2)] = 0 \tag{21}$$

$$m\mathcal{D}_t^{2\alpha}v_3 + k[(v_3-v_1)-(v_1-v_2)] = 0 \tag{22}$$

As $\alpha \to 1$, the above six equations are reduced to the classical Eqs. (10)–(15).

In the following, we aim to obtain numerical solution for Eqs. (17)–(22) for some fractional orders and initial conditions.

4 Simulation Results

In this section, we will discuss how to use the numerical procedure to obtain the approximate solution of the problem under study. To this end, let us take into account the following equation

$$\mathcal{D}_t^{2\alpha}\mathcal{Y}(t) = \mathcal{S}(t, \mathcal{Y}(t)). \qquad \alpha \in (0, 1] \tag{23}$$

By using the integral operator, one can write

$$\mathcal{Y}(t) - \mathcal{Y}(0) - t\mathcal{Y}'(0) = \frac{\alpha}{\Gamma(\alpha)} \int_0^t \mathcal{S}(\omega, \mathcal{Y}(\omega))(t-\omega)^{\alpha-1} d\omega. \tag{24}$$

Taking $t = t_n = n\hbar$ in (5.4), one achieves

$$\mathcal{Y}(t_n) = \mathcal{Y}(0) + t_n\mathcal{Y}'(0) + \frac{\alpha}{\Gamma(\alpha)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathcal{S}(\omega, \mathcal{Y}(\omega))(t_n-\omega)^{\alpha-1} d\omega. \tag{25}$$

Now, with the help of linear interpolation of $\mathcal{S}(t, \mathcal{Y}(t))$, one gets

$$\mathcal{S}(t, \mathcal{Y}(t)) \approx \mathcal{S}(t_{i+1}, \mathcal{Y}_{i+1}) + \frac{t - t_{i+1}}{\hbar}(\mathcal{S}(t_{i+1}, \mathcal{Y}_{i+1})) - \mathcal{S}(t_i, \mathcal{Y}_i)), \quad t \in [t_i, t_{i+1}], \tag{26}$$

where the notation of $\mathcal{Y}_i = \mathcal{Y}(t_i)$ is used.

Substituting (23) in (22), the approximate solution of the problem will obtain as [35]

$$\mathcal{Y}_n = \mathcal{Y}_0 + t_n\mathcal{Y}'_0 + \hbar^{2\alpha}\left(\eta_n\mathcal{S}(t_0, \mathcal{Y}_0) + \sum_{i=1}^{n} \theta_{n-i}\mathcal{S}(t_i, \mathcal{Y}_i)\right) \tag{27}$$

where

$$\eta_n = \frac{(n-1)^{2\alpha+1} - n^{2\alpha}(n - 2\alpha - 1)}{\Gamma(2\alpha+2)}, \quad \theta_j = \begin{cases} \dfrac{1}{\Gamma(2\alpha+2)}, & j = 0 \\ \dfrac{(j-1)^{2\alpha+1} - 2j^{2\alpha+1} + (j+1)^{2\alpha+1}}{\Gamma(2\alpha+2)}, & j = 1, 2, \ldots, n-1 \end{cases} \tag{28}$$

5 Results and Discussion

Using the numerical method presented above, the approximate solution of the main problem (17)–(22) will be achieved recursively as

$$\begin{aligned}
u_{1n} &= u_{10} + \hbar^{2\alpha}\left(k\eta_n[(u_{10}-u_{20})-(u_{30}-u_{10})] + \sum_{i=1}^{n} k\theta_{n-i}[(u_{1i}-u_{2i})-(u_{3i}-u_{1i})]\right),\\
u_{2n} &= u_{20} + \hbar^{2\alpha}\left(k\eta_n[(u_{20}-u_{30})-(u_{10}-u_{20})] + \sum_{i=1}^{n} k\theta_{n-i}[(u_{2i}-u_{3i})-(u_{1i}-u_{2i})]\right),\\
u_{3n} &= u_{30} + \hbar^{2\alpha}\left(k\eta_n[(u_{30}-u_{10})-(u_{20}-u_{30})] + \sum_{i=1}^{n} k\theta_{n-i}[(u_{3i}-u_{1i})-(u_{2i}-u_{3i})]\right),\\
v_{1n} &= v_{10} + \hbar^{2\alpha}\left(k\eta_n[(v_{10}-v_{20})-(v_{30}-v_{10})] + \sum_{i=1}^{n} k\theta_{n-i}[(v_{1i}-u_{2i})-(v_{3i}-v_{1i})]\right),\\
v_{2n} &= v_{20} + \hbar^{2\alpha}\left(k\eta_n[(v_{20}-v_{30})-(v_{10}-v_{20})] + \sum_{i=1}^{n} k\theta_{n-i}[(v_{2i}-u_{3i})-(v_{1i}-v_{2i})]\right),\\
v_{3n} &= v_{30} + \hbar^{2\alpha}\left(k\eta_n[(v_{30}-v_{10})-(v_{20}-v_{30})] + \sum_{i=1}^{n} k\theta_{n-i}[(v_{3i}-v_{1i})-(v_{2i}-v_{3i})]\right).
\end{aligned} \tag{29}$$

