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Soliton Solutions of Mathematical Physics Models Using the Exponential Function Technique

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Abstract: This paper is based on finding the exact solutions for Burger's equation, Zakharov-Kuznetsov (ZK) equation and Kortewegde vries (KdV) equation by utilizing exponential function method that depends on the series of exponential functions. The exponential function method utilizes the homogeneous balancing principle to find the solutions of nonlinear equations. This method is simple, wide-reaching and helpful for finding the exact solution of nonlinear conformable PDEs.

Keywords: exp-function method; conformable derivative; conformable ZK equation; conformable kdV equation; conformable Burger's equation

1. Introduction

When a physical system is modeled mathematically, nonlinear evolution equations are generated. Many physical phenomenon are inherently nonlinear. Consequently, modeling such phenomena mathematically gives the nonlinear evolution equation. Investigation of the wave solutions relating to the nonlinear PDEs in imperative for contemplating nonlinear physical occasions.

Recently, scientists and engineers have been attracted towards PDEs of non-integer orders. Ordinary Calculus does not satisfy the needs of solving such PDEs with non-integer problems. These needs extended the boundary of Ordinary calculus for solving these PDEs. In this paper, three nonlinear PDEs, i.e., Burger's equation, Kortewegde vries (Kdv) equation and Zakharov-Kuznetsov (ZK) equation are discussed. All of these nonlinear PDEs have symmetries in many ways. For example, Burger's equation is invariant, i.e., symmetric with respect to any space or time shift. Burger's equation also possesses scale symmetries. The said symmetry is based on the a priori premise that the division of variables into dependent and independent ones is a necessary part of a PDE. However, many arguments show that this point of view is limited. In particular, numerous tricks are available to find transformations which do not respect this division. Alexandre Vinogradov discussed four ideas to show the symmetries in nonlinear PDEs [1]. Readers are referred to these papers for understanding symmetry, its various variant of definitions, and application of symmetries on PDEs etc. [2–7].

Due to the applications of these PDEs in variety of practical problems like hydrodynamics, biomathematics, fluid dynamics, chemical physics, nonlinear optics, plasma physics, elastic media, optical fibers, geochemistry and chemical kinematics, there is a need to have general solutions to fit in for every kind of practical problem. We can explore hidden physical information extracted from nonlinear phenomenon by solving a nonlinear equation formulated against a complex event. In order to find the exact solutions of nonlinear PDEs, many powerful methods have been developed. A few of the methods are: the sub-equation method [8–10], the modified simple equation method [11], the $\exp(-\phi(\xi))$ method [12,13], homotopy analysis method [14,15] , first integral method [16,17], G/G' expansion method [18–20], the functional variable method [21,22], the improved $\tan\left(\frac{\phi}{2}\right)$-expansion method [23,24], etc.

In 2006, He and Wu presented an exp-function method to find the travelling wave solutions of nonlinear PDEs [25–28]. This method was reliable and effective to find a solution for such complex physical phenomena. The exp-function method is based on the homogeneous balancing principle. This is the reason that it can be applicable to nonlinear PDEs that satisfy the homogeneous balancing principle. The performance of the exponential function method is satisfactory. Moreover, it is not only reliable but also effective. A concise and promising tool for solving nonlinear PDEs could be built with the help of the Exp-function method. In this paper, the exp-function method is employed to extract the new exact solutions of the conformable ZK equation, conformable KdV equation and conformable Burger's equation.

This paper is divided into the following sections. The subsequent Section comprises of knowledge about preliminaries related to the conformable derivative, the same section deals with the properties of the derivatives, Section 3 comprises the exp-function method, Section 4 consists of implementation of exp-function for ZK equation, KdV equation and Burger's equation and finally conclusions and recommendations are presented in Section 5.

2. Groundwork for Conformable Derivative

Many scientists likes to have simple, effective and the most natural definitions; these definitions are preferred and have become popular due to ease of understanding. In this regard, khalil et al. described the definition of local derivative also known as conformable derivative of order $\zeta \in (0,1]$ [29], which became popular soon after publishing. For any value of ζ, the definition can be generalized, but the most important case is $\zeta \in (0,1]$. Once it is demonstrated for particular case of ζ, the other cases become simple.

In this section, conformable derivatives are explained in detail with examples. A conformable derivative of s of order ζ is defined for any generalized function S where $s : [0,\infty) \to \mathbb{R}$, as follows

$$(M_\zeta s)(t) = \lim_{\varepsilon \to 0} \frac{s\left(t + \varepsilon t^{1-\zeta}\right) - s(t)}{\varepsilon} \tag{1}$$

for all $t > 0$, $\zeta \in (0,1)$. If s is ζ-differentiable for the range $(0,a)$, $a > 0$ and $\lim_{t\to 0^+} s^{(\zeta)}(t)$ exists, then define

$$s^{(\zeta)}(0) = \lim_{t \to 0^+} s^{(\zeta)}(t).$$

It is obvious that a constant function has zero conformable derivative. Also, if $M_\zeta s(t) = 0$ then $s(x) = 0$ for some $x \in (c,d)$ using conformable fractional mean value theorem proved in [29].

