

Novel Alternative Methods to Romberg Integration and Richardson's Extrapolation with Matlab Package: Integral_Calculator

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(Received: 2020-01-04, Accepted: 2020-04-05)

Abstract

This paper introduces new integration methods for numerical integration problems in science and engineering applications. It is shown that the exact results of these integrals can be obtained by these methods with the use of only 2 segments. So no additional function and integrand evaluations are required for different levels of computation. This situation overcomes the computational inefficiency. A new Matlab Package; Integral_Calculator is presented. Integral_Calculator provides a user-friendly computational platform which requires only 3 data entries from the user and performs the integration and give the results for any functions to be integrated. This package has been tested for each numerical example considered below.

Keywords: Romberg Integration, Richardson's Extrapolation, New Algorithms, Matlab

Introduction

The Newton-Cotes formulas are the most common techniques for numerical integration of science and engineering problems. Although these methods can be employed for integrating functions numerically, the result is attained with certain accuracy and not always convenient [5, 7, 13, 24]. For this purpose, Richardson's extrapolation is employed by use of two Trapezoidal rule estimates to obtain a more accurate result. Richardson's extrapolation is a widespread technique used for integration in many science and engineering problems with the help of other methods such as Crank-Nicolson and Runge-Kutta schemes in literature (see e.g. [2-4, 9, 11, 18- 21, 23, 27, 28] Efficient implementation of Richardson's extrapolation leads to a numerical integration technique, called Romberg integration. This method is iterative and can be employed to attain an approximate result within a preestablished error tolerance [12, 22]. Romberg integration has been used for many applications in science, such as optics and fuzzy functions [1, 2, 6, 15, 16, 26, 30]. It was shown that the error in Trapezoidal rule for Romberg integration was expressed as a series in even powers of the step size provided that the Taylor Series for the governing function converges for every point in the prespecified interval [25]. Dutka [10] analyzed both Richardson extrapolation and Romberg integration at first in the literature. Alternative numerical integration method for the Romberg integration was proposed firstly by Youngberg [33]. Since computational algorithms for integration numerically cannot be performed without the aid of softwares [14, 17, 31, 32], computer programs are the indispensable part of these processes. Neither of the works dealt with new alternatives to Romberg integration and Richardson's extrapolation concurrently with implementation of a specified software. For these reasons, this study aims to introduce new methods, alternative to both Romberg integration and Richardson's extrapolation with the design of a calculator in Matlab. This calculator performs numerical integral computations and then present results for all methods, at the same time.

2 Overview of Richardson's extrapolation and Romberg integration

Both methods are explained briefly in the following subsections.



2.1 Richardson's extrapolation

Richardson's extrapolation is not only a more accurate numerical integration method than Trapezoidal rule but it also underlies the numerical integration technique called Romberg integration. In this technique, the integral

$$I = \int_a^b f(x)dx, \quad a, b \in R, \quad f \in C^k[a, b] \quad (1)$$

is approximated using the Composite Trapezoidal rule with step sizes $h_k = \frac{b-a}{2^k}$, here k is a nonnegative integer. Richardson extrapolation is employed $k-1$ times to previously calculated approximations in order to enhance the order of accuracy as much as possible. Using Composite Trapezoidal rule with one and two segments, respectively, the following approximations can be computed:

$$I_{1,1} = \frac{(b-a)}{2} (f(a) + f(b)) \quad (2a)$$

$$I_{2,1} = \frac{(b-a)}{4} \left(f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right) \quad (2b)$$

If $f(x)$ has continuous derivatives of all orders on $[a, b]$. Then Composite Trapezoidal rule for a general number of n segments, should satisfy the following:

$$\int_a^b f(x)dx = \frac{h}{2} \left(f(a) + 2 \sum_{j=1}^{n-1} f(x_j) + f(b) \right) + \sum_{i=1}^{\infty} K_i h^{2i} \quad (3)$$

ere $h=(b-a)/n$, $x_j = a + jh$ and the constants $\{K_i\}_{i=1}^{\infty}$ depend only on the derivatives of $f(x)$. It follows that one can employ Richardson's extrapolation to calculate an integral numerically with a higher order of accuracy. If true value of the integral is represented by I , then (2a) and (2b) become

$$I_{1,1} = I + K_1 h^2 + O(h^4) \quad (4a)$$

$$I_{2,1} = I + K_1 (h/2)^2 + O(h^4) \quad (4b)$$

The system of equations can be solved for K_1 and I by neglecting the $O(h^4)$ terms. Then, I takes the following form:

$$I = \frac{4}{3} I_{2,1} - \frac{1}{3} I_{1,1} \quad (5)$$

The result in (5) is based on accuracy $O(h^4)$. New approximate results may also be obtained by use of Composite Trapezoidal rule with higher orders of accuracy.