The approximate solutions to the problem can be determined using the numerical method described above. The dynamical behaviors of the FELEs for the planar system, which were expressed by Eqs. (19–22), have been investigated by considering different values of the fractional order α. To this aim, we consider the following parameters $m = 1$, and $K = 0.5,\ 1,$ and 3 with the following initial conditions

Case 1:

$$u_1(0) = u_2(0) = u_3(0) = 0,$$
$$u_1'(0) = u_2'(0) = u_3'(0) = 1.$$

Case 2:

$$u_1(0) = u_2(0) = -u_3(0) = 1,$$
$$u_1'(0) = u_2'(0) = u_3'(0) = 0,$$

Case 3:

$$u_1(0) = u_2(0) = u_3(0) = 1,$$
$$u_1'(0) = u_2'(0) = u_3'(0) = 0,$$

Case 4:

$$u_1(0) = u_2(0) = u_3(0) = 0,$$
$$u_1'(0) = u_2'(0) = -u_3'(0) = 1,$$

As can be seen in Figs. 2–5, the fractional order of α has a great impact on the numerical results of the considered model. Indeed, the characteristics of the output response such as overshoot, settling time, peak time, rise time, frequency, etc., are changed by changing the fractional order. Therefore, we have here a degree of flexibility due to the existence of the fractional order, a fact which provides a flexible model as well capable of extracting the hidden aspects of physical phenomena in an appropriate, precise manner.

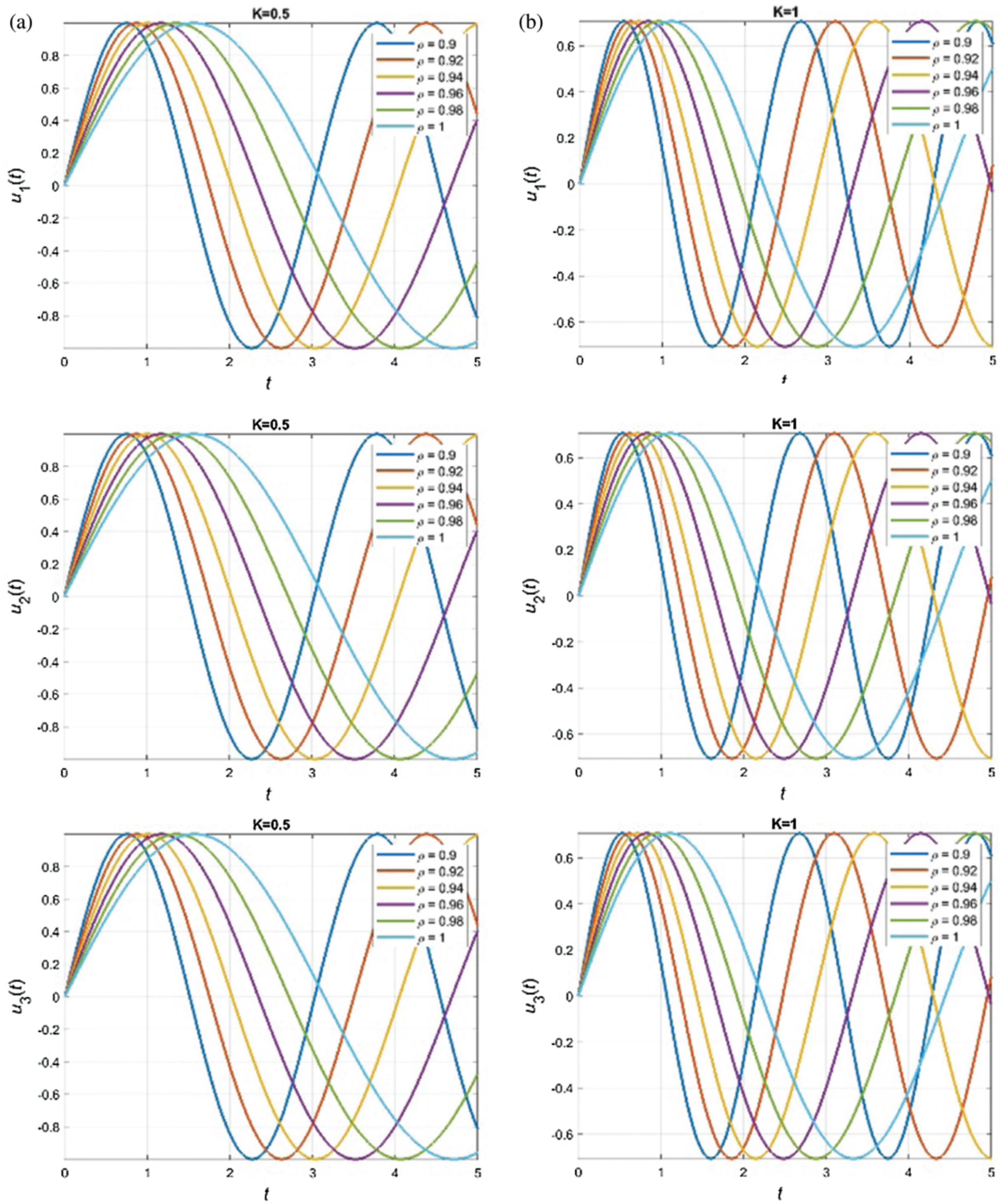

Figure 2: (continued)

