




A novel Jacobi operational matrix for numerical solution of multi-term variable-order fractional differential equations

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ABSTRACT

In this article, we introduce a numerical technique for solving a class of multi-term variable-order fractional differential equation. The method depends on establishing a shifted Jacobi operational matrix (SJOM) of fractional variable-order derivatives. By using the constructed (SJOM) in combination with the collocation technique, the main problem is reduced to an algebraic system of equations that can be solved numerically. The bound of the error estimate for the suggested method is investigated. Numerical examples are introduced to illustrate the applicability, generality, and accuracy of the proposed technique. Moreover, many physical applications problems that have the multi-term variable-order fractional differential equation formulae such as the damped mechanical oscillator problem and Bagley-Torvik equation can be solved via the presented method. Furthermore, the proposed method will be considered as a generalization of many numerical techniques.

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1. Introduction

In the past few decades, the fractional calculus has massive contributions in scientific fields [1–3]. Therefore, it plays the main role for modelling many problems via in studying the fractional-order differential equations and its applications in different areas [4–7]. These applications are used in several fields of engineering and science such as biology [8], dynamics [9], physics [10,11], medicine [12], fluid [13] and others [14–16]. For these reasons, there have been a great number of essential works for studying the fractional-order differential equations, see [17–27]. Moreover, the fractional variable-order calculus is considered as a regular filter to supply an influential mathematical building for the description of the dynamical problems in the complex form [28,29]. In addition to this essential introduction of fractional applications, no one can deny the scientific research revolution for many physical and mathematical problems that described through nonlinear models (see for instance [30–39]).

In all modelled problems via the fractional differential equations (FDEs), the main concern of the mathematician researchers is to obtain the analytical or the numerical solutions for these equations. But, in

many problems, obtaining the exact solution is more complicated. Consequently, there are great efforts of researchers to acquire accurate techniques for gaining a numerical solution for FDEs. Various techniques have been dragged special interest, such as in [40–42]. Also, one of the most familiar methods that is used is the spectral methods which lead to accurate approximate solutions because their basis is considered as a linear combination of the orthogonal polynomials (see, [43–45]).

Moreover, the spectral methods for solving the fractional-order differential equations basically depends on a set of orthogonal polynomials. One of these polynomials is the classical Jacobi polynomials that indicated by $P_n^{(\alpha,\beta)}(x)$ ($n \geq 0$, $\alpha > -1$ & $\beta > -1$), [46]. Jacobi polynomials have been used extensively in practical applications and mathematical analysis because it has the advantages of gaining the numerical solutions in α and β parameters. Thus, it would be beneficial to carry out a systematic study via Jacobi polynomials with general indexes α and β and this clearly considered one of the aims and the novelty of this manuscript in addition to the extension of the time interval $t \in [0, l]$. Furthermore, in recent decades, numerical study for VFDEs is widely used [47,48]. Therefore, it appeared a

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large number of techniques for obtaining the numerical solution of these equations (see for instance [49–51]).

Here, the basic idea of this article generalization of the orthogonal polynomials in the base of solution by introducing a new shifted Jacobi operational matrix of the fractional derivative for solving numerically the multi-term variable-order FDEs in the following form:

$$D^{\mu(t)}u(t) = F \left(t, u(t), D^{\gamma_1(t)}u(t), D^{\gamma_2(t)}u(t), \dots, D^{\gamma_k(t)}u(t) \right), \quad 0 < t \leq l. \tag{1}$$

where $D^{\mu(t)}u(t)$ and $D^{\gamma_i(t)}u(t)$ ($i = 1, 2, \dots, k$) are the variable-order fractional derivatives defined in the Caputo sense.

Note that: if $\mu(t)$ and $\gamma_i(t)$ ($i = 1, 2, \dots, k$) are constants, then Equation (1) have the form:

$$D^\mu u(t) = F \left(t, u(t), D^{\gamma_1}u(t), D^{\gamma_2}u(t), \dots, D^{\gamma_k}u(t) \right), \quad 0 < t \leq l. \tag{2}$$

Also note that: our new suggestion method considered as a generalization for every numerical solution of the multi-term fractional-order problem that can be solved by many polynomials such as all Chebyshev polynomials, Legendre polynomials, Gegenbauer polynomials, Lucas polynomials, Vieta- Lucas polynomials, and Fibonacci polynomials.

In what follows: begin by reviewing certain definitions and properties of the variable-order fractional derivatives in Section 2. In Section 3, the Jacobi polynomials and their shifted ones are introduced in addition to the matrix formula of the approximate solution via the shifted Jacobi polynomials. In Section 4, the shifted Jacobi operational matrix (SJOP) of the variable-order derivatives is derived. In Section 5, the bound error of the investigated method is obtained. The numerical results of the suggested method are obtained and compared with other methods 6. Conclusions are obvious in the last Section.

2. Preliminaries

In this preparative section, we introduce briefly main required definitions of the Caputo’s fractional derivative in variable-order functions which help us in what follows.

Definition 2.1 ([19,48]): The Caputo variable-order fractional derivatives definition for the function $u(t) \in C^m[0, b]$ can be defined as:

$$D^{\mu(t)}u(t) = \frac{1}{\Gamma(1 - \mu(t))} \int_{0^+}^t \frac{u'(\tau)}{(t - \tau)^{\mu(t)}} d\tau + \frac{u(0^+) - u(0^-)}{\Gamma(1 - \mu(t))} t^{-\mu(t)}. \tag{3}$$

At the beginning time and $0 < \mu(t) < 1$, we have:

Definition 2.2 ([19,48]):

$$D^{\mu(t)}u(t) = \frac{1}{\Gamma(1 - \mu(t))} \int_{0^+}^t \frac{u'(\tau)}{(t - \tau)^{\mu(t)}} d\tau. \tag{4}$$

Also,

$$D^{\mu(t)}(a h(t) + b q(t)) = a D^{\mu(t)}h(t) + b D^{\mu(t)}q(t). \tag{5}$$

where a and b constant amounts. According to Equation (3), as in [19,48] we have:

$$D^{\mu(t)} C = 0, \quad C \text{ is a constant.} \tag{6}$$

$$D^{\mu(t)} t^m = \begin{cases} 0, & \text{for } m = 0, \\ \frac{\Gamma(m + 1)}{\Gamma(m + 1 - \mu(t))} t^{m - \mu(t)}, & \text{for } m = 1, 2, \dots \end{cases} \tag{7}$$

3. Jacobi polynomials and their properties

The beginning of this section by introducing essential basic facts of the Jacobi polynomials and their shifted ones, in addition to deriving some important tools that help us for developing the suggested method.