2.2 Romberg integration

This technique approximates the integral in (1) based on the Euler-Maclaurin asymptotic error expansion formula and the Richardson's extrapolation [13]. Romberg [24] has formulated the Richardson's extrapolation technique for automatic computations. With use of Composite Trapezoidal rule, Romberg sequence ensures full overlapping of the nodes of integration from one extrapolation level to another. Precisely, as the number of segments and order of accuracy increase, the result of integral converges to the exact result and approximation improves [5, 7, 13, 24, 32]. The general formulation for Romberg integration can be expressed as follows [7]:

$$I_{j,k} \cong \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1} \quad (6)$$

where $I_{j+1,k-1}$ is the more accurate result of integral and $I_{j,k-1}$ is the less accurate result. The index k denotes the level of integration, here $k=1$ represents the original trapezoidal rule estimates, $k=2$ relates to $O(h^4)$, $k=3$ corresponds to $O(h^6)$ and so on.

In other respects, the complexity of Romberg method is mainly expressed by the number of function evaluations at the nodes of integration. In this integration, from level $k-1$ to k , 2^k additional integrand evaluations are needed. In higher dimensions, this causes too many function evaluations, and this technique becomes computationally inefficient. This inefficiency can be prevented by the new methods proposed in the following section.

3 Proposed methods

These new methods are proposed to improve Richardson's extrapolation and Romberg integration techniques, respectively. Main aim is to cope with the complexity of Richardson's extrapolation and Romberg techniques in terms of numerous function and integrand computations. Only use of $n=2$ segments and $h_2 = \frac{h_1}{3^n}$ are sufficient to attain the exact result of integral. The demonstration of these novel methods with error analysis are presented in the following subsection.

3.1 Error analysis

The estimate and error associated with a multiple-application Simpson's rule can be denoted generally as

$$I = I(h) + E(h) \quad (7)$$

Where I the exact value of the integral, $I(h)$ = the approximation from an n -segment application of the Simpson's rule with step size $h = (b-a)/n$ and $E(h)$ = the truncation error. If two separate estimates using step sizes of h_1 and h_2 are employed with the following values for the error,

$$I(h_1) + E(h_1) = I(h_2) + E(h_2) \quad (8)$$

Then the error of the multiple-application Simpson's rule is (with $n = (b-a)/h$) [7]

$$E \cong -\frac{(b-a)}{180} h^4 \bar{f}^{(4)} \quad (9)$$

Where $\bar{f}^{(4)}$ is the average fourth derivative for the interval.

If it is supposed that $\bar{f}^{(4)}$ is constant regardless of step size, (9) can be employed to designate that the ratio of the two errors will be

$$\frac{E(h_1)}{E(h_2)} \cong \frac{h_1^4}{h_2^4} \quad (10)$$

Relation (10) has the significant effect of removing the term $\bar{f}^{(4)}$ from the computation. So one can obtain $E(h_1)$ approximately without knowing the average fourth derivative for the interval.

$$E(h_1) \cong E(h_2) \left(\frac{h_1}{h_2}\right)^4 \quad (11)$$

Relation (11) can be inserted into (8):

$$I(h_1) + E(h_2) \left(\frac{h_1}{h_2}\right)^4 = I(h_2) + E(h_2) \quad (12)$$

By solving (12), $E(h_2)$ becomes



$$E(h_2) \cong \frac{I(h_1) - I(h_2)}{1 - \left(\frac{h_1}{h_2}\right)^4} \quad (13)$$

So, estimate of the truncation error in terms of the integral estimates and their step sizes have been developed. This estimate can then be put into

$$I = I(h_2) + E(h_2) \quad (14)$$

to obtain an improved approximation for the result of integral:

$$I \cong I(h_2) + \frac{I(h_2) - I(h_1)}{\left(\frac{h_1}{h_2}\right)^4 - 1} \quad (15)$$

So two Simpson's rule estimates of $O(h^2)$ are combined to attain a new improved result of $O(h^4)$ and so forth. For the special case where the interval is $h_2 = \frac{h_1}{3^n}$ this equation takes the following form:

$$I \cong I(h_2) + \frac{I(h_2) - I(h_1)}{3^{4n} - 1} \quad (16)$$

Or, arranging (16) becomes,

$$I \cong \frac{3^{4n} I(h_2)}{3^{4n} - 1} - \frac{I(h_1)}{3^{4n} - 1} \quad (17)$$

The formula in (17) is the alternative to Richardson's extrapolation. Relation (17) is arranged so that one can perform iterative computations by use of following formula:

$$I_{j,k} \cong \frac{(3^{4n})^{k-1} I_{j+1,k-1}}{(3^{4n})^{k-1} - 1} - \frac{I_{j,k-1}}{(3^{4n})^{k-1} - 1} \quad (18)$$

where $I_{j+1,k-1}$ and $I_{j,k-1}$ are the more and less accurate result of integral, respectively and $I_{j,k}$ is the improved result of the integral. The index k represents the level of integration. This is an alternative to Romberg integration method.