(c)

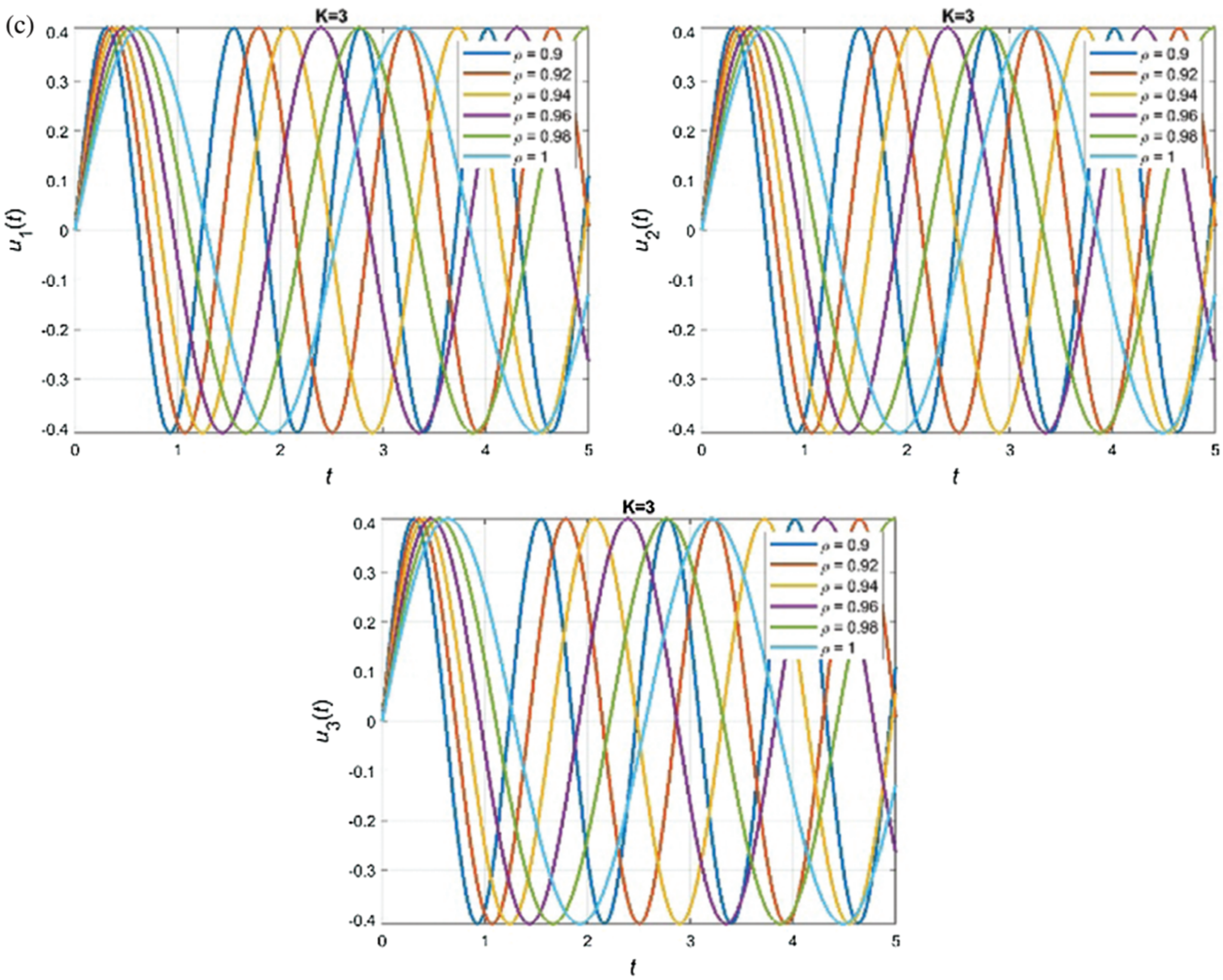

Figure 2: (a) Dynamical behaviors of u_1, u_2 and u_3 when $K = 0.5$ for different fractional orders of ρ. (b) Dynamical behaviors of u_1, u_2 and u_3 when $K = 1$ for different fractional orders of ρ. (c) Dynamical behaviors of u_1, u_2 and u_3 when $K = 3$ for different fractional orders of ρ

