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Research article

A note on (p, q)-analogue type of Fubini numbers and polynomials

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Abstract: In this paper, we introduce a new class of (p, q)-analogue type of Fubini numbers and polynomials and investigate some properties of these polynomials. We establish summation formulas of these polynomials by summation techniques series. Furthermore, we consider some relationships for (p, q)-Fubini polynomials associated with (p, q)-Bernoulli polynomials, (p, q)-Euler polynomials and (p, q)-Genocchi polynomials and (p, q)-Stirling numbers of the second kind.

Keywords: (p, q)-calculus; (p, q)-Bernoulli polynomials; (p, q)-Euler polynomials; (p, q)-Genocchi polynomials; (p, q)-Fubini numbers and polynomials; (p, q)-Stirling numbers of the second kind.
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1. Introduction

Throughout this presentation, we use the following standard notions: $\mathbb{N} = \{1, 2, 3, \cdots\}$, $\mathbb{N}_0 = \mathrm{N} \cup \{0\}$. Here as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The quantum calculus or q-calculus has several applications in different branches of physics and mathematics. It has attracted serious attention of researchers due to its different applications. At the beginning of 19^{th} century, Jackson initiated and developed the application of the q-calculus (see [7,8]). Later Chakrabarti and Jagannathan defined Jackson (p, q)-derivative as a generalization of q-derivative (see [3]). Sadjang [13] developed several properties of the (p,q)-derivatives and the (p,q)-integrals and as an application gave two (p,q)-Taylor formulas for polynomials.

For the expedience, we present some definitions and concepts of (p,q)-calculus that were used in this article by assuming as p and q are fixed number such that $0 < p < q \leq 1$.

The (p,q)-derivative of a function f (with respect to x) is defined by

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, (x \neq 0, p \neq q) \tag{1.1}$$

and $D_{p,q}f(0) = f'(0)$, where f' is the ordinary derivative of f.

The (p,q)-derivative operator holds the following properties

$$D_{p,q}(f(x)g(x)) = g(p(x))D_{p,q}f(x) + f(qx)D_{p,q}g(x), \tag{1.2}$$

and

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}. \tag{1.3}$$

The (p,q)-analogue of $(x+a)^n$ is given by

$$(x+a)^n_{p,q} = (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}), n \geq 1$$

$$= \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\binom{n}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}, \tag{1.4}$$

where the (p,q)-Gauss binomial coefficients $\binom{n}{k}_{p,q}$ and (p,q)-factorial $[n]_{p,q}!$ are defined by $\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}, (n \geq k)$ and $[n]_{p,q}! = [n]_{p,q}\cdots[2]_{p,q}[1]_{p,q}, (n \in \mathbb{N})$.

The (p,q)-exponential functions are defined by

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}x^n}{[n]_{p,q}!} \quad and \quad E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}x^n}{[n]_{p,q}!}, \tag{1.5}$$

under condition

$$e_{p,q}(x)E_{p,q}(-x) = 1. \tag{1.6}$$

It follows from (1.1) and (1.5) that

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \quad and \quad D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \tag{1.7}$$

The definite (p,q)-integral of a function f is defined by (see [3])

$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(a\frac{p^k}{q^{k+1}}\right),$$

with the following property

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x. \tag{1.8}$$

The generalized (p,q)-Bernoulli, the generalized (p,q)-Euler and the generalized (p,q)-Genocchi numbers and polynomials are defined by means of the following generating function as follows (see [5]):

$$\left(\frac{t}{e_{p,q}(t)-1}\right)^{\alpha} e_{p,q}(xt)E_{p,q}(yt) = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x,y:p,q)\frac{t^n}{[n]_{p,q}!}, \qquad |\, t \,|< 2\pi \tag{1.9}$$

$$\left(\frac{2}{e_{p,q}(t)+1}\right)^{\alpha} e_{p,q}(xt)E_{p,q}(yt) = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x,y:p,q)\frac{t^n}{[n]_{p,q}!}, \qquad |\, t \,|< \pi \tag{1.10}$$

and

$$\left(\frac{2t}{e_{p,q}(t)+1}\right)^{\alpha} e_{p,q}(xt)E_{p,q}(yt) = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x,y:p,q)\frac{t^n}{[n]_{p,q}!}, \qquad |\, t \,|< \pi \tag{1.11}$$