3.1. Jacobi polynomials and their shifted ones

The well-known Jacobi polynomials, $P_n^{(\alpha, \beta)}(x)$, are orthogonal polynomials of degree n in x defined on $[-1, 1]$ as the following analytical form [46]:

$$P_n^{(\alpha, \beta)}(x) = \sum_{i=0}^n \frac{\Gamma(\alpha + n + 1)\Gamma(\alpha + \beta + n + i + 1)}{\Gamma(\alpha + \beta + n + 1)\Gamma(\alpha + i + 1)\Gamma(i + 1)\Gamma(n - i + 1)} \times \left(\frac{x - 1}{2}\right)^i, \tag{8}$$

these polynomials given in Equation (8) can be generated by:

$$a_{1,n}^{\alpha, \beta} P_n^{(\alpha, \beta)}(x) = a_{2,n}^{\alpha, \beta} P_{n-1}^{(\alpha, \beta)}(x) - a_{3,n}^{\alpha, \beta} P_{n-2}^{(\alpha, \beta)}(x), \quad n = 2, 3, \dots \tag{9}$$

where

$$\begin{aligned} a_{1,n}^{\alpha, \beta} &= 2n(\alpha + \beta + n)(\alpha + \beta + 2n - 2), \\ a_{2,n}^{\alpha, \beta} &= (\alpha + \beta + 2n - 1) \\ &\quad \times (\alpha^2 - \beta^2 + (\alpha + \beta + 2n)(\alpha + \beta + 2n - 2)x), \\ a_{3,n}^{\alpha, \beta} &= 2(\alpha + n - 1)(\beta + n - 1)(\alpha + \beta + 2n). \end{aligned} \tag{10}$$

With the starting values

$$P_0^{(\alpha, \beta)}(x) = 1 \quad \text{and} \quad P_1^{(\alpha, \beta)}(x) = \frac{1}{2} [(\alpha + \beta + 2)x + (\alpha - \beta)]. \tag{11}$$

For using the polynomials of Equation (8) on the interval $t \in [0, l]$, modification of the variable $x = (2t/l - 1)$ will

be implemented. Hence, we have so-called the shifted Jacobi polynomials $P_n^{(\alpha,\beta)}(2t/l - 1)$ which indicated by $P_n^{*(\alpha,\beta)}(t)$. Thus, the analytical formula for these polynomials is given by:

$$P_n^{*(\alpha,\beta)}(t) = \sum_{i=0}^n (-1)^{n-i} \frac{\Gamma(n+\beta+1)\Gamma(n+i+\alpha+\beta+1)}{\Gamma(i+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n-i+1)\Gamma(i+1)^i} x^i \tag{12}$$

where

$$P_n^{*(\alpha,\beta)}(0) = (-1)^j \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)\Gamma(n+1)} \quad \text{and} \tag{13}$$

$$P_n^{*(\alpha,\beta)}(l) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}.$$

Also, $P_n^{*(\alpha,\beta)}(t)$ can be obtained via the recurrence relation:

$$b_{1,n}^{\alpha,\beta} P_n^{*(\alpha,\beta)}(t) = b_{2,n}^{\alpha,\beta} P_{n-1}^{*(\alpha,\beta)}(t) - b_{3,n}^{\alpha,\beta} P_{n-2}^{*(\alpha,\beta)}(t), \quad n = 2, 3, \dots \tag{14}$$

Where

$$b_{1,n}^{\alpha,\beta} = 2n(\alpha + \beta + n)(\alpha + \beta + 2n - 2),$$

$$b_{2,n}^{\alpha,\beta} = (\alpha + \beta + 2n - 1) \left[\alpha^2 - \beta^2 + (\alpha + \beta + 2n) \times (\alpha + \beta + 2n - 2) \left(\frac{2t}{l} - 1 \right) \right],$$

$$b_{3,n}^{\alpha,\beta} = 2(\alpha + n - 1)(\beta + n - 1)(\alpha + \beta + 2n). \tag{15}$$

With the starting values

$$P_0^{*(\alpha,\beta)}(t) = 1 \quad \text{and}$$

$$P_1^{*(\alpha,\beta)}(t) = (\alpha + \beta + 2) \frac{t}{l} - (\beta + 1). \tag{16}$$

Moreover, $P_n^{*(\alpha,\beta)}(t)$ satisfy the orthogonality relation on $[0, l]$ as:

$$\langle P_n^{*(\alpha,\beta)}(t), P_m^{*(\alpha,\beta)}(t) \rangle = \int_0^l P_n^{*(\alpha,\beta)}(t) P_m^{*(\alpha,\beta)}(t) \omega^{(\alpha,\beta)} dt = h_n$$

$$= \begin{cases} 0, & n \neq m, \\ \frac{l^{\alpha+\beta+1} \Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)}, & n = m. \\ \Gamma(n+\alpha+\beta+1) \end{cases} \tag{17}$$

where $\omega^{(\alpha,\beta)} = x^\beta(l-x)^\alpha$ is the weight function of $P_n^{*(\alpha,\beta)}(t)$.

Furthermore, since the shifted Jacobi polynomials that are given in Equation (12) with general indexes

$(\alpha > -1 \ \& \ \beta > -1)$. Then can claim of these polynomials; the most usually used orthogonal polynomials in numerical methods; the shifted Legendre polynomials $L_n^*(t)$; the shifted Gegenbauer polynomials $G_n^{*(\alpha,\beta)}(t)$. In addition to the shifted Chebyshev polynomials of the first, second, third and fourth kinds which denoted by $T_n^*(t)$, $U_n^*(t)$, $V_n^*(t)$ and $W_n^*(t)$, respectively. All these orthogonal polynomials are contacted with $P_n^{*(\alpha,\beta)}(t)$ through the relations:

$$L_n^*(t) = P_n^{*(0,0)}(t),$$

$$G_n^{*(\alpha,\beta)}(t) = \frac{\Gamma(n+1)\Gamma(\alpha+\frac{1}{2})}{\Gamma(n+\alpha+\frac{1}{2})} P_n^{*(\alpha-\frac{1}{2},\beta-\frac{1}{2})}(t),$$

$$T_n^*(t) = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})} P_n^{*(-\frac{1}{2},-\frac{1}{2})}(t),$$

$$U_n^*(t) = \frac{\Gamma(n+2)\Gamma(\frac{1}{2})}{2\Gamma(n+\frac{3}{2})} P_n^{*(\frac{1}{2},\frac{1}{2})}(t),$$

$$V_n^*(t) = \frac{2^{2n}[\Gamma(n+1)]^2}{\Gamma(2n+1)} P_n^{*(-\frac{1}{2},\frac{1}{2})}(t),$$

$$W_n^*(t) = \frac{2^{2n}[\Gamma(n+1)]^2}{\Gamma(2n+1)} P_n^{*(\frac{1}{2},-\frac{1}{2})}(t). \tag{18}$$

3.2. Function approximation via shifted Jacobi polynomials

The function $u(t) \in L_{\omega^{(\alpha,\beta)}}^2[0, l]$ can be expanded as the following expression: [46]

$$u(t) = \sum_{i=0}^{\infty} c_i P_i^{*(\alpha,\beta)}(t), \tag{19}$$

where c_i are the coefficients of the series. Hence, by taking n -terms of the series in Equation (19) we obtain:

$$u_n(t) = \sum_{i=0}^n c_i P_i^{*(\alpha,\beta)}(t) = C^T \phi(t). \tag{20}$$

where in Equation (20) $\phi(t) = [P_0^{*(\alpha,\beta)}(t), P_1^{*(\alpha,\beta)}(t), \dots, P_n^{*(\alpha,\beta)}(t)]^T$ and $C = [c_0, c_1, \dots, c_n]^T$ is a vector and it's enters elements have values given by

$$c_i = \frac{1}{h_i} \int_0^l u(t) P_i^{*(\alpha,\beta)}(t) \omega^{(\alpha,\beta)} dt, \quad i = 0, 1, 2, \dots, n. \tag{21}$$