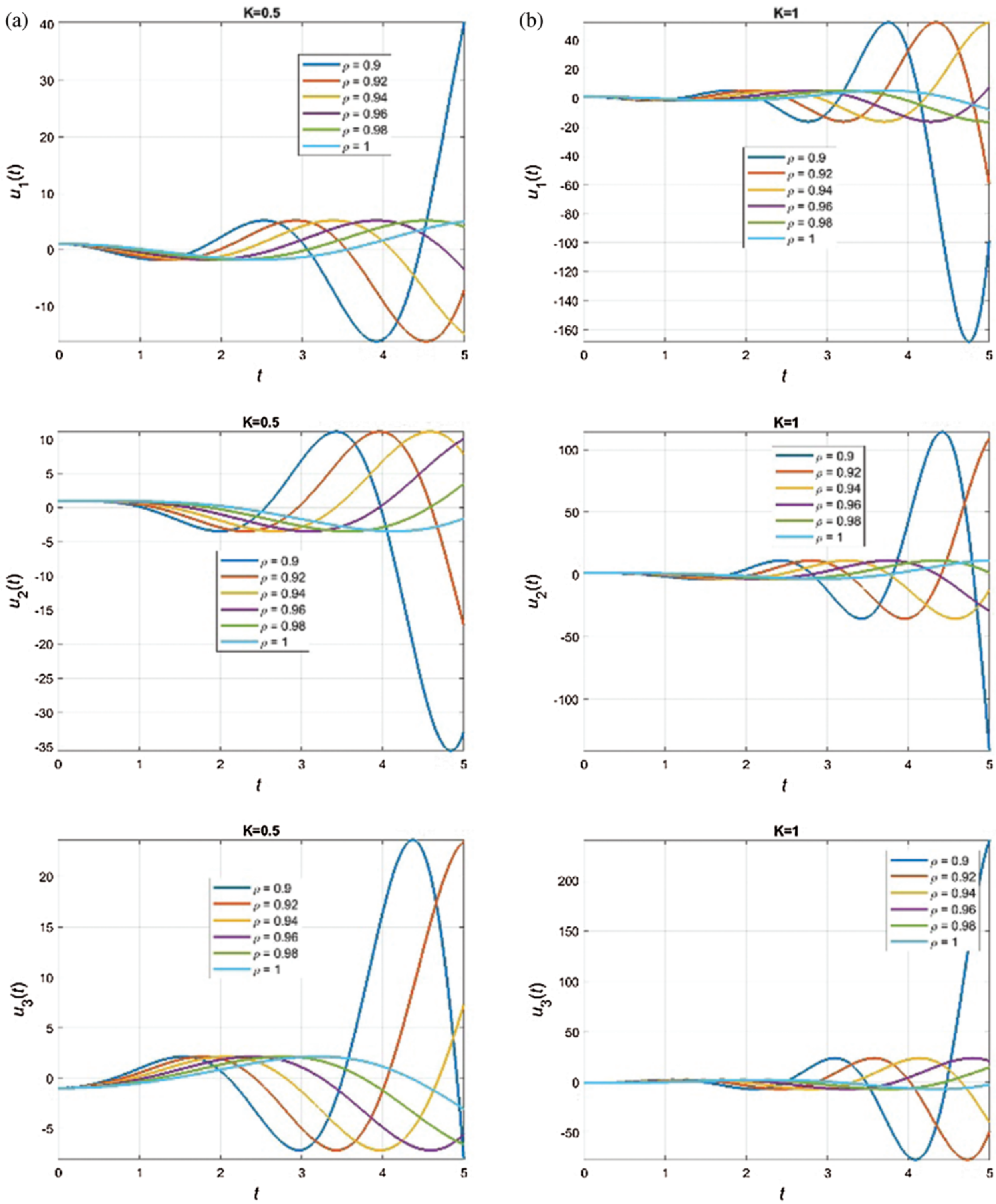

Figure 3: (continued)