It is clear that

$$B_n^{(\alpha)}(0,0:p,q) = B_n^{(\alpha)}(p,q)\ ,\ E_n^{(\alpha)}(0,0:p,q) = E_n^{(\alpha)}(p,q),$$

and

$$G_n^{(\alpha)}(0,0:p,q) = G_n^{(\alpha)}(p,q), \qquad (n \in \mathbb{N}).$$

Geometric polynomials (also known as Fubini polynomials) are defined as follows (see [1]):

$$F_n(x) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k!x^k, \tag{1.12}$$

where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is the Stirling number of the second kind (see [9]).
For $x = 1$ in (1.12), we get the n^{th} Fubini number (called Bell number or geometric number) F_n [2,4,14] in defined by

$$F_n(1) = F_n = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} k!. \tag{1.13}$$

The exponential generating functions of geometric polynomials are given by (see [1]):

$$\frac{1}{1-x(e^t-1)} = \sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!}, \tag{1.14}$$

and related to the geometric series (see [1]):

$$\left(x\frac{d}{dx}\right)^m \frac{1}{1-x} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x}F_m\left(\frac{x}{1-x}\right), |\, x \,|< 1.$$

Let us give a short list of these polynomials and numbers as follows:

$$F_0(x) = 1, F_1(x) = x, F_2(x) = x + 2x^2, F_3(x) = x + 6x^2 + 6x^3, F_4(x) = x + 14x^2 + 36x^3 + 24x^4,$$

and

$$F_0 = 1, F_1 = 1, F_2 = 3, F_3 = 13, F_4 = 75.$$

The out line of this paper is as follows: In section 2, we consider generating functions for (p, q)-analogue type of Fubini numbers and polynomials and give some properties of these numbers and polynomials. In section 3, we derive summation formulas for (p, q)-analogue type of Fubini numbers and polynomials. In section , we give relationships for (p, q)-Fubini polynomials associated with (p, q)-Bernoulli polynomials, (p, q)-Euler polynomials and (p, q)-Genocchi polynomials and (p, q)-Stirling numbers of the second kind.

2. (p, q)-analogue type of Fubini numbers and polynomials

In this section, we introduce (p, q)-Fubini polynomials and obtain some basic properties which give new formulas for $F_n(x, y; z : p, q)$.

Definition 2.1. Let $p, q \in \mathbb{C}$ with $0 <\mid q \mid<\mid p \mid\leq 1$, the three variable (p, q)-Fubini polynomials $F_n(x, y; z : p, q)$ are defined by means of the following generating function:

$$\frac{1}{1 - z(e_{p,q}(t) - 1)} e_{p,q}(xt) E_{p,q}(yt) = \sum_{n=0}^{\infty} F_n(x, y; z : p, q) \frac{t^n}{[n]_{p,q}!}. \tag{2.1}$$

For $x = y = 0$ and $z = 1$ in (2.1), we have

$$F_n(0, 0; 1 : p, q) = F_n(p, q), \tag{2.2}$$

where $F_n(p, q)$ are called the (p, q)-Fubini numbers.

Obviously that

$$F_n(0, 0; z : p, q) = F_n(z : p, q),$$

$$F_n(x, y; z : p, q)|_{p=1} = F_n(x, y; z : q), \quad \text{(see [6])}$$

$$\lim_{q \longrightarrow 1^-_{p=1}} F_n(x, 0 : p, q) = F_n(x; z), \quad \text{(see [1])}$$

$$\lim_{q \longrightarrow 1^-_{p=1}} F_n(0, 0; z : p, q) = F_n(z), \quad \text{(see [1,4])}.$$

Theorem 2.1. The following series representation for the (p, q)-Fubini polynomials $F_n(x, y; z : p, q)$ holds true:

$$F_n(x, y; z : p, q) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right)_{p,q} F_{n-m}(z : p, q)(x + y)^m_{p,q}. \tag{2.3}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} F_n(x, y; z : p, q) \frac{t^n}{[n]_{p,q}!} = \frac{1}{1 - z(e_{p,q}(t) - 1)} e_{p,q}(xt) E_{p,q}(yt)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} F_n(z : p, q) \frac{t^n}{[n]_{p,q}!} (x + y)^m_{p,q} \frac{t^m}{[m]_{p,q}!}.$$

Replacing n by $n-m$ in above equation and equating the coefficients of same powers of t in both sides of resultant equation, we get representation (2.3). □

Theorem 2.2. The following summation formula for the (p,q)-Fubini polynomials $F_n(x,y;z:p,q)$ holds true:

$$F_n(x,y;0:p,q)=\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)_{p,q}p^{\binom{m}{2}}q^{\binom{n-m}{2}}x^m y^{n-m}, \tag{2.4}$$

$$F_n(x,y;z:p,q)=\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)_{p,q}p^{\binom{n-m}{2}}F_m(0,y;z:p,q)x^{n-m}, \tag{2.5}$$

$$F_n(x,y;z:p,q)=\sum_{m=0}^{n}\left(\begin{array}{c}n\\m\end{array}\right)_{p,q}q^{\binom{n-m}{2}}F_m(x,0;z:p,q)y^{n-m}. \tag{2.6}$$