The conformable integral of the same function is defined in following equation:

$$I_\zeta^c(r)(x) = \int_c^x \frac{r(t)}{t^{1-\zeta}} dt \tag{2}$$

Here, $c \geq 0$ and $\zeta \in (0,1]$. The above integral is in the form of the famous Riemann integral in improper format.

For $\zeta \in (0,1]$ if r is a function which is differentiable with respect to ζ- at a point $t > 0$ then $(M_\zeta r)(t) = t^{1-\zeta \frac{dr}{dt}}(t)$.

Theorem 1. *(Chain rule) Suppose $r, s : (a, \infty) \to \mathbb{R}$ are (left) ζ-differentiable functions, where $\zeta \in (0,1]$. If $k(t) = r(s(t))$, then $k(t)$ is (left) ζ-differentiable and $\forall t$ with $t \neq a$ and $s(t) \neq 0$ we have*

$$\left(M_\zeta^a h\right)(t) = \left(M_\zeta^a r\right)(s(t)).\left(M_\zeta^a s\right)(t).s(t)^{\zeta-1}, \tag{3}$$

taking $t = a$, we have

$$\left(M_\zeta^a h\right)(a\ ) = \lim_{t \to a^+} \left(M_\zeta^a r\right)(s(t)).\left(M_\zeta^a s\right)(t).s(t)^{\zeta-1}. \tag{4}$$

This definition has gained the attention of many scientist and a lot of work has been done on it. It was introduced in 2014 then developed in 2015 and is currently under intensive investigations [29–31]. Abdel Jawad used this derivative to demonstrate exponential functions, by parts integration, chain rule, Taylor power series expansion [30] and Laplace transform.

3. Presentation of Exp-Function Method for Nonlinear Conformable Pdes

The primitive work on the exponential function method was explored and performed by He and Wu in 2006 [25]. This work was further studied with detailed analysis in [32–38]. Homogeneous balancing principle was the base of this proposed method. The only nonlinear conformable PDEs that can be solved by this method are those which satisfy the homogeneous balancing principle. As a result, the exact solutions can be obtained for these equations. In mathematical physics there are many types of nonlinear evolution equations which appear to present any practical phenomenon, and requires effective, concise and easy method to solve these equations. In this way, the Exp-function is an excellent candidate to be used as solver for these equations. Symmetries are involved in the transformations of space and time. Spatial symmetries are related to spatial geometry of a physical system, while spatio-temporal symmetries involve changes in both space and time. The spatio-temporal symmetries are used to transform PDEs into ODEs (see Equation (6)). These symmetries play a vital role in finding the exact solution. Various methods are available to reduce PDEs into ODEs using some special ansatz for obtaining travelling wave solutions [2]. Almost all of these methods can be viewed as a special case of a symmetry reduction, potentially known as a non-classical symmetry. This approach is used (cf. Equation (6)) in this paper to obtain the exact solutions of nonlinear PDEs. In this paper, wave transformation is used to obtain nonlinear ODEs from nonlinear PDEs and hence making it easy to find a solution.

A general nonlinear conformable PDE [25] is considered in the form of equation:

$$F\left(u, \frac{\partial^\zeta u}{\partial t^\zeta}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^{2\zeta} u}{\partial t^{2\zeta}}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}\right) = 0. \tag{5}$$

A new wave variable is introduced in the following equation

$$u(x,y,t) = u(\omega), \quad \omega = px + qy + r\frac{t^\zeta}{\zeta}, \tag{6}$$

where $q \neq 0, r \neq 0$,and $p \neq 0$ are arbitrary constants, and we will evaluate these constants later.

By using chain rule, we get

$$\frac{\partial^\zeta u}{\partial t^\zeta} = r\frac{du}{d\omega}, \quad \frac{\partial u}{\partial x} = p\frac{du}{d\omega}, \quad \frac{\partial u}{\partial y} = q\frac{du}{d\omega}, \ldots. \tag{7}$$

Putting Equation (7) into Equation (5), it becomes a nonlinear ODE as

$$Q(u, u_\omega, u_{\omega\omega}, u_{\omega\omega\omega}, u_{\omega\omega\omega\omega}, ...) = 0. \tag{8}$$

Suppose the solution of the above ODE is

$$u(\omega) = \frac{\sum_{i=-c}^{d} a_i e^{i\omega}}{\sum_{j=-m}^{n} b_j e^{j\omega}}, \tag{9}$$

where all the integers listed in the equation are positive integers and the values of these integers are determined by homogeneous balancing principle, a_i and b_j are unknown constants. The like linear term of the lowest order in Equation (8) is balanced with lowest order linear term, in order to have the values of c and m used in the same equation. Likewise, the highest order linear term of Equation (8) is balanced with the highest order nonlinear term, in order to find out the values of d and n variables present in the same equation. This procedure will lead to an exact solution of nonlinear PDEs.

4. Exact Solutions of Burger'S Equation, *Zk* Equation and Kdv Equation

In this section, three equations are selected for having the exact solutions. These equations are, conformable ZK, KdV and Burger's equations. The solutions of these equations are discussed in the following sections

4.1. Exact Solutions of Conformable Zk Equation

In this section, conformable ZK equations will be presented and then the solution of this equations will be discussed in details. The conformable ZK equation is given as

$$D_t^\zeta u + u^2 u_x + u_{xxx} + u_{xyy} = 0, \tag{10}$$

where D_t^ζ shows the conformable derivative for $\zeta \in (0, 1]$.