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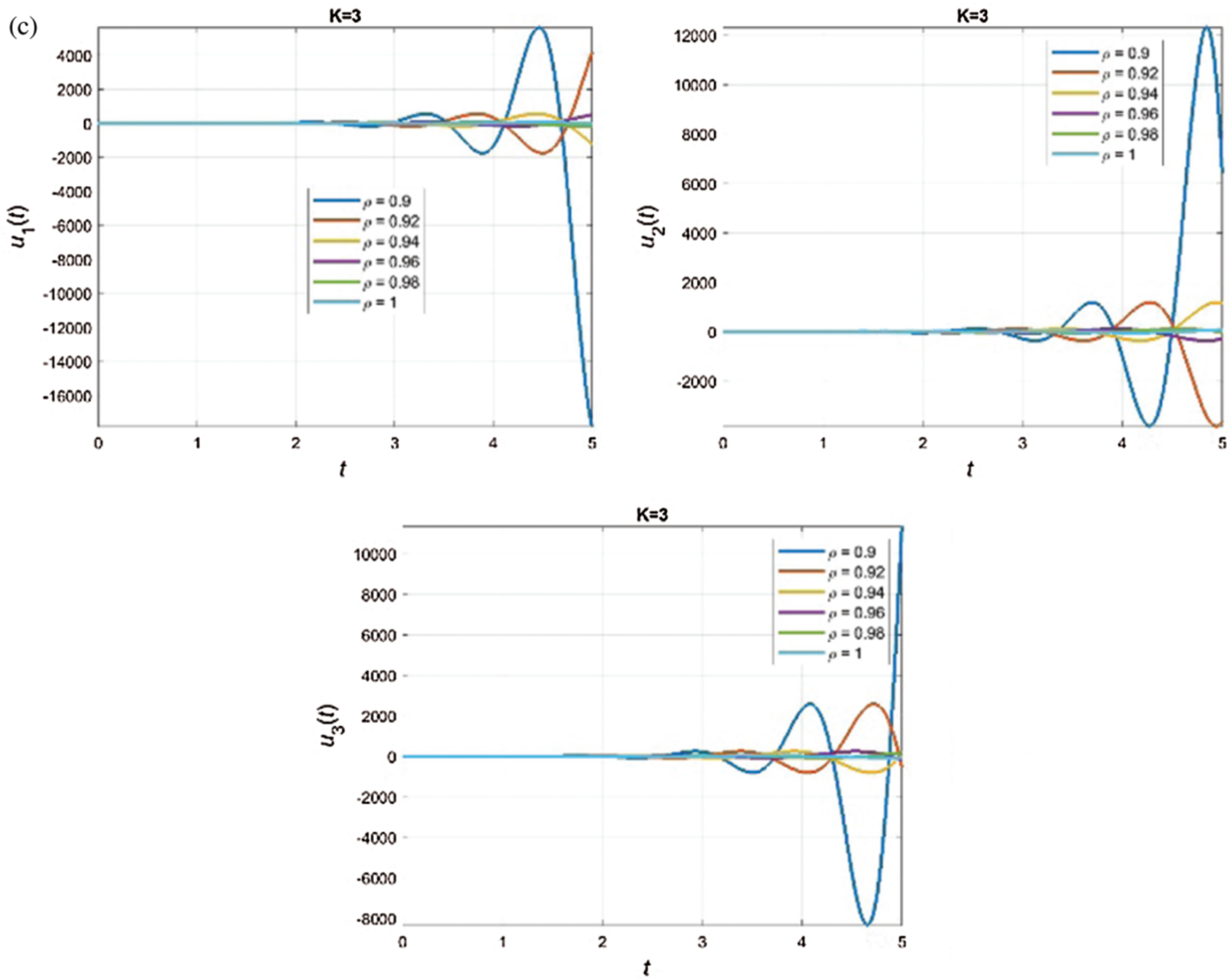

Figure 3: (a) Dynamical behaviors of u_1, u_2 and u_3 when $K = 0.5$ for different fractional orders of ρ. (b) Dynamical behaviors of u_1, u_2 and u_3 when $K = 1$ for different fractional orders of ρ. (c) Dynamical behaviors of u_1, u_2 and u_3 when $K = 3$ for different fractional orders of ρ

