SOME NEW TRENDS IN SUPERINTEGRABLE SYSTEMS

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SOME NEW TRENDS IN SUPERINTEGRABLE SYSTEMS

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ABSTRACT

SOME NEW TRENDS IN SUPERINTEGRABLE SYSTEMS

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The Hamiltonian integrability in classical mechanics and the explicit metrics for a class of two-dimensional cubically superintegrable systems are reviewed. Firstly, the integrals that are quadratic in moments corresponding to the natural Lagrangian systems are discussed with a special view focus to the two-dimensional case. The classical free Lagrangian admitting a constant of motion, in one and two dimensional space, was generalized by using the fractional Caputo derivative. The fractional Killing vectors and Killing-Yano tensors are presented in connection with the hidden symmetries of curved spaces. The Dunkl-Coulomb system in the plane was considered. The model that was defined in terms of the Dunkl Laplacian, involves reflection operators, with a r^{-1} potential. The system is shown to be maximally superintegrable and exactly solvable.

Keywords: Superintegrable systems, Hamiltonian systems, natural Lagrangian system, first integral, Killing vector, Killing tensors, Killing-Yano tensors, conformal Killing-Yano tensors, fractional Killing-Yano tensors.

ÖZ

Süperintegrallenebilir Sistemlerde Bazı Yeni Akımlar

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Klasik mekanikteki Hamilton integrallenebilme ve iki boyutlu kübik olarak süperintegrsllenebilir sistemler sınıfı için belirgin metrikler sunulmuştur. İlk olarak, doğal Lagrangian sistemlerine ilişkin momentlerine göre ikinci dereceden olan integraller iki boyuta özel bir bakışla tartışıldı. Bir ve iki boyutlu uzayda bir hareket sabitine izin veren serbest Lagrangian kesirli hesaplamanın Caputo türevi kullanılarak genellestirildi. Kesirli Killing vektörleri ve Killing-Yano tensörleri, eğimli uzayların saklı simetrileri ile bağlantılı olarak sunuldu. Dunkl-Coulomb sistemi düzlemde ele alinmış bulundu. Model, r^{-1} potansiyeli ile yansıma operatörleri içeren Dunkl-Laplace operatorü açısından tanımlanmış. Sistemin maksimal süperintegrallenebilir ve tam çözülebilir olduğu gösterilmiştir.

Anahtar Kelimeler: Süperintegrallenebilir sistemler, Hamiltonian sistemler, doğal Lagrangian sistemler, birinci integral, Killing vektörleri, Killing tensörleri, Killing-Yano tensörleri, konformal Killing-Yano tensörleri, kesirli Killing-Yano tensörleri.

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LIST OF ABBREVIATIONS

MSh	Matveev and Shevechin
KV	Killing vector
KY	Killing-Yano
ChS	Christoffel symbols
CO	Casimir operator
DC	Dunkl-Coulomb

CHAPTER 1

INTRODUCTION

A superintegrable system is an integrable system if it has more integrals of motion than the degrees of freedom, the Kepler system, a hydrogen atom and harmonic oscillator are well-known examples of such a system. The importance of these systems is their symmetry, which may result almost integrability for the highly symmetric once [1-4]. We can also offer some famous models of integrable systems some classical mechanics models, e.g. free particle, harmonic oscillator, spinning top, planetary motion, and some (1 + 1)-dimensional classical field theories or partial differential equations, e.g. KdV, sine-Gordon, Einstein gravity, classical magnets and string theory. Also in quantum mechanical there are some models, e.g. the quantum versions of the above classical mechanics models, and some (1 + 1)-dimensional quantum field theories [5].

The thesis is a review of some new trends in the area of superintegrable systems. The structure of my thesis is as follows:

In Chapter 2 the basic definitions for integrable systems and action- angle variables are reviewed [6-9].

In Chapter 3 the explicit form of two dimensional superintegrable system of Matveev and Shevechin in local coordinates, including cubic and linear integrals are given. That leads us to specific the parameters values that systems are in fact globally defined on $\mathbb{S}^2$ [10-16].

In Chapter 4 we reviewed the integral of motion generated by the fractional classical free Lagrangian in one and two dimensional space with the help of Killing-Yano tensor and Killing vector [17-55].

In Chapter 5 we reviewed the Dunkl-Coulomb (DC) system in the plane which is derived by using the Hamiltonian system [56-67].

Chapter 6 is dedicated to my concluding remarks.

CHAPTER 2

INTEGRABILITY AND CLASSICAL MECHANICS

In this chapter we are going to represent the integrability of ordinary differential equation.

2.1 Basic definitions

We can employ the Lagrangian mode to formulate the concept of integrability then the natural form of Lagrangian formulation is $L = L(q_j, \dot{q}_j, t)$ when q_j are refer to generalized coordinates and $j = 1, \dots \dots, n$ and $\dot{q}_j = \frac{dq_j}{dt}$ are refer to generalized momenta. The equations of motion are [1,2]

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{q_j} = 0, \quad j = 1, \dots\dots, n. \tag{2.1}$$

These equation are of 2^{nd} order differential equation which require $2n$ initial conditions as $q_j(t = 0)$and $\dot{q}_j(t = 0)$. Below we utilizedthe Hamiltonian approach [1,2].

2.1.1 Hamiltonian approach and integrable systems

Let a system possessing n degrees of freedom. The motion of this system is in a $2n$-dimensional phase space M is described by a trajectory, an open set of R^{2n} with the local coordinates such as $(p_{j\,,} q_j)$, where $= 1, \dots \dots, n$, (p_j is refer to the generalized momenta and q_j refers to the generalized positions). Consider the differentiable functions $: M \times \mathbb{R} \to \mathbb{R}$, therefore $F = F(p, q, t)$. Consider F and G being two dynamical variables so that the Poisson bracket of them is defined by [7]

$$\{F,G\}=\left\{\frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j}-\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j}\right\} \tag{2.2}$$

which fulfills the following properties [7]

$$\{F,G\}=-\{G,F\},$$
$$\{F,\{G,S\}\}+\{G,\{S,F\}\}+\{S,\{F,G\}\}=0. \tag{2.3}$$

Here $(p_{i\,,}q_i)$ are the coordinate functions which satisfy the following canonical commutation relations, namely

$$\{p_{i\,,}p_j\}=0,\quad \{q_{i\,,}q_j\}=0, \{q_{i\,,}q_j\}=\delta_{ij} \tag{2.4}$$

$H=H(p,q,t)$ is the given Hamiltonian. The dynamical variable F is determined by $\frac{dF}{dt}=\frac{\partial F}{\partial t}+\{H,F\}$ for any $F=F(p,q,t)$. By putting $F=p_i$ or $F=q_i$ the Hamiltonian equations are given by [7]

$$\dot{p}_i\ =-\frac{\partial H}{\partial q_i},\ \ \dot{q}_i\ =\frac{\partial H}{\partial p_i}\ ,\qquad i=1,\ldots\ldots,n. \tag{2.5}$$

It means that in the phase space the volume of the elements are preserved [7].

Definition 1 [7]: A function $f=f(p_j,q_j,t)$ satisfying $\frac{df}{dt}=0$ such that (2.5) is satisfied is a first integral or a constant of motion. Equivalently, $f(p(t),q(t),t)=const.$ if $p(t)$ then $q(t)$ denote the solutions of (2.5). The system (2.5) will be solvable provided that it possess sufficiently many first integrals and the order reduction can be utilized.