Proof. Using Eqs (1.4)-(1.6) in generating function (2.1), the proof can be easily proved. So we omit it. □

Theorem 2.3. For $n\geq 0$, the following formula for (p,q)-Fubini polynomials holds true:

$$(x+y)^n_{p,q}=F_n(x,y;z:p,q)-zF_n(x+1,y;z:p,q)+zF_n(x,y;z:p,q). \tag{2.7}$$

Proof. We begin with the definition (2.1) and write

$$e_{p,q}(xt)E_{p,q}(yt)=\frac{1-z(e_{p,q}(t)-1)}{1-z(e_{p,q}(t)-1)}e_{p,q}(xt)E_{p,q}(yt)$$

$$=\frac{e_{p,q}(xt)E_{p,q}(yt)}{1-z(e_{p,q}(t)-1)}-\frac{z(e_{p,q}(t)-1)}{1-z(e_{p,q}(t)-1)}e_{p,q}(xt)E_{p,q}(yt).$$

Then using the definition of (1.4) and (2.1), we have

$$\sum_{n=0}^{\infty}(x+y)^n_{p,q}\frac{t^n}{[n]_{p,q}!}$$

$$=\sum_{n=0}^{\infty}\left[F_n(x,y;z:p,q)-zF_n(x+1,y;z:p,q)+zF_n(x,y;z:p,q)\right]\frac{t^n}{[n]_{p,q}!}.$$

Finally, comparing the coefficients of $\frac{t^n}{n!}$, we get (2.7). □

Theorem 2.4. The following formula for (p,q)-Fubini polynomials holds true:

$$zF_n(x+1,y;z:p,q)=(1+z)F_n(x,y;z:p,q)-(x+y)^n_{p,q}. \tag{2.8}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}\left[F_n(x+1,y;z:p,q)-F_n(x,y;z:p,q)\right]\frac{t^n}{[n]_{p,q}!}$$

$$= \frac{e_{p,q}(xt)E_{p,q}(yt)}{1 - z(e_{p,q}(t) - 1)}(e_{p,q}(t) - 1)$$

$$= \frac{1}{z}\left[\frac{e_{p,q}(xt)E_{p,q}(yt)}{1 - z(e_{p,q}(t) - 1)} - e_{p,q}(xt)E_{p,q}(yt)\right]$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left[F_n(x, y; z : p, q) - (x + y)^n_{p,q}\right]\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we obtain (2.8). □

Theorem 2.5. The following recursive formulas for the (p, q)-Fubini polynomials $F_n(x, y; z : p, q)$ holds true:

$$D_{p,q;x}F_n(x, y; z : p, q) = [n]_{p,q}F_{n-1}(px, y; z : p, q), \tag{2.9}$$

$$D_{p,q;y}F_n(x, y; z : p, q) = [n]_{p,q}F_{n-1}(x, qy; z : p, q). \tag{2.10}$$

Proof. Differentiating generating function (2.1) with respect to x and y with the help of Eq. (1.7) and then simplifying with the help of the Cauchy product rule formulas (2.9) and (2.10) are obtained. □

Theorem 2.6. The following (p, q)-integral is valid

$$\int_a^b F_n(x, y; z : p, q)d_{p,q}x = p\frac{F_{n+1}(\frac{b}{p}, y; z : p, q) - F_{n+1}(\frac{a}{p}, y; z : p, q)}{[n + 1]_{p,q}}, \tag{2.11}$$

$$\int_a^b F_n(x, y; z : p, q)d_{p,q}y = p\frac{F_{n+1}(x, \frac{b}{q}; z : p, q) - F_{n+1}(x, \frac{a}{q}; z : p, q)}{[n + 1]_{p,q}}. \tag{2.12}$$

Proof. Since

$$\int_a^b \frac{\delta}{\delta_{p,q}x}F_n(x, y; z : p, q)d_{p,q}x = f(b) - f(a), \text{(see [13])},$$

in terms of Eq. (2.9) and Eqs (1.7) and (1.8), we arrive at the asserted result

$$\int_a^b \frac{\delta}{\delta_{p,q}x}F_n(x, y; z : p, q)d_{p,q}x = \frac{p}{[n + 1]_{p,q}}\int_a^b F_n\left(\frac{x}{p}, y; z : p, q\right)d_{p,q}x$$

$$= p\frac{F_{n+1}(\frac{b}{p}, y; z : p, q) - F_{n+1}(\frac{b}{p}, y; z : p, q)}{[n + 1]_{p,q}}.$$