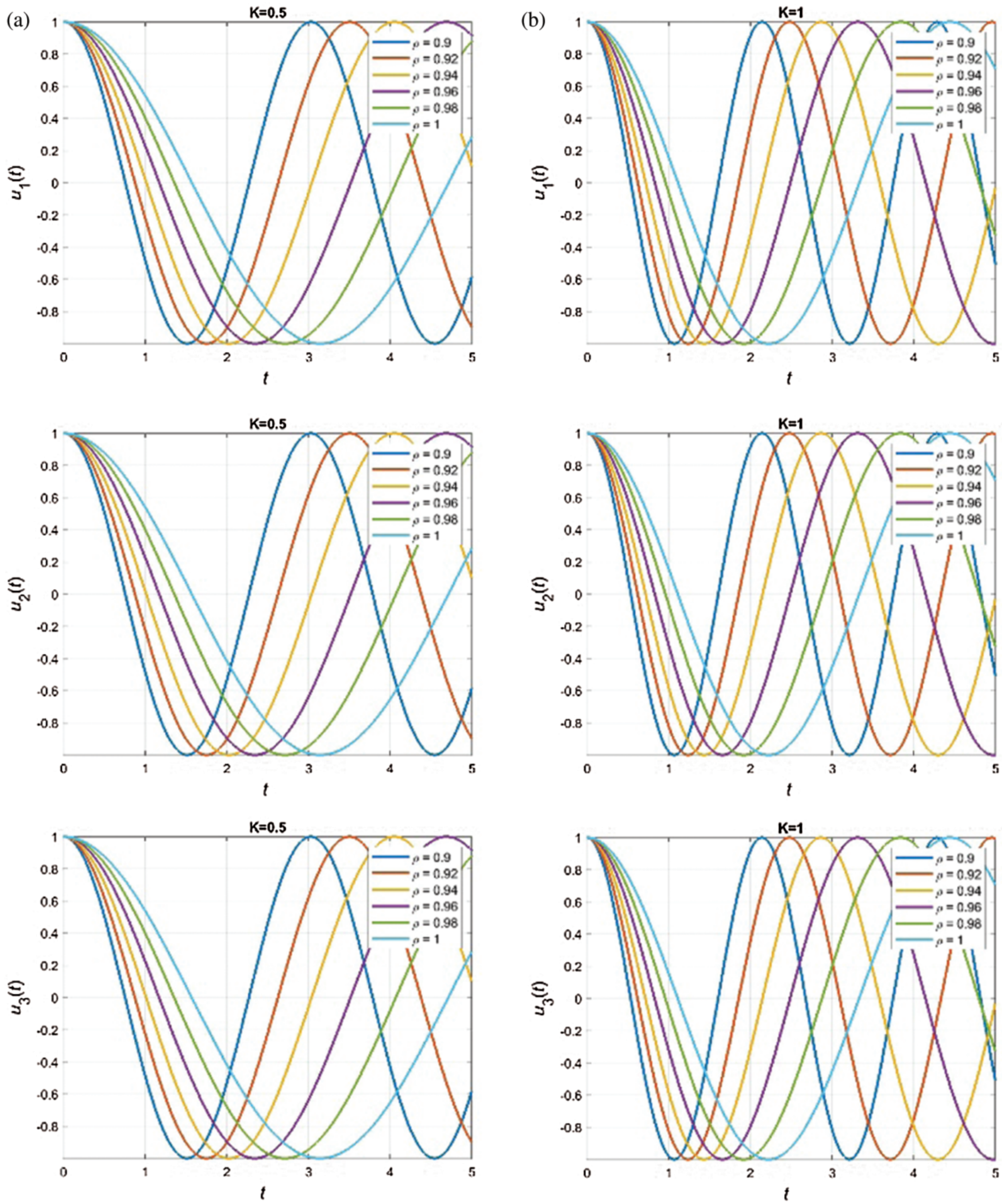

Figure 4: (continued)

(c)