4 Numerical examples

Numerical results are presented from civil, mechanical engineering and other integral problems. The error tolerance for each example is 10^{-10} . Two segments are employed for each example. All results are obtained by use of Integral_Calculator in Matlab.

4. 1 Civil engineering

The amount of mass transported via a pipe over a period of time can be computed as [7]

$$M = \int_{t_1}^{t_2} Q(t)c(t)dt \quad (19)$$

where M is mass (mg), $t_1=2$ min (initial time), $t_2=8$ min (final time), $Q(t) = 9 + 4\cos^2(0.4t)$ (flow rate, m^3/min and

$$c(t) = 5e^{-0.5t} + 2e^{0.15t}.$$

All integration results are presented in the following table.

Table 1. Results for integration of civil engineering example



	Classical Richardson's extrapolation	Classical Romberg method	Proposed method alternative to Richardson's extrapolation	Proposed method alternative to Romberg method	Integrated by analytically
Integration results	286.5328	315.1858	322.3484	322.3484	322.3484

4. 2 Mechanical engineering

The distance problem of how far the parachutist has fallen after a final time t_f ($t_f = 10$ min) is solved by performing the following integration [7].

$$d = \frac{gm}{c} \int_0^{t_f} (1 - e^{-(c/m)t}) dt \tag{20}$$

where d is the distance in meters, $c=12.5$ kg/s (the drag coefficient), $m=68.1$ kg (mass of the parachutist) and $g=9.8$ m/s² (the gravitational constant).

The numerical and exact integration results are exhibited in Table 2.

Table 2. Results for integration of mechanical engineering example

	Classical Richardson's extrapolation	Classical Romberg method	Proposed method alternative to Richardson's extrapolation	Proposed method alternative to Romberg method	Integrated by analytically
Integration results	257.2762	283.0036	289.4351	289.4351	289.4351

4. 3 Other integrals

Integration results by numerically and analytically for other problems are displayed in Table 3.

Table 3. Results for integration of other examples

Functions to be Integrated	Result of classical Richardson's extrapolation	Result of classical Romberg method	Result of proposed method alternative to Richardson's extrapolation	Result of proposed method alternative to Romberg method	Exact result
$\int_{-4}^4 \frac{dx}{1+x^2}$ [8]	2.357000	2.592700	2.651600	2.651600	2.651600



$\int_0^2 e^{\cos(x)} dx$ [32]	3.070500	3.377600	3.454400	3.454400	3.454400
$\int_0^1 2\cos(x^2) dx$ [32]	1.608000	1.768900	1.809000	1.809000	1.809000
$\int_{-1}^1 x^4 \sin^2(\pi x) dx$ [8]	0.1014000	0.1115400	0.1140800	0.1140800	0.1140800

The results presented in above Tables demonstrate that exact value of the integrals can be obtained by new methods with only 2 segments and 10^{-10} error tolerance as compared to the classical Richardson's extrapolation, Romberg method and also the studies in literature [7, 8, 32].

5 Matlab Package: Integral_Calculator and Implementation

The numerical computations begin with developing an algorithm. This algorithm consists of codes for classical Richardson's extrapolation, Romberg integration and new alternative methods to these techniques simultaneously. The algorithm is adjusted as application of Matlab package in a selected folder. This application is called as Integral_Calculator. Matlab R2016a (9.0.0.341360) was used for both creation of Integral_Calculator and its implementation. Once, one clicks the small icon (red arrow head) shown in Fig. 1, then computations start.



Figure 1. Screenshot of Integral_Calculator App

After clicking this icon, an input box appears which requires lower, upper limits and function to be integrated, from user. Fig. 2 demonstrates the display image of input box waiting for the data entries.



Figure 2. Display image of input box

User should enter these data into the corresponding fields in input box. As an illustration; the numerical data of the last example in Table 3 has been entered into the input box. The screenshot of this process is given in Fig. 3.