The other can be shown using similar method. Therefore, the complete the proof of this theorem. □

3. Main results

First, we prove the following result involving the (p,q)-Fubini polynomials $F_n(x,y;z:p,q)$ by using series rearrangement techniques and considered its special case:

Theorem 3.1. The following summation formula for (p,q)-Fubini polynomials $F_n(x,y;z:p,q)$ holds true:

$$F_{k+l}(w,y;z:p,q) = \sum_{n,s=0}^{k,l} \binom{k}{n}_{p,q} \binom{l}{s}_{p,q} p^{\binom{n+s}{2}} (w-x)^{n+s} F_{k+l-n-s}(x,y;z:p,q). \tag{3.1}$$

Proof. Replacing t by $t+u$ in (2.1) and then using the formula [12, p.52(2)]:

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.2}$$

in the resultant equation, we find the following generating function for the (p,q)-Fubini polynomials $F_n(x,y;z:p,q)$:

$$\frac{1}{1-z(e_{p,q}(t+u)-1)}E_{p,q}(y(t+u))$$
$$= e_{p,q}(-x(t+u))\sum_{k,l=0}^{\infty} F_{k+l}(x,y;z:p,q)\frac{t^k}{[k]_{p,q}!}\frac{u^l}{[l]_{p,q}!}, \text{(see [10,11])}. \tag{3.3}$$

Replacing x by w in the above equation and equating the resultant equation to the above equation, we find

$$e_{p,q}((w-x)(t+u))\sum_{k,l=0}^{\infty} F_{k+l}(x,y;z:p,q)\frac{t^k}{[k]_{p,q}!}\frac{u^l}{[l]_{p,q}!}$$
$$= \sum_{k,l=0}^{\infty} F_{k+l}(w,y;z:p,q)\frac{t^k}{[k]_{p,q}!}\frac{u^l}{[l]_{p,q}!}. \tag{3.4}$$

On expanding exponential function (3.4) gives

$$\sum_{N=0}^{\infty}\frac{[(w-x)(t+u)]^N p^{\binom{N}{2}}}{[N]_{p,q}!}\sum_{k,l=0}^{\infty} F_{k+l}(x,y;z:p,q)\frac{t^k}{[k]_{p,q}!}\frac{u^l}{[l]_{p,q}!}$$
$$= \sum_{k,l=0}^{\infty} F_{k+l}(w,y;z:p,q)\frac{t^k}{[k]_{p,q}!}\frac{u^l}{[l]_{p,q}!}, \tag{3.5}$$

which on using formula (3.2) in the first summation on the left hand side becomes

$$\sum_{n,s=0}^{\infty}\frac{(w-x)^{n+s}t^n u^s p^{\binom{n+s}{2}}}{[n]_{p,q}![s]_{p,q}!}\sum_{k,l=0}^{\infty} F_{k+l}(x,y;z:p,q)\frac{t^k}{[k]_{p,q}!}\frac{u^l}{[l]_{p,q}!}$$
$$= \sum_{k,l=0}^{\infty} F_{k+l}(w,y;z:p,q)\frac{t^k}{[k]_{p,q}!}\frac{u^l}{[l]_{p,q}!}. \tag{3.6}$$

Now replacing k by $k-n$, l by $l-s$ and using the lemma ([12, p.100(1)]):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A(n,k)=\sum_{k=0}^{\infty}\sum_{n=0}^{k}A(n,k-n), \tag{3.7}$$

in the l.h.s. of (3.6), we find

$$\sum_{k,l=0}^{\infty}\sum_{n,s=0}^{k,l}\frac{(w-x)^{n+s}p^{\binom{n+s}{2}}}{[n]_{p,q}![s]_{p,q}!}F_{k+l-n-s}(x,y;z"p,q)\frac{t^k}{(k-n)_{p,q}!}\frac{u^l}{(l-s)_{p,q}!}$$

$$=\sum_{k,l=0}^{\infty}F_{k+l}(w,y;z:p,q)\frac{t^k}{[k]_{p,q}!}\frac{u^l}{[l]_{p,q}!}. \tag{3.8}$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the assertion (3.1) of Theorem 3.1. □

Remark 3.1. Taking $l=0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. The following summation formula for (p,q)-Fubini polynomials $F_n(x,y;z:p,q)$ holds true:

$$F_k(w,y;z:p,q) = \sum_{n=0}^{k}\binom{k}{n}_{p,q}p^{\binom{n+s}{2}}(w-x)^nF_{k-n}(x,y;z:p,q). \tag{3.9}$$

Remark 3.2. Replacing w by $w+x$ in (3.9), we obtain

$$F_k(w+x,y;z:p,q) = \sum_{n=0}^{k}\binom{k}{n}_{p,q}p^{\binom{n+s}{2}}w^nF_{k-n}(x,y;z:p,q). \tag{3.10}$$