Now, if we suppose

$$R(t) = [1, t, t^2, \dots, t^n]^T. \tag{22}$$

Then, according to Equation (22), the vector $\phi(t)$ can be expressed as:

$$\phi(t) = A_{(\alpha,\beta)} R(t). \tag{23}$$

where $A_{(\alpha,\beta)}$ is $(n + 1) \times (n + 1)$ of Equation (23) is a square matrix that specified by:

$$(a_{ij})_{0 \leq i,j \leq n} = \begin{cases} \Gamma(n + \beta + 1)\Gamma \\ (-1)^{n-i} \frac{(n + i + \alpha + \beta + 1)}{\Gamma(i + \beta + 1)\Gamma}, & i \geq j, \\ (n + \alpha + \beta + 1) \\ \Gamma(n - i + 1)\Gamma(i + 1)^i & \text{otherwise.} \\ 0, & \end{cases} \quad (24)$$

For instance, if $n = 4, \alpha = \beta = 0$ then A (matrix of shifted Legendre polynomials) is given by:

$$A_{(0,0)} = \frac{1}{j!} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 1 & -6 & 6 & 0 & 0 \\ -1 & 12 & -30 & 20 & 0 \\ 1 & -20 & 90 & -140 & 70 \end{pmatrix}.$$

Therefore, using Equation (23), we obtain

$$R(t) = A_{(\alpha,\beta)}^{-1} \phi(t). \quad (25)$$

Remark 3.1: For obtaining the square matrix A for all other orthogonal polynomials that related to the shifted Jacobi polynomials such as shifted Legendre polynomials, shifted Gegenbauer polynomials and shifted Chebyshev polynomials of all kinds, the coefficient of these polynomials according to the relation given in Equation (18) must be taken into consideration.

For example, if $n = 4, \alpha = \frac{1}{2}, \beta = \frac{-1}{2}$ then the square matrix A that derived in Equation (24) became (matrix of the fourth kind shifted Chebyshev polynomials) as follows:

$$A_{(\frac{1}{2},-\frac{1}{2})} = \frac{1}{j!} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 \\ 1 & -12 & 16 & 0 & 0 \\ -1 & 24 & -80 & 64 & 0 \\ 1 & -40 & 240 & -448 & 256 \end{pmatrix}. \quad (26)$$

4. Shifted Jacobi polynomials operational matrix (SJOM)

The shifted Jacobi operational matrix of fractional variable-order for supporting the numerical solution of the main problem will be investigated in this section. Hence, the problem will be transformed into the algebraic system of equations which solved numerically at the collocation points.

Firstly, $D^{\mu(t)}\phi(t)$ can be deduced as the following: since $\phi(t) = A_{(\alpha,\beta)} R(t)$, then we obtain

$$D^{\mu(t)}\phi(t) = D^{\mu(t)}(A_{(\alpha,\beta)} R(t)) = A_{(\alpha,\beta)} D^{\mu(t)}[1, t, t^2, \dots, t^n]^T. \quad (27)$$

Using Equation (7) in Equation (27), we obtain:

$$D^{\mu(t)}\phi(t) = A_{(\alpha,\beta)} \begin{bmatrix} 0, \frac{\Gamma(2)}{\Gamma(2 - \mu(t))} t^{(1-\mu(t))}, \\ \frac{\Gamma(3)}{\Gamma(3 - \mu(t))} t^{(2-\mu(t))}, \dots, \\ \frac{\Gamma(n + 1)}{\Gamma(n + 1 - \mu(t))} t^{(n-\mu(t))} \end{bmatrix}^T = A_{(\alpha,\beta)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2 - \mu(t))} t^{-\mu(t)} & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3 - \mu(t))} t^{-\mu(t)} \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \dots & 0 & \\ \dots & 0 & \\ \dots & 0 & \\ \vdots & \vdots & \\ \dots & \frac{\Gamma(n + 1)}{\Gamma(n + 1 - \mu(t))} t^{-\mu(t)} & \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^n \end{bmatrix} = A_{(\alpha,\beta)} G(t) R(t). \quad (28)$$

Where

$$G(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2 - \mu(t))} t^{-\mu(t)} & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3 - \mu(t))} t^{-\mu(t)} \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \dots & 0 & \\ \dots & 0 & \\ \dots & 0 & \\ \vdots & \vdots & \\ \dots & \frac{\Gamma(n + 1)}{\Gamma(n + 1 - \mu(t))} t^{-\mu(t)} & \end{bmatrix}. \quad (29)$$

Using Equation (25), we have

$$D^{\mu(t)}\phi(t) = A_{(\alpha,\beta)} G(t) A_{(\alpha,\beta)}^{-1} \phi(t). \quad (30)$$

$A_{(\alpha,\beta)} G(t) A_{(\alpha,\beta)}^{-1}$ is the operational matrix of $D^{\mu(t)}\phi(t)$.

Now, we can obtain the fractional variable order the approximated function that given in Equation (20) as

the following:

$$\begin{aligned}
 D^{\mu(t)} u(t) &\simeq D^{\mu(t)} (C^T \phi(t)) = C^T D^{\mu(t)} \phi(t) \\
 &= C^T A_{(\alpha,\beta)} G(t) A_{(\alpha,\beta)}^{-1} \phi(t).
 \end{aligned}
 \tag{31}$$

Secondly, $D^{\gamma_i(t)} \phi(t)$, $i = 1, 2, \dots, k$ can be obtained by follow the same way that proposed for obtaining the (SJOP) of $D^{\mu(t)} \phi(t)$, as follows:

$$D^{\gamma_i(t)} \phi(t) = (A_{(\alpha,\beta)} Q_i(t) A_{(\alpha,\beta)}^{-1}) \phi(t). \tag{32}$$

Where

$$Q_i(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\gamma_i(t))} t^{-\gamma_i(t)} & 0 \\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\gamma_i(t))} t^{-\gamma_i(t)} \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ \dots & 0 & \\ \dots & 0 & \\ \dots & 0 & \\ \vdots & \vdots & \\ \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\gamma_i(t))} t^{-\gamma_i(t)} & \end{bmatrix}. \tag{33}$$

The square matrix $A_{(\alpha,\beta)} Q_i(t) A_{(\alpha,\beta)}^{-1}$ is the operational matrix of $D^{\gamma_i(t)} \phi(t)$. Therefore,

$$\begin{aligned}
 D^{\gamma_i} u(t) &\simeq D^{\gamma_i(t)} (C^T \phi(t)) = C^T D^{\gamma_i(t)} \phi(t) \\
 &= C^T (A_{(\alpha,\beta)} Q_i(t) A_{(\alpha,\beta)}^{-1}) \phi(t).
 \end{aligned}
 \tag{34}$$