Presenting a transformation $\omega = px + qy + r\frac{t^\zeta}{\zeta}$, and then by chain rule we get

$$\frac{\partial^\zeta u}{\partial t^\zeta} = r\frac{du}{d\omega}, \quad \frac{\partial u}{\partial x} = p\frac{du}{d\omega}, \quad \frac{\partial^3 u}{\partial x^3} = p^3\frac{du}{d\omega}, \quad \frac{\partial^2}{\partial y^2}\frac{\partial u}{\partial x} = pq^2\frac{du}{d\omega}, \tag{11}$$

an ODE in the following form can be obtained by putting Equation (11) into Equation (10),

$$ru_\omega + pu^2 u_\omega + \left(p^3 + pq^2\right)u_{\omega\omega\omega} = 0. \tag{12}$$

The solution of Equation (12) will be considered with unknown variables in the form of

$$u(\omega) = \frac{a_d e^{d\omega} + ... + a_{-c} e^{-c\omega}}{b_n e^{n\omega} + ... + b_{-m} e^{-m\omega}}. \tag{13}$$

The method to have the values of different variables in the above equation is described in the previous section. To evaluate the value of variables, a linear term of highest order of Equation (12) is compared with the highest order nonlinear term

$$u'''(\omega) = \frac{... + k_1 e^{(d+7n)\omega} + ...}{... + k_2 e^{(8n\omega)} + ...}, \tag{14}$$

and

$$u^2 u' = \frac{... + k_3 e^{(n+3d)\omega} + ...}{... + k_4 e^{(4n\omega)} + ...} = \frac{... + k_3 e^{(5n+3d)\omega} + ...}{... + k_4 e^{(8n\omega)} + ...}, \tag{15}$$

where k_i are coefficients, obtained by highest order term balancing in Equations (14) and (15), we get

$$d + 7n = 5n + 3d,$$

which results

$$d = n.$$

Comparing the linear term of lowest order of Equation (12) with the lowest order nonlinear term can provide the values of m and c, hence we have

$$u'''(\omega) = \frac{... + l_1 e^{-(c+7m)\omega} + ...}{l_2 e^{-(8m\omega)} + ...}, \tag{16}$$

$$u^2 u' = \frac{... + l_3 e^{-(m+3c)\omega} + ...}{... + l_4 e^{-(4m\omega)} + ...} = \frac{... + l_3 e^{-(5m+3c)\omega} + ...}{... + l_4 e^{-(8m\omega)} + ...}. \tag{17}$$

l_i are coefficients. Lowest order terms in Equations (16) and (17) are Balanced to have

$$-c - 7m = -5m - 3c,$$

which gives

$$c = m.$$

Taking $d = n = 1.0$ and $c = m = 1.0$, then the predicted solution Equation (13) reduces to

$$u(\omega) = \frac{a_1 e^{\omega} + a_0 + a_{-1} e^{-\omega}}{b_1 e^{\omega} + b_0 + b_{-1} e^{-\omega}}. \tag{18}$$

Switching Equation (18) into Equation (12), we have

$$\frac{1}{B}\Big[h_3 e^{(3\omega)} + h_2 e^{(2\omega)} + h_1 e^{(\omega)} + h_0 + h_{-1} e^{-(\omega)} + h_{-2} e^{-(2\omega)} + h_{-3} e^{-(3\omega)}\Big] = 0, \tag{19}$$

where $B = \left(b_1 e^{(\omega)} + b_0 + b_{-1} e^{-(\omega)}\right)^4$, $h_i (i = -3, ..., 0, ..., 3)$ are unknown constants that are obtained by software Maple 13.0.

Following system of algebraic equations can be obtained by equating coefficients of $e^{i\omega}$ equals to zero,

$$h_3 = 0.0, \;\; h_2 = 0.0, \;\; h_1 = 0.0, \;\; h_0 = 0.0, \;\; h_{-1} = 0.0, \;\; h_{-2} = 0.0, \;\; h_{-3} = 0.0. \tag{20}$$

Solving algebraic equations for this system, following equation can be obtained

$$a_{-1} = 0.0, \;\; b_0 = b_0, \;\; b_1 = b_1, \;\; b_{-1} = \frac{a_0^2}{24 b_1 (p^2 + q^2)}, \;\; a_1 = 0, \;\; r = -\left(p^3 + pq^2\right), \;\; a_0 = a_0. \tag{21}$$

Hence, ZK equation has the following exact solution

$$u(x, y, t) = \frac{a_0}{b_1 e^{(px + qy - (p^3 + pq^2)\frac{t^{\zeta}}{\zeta})} + \frac{a_0^2}{24 b_1 (p^2 + q^2)} e^{-(px + qy - (p^3 + pq^2)\frac{t^{\zeta}}{\zeta})}}, \tag{22}$$

where p, b_1, a_0 and q are real numbers.