Example 1 [7]: In regard to a system with $n=1$ and $M=\mathbb{R}^2$, where (n is the degrees of freedom, M is phase space) the form of Hamiltonian is

$$H(p,q)=\frac{1}{2}p^2+V(q)\equiv E. \tag{2.6}$$

Then, by equations (2.5) we get

$$\dot{q}=\frac{\partial H}{\partial p}=\frac{1}{2}\,.\,2p=p,\quad \text{and}\quad \dot{q}=p,\ \dot{p}=-\frac{dV}{dq}\,. \tag{2.7}$$

Using the fact that the Hamiltonian is a constant of motion namely $\{H,H\}=0$, then

$$H\big(p(t), q(t)\big) = \frac{1}{2}p^2(t) + V\big(q(t)\big) \equiv E. \tag{2.8}$$

E is called the Hamiltonian energy, where $E = H\big(p(0), q(0)\big)$. So, we have [7]

$$\frac{1}{2}p^2 + V(q) = E, \tag{2.9}$$

or

$$p = \pm\sqrt{2(E - V(q)},\ \text{and} \quad \dot{q} = p. \tag{2.10}$$

By integrating $\frac{dt}{dq} = \frac{1}{p}$ following solution is obtained in an implicit form

$$t = \pm \int \frac{dq}{\sqrt{2\,(\mathrm{E} - \mathrm{V(q)})}}. \tag{2.11}$$

The exact solution can be obtain if we evaluate the integral on the right hand side and invert the relation $t = t(q)$ to obtain $q(t)$. Sometimes the two steps are impossible to take but anyway these systems would be considered as integrable system [7].

Definition 2 [9]: Two functions F_1 and F_2 on a symplectic manifold are in involution if $\{F_1, F_2\} = 0$.

2.2. The integrability with action angle variables

The equations (2.5) are usually enough to identity n constants of motion [7].

Definition 3 [7]: An integrable system means a $2n$-dimensional phase space M equipped with n independent functions $g_1, \cdots\cdots, g_n: M \to \mathbb{R}$ fulfilling

$$\{\, g_j\ , g_k \,\} = 0,\ \text{and} \quad j\,, k = 1, \cdots\cdots\cdots, n, \tag{2.12}$$

which says that the first integrals are in involution. Thus, the transformation

$$Q_k = Q_k(p, q), P_k = P_k(p, q), \tag{2.13}$$

is named canonical if the Poisson bracket remains invariant [7]

$$\sum_{k=1}^{n} \frac{\partial g}{\partial q_k}\frac{\partial h}{\partial p_k} - \frac{\partial g}{\partial p_k}\frac{\partial h}{\partial q_k} = \sum_{k=1}^{n} \frac{\partial g}{\partial Q_k}\frac{\partial h}{\partial P_k} - \frac{\partial g}{\partial P_k}\frac{\partial h}{\partial Q_k} \tag{2.14}$$

for all $g, h: M \to \mathbb{R}$. The canonical transformations preserve (2.5). For a function $S(q, P, t)$ fulfilling $\det(\frac{\partial^2 S}{\partial q_j \partial P_n}) \neq 0$. Thus, we are able to build a canonical transformation, namely [7]

$$p_k = \frac{\partial S}{\partial q_n}, \qquad Q_k = \frac{\partial S}{\partial P_n}, \qquad \widehat{H} = H + \frac{\partial S}{\partial t}. \tag{2.15}$$

S denotes a generating function [7,8]. We concentrate to seek a canonical transformation such that in the new variables $H = H(P_1, \dots\dots, P_k)$, namely [7]

$$Q_n(t) = Q_n(0) + t\frac{\partial H}{\partial P_n}, P_n(t) = P_n(0) = const. \tag{2.16}$$

Theorem 1 [7,9]: Assume $M = f_1, \dots\dots, f_n$ be Liouville-Arnold integrable system together with $H = f_1$, and let

$$M_f = \{(p, q) \in M;\ f_k(p, q) = c_k,\ k = 1, \cdots\cdots\cdots, n,\},$$

where $c_1 \dots\dots c_n$ are constants and be an n dimensional level surface of 1^{st} integrals f_k. Then, if M_f is compact and connected then it is diffeomorphic to a torus

$$T^n = S^1 \times S^1 \times \cdots \times S^1,$$

and someone can offer the action-angle coordinates

$$I_1, \cdots\cdots, I_n,\ \emptyset_1, \cdots\cdots \emptyset_n,\ \emptyset_k \in [0,2\pi],$$

such that angles $\emptyset_k$ are coordinates on M_f and actions I_k are first integrals [7,9]. The Hamiltonian equation's (2.5) will be

$$\dot{I}_k = 0, \dot{\emptyset}_k = \omega_k(I_1, \cdots\cdots\cdots, I_n), \quad k = 1, \cdots\cdots\cdots, n, \tag{2.17}$$

So, via quadratures the integrable system are solvable [7,9].

Proof : See [7,9].

The motion is given on

$$f_1(p,q) = c_1, \qquad f_2(p,q) = c_2 \dots\dots, \qquad f_n(p,q) = c_n$$

of dimension n. The 1^{st} portion of the theorem implies that this surface denotes a torus. For any point in M there are strictly 1 torus T^n passing through that point. This implies that M admits a foliation via n-dimensional leaves. Any leaf is a torus and different tori coincide to different choices of $c_1, \cdots\cdots, c_n$ [7,9].

Suppose $det\ (\frac{\partial f_j}{\partial p_k}) \neq 0$, thus, $f_k(p,q) = c_k$ can be solved for $p_i = p_i(q,c)$ and $f_i\big(q, p(q,c)\big) = c_i$. By differentiating with respect to q_j we conclude [7,9]

$$\frac{\partial f_i}{\partial q_j} + \sum_k \frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_j} = 0,$$

$$\sum_j \frac{\partial f_m}{\partial p_j}\frac{\partial f_i}{\partial q_j} + \sum_k \frac{\partial f_m}{\partial p_j}\frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_j} = 0. \qquad (2.18)$$

Thus, we conclude that

$$\{f_i, f_m\} + \sum_{j,k} \left(\frac{\partial f_m}{\partial p_j}\frac{\partial f_i}{\partial p_k}\frac{\partial p_k}{\partial q_j} - \frac{\partial f_i}{\partial p_j}\frac{\partial f_m}{\partial p_k}\frac{\partial p_k}{\partial q_j}\right) = 0. \qquad (2.19)$$

The 1^{st} term vanishes since the 1^{st} integrals are in involution. After some calculations we conclude that [7,9]

$$\sum_{j,k} \frac{\partial f_i}{\partial p_k}\frac{\partial f_m}{\partial p_j}\left(\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k}\right) = 0 \qquad (2.20)$$

and, as the matrices $\frac{\partial f_i}{\partial p_k}$ are invertible, we conclude that

$$\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k} = 0. \qquad (2.21)$$

This stipulation means that $\oint \sum_j p_j dq_j = 0$ for each closed contractible curve on T^n which is valid due to the theorem of Stokes. To see it recall that when $n = 3$ we have $\oint_{\delta D} p.dq = \int_D (\nabla \times p)$. Here δD denotes a boundary of a surface D and we have [7,9]

$$(\nabla \times p)_m = \frac{1}{2} \epsilon_{jkm} \left(\frac{\partial p_k}{\partial q_j} - \frac{\partial p_j}{\partial q_k} \right). \tag{2.22}$$