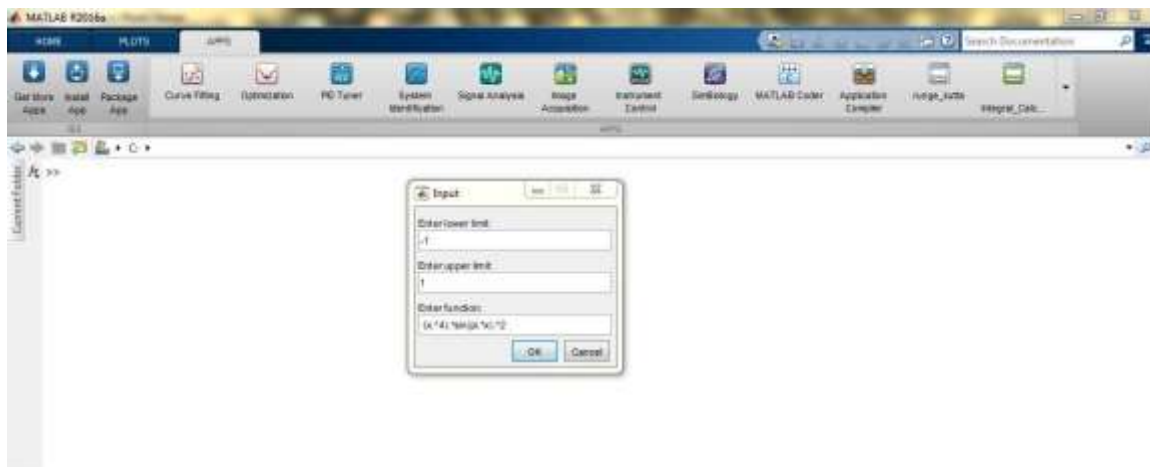


Figure 3. Screenshot of filled input box

Once clicking OK, the results are appeared immediately in command window. Fig. 4 indicates all inputs and outputs including results for each method, explicitly.

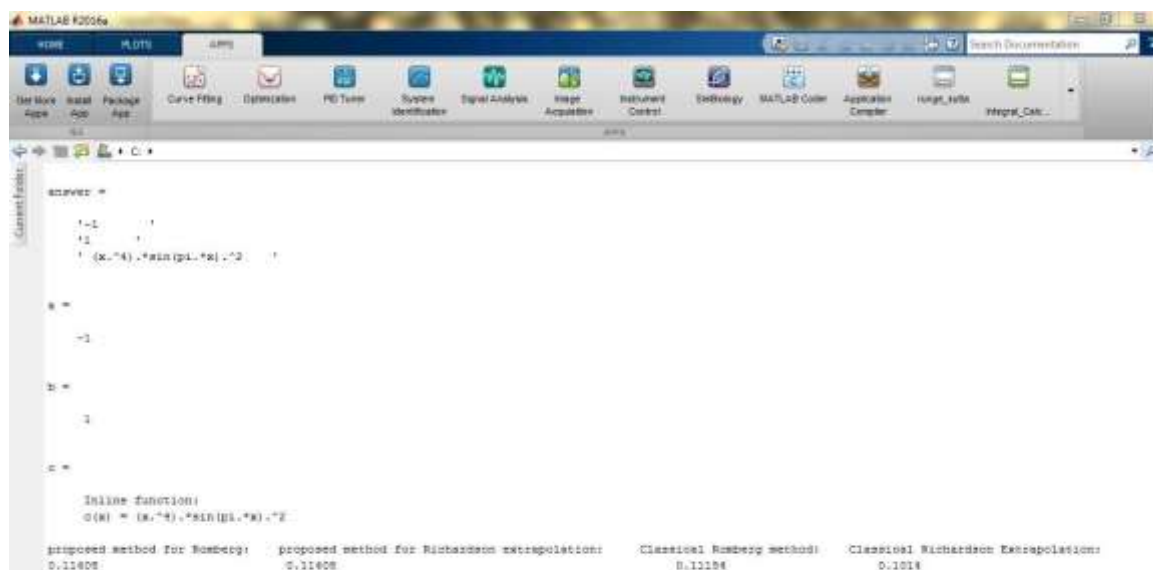


Figure 4. Display of results for numerical computations

Conclusions

In this paper, novel alternative methods to Richardson's extrapolation and Romberg integration are presented. These methods can be used for any integration problem of any complicated functions in science and engineering applications, comfortably. Only 2 segments for computations is sufficient to attain exact value of the integrals by the proposed techniques. Therefore, there is no need to perform additional function and integrand computations. Furthermore, another important contribution of this study is to design a new Matlab Package namely, Integral_Calculator. This ensures a user-friendly computational platform including classical Richardson's extrapolation, Romberg methods and proposed methods alternatives to them, simultaneously with high accuracy. This package allows the user to obtain the integration results from each numerical integration method at the same time by entering lower, upper limits and function only.

Data Availability (excluding Review articles)

Romberg Integration, Richardson's Extrapolation, New alternative algorithms to Richardson's Extrapolation and Romberg Integration, Matlab Package; Integral_Calculator

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