Figures 1–4 demonstrates the solution of the function $u(x, t)$ of ZK equation by assigning different values to ζ. These figures shows that for different values of ζ, amplitude of solitons is decreasing toward left and for large distances these solitons are asymptotically zero.

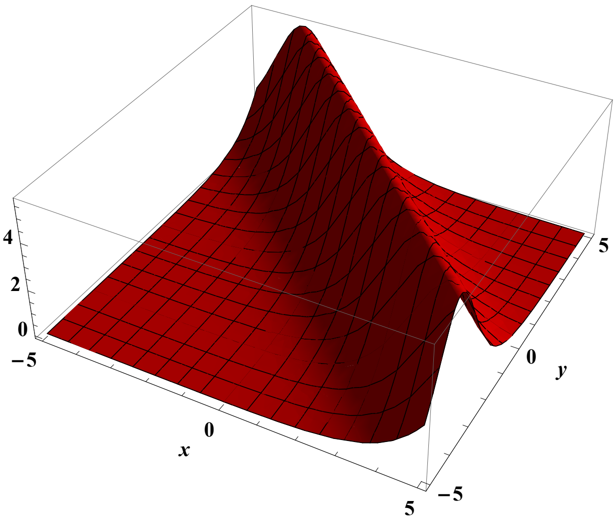

Figure 1. The 3D solution in pictorial form for the function $u(x, y, t)$, and for the parameter values $p = q = t = 1.0$ and $\zeta = 0.50$.

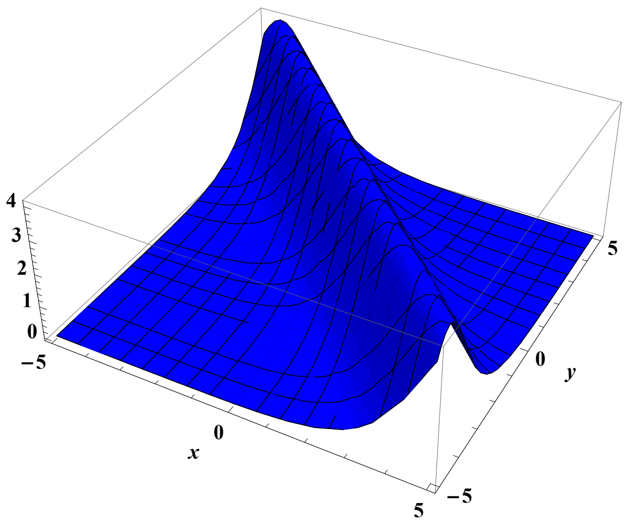

Figure 2. The 3D solution in pictorial form for the function $u(x, y, t)$, and for the parameter values $p = q = t = 1.0$ and $\zeta = 0.75$.

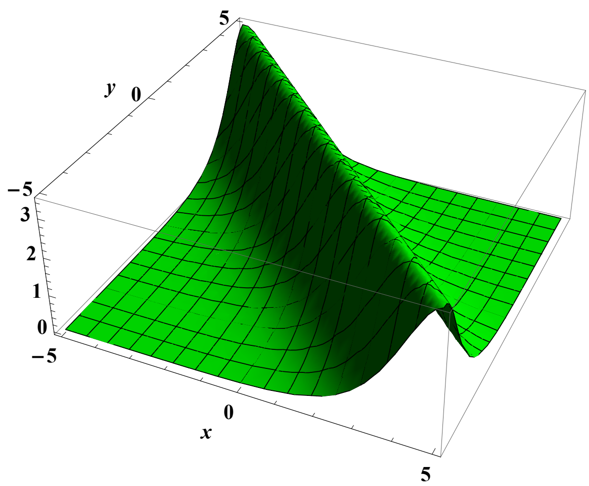

Figure 3. The 3D solution in pictorial form for the function $u(x, y, t)$, and for the parameter values $p = q = t = 1.0$ and $\zeta = 1.00$.

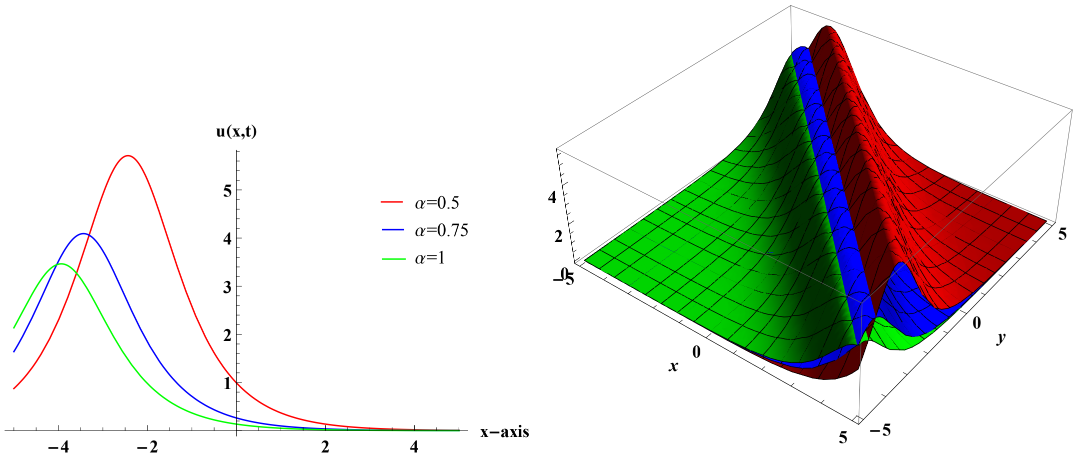

Figure 4. 2D and 3D solutions in pictorial form for function $u(x, y, t)$, for the parameter values $p = q = t = 1.0$ and $\zeta = 0.25, \zeta = 0.50, \zeta = 0.75, \zeta = 1.00$