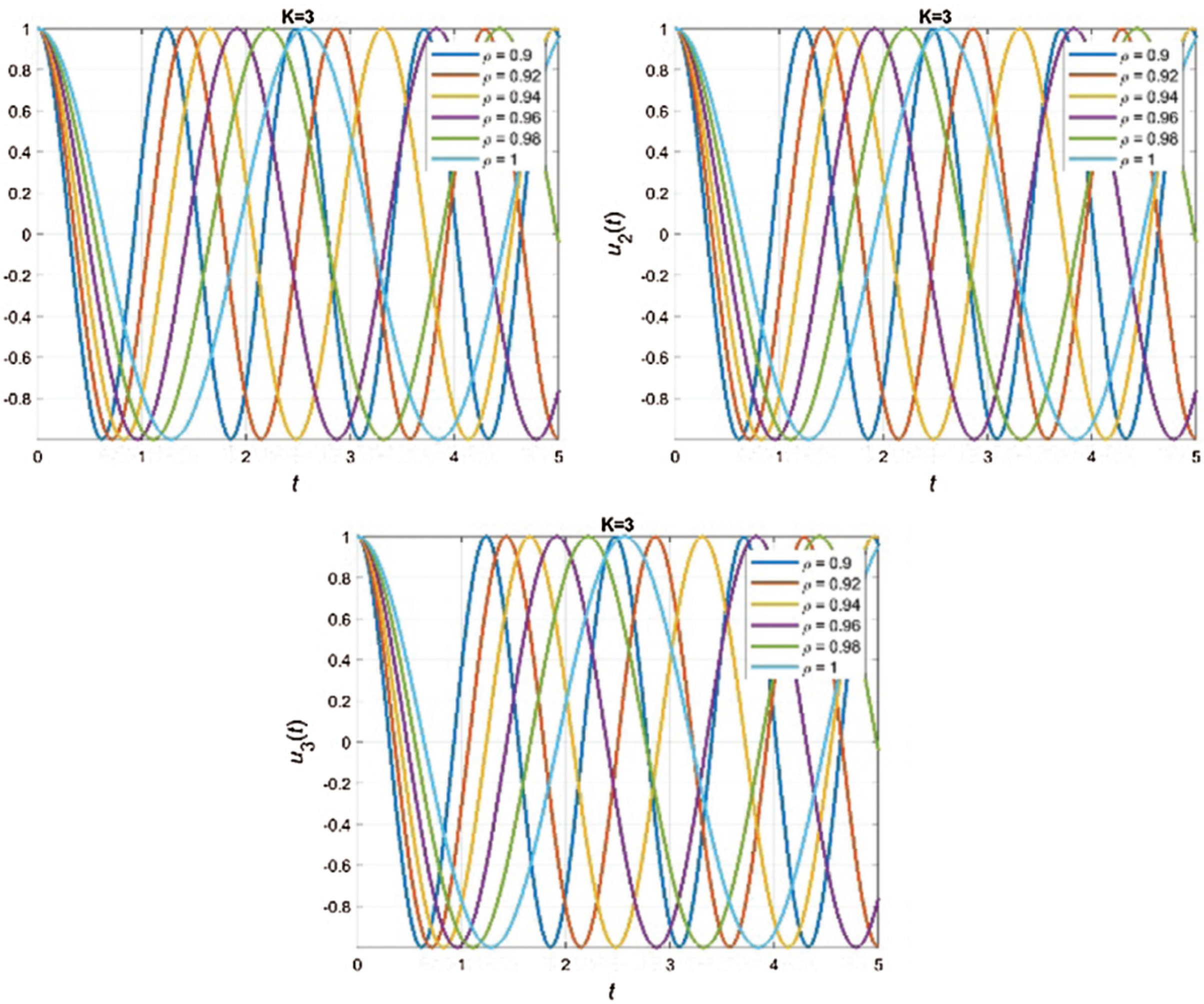

Figure 4: (a) Dynamical behaviors of u_1, u_2 and u_3 when $K = 0.5$ for different fractional orders of ρ. (b) Dynamical behaviors of u_1, u_2 and u_3 when $K = 1$ for different fractional orders of ρ. (c) Dynamical behaviors of u_1, u_2 and u_3 when $K = 3$ for different fractional orders of ρ

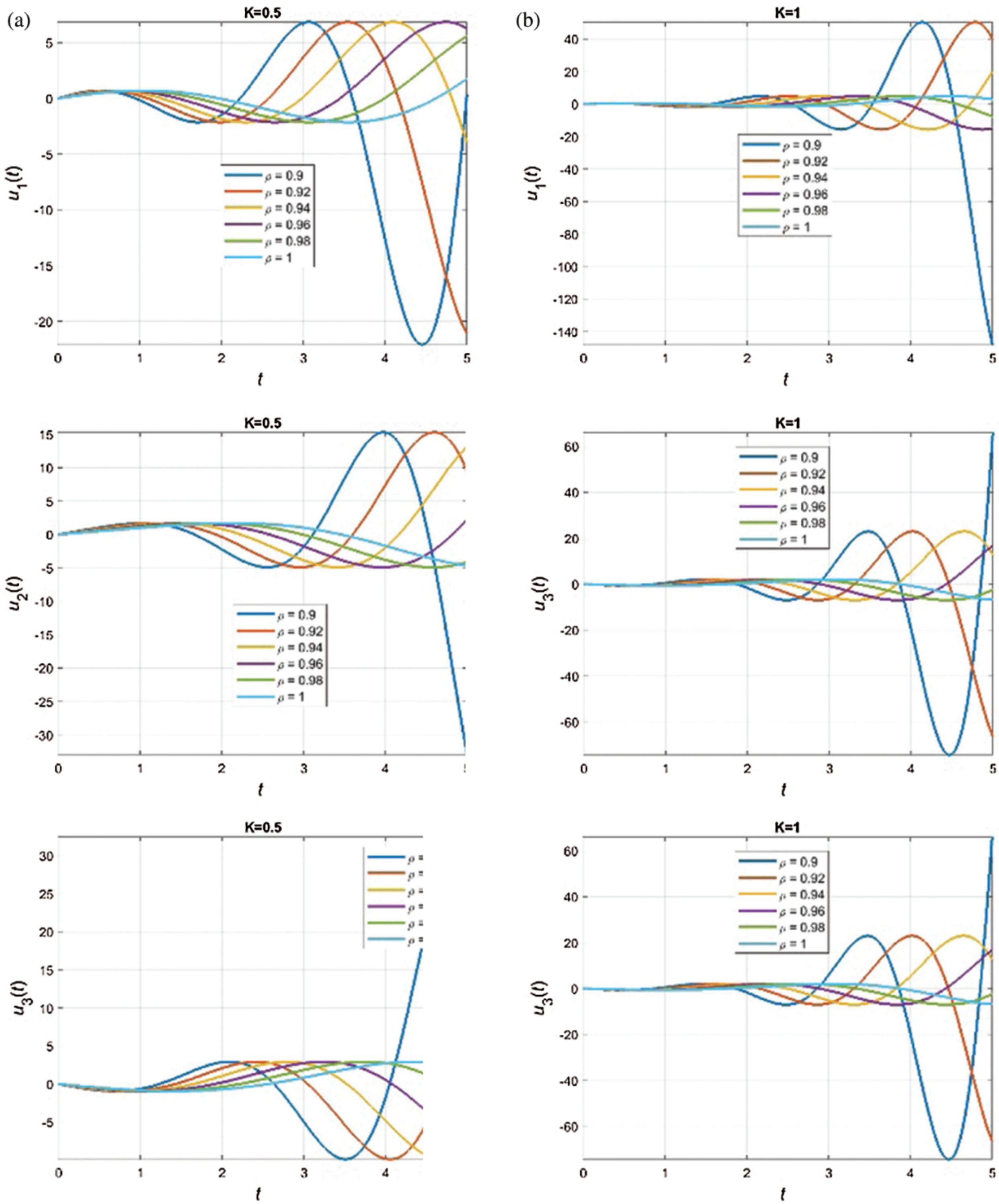

Figure 5: (continued)