Theorem 3.2. The following summation formula for (p,q)-Fubini polynomials $F_n(x,y;z:p,q)$ holds true:

$$F_n(w,u;z:p,q)F_m(W,U;Z:p,q)=\sum_{r,k=0}^{n,m}\binom{n}{r}_{p,q}\binom{m}{k}_{p,q}(w-x+u-y)^r_{p,q}$$

$$\times F_{n-r}(x,y;z:p,q)(W-X+U-Y)^k_{p,q}F_{m-k}(X,Y;Z:p,q). \tag{3.11}$$

Proof. Consider the product of the (p,q)-Fubini polynomials, we can be written as generating function (2.1) in the following form:

$$\frac{1}{1-z(e_{p,q}(t)-1)}e_{p,q}(xt)E_{p,q}(yt)\frac{1}{1-Z(e_{p,q}(T)-1)}e_{p,q}(XT)E_{p,q}(YT)$$

$$=\sum_{n=0}^{\infty}F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}\sum_{m=0}^{\infty}F_m(X,Y;Z:p,q)\frac{T^m}{[m]_{p,q}!}. \tag{3.12}$$

Replacing x by w, y by u, X by W and Y by U in (3.12) and equating the resultant to itself,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_n(w,u;z:p,q)F_m(W,U;Z:p,q)\frac{t^n}{[n]_{p,q}!}\frac{T^m}{[m]_{p,q}!}$$
$$= e_{p,q}\left((w-x)t\right)E_{p,q}((u-y)t)e_{p,q}\left((W-X)T\right)E_{p,q}\left((U-Y)t\right)$$
$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_n(x,y;z:p,q)F_m(X,Y;Z:p,q)\frac{t^n}{[n]_{p,q}!}\frac{T^m}{[m]_{p,q}!},$$

which on using the generating function (3.7) in the exponential on the r.h.s., becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_n(w,u;z:p,q)F_m(W,U;Z:p,q)\frac{t^n}{[n]_{p,q}!}\frac{T^m}{[m]_{p,q}!}$$
$$=\sum_{n,r=0}^{\infty}(w-x+u-y)^r_{p,q\,H}F_n(x,y;z)\frac{t^{n+r}}{[n]_{p,q}![r]_{p,q}!}$$
$$\times\sum_{m,k=0}^{\infty}(W-X+U-Y)^k_{p,q\,H}F_m(X,Y;Z)\frac{T^{m+k}}{[m]_{p,q}![k]_{p,q}!}. \tag{3.13}$$

Finally, replacing n by $n-r$ and m by $m-k$ and using Eq. (3.7) in the r.h.s. of the above equation and then equating the coefficients of like powers of t and T, we get assertion (3.11) of Theorem 3.2. □

Remark 3.3. Replacing u by y and U by Y in assertion (3.11) of Theorem 3.2, we deduce the the following consequence of Theorem 3.2.

Corollary 3.2. The following summation formula for (p,q)-Fubini polynomials $F_n(x,y;z:p,q)$ holds true:

$$F_n(w,y;z:p,q)F_m(W,Y;Z:p,q)=\sum_{r,k=0}^{n,m}\binom{n}{r}_{p,q}\binom{m}{k}_{p,q}(w-x)^r_{p,q}F_{n-r}(x,u;z:p,q)$$
$$\times(W-X)^k_{p,q}F_{m-k}(X,U;Z:p,q). \tag{3.14}$$

Theorem 3.3. The following summation formula for (p,q)-Fubini polynomials $F_n(x,y;z:p,q)$ holds true:

$$F_n(x+1,y;z:p,q)=\sum_{r=0}^{n}\binom{n}{r}_{p,q}F_{n-r}(x,y;z:p,q). \tag{3.15}$$

Proof. Using the generating function (2.1), we have

$$\sum_{n=0}^{\infty}F_n(x+1,y;z:p,q)\frac{t^n}{[n]_{p,q}!}-\sum_{n=0}^{\infty}F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}$$
$$=\left(\frac{1}{1-z(e_{p,q}(t)-1)}\right)(e_{p,q}(t)-1)e_{p,q}(xt)E_{p,q}(yt)$$

$$= \sum_{n=0}^{\infty} F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}\left(\sum_{r=0}^{\infty}\frac{t^r}{[r]_{p,q}!}-1\right)$$

$$= \sum_{n=0}^{\infty} F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}\sum_{r=0}^{\infty}\frac{t^r}{[r]_{p,q}!} - \sum_{n=0}^{\infty} F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}_{p,q} F_{n-r}(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!} - \sum_{n=0}^{\infty} F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}.$$