Via, Equations (31) and (34), then Equation (1) converted to:

$$\begin{aligned}
 &C^T A_{(\alpha,\beta)} G(t) A_{(\alpha,\beta)}^{-1} \phi(t) \\
 &= F[t, C^T \phi(t), C^T (A_{(\alpha,\beta)} Q_1(t) A_{(\alpha,\beta)}^{-1}) \phi(t), \\
 &\dots, C^T (A_{(\alpha,\beta)} Q_k(t) A_{(\alpha,\beta)}^{-1}) \phi(t)], \quad 0 \leq t \leq l.
 \end{aligned}
 \tag{35}$$

Now, using $t_i = l(2i + 1)/(2n + 2)$, $i = 0, 1, \dots, n$. Then Equation (35) can be turned into the following system:

$$\begin{aligned}
 &C^T A_{(\alpha,\beta)} G(t_i) A_{(\alpha,\beta)}^{-1} \phi(t_i) \\
 &= F[t_i, C^T \phi(t_i), C^T (A_{(\alpha,\beta)} Q_1(t_i) A_{(\alpha,\beta)}^{-1}) \phi(t_i), \\
 &\dots, C^T (A_{(\alpha,\beta)} Q_k(t_i) A_{(\alpha,\beta)}^{-1}) \phi(t_i)].
 \end{aligned}
 \tag{36}$$

Here, the system in Equation (36) in addition to the initial and boundary conditions can be solved numerically to determine the unknown vector C . Hence, the numerical solution that defined in Equation (20) can be calculated.

5. Error estimate

Theorem 5.1: Consider $u(t) \in [0, l]$ be n -th times continuously differentiable and $u_n(t)$ be the best square approaches of $u(t)$ given in Equation (20) then, we claim

$$\|u(t) - u_n(t)\| \leq \frac{MH^{n+1}}{(n+1)!} \sqrt{l}, \tag{37}$$

where

$$\begin{aligned}
 M &= \max_{t \in [0,l]} u^{(n+1)}(t), \quad H = \max \{l - t_0, t_0\} \quad \text{and} \\
 I &= \int_0^l t^\beta (l-t)^\alpha dt.
 \end{aligned}
 \tag{38}$$

Proof: Using Taylor expansion for $u(t)$ as follows:

$$\begin{aligned}
 u(t) &= u(t_0) + (t - t_0)u'(t_0) + \dots + \frac{(t - t_0)^n}{n!} u^{(n)}(t_0) \\
 &+ \frac{(t - t_0)^{n+1}}{(n+1)!} u^{(n+1)}(\xi),
 \end{aligned}
 \tag{39}$$

where $t_0 \in [0, l]$ and $\xi \in]t_0, t[$. Assume

$$\tilde{u}_n(t) = u(t_0) + (t - t_0)u'(t_0) + \dots + \frac{(t - t_0)^n}{n!} u^{(n)}(t_0), \tag{40}$$

then

$$\|u(t) - \tilde{u}_n(t)\| = \left| \frac{(t - t_0)^{n+1}}{(n+1)!} u^{(n+1)}(\xi) \right|. \tag{41}$$

According to, $u_n(t)$ that given in Equation (20), we obtain

$$\begin{aligned}
 \|u(t) - u_n(t)\|^2 &\leq \|u(t) - \tilde{u}_n(t)\|^2 \\
 &= \int_0^l \omega^{(\alpha,\beta)} [u(t) - \tilde{u}_n(t)]^2 dt \\
 &= \int_0^l \omega^{(\alpha,\beta)} \left[\frac{(t - t_0)^{n+1}}{(n+1)!} u^{(n+1)}(\xi) \right]^2 dt \\
 &\leq \frac{M^2}{[(n+1)!]^2} \int_0^l \omega^{(\alpha,\beta)} [(t - t_0)^{n+1}]^2 dt.
 \end{aligned}
 \tag{42}$$

Now, let $H = \max \{l - t_0, t_0\}$, thus Equation (42) can rewritten as:

$$\|u(t) - u_n(t)\|^2 \leq \frac{M^2 [(H)^{n+1}]^2}{[(n+1)!]^2} \int_0^l \omega^{(\alpha,\beta)} dt. \tag{43}$$

Since, $\omega^{(\alpha,\beta)} = t^\beta (l-t)^\alpha$ then,

$$\|u(t) - u_n(t)\|^2 \leq \frac{M^2 [(H)^{n+1}]^2}{[(n+1)!]^2} \int_0^l t^\beta (l-t)^\alpha dt. \tag{44}$$

Hence, the proof is completed. ■

Corollary 5.1: Using Equation (37) in case of $\alpha = \beta = 0$, then $l = l$. This implies that we can obtain the error bound in case of using the shifted Legendre polynomials [50].

Corollary 5.2: Via Equation (37) if $\alpha = \beta = \frac{1}{2}$, then $l = l^2\pi/8$ which is in complete agreement with the error bound that obtained via the second kind shifted Chebyshev polynomials which introduced by Liu and et al. [48].

Corollary 5.3: Using Equation (37) in case of $\alpha = \frac{1}{2}, \beta = \frac{-1}{2}$, then $l = l\pi/2$ which is in complete agreement with the error bound that obtained via the fourth kind shifted Chebyshev polynomials which investigated by Nagy and et al. [49].

Remark 5.1: The error bound that obtained by all orthogonal related functions with the shifted Jacobi polynomials can be obtained by using the suggested technique.

6. Numerical experiments

To illustrate the accuracy, applicability, and generality of the presented method, several test examples are carried out in our hand section. Also, these examples support our theoretical discussion and give more general results where the indexes of the shifted Jacobi polynomials α, β carried out the results at different values of them. For comparing our computational results, the maximum absolute error E_{max} between the analytical and numerical solutions will be used. Where E_{max} defined as the following:

$$E_{max} = \max_{i=0,1,\dots,n} |u(t_i) - u_n(t_i)|, \quad t_i = \frac{l(2i + 1)}{(2n + 2)}, \quad i = 0, 1, \dots, n. \quad (45)$$

Example 6.1 ([48,49]): Consider the multi-term fractional variable-order differential equation

$$a D^{\mu(t)} u(t) + b(t) D^{\gamma_1(t)} u(t) + c(t) D^{\gamma_2(t)} u(t) + e(t) D^{\gamma_3(t)} u(t) + k(t) u(t) = g(t), \quad u(0) = 2, \quad u'(0) = 0, \quad t \in [0, l]. \quad (46)$$

Where

$$a = 1, \quad b(t) = t^{\frac{1}{2}}, \quad c(t) = t^{\frac{1}{3}}, \quad e(t) = t^{\frac{1}{4}}, \quad k(t) = t^{\frac{1}{5}}, \quad g(t) = -a \frac{t^{2-\mu(t)}}{\Gamma(3-\mu(t))} - b(t) \frac{t^{2-\gamma_1(t)}}{\Gamma(3-\gamma_1(t))} - c(t) \frac{t^{2-\gamma_2(t)}}{\Gamma(3-\gamma_2(t))} - e(t) \frac{t^{2-\gamma_3(t)}}{\Gamma(3-\gamma_3(t))} + k(t) \left(2 - \frac{t^2}{2}\right). \quad (47)$$

The analytical solution is $u(t) = (2 - t^2/2)$. This example will be solved in different cases as follows:

Table 1. c_0, c_1 and c_2 at distinct values of α and β for Example 6.1 Case 1.