So, we have n closed curves which cannot be contracted to a point, so that the conforming integrals do not vanish. Thus, we can define the action coordinates as [7,9]

$$I_k = \frac{1}{2} \int_{\Gamma_k} \sum_j p_j \, dq_j, \tag{2.23}$$

where the closed curve Γ_k is the kth basic cycle of the torus T^n. So, we conclude

$$\Gamma_k = \{ (\widetilde{\emptyset}, \dots \dots \dots, \widetilde{\emptyset}_n) \in T^n, \ \widetilde{\emptyset}_k \in [0, 2\pi], \ \widetilde{\emptyset}_j = const. \ \ for \ \ j \neq k \}, \tag{2.24}$$

where $\widetilde{\emptyset}$ are some coordinates on T^n. The Stokes theorem means that the actions (2.23) are independent regarding the choice of Γ_k. Given two such cycles Γ_k and $\acute{\Gamma}_k$ of opposite orientations [7,9] we have

$$\oint_{\Gamma_k} \sum_j p_j \, dq_j + \oint_{\acute{\Gamma}_k} \sum_j p_j dq_j = \int \left(\frac{\partial p_i}{\partial q_j} - \frac{\partial p_j}{\partial q_i} \right) dq_j \wedge dq_i = 0. \tag{2.25}$$

The actions (2.23) are the first integrals as $\oint p(q,c) dq$ only depending on $c_k = f_k$ and f_{k^s} are the 1^{st} integrals. The actions are Poisson are commuting, namely [7,9]

$$\{I_i, I_j\} = \sum_{r,s,k} \frac{\partial I_i}{\partial f_r} \frac{\partial f_r}{\partial q_k} \frac{\partial I_j}{\partial f_s} \frac{\partial f_s}{\partial p_k} - \frac{\partial I_i}{\partial f_r} \frac{\partial f_r}{\partial p_k} \frac{\partial I_j}{\partial f_s} \frac{\partial f_s}{\partial q_k} = \sum_{r,s} \frac{\partial I_i}{\partial f_r} \frac{\partial I_j}{\partial f_s} \{f_r, f_s\} = 0. \tag{2.26}$$

The torus M_f can be equivalently introduced by

$$I_1 = \tilde{c}_1, \dots \dots, I_1 = \tilde{c}_n, \tag{2.27}$$

for several constants $\tilde{c}_1, \cdots \cdots, \tilde{c}_n$, [7,9].

We structure the angle coordinates $\emptyset_k$ canonically conjugate to the actions using a generating function $S(q, I) = \int_{q_0}^{q} \sum_j p_j \, dq_j$, where q_0 is a chosen point on the torus. We recall that definition does not depend on a path joining q_0 and q due to (2.21) and Stokes's theorem, respectively. Choosing a different q_0 merely adds a scalar to S but it leaves the angles $\emptyset_i = \frac{\partial S}{\partial I_i}$ invariant [7,9]. The angles are periodic coordinates possessing a period 2π. To see it regard two paths C and $C \cup C_k$ between q_0 and q and calculate [7,9].

$$S(q,I) = \int_{\mathrm{C} \cup \mathrm{Ck}} \sum_j p_j d\, q_j = \int_{\mathrm{C}} \sum_j p_j\, dq_j + \int_{\mathrm{Ck}} \sum_j p_j\, dq_j = \mathrm{S}\,(\mathrm{q}\,,\mathrm{I}) + 2\pi\, I_k,$$

$$\phi_k = \frac{\partial S}{\partial I_k} = \phi_k + 2\pi. \qquad (2.28)$$

The transformations $q = q(\emptyset, I)$, $p = p(\emptyset, I)$, and $\emptyset = \emptyset(q,p), I = I(p,q)$, are canonical and invertible. So, we conclude that [7,9]

$$\{I_j, I_k\} = 0, \{\emptyset_j, \emptyset_k\} = 0, \{\emptyset_j, I_k\} = \partial_{jk}, \qquad (2.29)$$

and the dynamics is given via $\dot{\emptyset}_k = \{\emptyset_k, \widetilde{H}\}$, $\dot{I}_k = \{I_k, \widetilde{H}\}$, when [7,9]

$$\widetilde{H}(\emptyset, I) = H\big(q(\emptyset, I), p(\emptyset, I)\big). \qquad (2.30)$$

We recall that I_{K^s} are 1^{st} integrals, thus,

$$\dot{I}_k = -\frac{\partial \widetilde{H}}{\partial \emptyset_k} = 0, \text{ so } \widetilde{H} = \widetilde{H}(I) \text{ and } \dot{\emptyset}_k = \frac{\partial \widetilde{H}}{\partial I_k} = \omega_k(I),$$

where the ${\omega_k}^s$ are also 1^{st} integrals. As a result we proved (2.17). By integration we obtain [7,9]

$$\emptyset_k(t) = \omega_k(I)t + \emptyset_k(0), \;\; I_k(t) = I_k(0), \quad k = 1, \dots\dots, n. \qquad (2.31)$$

The proof is complete.

Example 2 [7]: All Hamiltonian system time independent are integrable system in two dimensional phase space. Hamiltonian with Harmonic oscillator is given as

$$H(p,q) = \frac{1}{2}(p^2 + \omega^2 q^2).$$

We get a foliation of M_f by ellipses through different options of the energy

$$E = \frac{1}{2}(p^2 + \omega^2 q^2).$$

We take $\Gamma = M_f$ for a constant value of energy E. Thus,

$$I = \frac{1}{2\pi}\oint_{M_f} pdq = \frac{1}{2\pi}\iint_S dpdq = \frac{E}{\omega},$$

where S is the area that enclosed by M_f and using the Stokes theorem we get a moving from an integral on the boundary to an area integral inside. Thus, the new Hamiltonian form is

$$\widehat{H} = \omega I \text{ and } \quad \dot{\emptyset} = \frac{\partial \widehat{H}}{\partial I} = \omega, \emptyset = \omega t + \emptyset_0.$$

Then we have

$$I = \frac{1}{2}\left(\frac{1}{\omega}p^2 + \omega q^2\right).$$

Also the generating function is as [7]

$$S(q, I) = \int pdq = \pm \int \sqrt{2I\omega - \omega^2 q^2}\, dq. \qquad (2.32)$$

After that we choose a sign+, namely

$$\emptyset = \frac{\partial S}{\partial I} = \int \frac{\omega dq}{\sqrt{2I\omega - \omega^2 q^2}} = \arcsin\left(q\sqrt{\frac{\omega}{2I}}\right) - \emptyset_0, \qquad (2.33)$$

thus, we get $q = \sqrt{\frac{\omega}{2I}} \sin\ (\emptyset + \emptyset_0)$ when the familiar solution are recovered as

$$p = \sqrt{2E}\cos(\omega t + \emptyset_0), \qquad q = \sqrt{\frac{2E}{\omega^2}}\sin(\omega t + \emptyset_0). \qquad (2.34)$$