Finally, equating the coefficients of the like powers of t on both sides, we get (3.15). □

Theorem 3.4. For $n \geq 0$ and $z_1 \neq z_2$, the following formula for(p,q)-Fubini polynomials holds true:

$$\sum_{k=0}^{n}\binom{n}{k}_{p,q} F_{n-k}(x_1,y_1;z_1:p,q)F_k(x_2,y_2;z_2:p,q)$$

$$= \frac{z_2 F_n(x_1+x_2,y_1+y_2;z_2:p,q) - z_1 F_n(x_1+x_2,y_1+y_2;z_1:p,q)}{z_2-z_1}. \tag{3.16}$$

Proof. The products of (2.1) can be written as

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} F_n(x_1,y_1;z_1:p,q)F_k(x_2,y_2;z_2:p,q)\frac{t^n}{[n]_{p,q}!}\frac{t^k}{[k]_{p,q}!}$$

$$= \frac{e_{p,q}(x_1t)E_{p,q}(y_1t)}{1-z_1(e_{p,q}(t)-1)}\frac{e_{p,q}(x_2t)E_{p,q}(y_2t)}{1-z_2(e_{p,q}(t)-1)}$$

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_{p,q} F_{n-k}(x_1,y_1;z_1:p,q)F_k(x_2,y_2;z_2:p,q)\right)\frac{t^n}{[n]_{p,q}!}$$

$$= \frac{z_2}{z_2-z_1}\frac{e_{p,q}[(x_1+x_2)t]E_{p,q}[(y_1+y_2)t]}{1-z_1(e_{p,q}(t)-1)} - \frac{z_1}{z_2-z_1}\frac{e_{p,q}[(x_1+x_2)t]E_{p,q}[(y_1+y_2)t]}{1-z_2(e_{p,q}(t)-1)}$$

$$= \sum_{n=0}^{\infty}\left(\frac{z_2 F_n(x_1+x_2,y_1+y_2;z_2:p,q) - z_1 F_n(x_1+x_2,y_1+y_2;z_1:p,q)}{z_2-z_1}\right)\frac{t^n}{[n]_{p,q}!}.$$

By equating the coefficients of $\frac{t^n}{[n]_{p,q}!}$ on both sides, we get (3.16). □

Theorem 3.5. The following relation for the (p,q)-Fubini polynomials $F_n(x,y;z:p,q)$ holds true:

$$(1+z)F_n(x,y;z:p,q) = z\sum_{k=0}^{n}\binom{n}{k}_{p,q} F_{n-k}(x,y;z:p,q) + (x+y)^n_{p,q}. \tag{3.17}$$

Proof. Consider the following identity

$$\frac{1+z}{(1-z(e_{p,q}(t)-1))ze_{p,q}(t)} = \frac{1}{1-z(e_{p,q}(t)-1)} + \frac{1}{ze_{p,q}(t)}.$$

Evaluating the following fraction using above identity, we find

$$\frac{(1+z)e_{p,q}(xt)E_{p,q}(yt)}{(1-z(e_{p,q}(t)-1))ze_{p,q}(t)} = \frac{e_{p,q}(xt)E_{p,q}(yt)}{1-z(e_{p,q}(t)-1)} + \frac{e_{p,q}(xt)E_{p,q}(yt)}{ze_{p,q}(t)}$$

$$(1+z)\sum_{n=0}^{\infty} F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}$$

$$= z\sum_{n=0}^{\infty} F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}\sum_{k=0}^{\infty}\frac{t^k}{[k]_{p,q}!} - \sum_{n=0}^{\infty}(x+y)^n_{p,q}\frac{t^n}{[n]_{p,q}!}.$$

Applying the Cauchy product rule in the above equation and then equating the coefficients of like powers of t in both sides of the resultant equation, assertion (3.17) follows. □

4. Relationship between (p,q)-Bernoulli, (p,q)-Euler and (p,q)-Genocchi polynomials and (p,q)-Stirling numbers of the second kind

In this section, we prove some relationships for (p,q)-Fubini polynomials related to (p,q)-Bernoulli polynomials, (p,q)-Euler polynomials and (p,q)-Genocchi polynomials and (p,q)-Stirling numbers of the second kind. We start a following theorem.

Theorem 4.1. Each of the following relationships holds true:

$$F_n(x,y;z:p,q)$$
$$= \sum_{s=0}^{n+1}\binom{n+1}{s}_{p,q}\left[\sum_{k=0}^{s}\binom{s}{k}_{p,q}B_{s-k}(x:p,q)p^{\binom{k}{2}} - B_s(x:p,q)\right]\frac{F_{n+1-s}(0,y;z:p,q)}{[n+1]_{p,q}}, \quad (4.1)$$

where $B_n(x:p,q)$ are called the (p,q)-Bernoulli polynomials.