α	β	c_0	c_1	c_2
0	0	1.8333333333333333	-0.25	-0.0833333333333333
1	1	1.84375	-0.25	-0.09375
0.5	0.5	1.84375	-0.125	-0.03125
-0.5	0.5	1.6875	-0.15625	-0.03125
0.5	-0.5	1.93750	-0.09375	-0.03125

Case 1: Consider the fractional variable-order parameters defined as the following:

$$\mu(t) = 2t, \quad \gamma_1(t) = \frac{t}{3}, \quad \gamma_2(t) = \frac{t}{4}, \quad \gamma_3(t) = \frac{t}{5}. \quad (48)$$

Through the investigated technology in Section 4 with $(n + 1)$ finite terms of the approximate solution that given in Equation (20) then we obtain

$$aC^T A_{(\alpha,\beta)} G(t) A_{(\alpha,\beta)}^{-1} \phi(t) + b(t) C^T A_{(\alpha,\beta)} Q_1(t) A_{(\alpha,\beta)}^{-1} \phi(t) + c(t) C^T A_{(\alpha,\beta)} Q_2(t) A_{(\alpha,\beta)}^{-1} \phi(t) + e(t) C^T A_{(\alpha,\beta)} Q_3(t) A_{(\alpha,\beta)}^{-1} \phi(t) + k(t) \phi(t) = g(t). \quad (49)$$

Here, using the collection points t_i that defined in Equation (45) for solving numerically the system of equations that given in Equation (49) via the collocation method. Then, we claim

$$aC^T A_{(\alpha,\beta)} G(t_i) A_{(\alpha,\beta)}^{-1} \phi(t_i) + b(t_i) C^T A_{(\alpha,\beta)} Q_1(t_i) A_{(\alpha,\beta)}^{-1} \phi(t_i) + c(t_i) C^T A_{(\alpha,\beta)} Q_2(t_i) A_{(\alpha,\beta)}^{-1} \phi(t_i) + e(t_i) C^T A_{(\alpha,\beta)} Q_3(t_i) A_{(\alpha,\beta)}^{-1} \phi(t_i) + k(t_i) \phi(t_i) = g(t_i), \quad i = 0, 1, \dots, n. \quad (50)$$

Hence, the system of Equations (50) solved numerically via Newton's iteration method and the vector $C = [c_0, c_1, \dots, c_n]^T$ is obtained. Therefore, the numerical solution in Equation (20) is calculated.

Now, solving the algebraic system of Equations (50) with $n = 2$ and $l = 1$. Thus, we can write

$$u_n(t) = \sum_{i=0}^2 c_i P_i^{*(\alpha,\beta)}(t) = \left(2 - \frac{t^2}{2}\right). \quad (51)$$

Table of the numerical results will not be presented in Case 1 because the exact solution is obtained. The three unknown coefficients with several choices of α and β are presented in Table 1 (Figure 1).

Moreover, Table 2 shows E_{max} that obtained via the suggested method in the case of $\alpha = \beta = 0$ and those given in [48] with several values of l and n for the fractional variable-order derivatives $\mu(t), \gamma_1(t), \gamma_2(t)$ and $\gamma_3(t)$ that given in Case 1. From these results, we can deduce that these results identically the analytical

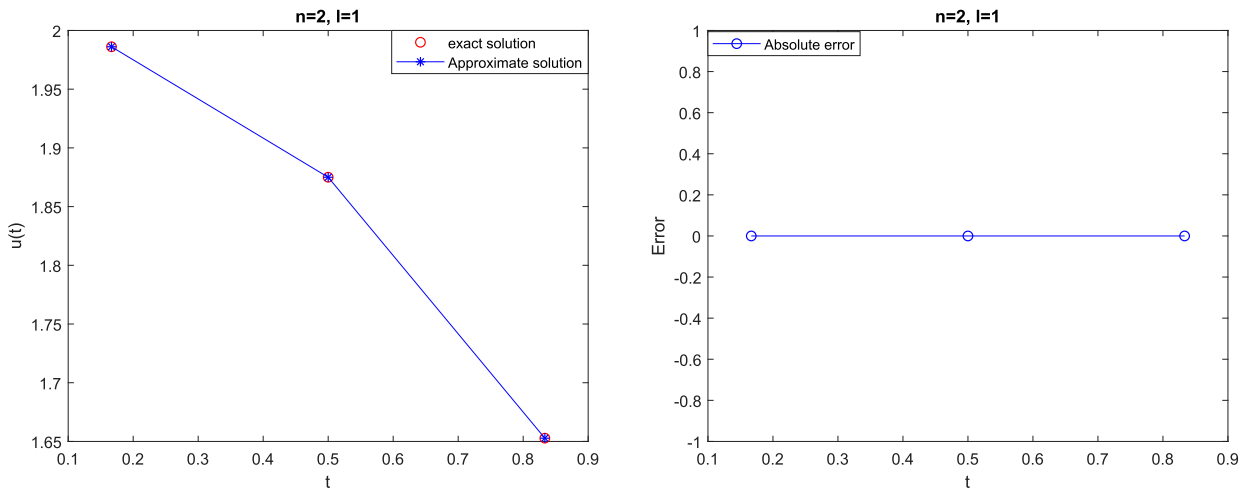


Figure 1. Exact solution and numerical solutions (left Figure), absolute error (right Figure) for Example 6.1 Case 1 in case of $n = 2$, $l = 1$, and $\alpha = \beta = 0$.

Table 2. Comparison E_{\max} between technique given in [48] and proposed technique at $\alpha = \beta = 0$ for Example 6.1 Case 1 with disjoint of l and n .

l	Method in [48]			Our proposed method		
	$n = 3$	$n = 4$	$n = 5$	$n = 3$	$n = 4$	$n = 5$
1	0	2.2204×10^{-16}	2.2204×10^{-16}	0	0	0
2	0	4.4409×10^{-16}	1.3323×10^{-15}	0	0	0
4	2.2204×10^{-16}	3.5527×10^{-15}	3.1974×10^{-14}	0	0	0

solution and more accurate than these results that obtained by Liu and et al. [48]. Also, the method in [48] depends on the second kind shifted Chebyshev polynomials which consider as a particular case of our suggested technique that depends on the shifted Jacobi polynomials. Furthermore, as seen from Table 3, the vector $C^T = [c_0, c_1, \dots, c_n]$ obtained is mainly composed of three terms, namely, $[c_0, c_1, c_2]$. Also, E_{\max} at $l = 1$ and distinct choices of n, α and β for the fractional variable-order derivatives $\mu(t), \gamma_1(t), \gamma_2(t)$ and $\gamma_3(t)$ is shown in the same Table. Table 3, shows achieving a full agreement with the analytical solution of the given problem via a few terms of $P_n^{*(\alpha, \beta)}(t)$.