CHAPTER 3

TWO–DIMENSIONAL CUBICALLY SUPERINTEGRABLE SYSTEMS

3.1 Preliminaries:

Integrable systems demand a set of independent functions such as $(H, F_1, F_2 \ldots., F_{n-1})$ that they are all in involution to the Poisson bracket $\{\ \cdot\ ,\ \cdot\ \}$, on the cotangent bundle T^*M of a n dimensional manifold M. The superintegrable systems are obtained by a set of independent functions observables such as $H, F_1, F_2 \ldots., F_{\gamma-1}$, where $\gamma \geq n$ and satisfying [1,2,10]

$$\{ H, F_i\} = 0, \quad \text{to each} \quad i = 1,2, \ldots.., \gamma - 1. \tag{3.1}$$

The maximal value of γ is $2n - 1$ because in the system (3.1) $dH(X_{F_i}) = 0$, meaning that the span of the Hamiltonian vector fields, X_{F_i}, is at every point of cotangent bundle T^*M, subspace of the annihilator of the 1-form dH, the second one having the $2n - 1$. It is noted that in two-dimensional manifolds, a superintegrable system is maximal because $\gamma = 3$. According to [11] the huge quantity of outcomes for the models of superintegrable systems is constrained to quadratically superintegrable ones, implies that, the integrals F_i in the momenta are either quadratic or linear, and the metrics for all these systems are either constant curvature or flat. On surfaces of revolution MSh gave a full classification of all (local) Riemannian metrics [12] that is

$$G = \frac{dx^2+dy^2}{h_x^2}, \quad h = h(x), \quad h_x = \frac{dh}{dx} \tag{3.2}$$

that possess a superintegrable geodesic flow, having integral $L = P_y$ and S are linear and cubic integral in momenta. In [12] it was emphasized that whether the metric G does not belong to the constant curvature, thus $\ell^{\,3}(G)$, the span linear of the cubic integral, here possess four dimensional with L^3, LH, S_1, S_2 as a natural footing also with the following constructer. Here $\mathcal{L}: S \to \{L, S\}$ is denoting the endomorphism linear that of $\ell^3(g)$ [10] such that:

(i) S_1, S_2 are the coinciding eigenvectors when $\mathcal{L}$ has real eigenvalues $\pm\mu$ to the several real $\mu > 0$, or [10]

(ii) $S_1 \pm iS_2$ are the coinciding eigenvectors when $\mathcal{L}$ has imaginary eigenvalues $\pm i\mu$ to the several real $\mu > 0$, or [10]

(iii) The eigenvalue which is $\mu = 0$ also with 1- Jordan block of size three are in $\mathcal{L}$, in this state $\{L, S_1\} = \frac{A_3}{2} + A_1 LH,\ \{L, S_2\} = S_1,$ for A_1, A_3 that are real constants.

Then the superintegrability is satisfied if the function h becomes a solution of the next non-linear 1^{st} order differential equation, such that [10]

a) $h_x(A_0 h_x^2 + \mu^2 A_0 h^2 - A_1 h + A_2) = A_3 \frac{\sin(\mu x)}{\mu} + A_4 \cos(\mu x).$

b) $h_x(A_0 h_x^2 + \mu^2 A_0 h^2 - A_1 h + A_2) = A_3 \frac{\sinh(\mu x)}{\mu} + A_4 \cosh(\mu x).$ (3.3)

c) $h_x(A_0 h_x^2 - A_1 h + A_2) = A_3 x + A_4.$

Through all of these cases the explicit formula had given of the cubical integrals. For example, when $\mu = 1$ or $\mu = i$, their form are written as [10]

$$S_{1,2} = e^{\pm\mu y}\left(a_0(x)P_x^3 + a_1(x)P_x^2 P_y + a_2(x)P_x P_y^2 + a_3(x)P_y^3\right). \tag{3.4}$$

$a_i(x)$ are expressed explicitly through the terms of h and also through its derivatives [12]. These equations for $A_0 = 0$ are simply integrated and we get the metrics of Koenigs [11]. The cubic integrals possess the reducible constructer $S_{1,2} = P_y Q_{1,2}$ such that $Q_{1,2}$ are exactly those provided by Koenigs. It also proved that in the state (b), under the restrictions [10]

$$\mu > 0,\ A_0 > 0, \mu A_4 > |A_3|, \tag{3.5}$$

the metric together with the cubic integrals are real-analytic and globally defined on $\mathbb{S}^2$ [10].

Theorem 1 [10]: The metric $G = \rho^2 \frac{dv^2}{D} + \frac{4D}{P} d\emptyset^2$, $v \in (a, 1)$, $\emptyset \in \mathbb{S}^1$ such that

$$D = (v-a)(1-v^2),\ P = (v^2 - 2av + 1)^2, \quad -\rho = 1 + 4\frac{(v-a)D}{P}, \tag{3.6}$$

is globally defined on $\mathbb{S}^2$.The Hamiltonian is given by

$$H = \frac{1}{2} G^{ij} P_i P_j = \frac{1}{2}\left(\Pi^2 + \frac{P}{4D} P_\phi^2\right), \Pi = \frac{\sqrt{D}}{\rho} P_v,\ \text{iff} \quad a \in (-1, +1). \tag{3.7}$$

The S_1 and S_2 cubic integrals, are globally defined on $\mathbb{S}^2$, namely

$$S_1 = \cos\phi \mathcal{A} + \sin\phi \mathfrak{B}, \quad S_2 = -\sin\phi \mathcal{A} + \cos\phi \mathfrak{B}, \tag{3.8}$$

$$\mathcal{A} = \Pi^3 - f\ddot{f}\Pi P_\phi^3,\ \mathfrak{B} = \dot{f}\Pi^2 P_\phi - f\left(1 + \dot{f}\ddot{f}\right)P_\phi^3,\ f = \sqrt{D}. \tag{3.9}$$

Theorem 2 [10]: Consider the metric G

$$G = \rho^2 \frac{dv^2}{D} + \frac{4D}{P} d\emptyset^2 \ ,\ \rho = \frac{Q}{P},\ x \in (-1, +1),\ \emptyset \in \mathbb{S}^1, \tag{3.10}$$

such that

$$\begin{cases} D = (x+m)(1-x^2) \\ P = \left(L_+(1-x^2) + (m+x)\right)\left(L_-(1-x^2) + 2(m+x)\right), L_\pm = \mathfrak{l} \pm \sqrt{\mathfrak{l}^2 - 1}, \\ Q = 3x^4 + 4mx^3 - 6x^2 - 12mx - 4m^2 - 1 \end{cases} \tag{3.11}$$

and

$$H = \frac{1}{2} G^{ij} P_i P_j = \frac{1}{2}\left(\Pi^2 + \frac{P}{4D} P_\phi^2\right), \Pi = \frac{\sqrt{D}}{\rho} P_x,\ \text{iff} \quad m > 1,\ \text{and} \quad \mathfrak{l} > -1.$$

The cubic integrals S_1 and S_2 given by (3.8) and (3.9), are globally defined on $\mathbb{S}^2$ too [10].