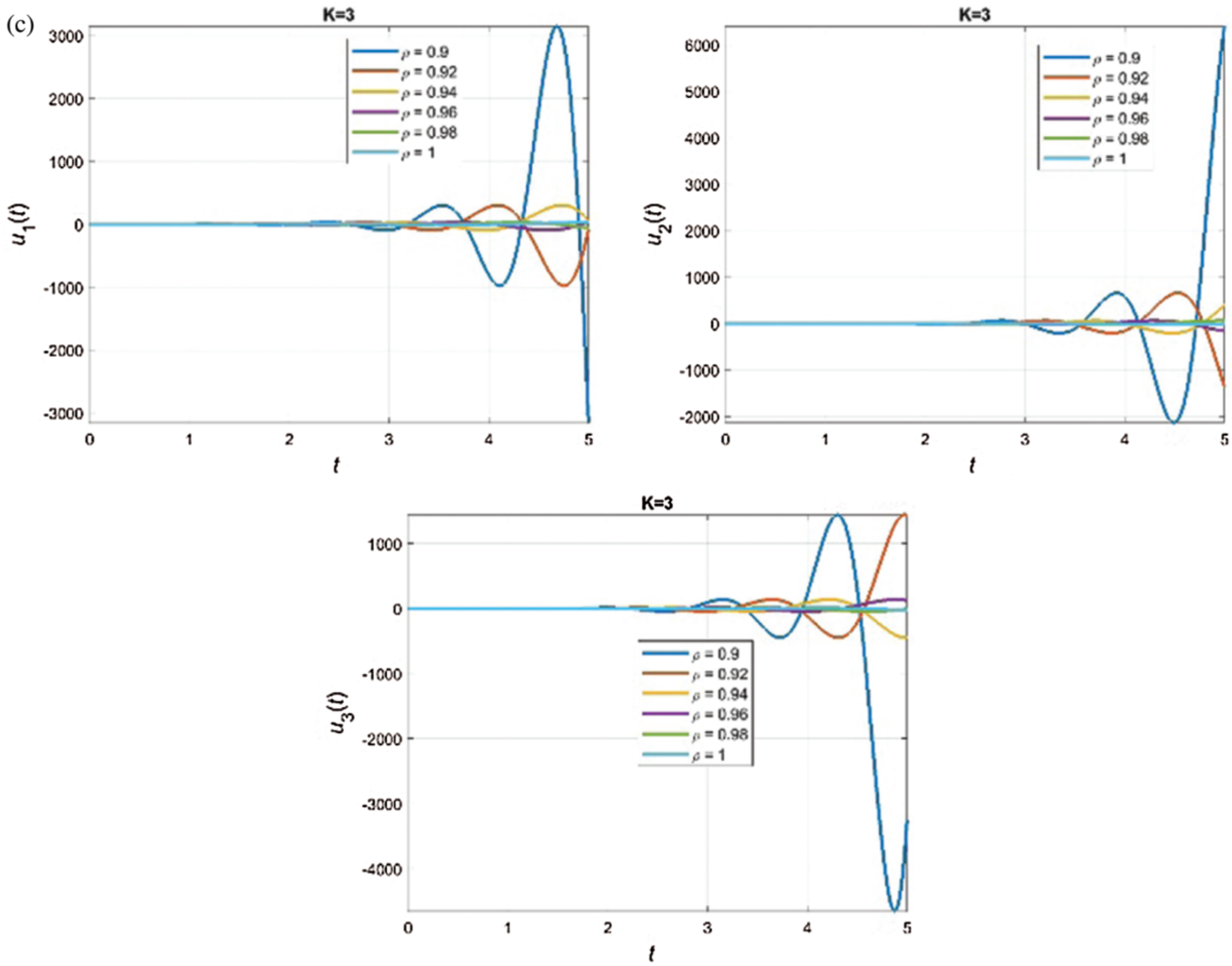

Figure 5: (a) Dynamical behaviors of u_1, u_2 and u_3 when $K = 0.5$ for different fractional orders of ρ. (b) Dynamical behaviors of u_1, u_2 and u_3 when $K = 1$ for different fractional orders of ρ. (c) Dynamical behaviors of u_1, u_2 and u_3 when $K = 3$ for different fractional orders of ρ

6 Conclusion

In this paper, we have considered the arrangement of three masses on the corners of equilateral triangle in a fractional point of view. The derivative used in this model is of Caputo type. One of the benefits of this fractional modelling is the use of the concept of memory. This means that important information is retained over time. Various numerical simulations were presented in this paper to investigate the role of the fractional order on the response behavior. In each case, different choices of this parameter can lead to interesting and different behaviors in the system response. The idea used in this paper can be applied to generalize fractional derivatives to other similar problems.

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