Remark 6.1: If $\alpha = \frac{1}{2}, \beta = \frac{-1}{2}$ and $l = 1, 2 \& 4$, then the numerical results that obtained by our technique for

Example 6.1 Case 1 is in complete agreement with the results that obtained by authors in [49].

Case 2: In the case of:

$$\mu = 2, \quad \gamma_1 = 1.234, \quad \gamma_2 = 1, \quad \gamma_3 = 0.333. \quad (52)$$

This case considered as a particular case of Case 1 because the multi-term fractional variable-order differential equation that given in Equation (46) becomes multi-term constant-order fractional differential equation.

Table 4 gives E_{\max} that gained through the introduced technique for Example 6.1 Case 2 at distinct values of $n, \alpha = \beta = 0$ and $l = 1$. From the first look at Table 4, the suggested gives highly accurate results comparing with others [40,44,48]. Table 5, presented

Table 3. Computational results of Example 6.1 Case 1 for disjoint choices of α and β at $l = 1$..

n	α	β	C^T	E_{\max}
3			[1.8333, -0.2500, -0.0833, 0.0000]	0
4	0	0	[1.8333, -0.2500, -0.0833, 0.0000, -0.0000]	0
5			[1.8333, -0.2500, -0.0833, 0.0000, -0.0000, 0.0000]	0
6			[1.8333, -0.2500, -0.0833, 0.0000, -0.0000, 0.0000, 0.0000]	0
3			[1.8438, -0.2500, -0.0938, -0.0000]	0
4	1	1	[1.8438, -0.2500, -0.0938, 0.0000, -0.0000]	0
5			[1.8438, -0.2500, -0.0938, 0.0000, -0.0000, 0.0000]	0
6			[1.8438, -0.2500, -0.0938, -0.0000, -0.0000, -0.0000, 0.0000]	0
3			[1.8438, -0.1250, -0.0313, 0.0000]	0
4	$\frac{1}{2}$	$\frac{1}{2}$	[1.8438, -0.1250, -0.0313, 0.0000, 0.0000]	0
5			[1.8438, -0.1250, -0.0313, 0.0000, -0.0000, 0.0000]	0
6			[1.8438, -0.1250, -0.0313, 0.0000, 0.0000, 0.0000, -0.0000]	0
3			[1.9375, -0.0937, -0.0312, 0.0000]	0
4	$\frac{1}{2}$	$\frac{-1}{2}$	[1.9375, -0.0938, -0.0313, 0.0000, -0.0000]	0
5			[1.9375, -0.0937, -0.0312, 0.0000, -0.0000, 0.0000]	0
6			[1.9375, -0.0937, -0.0312, 0.0000, 0.0000, -0.0000, 0.0000]	3.997×10^{-14}

Table 4. Comparison the E_{\max} for the methods that were given in [40,44,48], and the proposed technique for Example 6.1 Case 2 where $l = 1$ $\alpha = -0.5$ and $\beta = 0.5$.

Method in [40]		Method in [44]		Method in [48]		Ours $\alpha = \beta = 0$	
n	E_{\max}	n	E_{\max}	n	E_{\max}	n	E_{\max}
10	0.023659	5	6.88384×10^{-5}	3	4.4409×10^{-16}	2	0
100	0.000986	10	3.00351×10^{-6}	4	1.4633×10^{-13}	3	0
1000	0.000044	20	1.67837×10^{-7}	5	3.2743×10^{-12}	4	0
		40	1.02241×10^{-8}	6	1.0725×10^{-13}	5	0

Table 5. Comparison E_{\max} between results in [48] and our results at $\alpha = -\beta = \frac{-1}{2}$ for Example 6.1 Case 2 with disjoint choices of l and n .

l	Method in [48]			Proposed method		
	$n = 3$	$n = 4$	$n = 5$	$n = 3$	$n = 4$	$n = 5$
2	8.8818×10^{-16}	9.1038×10^{-15}	2.2959×10^{-13}	0	0	4.8849×10^{-14}
4	8.8818×10^{-16}	1.0214×10^{-14}	7.3275×10^{-15}	0	0	1.9007×10^{-13}

the comparison of E_{\max} between numerical results given in [48] and our method at $\alpha = -\beta = \frac{-1}{2}$ for Example 6.1 Case 2 with different values of l and n . The values of E_{\max} are better than that given in [48], in addition to the extension the interval from $[0, 1]$ to $[0, 2]$ and $[0, 4]$ which is not solved in [44]. Moreover, Figure 2 shows the analytical and the approximate solutions (left figure), the absolute error (right figure) for Example 6.1 Case 2 at $n = 5, l = 6$ and $\alpha = \beta = 0$. These results in a whole compact with the numerical results in [50].

Remark 6.2: At $\alpha = \frac{1}{2}, \beta = \frac{-1}{2}$ and $l = 1, 2$ & 4 , numerical results that obtained by our technique coincidence with the results that obtained by the authors in [49]. Moreover, we can obtain numerical results in Case 2 for several choices of the indexes (α, β) at disjoint values of n and the length of the interval $[0, l]$. These results approximately give the analytic solution. Moreover when the values of the fractional parameters are integers or more than one, we must have this into our accounts in the calculating of the square A (became matrix of the ordinary differentiate in the integer case).

Example 6.2: Assume the following equation:

$$a D^{\mu(t)}u(t) + b D^{\gamma(t)}u(t) + c u(t) = g(t), \quad u(0) = u_0, \quad u'(0) = u_1, \quad t \in [0, l]. \quad (53)$$

This example will take several forms depends on the known coefficients a, b and c , the initial conditions u_0, u_1 , the known function $g(t)$, and the variable of fractional order derivatives $\mu(t), \gamma(t)$ as the following cases:

Case 1: [49] Suppose that

$$a = 1, \quad b = 2, \quad c = 4, \quad u_0 = u_1 = 0,$$

$$\mu(t) = 2t, \quad \gamma(t) = \frac{1+t}{2},$$

$$g(t) = \frac{\Gamma(3)}{\Gamma(3-\mu(t))} t^{2-\mu(t)} + \frac{2\Gamma(3)}{\Gamma(3-\gamma(t))} t^{2-\gamma(t)} + 4t^2. \quad (54)$$

The analytical solution of Example 6.2 Case 1 is $u(t) = t^2$.