3.2. The trigonometric states

3.2.1. The explicit formula of the metric

The equation (3.3) (a) that got from [12] is

$$h_x(A_0 h_x^2 + \mu^2 A_0 h^2 - A_1 h + A_2) = A_3 \frac{\sin(\mu x)}{\mu} + A_4 \cos(\mu x). \tag{3.12}$$

For the Koenigs metrics $A_0 = 0$. Through a scaling of x we can put $\mu = 1$. Through a scaling of h and a translation of x the right hand side will be $\lambda\, sinx$, with a real free parameter λ. Through a translation of h, one can put $A_2 = 0$ and $A_2 = a$. Therefore, we should solve [10]

$$h_x(h_x^2 + h^2 + a\ ) = \lambda \sin(x), a \in \mathbb{R},\ \mathbb{R} \setminus \{0\}. \tag{3.13}$$

We consider now $u = h_x$ like a function of h and we denote

$$U = u(u^2 + h^2 + a) = 0\ with\ \frac{d^2U}{dx^2} + U = 0. \tag{3.14}$$

Thus we conclude that [10]

$$\frac{d}{dh}\left(u\frac{dU}{dh}\right) + u^2 + h^2 + a = 0, \qquad a \in \mathbb{R}, \tag{3.15}$$

also can be integrated, is gotten

$$4hu\frac{dU}{dh} = c + (u^2 + h^2 + a)(3u^2 - h^2 - a). \tag{3.16}$$

Due to the fact that $U = \lambda \sin(x)$, the 1^{st} order equation is found as [10]

$$\dot{U}^2 = \lambda^2 - U^2 \text{ which implies that } \left(4hu\frac{dU}{dh}\right)^2 = 16h^2(\lambda^2 - U^2). \tag{3.17}$$

By using (3.16) the quartic equation for u become

$$[c + (u^2 + h^2 + a)(3u^2 - h^2 - a)]^2 = 16h^2[\lambda^2 - u^2(u^2 + h^2 + a)]. \tag{3.18}$$

For $\upsilon = u^2 + h^2$, this equation is still a quartic but in h^2 it happens to become a linear one. In terms of υ by solving for h^2, we conclude that [10]

$$v = u^2 + h^2, \quad h^2 = \frac{\dot{D}^2}{8D}, \; D(v) = (v+a)(\; v^2 - \; a^2 + c) + 2\lambda^2. \quad (3.19)$$

Now, we define [10]

$$f = \sqrt{D} = \sqrt{(v+a)(\; v^2 - \; a^2 + c) + 2\lambda^2} \;\; and \;\; g = 2v - \dot{f}^2, \qquad (3.20)$$

such that $\dot{f} = \frac{df}{dv}$. In the coordinates $(\,x, y)$ the metric becomes [10]

$$\frac{1}{2} G = \frac{1}{2h_x^2}(dx^2 + dy^2) = (\frac{\ddot{f}}{g})^2 dv^2 + \frac{dy^2}{g} \qquad (3.21)$$

which implies the Hamiltonian as

$$H \equiv G^{ij} P_i P_j = \frac{1}{2}\left(\Pi^2 + g P_y^2\right), \;\; \Pi = \frac{g}{\ddot{f}} P_v. \qquad (3.22)$$

3.2.2. The case of the cubic integrals

They are presented in (3.4), and quoted from [12], namely

$$S_{\pm} = e^{\pm y}\left(\Pi^3 \pm \dot{f}\Pi^2 P_y + f\ddot{f}\Pi P_y^2 \pm f\left(1 - \dot{f}\ddot{f}\right) P_y^3\right). \qquad (3.23)$$

Due to the fact that [10]

$$dH \wedge dP_y \wedge dS_+ \wedge dS_- = 0,$$

the four observables are not dependent . We have [10]

$$S_+ S_- = \; 8\,\mathrm{H}^3 \; + \; 8\,a\mathrm{H}^2 P_y^2 \; + 2\,cHP_y^4 \; - 2\,\lambda^2 P_y^6. \qquad (3.24)$$

So, we consider two different superintegrable systems, namely [10]

$$\ell_+ = \left(H, P_y, S_+\right) \;\; and \;\; \ell_- = \left(H, P_y, S_-\right), \qquad (3.25)$$

respectively.

Proposition 1 [10]: S_+ and S_- denote the integrals and the set $\left(H, P_y, S_+, S_-\right)$ generates a Poisson algebra.

Proof [10]: We have

$$\{H, S_\pm\} = e^{\pm y}\, \frac{g}{\ddot{f}}\, \Pi\, P_y^2\big(\Pi \pm \dot{f} P_y\big)\Big(f\dddot{f} - 3\big(1 - \dot{f}\ddot{f}\big)\Big). \tag{3.26}$$

The following equation

$$\Big(f\dddot{f} - 3\ \big(1 - \dot{f}\ddot{f}\big)\Big) = 0 \tag{3.27}$$

does linearize upon the substitution $f = \sqrt{D}$ due to the fact

$$2\left(f\dddot{f} - 3\left(1 - \dot{f}\ddot{f}\right)\right) = \dddot{D} - 6 = 0, \tag{3.28}$$

which provides for D the most general monic polynomial of degree third

$$D(v) = v^3 - S_1 v^2 + S_2 v - S_3, \tag{3.29}$$

The function D from (3.19) displays exactly three parameters, namely a, c, λ. In (3.26) and (3.27) insure then conservation of both cubic integrals S_+ and S_-.
In addition we have

$$\{S_+, S_-\} = -16aH^2P_y - 8cHP_y^3 + 12\lambda^2 P_y^5, \tag{3.30}$$

$$S_+S_- = 8\,H^3 + 8\,a\,H^2P_y^2 + 2\,c\,HP_y^4 + 2\,\lambda^2 P_y^6, \tag{3.31}$$

and it is generated by four observables.

3.2.3. The metric transformation and its curvature

Through the expression (3.29) for variable D, we define the quartic polynomials Q and P as

$$P = 8vD - \dot{D}^2, \;\; Q = 2D\ddot{D} - \dot{D}^2 = P + 4(v - S_1)D, \quad \dot{Q} = 12D. \tag{3.32}$$

After that the metric (3.21) can be written in the formula [10]

$$\frac{1}{2}G = \rho^2\frac{dv^2}{D} + \frac{4D}{P}dy^2, \qquad \rho \equiv \frac{Q}{P} = 1 + (v - S_1)\frac{4D}{P} \tag{3.33}$$

such that the scalar curvature existence has the following form [10]

$$R_G = \frac{1}{4Q^3}\big(2PQ\dot{W} - \big(Q\dot{P} + 2P\dot{Q}\big)W\big), \quad W \equiv D\dot{P} - P\dot{D} = 8D^2 - Q\dot{D}. \tag{3.34}$$

The following conditions are imposed, namely [10]:

1. $\upsilon = u^2 + h^2$ implies $\upsilon > 0$,
2. h to be real implies $D > 0$,
3. The metric G requires $P > 0$, to be Riemannian [10].

3.2.4. Global characteristics

We start with a metric [10]

$$G = dr^2 + r^2 d\emptyset^2, \quad r \in (0, +\infty), \quad \emptyset \in \mathbb{S}^1, \tag{3.35}$$

thus $r = 0$ is considered as an obvious singularity that can be erased, using the back coordinates of Cartesian [10].

Lemma 1 [10]: Let us consider the interval $I = (a, b)$ such that $D(\upsilon) > 0$ and $P(\upsilon) > 0$ to all $\upsilon \in I$. Assume that Q possess a simple real zero $\upsilon_* \in I$; therefore $\upsilon = \upsilon_*$ denotes a curvature singularity precluding any manifold construction related with G.