4.2. Exact Solution of Conformable Kdv Equation

Consider conformable kdV equation as

$$D_t^{\zeta} u + \gamma u_x^2 + \beta u_{xxx} = 0. \tag{23}$$

Introducing a transformation $\omega = px + r\frac{t^{\zeta}}{\zeta}$, and then by chain rule, Equation (23) is reduced into an ODE of the form

$$r u_{\omega} + \gamma p^2 u_{\omega}^2 + \beta p^3 u_{\omega\omega\omega} = 0 \tag{24}$$

assuming the solution of Equation (24) in the following form using exp-function method,

$$u(\omega) = \frac{a_d e^{d\omega} + ... + a_{-c} e^{-c\omega}}{b_n e^{n\omega} + ... + b_{-m} e^{-m\omega}}. \tag{25}$$

where d, c, m and n are unknowns. These constants are obtained by comparing linear term of highest order of Equation (24) and balancing it with the highest order nonlinear term, i.e., u_{ω} with u^2. As a result, we obtain

$$d = n,$$

and

$$c = m.$$

We take $d = n = 1$ and $c = m = 1$. Then, Equation (25) takes the form

$$u(\omega) = \frac{a_1 e^{\omega} + a_0 + a_{-1} e^{-\omega}}{b_1 e^{\omega} + b_0 + b_{-1} e^{-\omega}}. \tag{26}$$

Substituting Equation (26) into Equation (24), folloiwng equation can be obtained

$$\frac{1}{B}\Big[h_3 e^{(3\omega)} + h_2 e^{(2\omega)} + h_1 e^{(\omega)} + h_0 + h_{-1} e^{-(\omega)} + h_{-2} e^{-(2\omega)} + h_{-3} e^{-(3\omega)}\Big] = 0, \tag{27}$$

where $B = \left(b_1 e^{(\omega)} + b_0 + b_{-1} e^{-(\omega)}\right)^4, h_i (i = -3, ..., 0, ..., 3)$ are unknown constants and can be obtained by software Maple 13.0.

Like previous subsection, following system of the algebraic equations can be obtained by equating coefficients of $e^{i\omega}$ equals zero,

$$h_3 = 0.0,\ \ h_2 = 0.0,\ \ h_1 = 0.0,\ \ h_0 = 0.0,\ \ h_{-1} = 0.0,\ \ h_{-2} = 0.0,\ \ h_{-3} = 0.0. \tag{28}$$

following result can be obtained by solving this system of algebraic equations,

$$a_{-1} = 0.0,\ \ b_0 = b_0,\ \ b_1 = b_1,\ \ b_{-1} = 0.0,\ \ a_1 = a_1,\ \ r = -\beta p^3,\ \ a_0 = -\frac{b_0(6pb_1\beta - a_1\gamma)}{\gamma b_1}. \tag{29}$$

Therefore, exact solution can be obtained for time-fractional kdv equation of the following form

$$u(x,t) = \frac{a_1 e^{(px - \beta p^2 \frac{t^\zeta}{\zeta})} - \frac{b_0(6pb_1\beta - a_1\gamma)}{\gamma b_1}}{b_1 e^{(px - \beta p^2 \frac{t^\zeta}{\zeta})} + b_0}. \tag{30}$$

Figures 5–8 represents graphs for the various values of ζ in the function $u(x,t)$. These are known as kink solutions for different values of ζ. These solutions are ascending from one asymptotic state to another. For varying ζ, these kinks are moving with constant shape and speed.

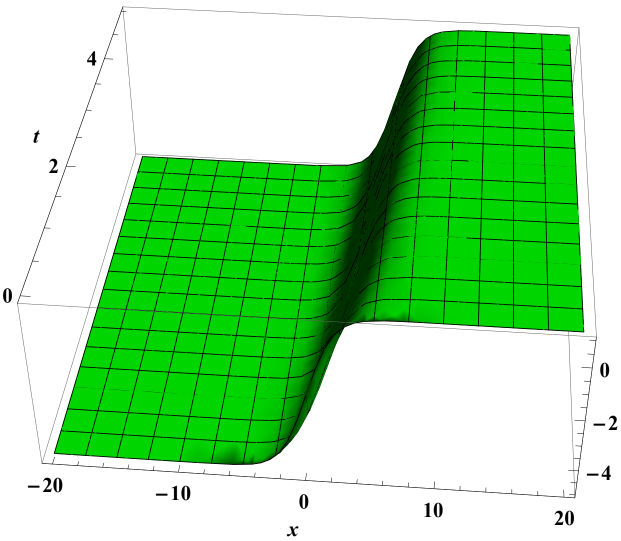

Figure 5. 3D solution in the pictorial form for the function $u(x,t)$ for $\gamma = \beta = p = 1.00$ and for $\zeta = 0.50$.