Now, take only three terms of the numerical series of the solution and follow the same steps that followed

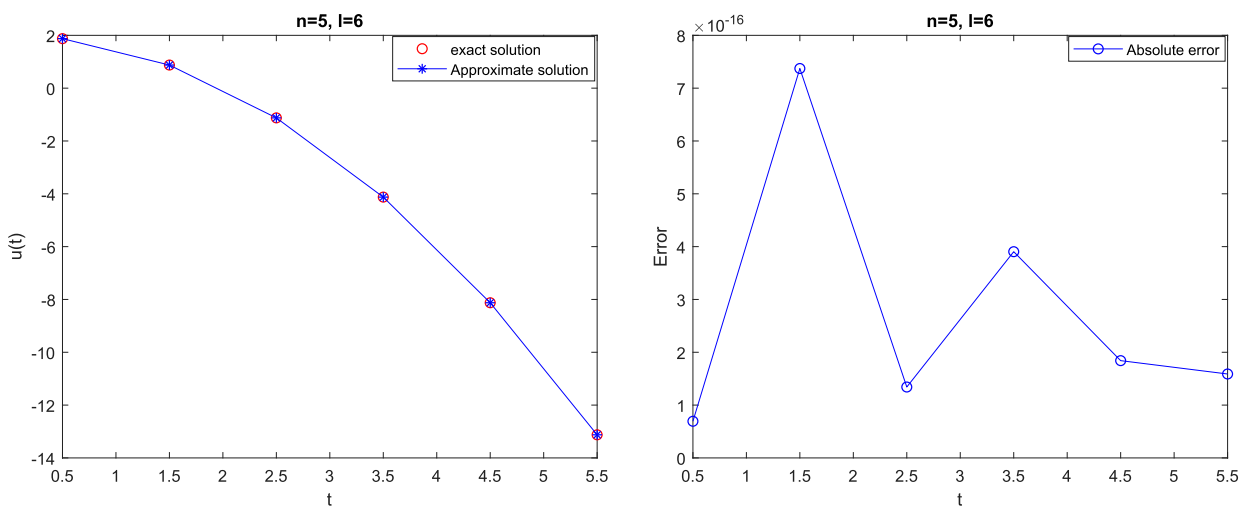


Figure 2. Exact solution and numerical solutions (left Figure), absolute error (right Figure) for Example 6.1 Case 2 in case of $n = 5, l = 6$, and $\alpha = \beta = 0$.

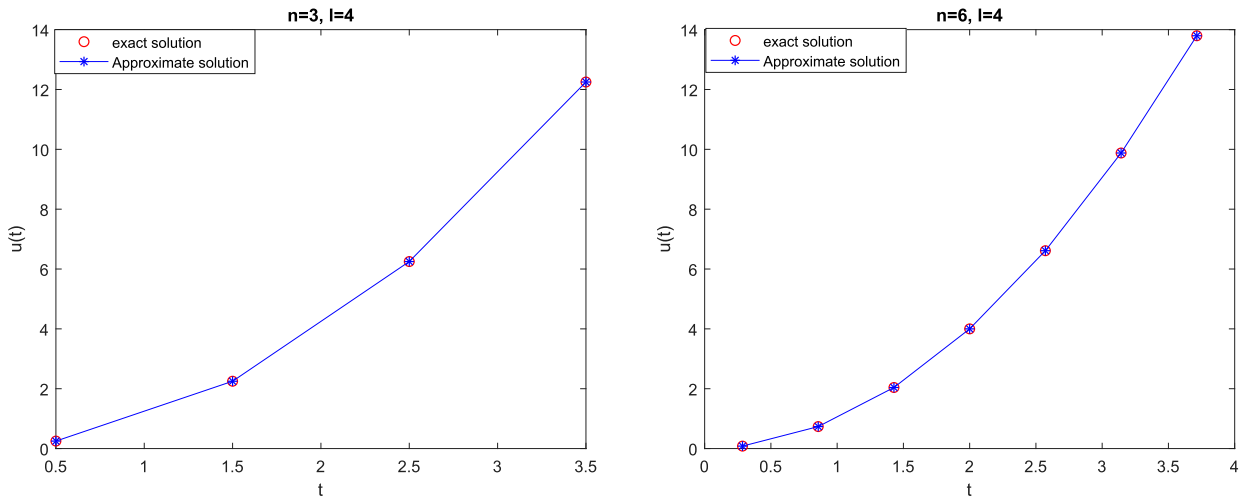


Figure 3. Exact solution and numerical solutions of Example 6.2 Case 1 for different values of n, l and $\alpha = -\beta = 0.5$.

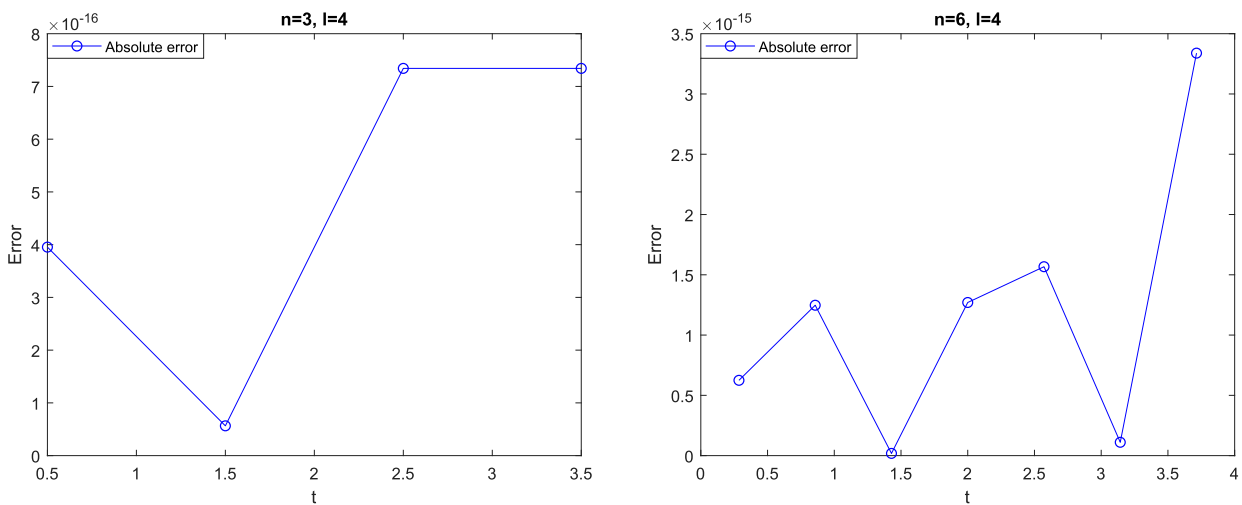


Figure 4. Absolute error of Example 6.2 Case 1 for different values of n, l and $\alpha = -\beta = 0.5$.

Table 6. c_0, c_1 and c_2 of Example 6.2 Case 1 for several choices of α and β at $n = 2, l = 1$.

α	β	c_0	c_1	c_2
0	0	0.3333333333333333	0.50	0.1666666666666667
1	1	0.3125	0.50	0.1875
0.5	0.5	0.3125	0.250	0.0625
-0.5	0.5	0.6250	0.3125	0.0625
0.5	-0.5	0.1250	0.1875	0.0625

in example 6.1 then, we claim: Here, the approximate solution

$$u_2(t) = \sum_{i=0}^2 c_i P_n^{*(\alpha, \beta)}(t) = t^2. \tag{55}$$

Which is in full accordance with the exact solution. Also, Figures 3 and 4 support the results (Table 6).