Proof [10]: From (3.34) we have that

$$\lim_{\upsilon \to \upsilon_*} Q^3(\upsilon)\, R_G(\upsilon) = -4P(\upsilon_*)D^2(\upsilon_*)\dot{Q}(\upsilon_*) \tag{3.36}$$

and the right hand side of the equation (3.30) is different from zero. The presence of a curvature singularity for $\upsilon_* \in I$ rules out the probability of a manifold construction.

Lemma 2 [10]: If υ takes its values in $I = (a, b)$ and if one of the end-points is a zero of P (and not of Q), then the manifold has infinite measure and it cannot be closed.

Proof [10]: We consider the allowed interval for υ be denoted by $I = (a, b)$. Then we have

$$\mu_G = 4\int_a^b \frac{Q(v)}{P^{2/3}(v)}dv \int dy. \tag{3.37}$$

Now, if P possess a zero at one end-point where Q does not vanish, then this integral diverges. Given any polynomial P we utilize the notation $\Delta(P)$ for its discriminant.

Proposition 2 [10]: If $\Delta(D) = 0$ the superintegrable systems ℓ_+ and ℓ_- given by (3.25) are either trivial or are not defined on a closed manifold.

Proof [10]: If $\Delta(D) = 0$, it implies that $D = (v - v_0)^3$. The scalar curvature from (3.34) is a constant. The theorem from [15] explains that for Riemannian spaces of constant curvature, namely $\mathbb{S}^n, \mathbb{R}^n, \mathbb{H}^n$ with $n \geq 2$, each Stäckel-Killing tensor of any degree is completely reducible to symmetrized tensor products of KV. It means that the cubic integrals here are reducible. For $\Delta(D) = 0$ it might also have $D = (v - v_0)(v - v_1)^2$ with $v_0 \neq v_1$, that yields [10]

$$\begin{cases} P(v) = -(v - v_1)^2 p(v),\ p(v) = v^2 - 2(2v_0 + 3v_1)v + (2v_0 + v_1)^2, \\ Q = 3(v - v_1)^3(v - v_*), \quad v_* = v_0 + \frac{v_0 - v_1}{3}\ . \end{cases} \tag{3.38}$$

Firstly notice that for the metric G [10]

$$\frac{1}{2}\,G \;=\; \frac{g(v-v_*)^2}{p(v)^2}\frac{dv^2}{(v-v_0)} \;+\; \frac{4(v-v_0)}{(-p(v))}\ dy^2, \tag{3.39}$$

to be Riemannian it should obey $v > v_0$ and $p(v) < 0$. If the roots $\omega_\pm$ of p are ordered as $\omega_- < \omega_+$, the positivity of the metric is achieved iff [10]

$$v \in I \;=\; (v_0, +\infty) \cap (\omega_-, \omega_+),$$

the upper bound of I being ω_+. Since $P(\omega_+) = 0$ and $Q(\omega_+) \neq 0$ the expected manifold cannot be closed utilizing the Lemma 2 [10].

Proposition 3 [10]: The superintegrable systems ℓ_+ and ℓ_- given by (3.25) are never globally defined on closed manifold provided that $\Delta(D) = 0$.

Proof [10]: If $\Delta(D) = 0$ the polynomial D possess just a simple real zero. Utilizing (a, b) as new parameters it can be written

$$D = (v - v_0)((v-a)^2 + b^2), \quad v \in (v_0, +\infty), \quad a \in \mathbb{R}, \quad b \in \mathbb{R} \setminus \{0\}, \tag{3.40}$$

such that

$$\Delta(D) = -4b^2((v_0 - a)^2 + b^2)^2. \tag{3.41}$$

For P and Q [10] we have

$$\begin{gathered} \Delta(P) = 16384a^2((v_0 + a)^2 + b^2)^2 \Delta(D), \\ \Delta(Q) = 27648b^2((v_0 - a)^2 + b^2)^2 \Delta(D). \end{gathered} \tag{3.42}$$

It must exclude $a = 0$ due to the fact that $P(v) = -(v^2 - 2v_0 v - b^2)^2$ is non-positive. Then, the former discriminants are precisely non-positive, meaning that both polynomials P and Q possess two simple real zeros. The relation $\dot{Q} = 12D$ shows that Q is increasing, namely [10]

$$Q(v_0) = -[((v_0 - a)^2 + b^2)^2] \text{ to } Q(+\infty) = +\infty. \tag{3.43}$$

Hence there exists a simple zero v_* of Q such that $v_* > v_0$ while the other one lies to the left of v_0 because $(-\infty) = +\infty$ .The polynomial P is [10]

$$P(v) = -(v^2 - 2(v_0 + 2a)v - a^2 - b^2 - 2av_0)^2 + 16a((v_0 + a)^2 + b^2\,)\, v, \tag{3.44}$$

reviling that for $a < 0$ it is never non-negative as it should; so, it will be left with the case $a > 0$. Utilizing

$$P(v) = Q(v) + 4(v_0 + 2a - v)D(v), \ \dot{P}(v) = 8D(v) + 4(2a + v_0 - v)\dot{D}(v), \tag{3.45}$$

it shows that $P(v_0)$ is precisely non-positive and that $\dot{P}(v)$ is non-negative from $v = v_0$ to $v = v_0 + 2a$. As a result P increases to its first zero $v = \omega_- < v_*$, (since $P(v_*) = 4(2a + v_0 - v_*)D(v_*) > 0$), is equal to Q for [10]

$$v = v_0 + 2a > v_*. \tag{3.46}$$

Then vanishes at its second zero ω_+ such that $\omega_+ > v_0 + 2a$ and finally, decreases to $-\infty$. Therefore, we end up with $v_0 < \omega_- < v_* < v_0 + 2a < \omega_+$.

So, $D > 0$ and $P > 0$ iff $\upsilon \in (\omega_-, \omega_+)$, so within this interval Q possess a simple zero for $\upsilon = \upsilon_*$. Then, there is no underlying manifold construction by Lemma1 [10].

Proposition 4 [10]: If $\Delta(D) = 0$, then the superintegrable systems ℓ_+ and ℓ_- provided by (3.25), on a closed manifold, are never globally defined.

Proof [10]: Let the roots of D be ordered according to $0 \leq \upsilon_0 < \upsilon_1 < \upsilon_2$. Thus

$$D = (\upsilon - \upsilon_0)(\upsilon - \upsilon_1)(\upsilon - \upsilon_2) = \upsilon^3 - S_1\upsilon^2 + S_2\upsilon - S_3, \text{ and } D > 0 \quad (3.47)$$

for

$$\upsilon \in (\upsilon_0, \upsilon) \cup (\upsilon_2, +\infty). \quad (3.48)$$

the positivity interval for *P* should be determined now. But [10]

$$\Delta(P) = 4096\sigma^2\Delta(D) > 0, \sigma = (\upsilon_0 + \upsilon_1)(\upsilon_1 + \upsilon_2)(\upsilon_2 + \upsilon_0) > 0 \quad (3.49)$$

thus, there will be either no real root or four real simple roots for *P*. The final is excluded since $P = 8\upsilon D - (\dot{D})^2$ at the zeros of D is non-positive, and non-negative at those of $\dot{D}$. Observing that $\Delta(Q) = -6912\Delta^2 < 0$ implies that Q possess two simple real roots and $\upsilon_* > \upsilon_2$ is one of them. This is so due to the fact that [10]

$$Q(\upsilon) = P(\upsilon) + 4(\upsilon - S_1)D(\upsilon), \quad (3.50)$$

which illustrates that

$$Q(\upsilon_2) = P(\upsilon_2) = -(\upsilon_0 - \upsilon_2)^2(\upsilon_1 - \upsilon_2)^2 < 0, \text{ but } \dot{Q} = 12D \quad (3.51)$$

entails that, for positive D, the function Q is increasing with $Q(+\infty) = +\infty$. Hence $\upsilon = \upsilon_*$ is a simple zero of Q and any manifold construction is prohibiting via Lemma 1. Just when $D > 0$ the zeros of P might appear.