Proof. By using definition (2.1), we have

$$\left(\frac{1}{1-z(e_{p,q}(t)-1)}\right)e_{p,q}(xt)E_{p,q}(yt)$$
$$= \left(\frac{1}{1-z(e_{p,q}(t)-1)}\right)\frac{t}{e_{p,q}(t)-1}\frac{e_{p,q}(t)-1}{t}e_{p,q}(xt)E_{p,q}(yt)$$
$$= \frac{1}{t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{s}\binom{s}{k}_{p,q}B_{s-k}(x:p,q)p^{\binom{k}{2}}\right)\frac{t^s}{[s]_{p,q}!}\sum_{n=0}^{\infty}F_n(0,y;z:p,q)\frac{t^n}{[n]_{p,q}!}$$
$$- \frac{1}{t}\sum_{s=0}^{\infty}B_s(x:p,q)\frac{t^s}{[s]_{p,q}!}\sum_{n=0}^{\infty}F_n(0,y;z:p,q)\frac{t^n}{[n]_{p,q}!}$$
$$= \frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\binom{n}{s}_{p,q}\sum_{k=0}^{s}\binom{s}{k}_{p,q}B_{s-k}(x:p,q)p^{\binom{k}{2}}\right]F_{n-s}(0,y;z:p,q)\frac{t^n}{[n]_{p,q}!}$$
$$- \frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\binom{n}{s}_{p,q}B_s(x:p,q)\right]F_{n-s}(0,y;z:p,q)\frac{t^n}{[n]_{p,q}!}.$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we arrive at the required result (4.1). □

Theorem 4.2. Each of the following relationships holds true:

$$\begin{aligned}&F_n(0,y;z;u:p,q)\\&=\sum_{s=0}^{n}\binom{n}{s}_{p,q}\left[\sum_{k=0}^{s}\binom{s}{k}_{p,q}E_{s-k}(x:p,q)p^{\binom{k}{2}}+E_s(x:p,q)\right]\frac{F_{n-s}(0,y;z:p,q)}{[2]_{p,q}},\end{aligned}\tag{4.2}$$

where $E_n(x;p,q)$ are called the (p,q)-Euler polynomials.

Proof. By using definition (2.1), we have

$$\begin{aligned}&\left(\frac{1}{1-z(e_{p,q}(t)-1)}\right)e_{p,q}(xt)E_{p,q}(yt)\\&=\left(\frac{1}{1-z(e_{p,q}(t)-1)}\right)\frac{[2]_{p,q}}{e_{p,q}(t)+1}\frac{e_{p,q}(t)+1}{[2]_{p,q}}e_{p,q}(xt)E_{p,q}(yt)\\&=\frac{1}{[2]_{p,q}}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_{p,q}E_{n-k}(x:p,q)p^{\binom{k}{2}}\right)\frac{t^n}{[n]_{p,q}!}+\sum_{n=0}^{\infty}E_n(x:p,q)\frac{t^n}{[n]_{p,q}!}\right]\\&\times\sum_{n=0}^{\infty}F_n(0,y;z:p,q)\frac{t^n}{[n]_{p,q}!}\\&=\frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\binom{n}{s}_{p,q}\sum_{k=0}^{s}\binom{s}{k}_{p,q}E_{s-k}(x:p,q)p^{\binom{k}{2}}+\sum_{s=0}^{n}\binom{n}{s}_{p,q}E_s(x:p,q)\right]\\&\times F_{n-s}(0,y;z:p,q)\frac{t^n}{[n]_{p,q}!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we arrive at the desired result (4.2). □

Theorem 4.3. Each of the following relationships holds true:

$$\begin{aligned}&F_n(x,y;z:p,q)\\&=\sum_{s=0}^{n}\binom{n+1}{s}_{p,q}\left[\sum_{k=0}^{s}\binom{s}{k}_{p,q}G_{s-k}(x:p,q)p^{\binom{k}{2}}+G_s(x:p,q)\right]\frac{F_{n-s}(0,y;z:p,q)}{[2]_{p,q}[n+1]_{p,q}},\end{aligned}\tag{4.3}$$

where $G_n(x;p,q)$ are called the (p,q)-Genocchi polynomials.