Remark 6.3: If $\alpha = -\beta = \frac{1}{2}$ and $l = 1, 2$ & 4, the numerical results that obtained by our technique for Example 6.2 Case 1 is congruent to the results that obtained by the authors in [49].

Case 2: [20,49] Suppose that

$$\begin{aligned}
 a &= 1, \quad b = -10, \quad c = 1, \quad u_0 = 5, \\
 u_1 &= 10, \quad \mu(t) = \frac{t + 2 \exp(t)}{7}, \quad \gamma(t) = 1, \\
 g(t) &= 10 \left(\frac{t^{2-\mu(t)}}{\Gamma(3 - \mu(t))} + \frac{t^{1-\mu(t)}}{\Gamma(2 - \mu(t))} \right) \\
 &\quad + 5t^2 - 90t - 95. \tag{56}
 \end{aligned}$$

The analytical solution of Example 6.2 Case 2 is $u(t) = 5(1 + t)^2$.

Again, we apply the proposed technique that proposed Section 4 and follows the same steps that followed in Example 6.1, we have the numerical results of this Example, Example 6.2, in this case (Case 2).

The absolute error that obtained via the suggested method for Example 6.2 Case 2 at $\alpha = \beta = 1$ and $n = 2$ are given Table 7. By looking at Table 7, it transparent that the claimed results through the proposed technique are more accurate than others [20,49]. Moreover, the numerical results can be obtained for various values of α and β .

Table 7. Comparison E_{\max} of the methods were given in [20,49] and the suggested method for Example 6.2 Case 2 at $\alpha = \beta = 1$.

$t \in [0, 1]$	ln [20], $n = 4$	ln [49], $n = 2$	The proposed Method, $n = 2$
0.2	8.091305×10^{-12}	1.818101×10^{-12}	0
0.4	2.024535×10^{-9}	1.817213×10^{-12}	8.881784×10^{-16}
0.6	9.564669×10^{-10}	1.820765×10^{-12}	1.776356×10^{-15}
0.8	1.696030×10^{-10}	1.818989×10^{-12}	1.776356×10^{-15}
1.0	1.734222×10^{-10}	1.818989×10^{-12}	0

Case 3: [10,49] (The damped mechanical oscillator)

$$a = 1, \quad b = 2, \quad c = 4, \quad u_0 = u_1 = 0, \\ \mu(t) = 2, \quad \gamma(t) = 1, \quad \text{and} \quad g(t) = 2 + 4t + 4t^2. \tag{57}$$

The exact solution of Example 6.2 Case 3 is $u(t) = t^2$. In this case, the physical system whose behaviour is governed by an (ODE) given by:

$$a D^2 u(t) + b Du(t) + c u(t) = g(t), \quad u(0) = 0, \quad u'(0) = 0, \quad t \in [0, 1], \tag{58}$$

where, a symbolize the mass of the particle linked to the spiral, b is a measurement of the power of the damper, c refers to the spring stiffness, and $g(t)$ is the applied outer force. t time and $u(t)$ is the displacement of the mass from its rest position.

Now, using technique given in Section 4 at $\alpha = \beta = 0$ and $n = 2$. Then the exact solution of the damped mechanical oscillator Equation (58) that given in Case 3 of Example 6.2 is obtained. Also, we can claim the numerical results of Example 6.2 Case 3 for several values of the indexes α, β such as $\alpha = \beta = 1, \alpha = \beta = \frac{1}{2}$ and $\alpha = -\beta = \mp \frac{1}{2}$ at $n = 2$. These results are comparable with the exact solution.

Remark 6.4: If $\alpha = -\beta = \frac{1}{2}$ and $n = 2, l = 1$. Then, the numerical results that obtained by our technique for Example 6.2 Case 3 is coincident with the results obtained by the authors in [49]. Therefore, the method that introduced in paper [49] considered as a special case of our suggested method in this article.

Case 4: [21] (Bagley-Torvik equation)

which arises in the modelling of the motion of a rigid plate immersed in Newtonian fluid, it obtained in case

of:

$$a = b = c = 1, \quad u_0 = u_1 = 0, \quad \mu(t) = 2, \\ \gamma(t) = \frac{3}{2}, \quad \text{and} \quad g(t) = t^2 + 2 + \frac{4}{\sqrt{\pi}} t^{0.5}. \tag{59}$$

The exact solution of Example 6.2 case 4 is $u(t) = t^2$.

The three unknown coefficients will be obtained in Table 8. These unknowns are calculated by applying the suggested method that proposed in Section 4 in the case of $n = 2$.

Consequently, we can rewrite the approximate solution as

$$u_2(t) = (c_0, \quad c_1, \quad c_2) \begin{pmatrix} 1 \\ P_1^{*(\alpha,\beta)}(t) \\ P_2^{*(\alpha,\beta)}(t) \end{pmatrix} = t^2. \tag{60}$$

Which is the exact solution.

It is clear that with the aid of Table 8, the exact solution for Example 6.2 Case 4 is obtained at $n = 2, l = 1$ via the steps of a solution by the introduced method. Although, the best results of the same problem which obtained by the authors given in [21] are 6×10^{-10} and 2×10^{-10} . Moreover, the authors of the same refereed article obtained the exact solution when $n \rightarrow \infty$. It is obvious that the suggested technique is more accurate than other methods given in [21].

Remark 6.5: If $\alpha = \beta = 0$ and $l = 1$. Then, the approximate results that gained through the presented technique for Example 6.2 Case 4 is exactly like the results that obtained by the authors in [43] which considered as a special case of our suggested technique.

7. Conclusions

In this article, the operational matrix of shifted Jacobi polynomials is investigated to solve the multi-term FDEs and multi-term variable order FDEs numerically. The basic idea behind this method is converted the main problem into a system of algebraic equations that can be solved numerically. Also, the presented technique is applied for several choices of the indexes (α, β) and disjoint fractional variable-order fractional derivatives $\mu(t), \gamma(t)$ which implies us with the flexibility to solve many fractional order problems. Therefore, we enabled to solve many different applications in physics such as the damped mechanical oscillator and Bagley-Torvik equation. Moreover, the error estimate

Table 8. c_0, c_1 and c_2 of Example 6.2 Case 4 for several choices of α and β at $n = 2, l = 1$.

α	β	c_0	c_1	c_2
0	0	0.33333333333329	0.500000000000003	0.166666666666667
1	1	0.312500000000004	0.499999999999991	0.187500000000001
0.5	0.5	0.312500000000000	0.249999999999998	0.06250
-0.5	0.5	0.624999999999996	0.312500000000002	0.06250
0.5	-0.5	0.1250	0.1875	0.0625

has been discussed. Furthermore, numerical examples are introduced in many cases for several choices of the parameters of the original FDEs to demonstrate the accuracy, efficiency, generality, and applicability of the introduced technique. From the obtained results, only a few terms of the shifted Jacobi polynomials are required to acquire satisfactory results. Finally, in future work, we can use the proposed method for solving non-linear problems in one and two dimensional in addition to the linear problems in two dimensional. All computed numerical results obtained via the MATLAB program and +e data used to support the findings of this study are included within the article.

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