Let us consider $\upsilon \in (\upsilon_0, \upsilon_1)$. We notice that [10]

$$P(\upsilon_0) = -(\upsilon_0 - \upsilon_1)^2(\upsilon_0 - \upsilon_2)^2 \text{ and } P(\upsilon_1) = -(\upsilon_1 - \upsilon_0)^2(\upsilon_1 - \upsilon_2)^2 \quad (3.52)$$

are negative and that there does exist $\upsilon = \upsilon{-} \in (\upsilon_0, \upsilon_1)$. For $\dot{D}(\upsilon -) = 0$ it will imply that $P(\upsilon-) > 0$

$$\upsilon_0 < \omega_0 < \upsilon{-} < \omega_1 < \upsilon_1,$$

where the 1^{st} pair of simple zeros of P is (ω_0, ω_1). Positivity of both D and P is therefore reported for $\upsilon\epsilon(\omega_0, \omega_1)$. The function Q remains strictly negative for $\upsilon\epsilon[\upsilon_0, \upsilon_1]$, and it can conclude by Lemma 2 that the assumed manifold cannot be closed. Then the remaining two zeros of P defined via $\omega_2 < \omega_3$ must lie in $(\upsilon_2, +\infty)$. Since [10]

$$Q(\upsilon_2) = -(\upsilon_2 - \upsilon_0)^2(\upsilon_2 - \upsilon_1)^2 < 0 \tag{3.53}$$

and then it increases to $Q(+\infty) = +\infty$, thus it will have a simple zero $\upsilon = \upsilon_* > \upsilon_2$, then at this point we have $P(\upsilon_*) = 4(S_1 - \upsilon_*)D(\upsilon_*)$. Let us discuss the following cases [10]:

a) If $\upsilon_* < S_1$, it implies that $P(\upsilon_*) > 0$, and using the fact that $P(+\infty) = -\infty$ we obtain

$$\upsilon_2 < \omega_2 < \upsilon_* < \omega_3.$$

b) If $\upsilon_* \geq S_1$, it implies that $P(P(\upsilon_*) < 0$, hence $\upsilon_2 < \upsilon_* < \omega_2 < \omega_3$, and the positivity of D and P requires $\upsilon \in (\omega_2, \omega_3)$. Since $Q(\omega_3) > 0$ the assumed manifold cannot be closed via Lemma 2 [10].

3.3. The hyperbolic state

3.3.1. The metric explicit formula

The equation (3.3) (b) has the following form [12]

$$h_x(A_0 h_x^2 - \mu^2 A_0 h^2 - A_1 h + A_2) = A_3 \frac{\sinh(\mu x)}{\mu} + A_4 \cosh(\mu x). \tag{3.54}$$

It might again put $A_0 = 1$, $\mu = 1$, $A_1 = 0$, $A_2 = -a$. Thus, the right hand side of the preceding equation will lead to three different states described according to

$$h_x(h_x^2 - h^2 - a) = \frac{\lambda}{2}(e^x + \epsilon e^{-x}), \quad \epsilon = 0, \pm 1. \tag{3.55}$$

Here λ is a free parameter. For $\epsilon = 0$, the shifts $x \to -x$ and $\lambda \to -\lambda$ reveal that there is no require to regard e^{-x} in (3.48). By the following definitions [10]

$$u = h_x, \quad U = u(u^2 - h^2 - a), \quad a \in \mathbb{R}, \tag{3.56}$$

we have

$$\ddot{U} - U = 0 \Rightarrow \frac{d}{dh}\left(u\frac{dU}{dh}\right)(u^2 - h^2 - a) = 0. \tag{3.57}$$

By integration it implies that [10]

$$4hu\frac{dU}{dh} = c + (u^2 - h^2 - a)(3u^2 + h^2 + a), \quad c \in \mathbb{R}. \tag{3.58}$$

When $U = \frac{\lambda}{2}(e^x + \epsilon e^{-x})$ the related first order equation becomes [10]

$$\dot{U}^2 = U^2 - \epsilon\lambda^2 \text{ which implies that } \left(4hu\frac{dU}{dh}\right)^2 = 16h^2(U^2 - \epsilon\lambda^2). \tag{3.59}$$

Supposing that $v = h^2 - u^2$, we have [10]

$$v = u^2 - h^2, \quad h^2 = \frac{\dot{D}^2}{8D}, \quad D(v) = (a - v)(v^2 - a^2 + c) - 2\epsilon\lambda^2 \tag{3.60}$$

It gives an interesting result that is similar to the state (a), unless the variable v needs to be not be positive. Defining [10]

$$f = \sqrt{D} = \sqrt{(a - v)(v^2 - a^2 + c) - 2\epsilon\lambda^2} \text{ and } g = \dot{f}^2 + 2v, \tag{3.61}$$

the following metric is reported, namely[10]

$$\frac{1}{2}G = \frac{1}{2h_x^2}(dx^2 + dy^2) = \left(\frac{\ddot{f}}{g}\right)^2 dv^2 + \frac{dy^2}{g}. \tag{3.62}$$

Then corresponding Hamiltonian becomes [10]

$$H \equiv G^{ij} P_i P_j = \frac{1}{2}\left(\Pi^2 + g P_y^2\right), \quad \Pi = \frac{g}{\ddot{f}} P_v. \tag{3.63}$$

3.3.2. The cubic integrals

Let us start with [10]

$$S_1 = \cos y\,\mathcal{A} + \sin y\,\mathfrak{B},\ S_2 = -\sin y\,\mathcal{A} + \cos y\,\mathfrak{B}, \qquad (3.64)$$

such that

$$\mathcal{A} = \Pi^3 - f\dddot{f}\Pi P_y^2, \quad \mathfrak{B} = \dot{f}\Pi^2 P_y - f\big(1 + \dot{f}\ddot{f}\big)P_y^3. \qquad (3.65)$$

Proposition 5 [10]: The observables S_1 and S_2 are integrals of the geodesic flow.