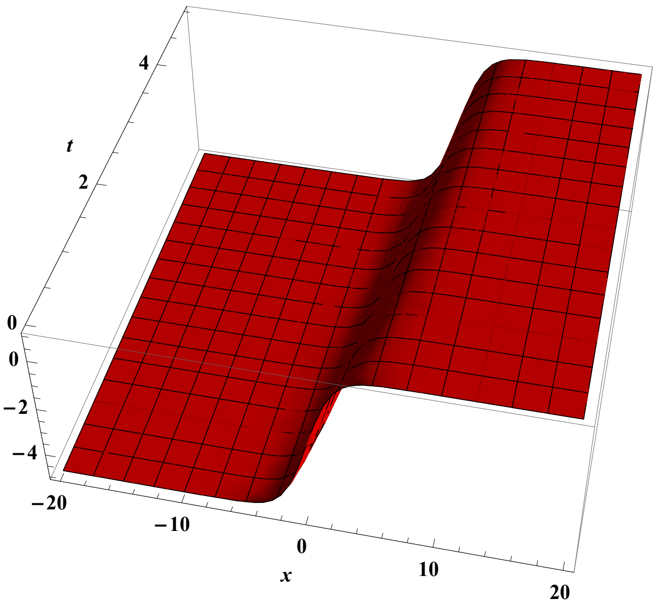

Figure 6. 3D solution in the pictorial form for the function $u(x,t)$ for $\gamma = \beta = p = 1.00$ and for $\zeta = 0.75$.

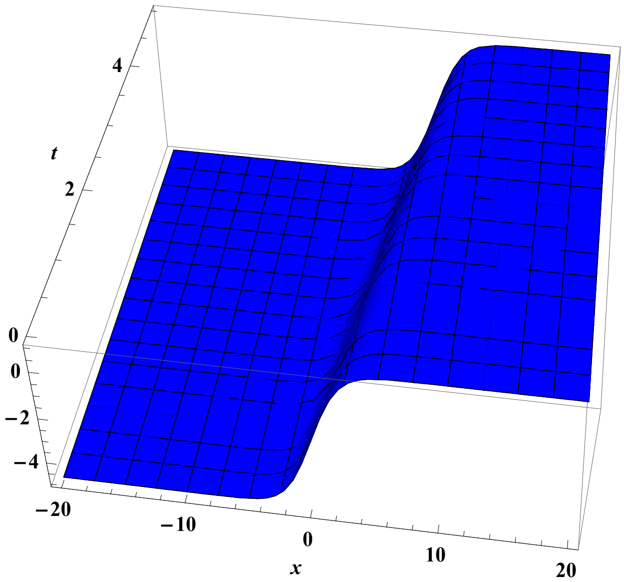

Figure 7. 3D solution in pictorial form for the function $u(x,t)$ for $\gamma = \beta = p = 1.00$ and for $\zeta = 1.00$.

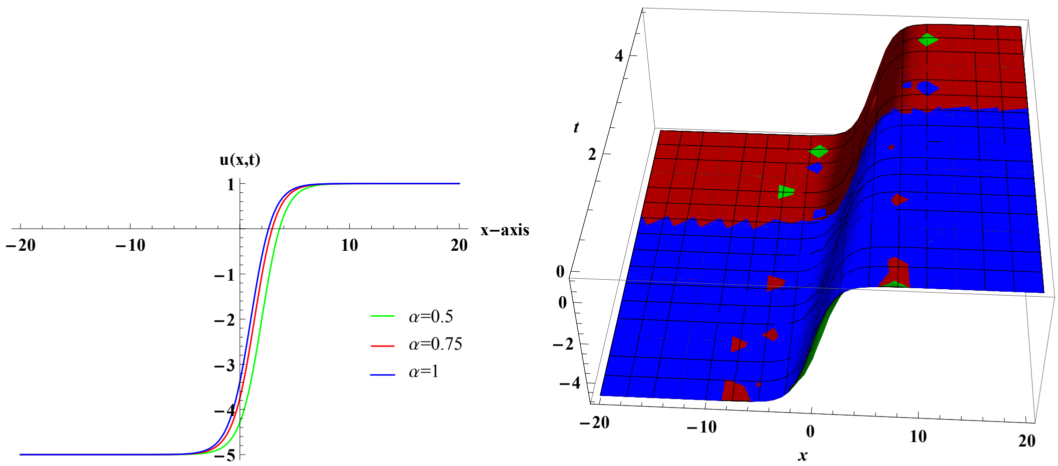

Figure 8. 2D and 3D solutions in the pictorial form for the function $u(x,t)$ and for various values of ζ.

4.3. Exact Solution of Conformable Burgers Equation

The nonlinear conformable Burgers equation is as follows

$$\frac{\partial^{\zeta} u}{\partial t^{\zeta}} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \tag{31}$$

where $\zeta \in (0,1)$, $u(x,t)$ is the function of velocity having spatial dimensions t and x, $\nu > 0.0$. $\nu > 0.0$ is the kinematic fluid viscosity, and $\frac{\partial^{\zeta} u}{\partial t^{\zeta}}$ shows the derivative of function $u(x,t)$ in conformable form.