Proof. By using definition (2.1), we have

$$\left(\frac{1}{1-z(e_{p,q}(t)-1)}\right)e_{p,q}(xt)E_{p,q}(yt)$$
$$=\left(\frac{1}{1-z(e_{p,q}(t)-1)}\right)e_{p,q}(xt)E_{p,q}(yt)\frac{[2]_{p,q}t}{e_{p,q}(t)+1}\frac{e_{p,q}(t)+1}{[2]_{p,q}t}e_{p,q}(xt)E_{p,q}(yt)$$
$$=\frac{1}{[2]_{p,q}t}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_{p,q}G_{n-k}(x:p,q)p^{\binom{k}{2}}\right)\frac{t^n}{[n]_{p,q}!}+\sum_{n=0}^{\infty}G_n(x:p,q)\frac{t^n}{[n]_{p,q}!}\right]$$
$$\times\sum_{n=0}^{\infty}F_n(0,y;z:p,q)\frac{t^n}{[n]_{p,q}!}$$
$$=\frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_{p,q}\sum_{k=0}^{s}\left(\begin{array}{c}s\\k\end{array}\right)_{p,q}G_{s-k}(x:p,q)p^{\binom{n}{2}}+\sum_{s=0}^{n}\left(\begin{array}{c}n\\s\end{array}\right)_{p,q}G_s(x:p,q)\right]$$
$$\times F_{n+1-s}(0.y;z:p,q)\frac{t^n}{[n+1]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, then we have the asserted result (4.3). □

Theorem 4.4. For $n \geq 0$, the following formula for (p,q)-Fubini polynomials holds true:

$$F_n(x,y;z:p,q)=\sum_{l=0}^{n}\binom{n}{l}_{p,q}(x+y)_{p,q}^{n-l}\sum_{k=0}^{l}z^k k!S_2(l,k:p,q). \tag{4.4}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}=\frac{1}{1-z(e_{p,q}(t)-1)}e_{p,q}(xt)E_{p,q}(yt)$$
$$=e_{p,q}(xt)E_{p,q}(yt)\sum_{k=0}^{\infty}z^k(e_{p,q}(t)-1)^k$$
$$=e_{p,q}(xt)E_{p,q}(yt)\sum_{k=0}^{\infty}z^k\sum_{l=k}^{\infty}k!S_2(l,k:p,q)\frac{t^l}{[l]_{p,q}!}$$
$$=\sum_{n=0}^{\infty}(x+y)_{p,q}^{n}\frac{t^n}{[n]_{p,q}!}\sum_{l=0}^{\infty}z^k\sum_{k=0}^{l}k!S_2(l,k:p,q)\frac{t^l}{[l]_{p,q}!}.$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty}F_n(x,y;z:p,q)\frac{t^n}{[n]_{p,q}!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\left(\begin{array}{c}n\\l\end{array}\right)_{p,q}(x+y)_{p,q}^{n-l}\sum_{k=0}^{l}z^k k!S_2(l,k:p,q)\right)\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$ in both sides, we get (4.4). □

Theorem 4.5. For $n \geq 0$, the following formula for (p,q)-Fubini polynomials holds true:

$$F_n(x+r,y;z:p,q) = \sum_{l=0}^{n} \binom{n}{l}_{p,q} (x+y)_{p,q}^{n-l} \sum_{k=0}^{l} z^k k! S_2(l+r,k+r:p,q). \tag{4.5}$$

Proof. Replacing x by $x+r$ in (2.1), we have

$$\sum_{n=0}^{\infty} F_n(x+r,y;z:p,q)\frac{t^n}{[n]_{p,q}!} = \frac{1}{1-z(e_{p,q}(t)-1)} e_{p,q}((x+r)t)E_{p,q}(yt)$$

$$= e_{p,q}(xt)E_{p,q}(yt)e_{p,q}(rt)\sum_{k=0}^{\infty} z^k (e_{p,q}(t)-1)^k$$

$$= e_{p,q}(xt)E_{p,q}(yt)e_{p,q}(rt)\sum_{k=0}^{\infty} z^k \sum_{l=k}^{\infty} k! S_2(l,k:p,q)\frac{t^l}{[l]_{p,q}!}$$

$$= \sum_{n=0}^{\infty}(x+y)_{p,q}^n \frac{t^n}{[n]_{p,q}!}\sum_{l=0}^{\infty} z^k \sum_{k=0}^{l} k! S_2(l+r,k+r:p,q)\frac{t^l}{[l]_{p,q}!}.$$

Replacing n by $n-l$ in above equation, we get

$$\sum_{n=0}^{\infty} F_n(x+r,y;z:p,q)\frac{t^n}{[n]_{p,q}!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{pmatrix} n \\ l \end{pmatrix}_{p,q}(x+y)_{p,q}^{n-l}\sum_{k=0}^{l} z^k k! S_2(l+r,k+r:p,q)\right)\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$ in both sides, we get (4.5). □

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Conflict of interest

The authors declare no conflict of interest.

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