Introducing a transformation $\omega = x - r\frac{t^{\zeta}}{\zeta}$, then using the chain rule we have

$$\frac{\partial^{\zeta} u}{\partial t^{\zeta}} = -r\frac{du}{d\omega}, \ \frac{\partial u}{\partial x} = \frac{du}{d\omega}, \ \frac{\partial^2 u}{\partial x^2} = \frac{d^2 u}{d\omega^2}u \tag{32}$$

Thus Equation (31) is transformed into an ODE, which is of the form

$$-r\frac{du}{d\omega} + u\frac{du}{d\omega} - \nu\frac{d^2 u}{d\omega^2} = 0. \tag{33}$$

By our proposed method, Equation (33) has a solution of the form

$$u(\omega) = \frac{a_d e^{d\omega} + ... + a_{-c} e^{-c\omega}}{b_n e^{n\omega} + ... + b_{-m} e^{-m\omega}}, \tag{34}$$

where d, c, m and n are unknowns. These constants are obtained through the method described previously in the same section. The linear term of highest order of Equation (33) is balanced with the highest order nonlinear term, i.e u_ω with u^2. As a result,

$$d = n,$$

and

$$c = m$$

is obtained. Equation (34) can be converted in following equation by considering the values $d = n = 1.0$ and $c = m = 1.0$

$$u(\omega) = \frac{a_1 e^{\omega} + a_0 + a_{-1} e^{-\omega}}{b_1 e^{\omega} + b_0 + b_{-1} e^{-\omega}}. \tag{35}$$

following system of algebraic equations can be obtained by using software Maple 13.0 after Substituting Equation (35) into Equation (33),

$$h_3 = 0.0, \quad h_2 = 0.0, \quad h_1 = 0.0, \quad h_0 = 0.0, \quad h_{-1} = 0.0, \quad h_{-2} = 0.0, \quad h_{-3} = 0.0. \tag{36}$$

following solution can be obtained by solving this system of algebraic equations,

$$\begin{gathered} r = -\tfrac{-a_{-1} + 2b_{-1}\nu}{b_{-1}}, \quad a_0 = 0, \quad a_1 = -\tfrac{b_1(-a_{-1} + 4b_{-1}\nu)}{b_{-1}}, \quad a_{-1} = a_{-1}, \\ b_0 = 0, \quad b_1 = b_1, \quad b_{-1} = b_{-1}. \end{gathered} \tag{37}$$

Substituting Equation (38) into Equation (35), we have

$$u(x,t) = \frac{-\frac{b_1(-a_{-1}+4b_{-1}\nu)}{b_{-1}} e^{(x + (\frac{-a_{-1}+2b_{-1}\nu}{b_{-1}})\frac{t^\zeta}{\zeta})} + a_{-1} e^{-(x + (\frac{-a_{-1}+2b_{-1}\nu}{b_{-1}})\frac{t^\zeta}{\zeta})}}{b_1 e^{(x + (\frac{-a_{-1}+2b_{-1}\nu}{b_{-1}})\frac{t^\zeta}{\zeta})} + b_{-1} e^{-(x + (\frac{-a_{-1}+2b_{-1}\nu}{b_{-1}})\frac{t^\zeta}{\zeta})}} \tag{38}$$

Substituting $a_{-1} = b_1 = b_{-1} = 1.0$, solution becomes

$$u(x,t) = \frac{(1 - 4\nu) e^{(x + (2\nu - 1)\frac{t^\zeta}{\zeta})} + e^{-(x + (2\nu - 1)\frac{t^\zeta}{\zeta})}}{e^{(x + (2\nu - 1)\frac{t^\zeta}{\zeta})} + e^{-(x + (2\nu - 1)\frac{t^\zeta}{\zeta})}}. \tag{39}$$

Figures 9–12 demonstrates the solution of function $u(x,t)$ of Burger's equation by assigning different values to ζ. These are the kink solutions for different values of ζ which are descending from one asymptotic state to another. For varying ζ these kinks are moving towards right with constant shape and speed.

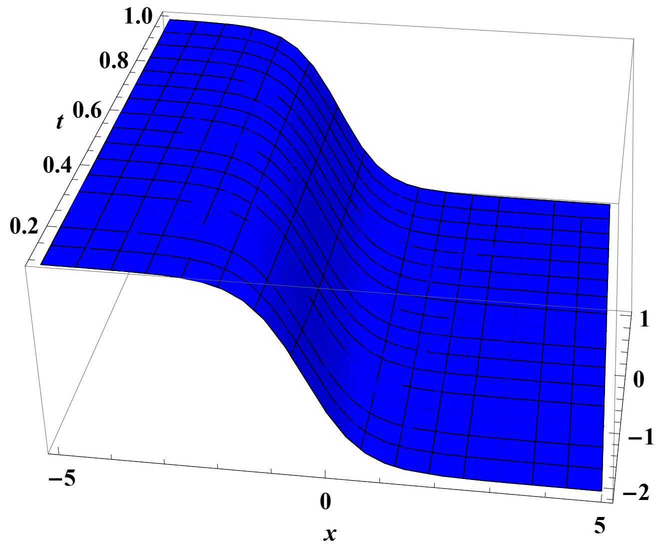

Figure 9. The graph of the function $u(x,t)$ for the values $\nu = 2.0, \zeta = 0.50$.

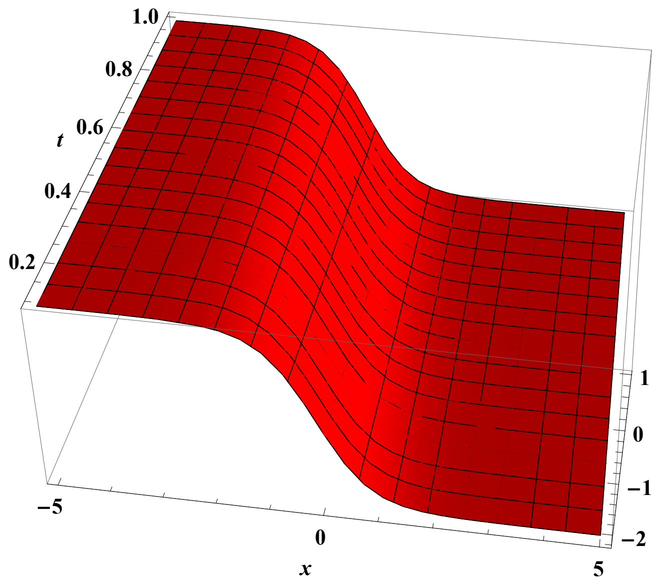

Figure 10. The graph of the function $u(x,t)$ for the values $\nu = 2.0, \zeta = 0.75$.

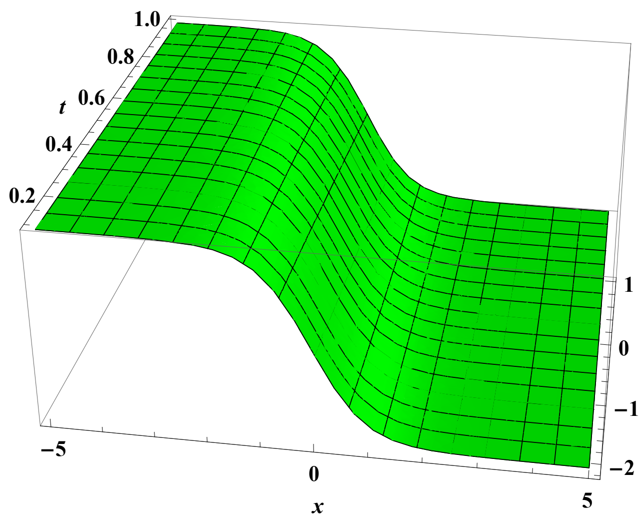

Figure 11. The graph of the function $u(x,t)$ for the values $\nu = 2.0, \zeta = 1.00$

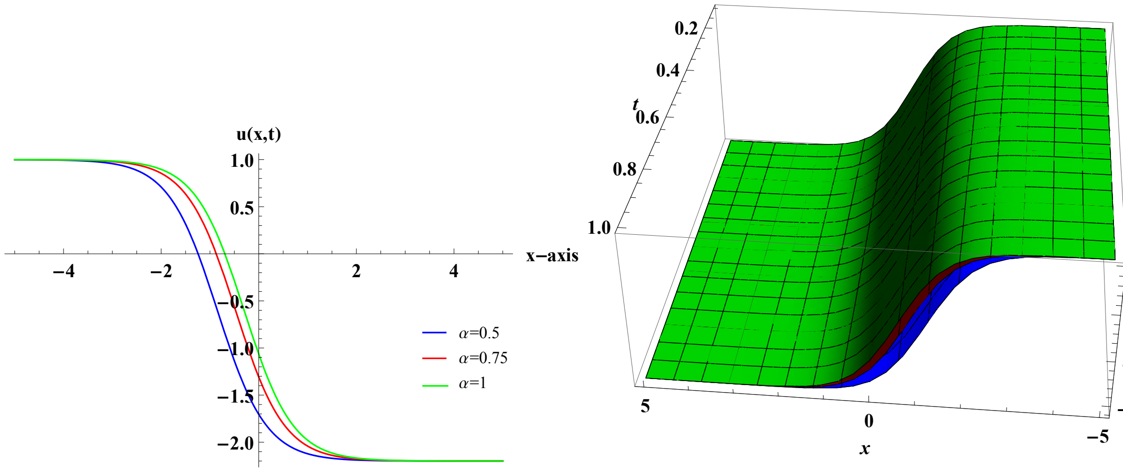

Figure 12. Graphs of the function $u(x,t)$ for the values $\nu = 2.0$ and for various values of ζ.

5. Conclusions

The aim of this work was to find the exact solutions of nonlinear conformable PDEs. ZK equation, KdV equation and Burgers equation are selected to present the validity of the Exp-function method for obtaining the exact solutions of these equations. The proposed software assisted method allows us to perform complicated nonlinear conformable algebraic calculations. The solving procedure described in this paper for nonlinear conformable PDEs reveals that the exp-function method is a succinct tool for solving such equations, and we expect it to be successful in the future.

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