Proof [10]: The complex object can be defined below, namely

$$\mathcal{s} = S_1 + i\,S_2 = \mathrm{e}^{-iy}(\mathcal{A} + i\,\mathfrak{B})\,. \qquad (3.66)$$

Thus, the Poisson brackets are

$$\{\mathrm{H}, \mathcal{s}\} = -\mathrm{e}^{-iy}\frac{g}{\ddot{f}}\,\Pi\,P_y^2\big(\Pi + i\dot{f}P_y\big)\left(f\dddot{f} + 3\,\big(1 + \dot{f}\ddot{f}\big)\right). \qquad (3.67)$$

The transformation $f = \sqrt{D}$ gives the next linearization, namely [10]

$$2\left(f\dddot{f} + 3\big(1 + \dot{f}\ddot{f}\big)\right) = \dddot{D} + 6 = 0$$

which implies $$D = -(v^3 - S_1 v^2 + S_2 v - S_3). \qquad (3.68)$$

From (3.61) and (3.67) $\mathcal{s}$ is an integral. Since $dH \wedge dP_y \wedge dS_1 \wedge dS_2 = 0$, it reveals that these four observables which are not independent functions. We notice that [10]

$$S_1^2 + S_2^2 = \mathcal{A}^2 + \mathfrak{B}^2 = 8\mathrm{H}^3 + 8a\mathrm{H}^2 P_y^2 + 2cHP_y^4 - 2\epsilon\lambda^2 P_y^6, \qquad (3.69)$$

which lead us to two various superintegrable systems,

$$\ell_1 = \big(H, P_y, S_1\big)\text{and}\ \ell_2 = \big(H, P_y, S_2\big). \qquad (3.70)$$

Also we have [10]

$$\{S_1, S_2\} = -\,8\,a\,H^2 P_y - 4\,c\,HP_y^3 + 6\,\epsilon\,\lambda^2 P_y^5, \qquad (3.71)$$

as in (3.62) for $S_1^2 + S_2^2$, but it is not satisfied for the product [10]

$$S_1\, S_2 = \cos(2y)\mathcal{A}\,\mathfrak{B} + \sin(2y\,)\frac{\mathfrak{B}^2 - \mathcal{A}^2}{2}, \tag{3.72}$$

which represent a new, independent, observable. (H, P_y, S_1, S_2) of 1^{st} integrals of the geodesic flow clearly does not generate a Poisson algebra [10].

3.3.3. The curvature and metric transformation

Using (3.68) for D, the related polynomials are [10]

$$P = 8\upsilon D + \dot{D}^2, \quad Q = 2D\ddot{D} - \dot{D}^2 = -P - 4(\upsilon - S_1)D, \quad \dot{Q} = -12D, \tag{3.73}$$

which lead us to the metric [10]

$$\frac{1}{2}G = \rho^2\frac{d\upsilon^2}{D} + \frac{4D}{P}dy^2, \quad -\rho \equiv -\frac{Q}{P} = 1 + (\upsilon - S_1)\frac{4D}{P}, \tag{3.74}$$

with the constraints $D > 0$ and $P > 0$ which include its Riemannian signature. The related scalar curvature becomes [10]

$$R_G = \frac{1}{4Q^3}\left(2PQ\dot{W} - \left(Q\dot{P} + 2P\dot{Q}\right)W\right), \quad W \equiv D\dot{P} - P\dot{D} = 8D^2 - Q\dot{D}. \tag{3.75}$$

Lemma 3 [10]: Assume $I = (-\infty, \upsilon_0)$ be the admitted interval to $\upsilon$when υ_0 is a simple zero of D. If for all $\upsilon \in I$ one has $P(\upsilon) > 0$ and $Q(\upsilon) > 0$, then the metric G reveals a conical singularity which precludes any manifold construction.

Proof [10]: With the help of (3.66), when $\upsilon \to \upsilon_0 +$ the metric approximates as given below, namely

$$\frac{1}{2}G \approx \frac{4}{\dot{D}(\upsilon_0)}(dr^2 + r^2dy^2), \quad r = \sqrt{\upsilon - \upsilon_0} \to 0+. \tag{3.76}$$

Thus, for this singularity to be apparent, assuming $y = \emptyset \in \mathbb{S}^1, \upsilon \to -\infty$ it will get [10]

$$\frac{1}{2}G \approx dr^2 + r^2\left(\frac{d\emptyset}{3}\right)^2, \qquad r = \frac{1}{\sqrt{-\upsilon}} \to 0+, \tag{3.77}$$

and it cannot have $\frac{\emptyset}{3} \in \mathbb{S}^1$. This type of singularity is called conical [10].

Lemma 4 [10]: Suppose that the metric G

$$G = A(v)dv^2 + B(v)d\phi^2,\ v \in I = [a,b],\quad \emptyset \in \mathbb{S}^1, \tag{3.78}$$

on a closed manifold is globally defined. Thus, its Euler property is given as

$$\chi\,(M) = \gamma(b) - \gamma(a),\qquad \gamma = -\frac{\dot{B}}{2\sqrt{AB}}. \tag{3.79}$$

Proof [10]: Utilizing the orthonormal frame $e_1 = \sqrt{Adv}$, $e_2 = \sqrt{Bd\phi}$, then the connection 1-formula reads $\omega_{12} = \frac{\gamma}{\sqrt{B}}\, e_2$, where γ is as in (3.73). The curvature 2-formula is $R_{12} = d\omega_{12} = \frac{\dot{\gamma}}{\sqrt{AB}} e_1 \wedge e_2$, from which it will get

$$\chi(M) = \frac{1}{2\pi}\int_M R_{12} = \int_I \dot{\gamma}(v)\ dv = \gamma(b) - \gamma(a). \tag{3.80}$$

3.3.4. The global construct for $\epsilon = 0$

3.3.4.1. First state: $\Delta(D) = 0$

Proposition 6 [10]: For $a \neq 0$ and $c = 0$there exists no closed manifold.

Proof [10]: In this state we have

$$D(v) = (a-v)(v^2-a^2),\qquad P(v) = (v-a)^4, \tag{3.81}$$

$$Q(v) = 3(v-a)^3(v-v_*),\qquad v_* = -\frac{5}{3}\,a, \tag{3.82}$$

then the metric G becomes [10]

$$\frac{1}{2}G = 9\,\frac{(v-v_*)^2}{(a-v)^4}\frac{dv^2}{-a-v} + \frac{4}{3}\frac{(a-v)}{(v-v_*)}dy^2. \tag{3.83}$$

For $a > 0$ the $D > 0$ and $P > 0$ iff $v \in I = (-\infty, -a)$ but since $v_* \in I$ it will obtain no manifold construction via Lemma 2 . For $a < 0$ the positivity of metric

G is investigated for $\upsilon \in (-\infty, -a) \cap (a, -a)$. During this two states, a is a zero for P but the Lemma 2 cannot be used because $Q(a) = 0$. Indeed, the measure of the sought manifold [10]

$$\mu_G = 12 \int \frac{(\upsilon - \upsilon_*)}{(\upsilon - a)^3} d\upsilon \int dy, \tag{3.84}$$

is divergent forbidding a closed manifold [10].

Proposition 7 [10]: For $c = a^2 > 0$ there exists no closed manifold.

Proof : See [10].

3.3.4.2. The 2^{nd} state $\Delta(\boldsymbol{D}) < 0$

We consider the following expressions, namely:

$$D = (a - \upsilon)(\upsilon^2 + c - a^2), \; c > a^2,$$
$$\mathrm{P} = (\upsilon - \omega_-)^2(\upsilon - \omega_+)^2,$$
$$\omega_\pm = a \pm \sqrt{c}\,, \tag{3.85}$$
$$Q = -P + 4(a - \upsilon)D, \quad \dot{Q} = -12D. \tag{3.86}$$

Proposition 8 [10]: For $\Delta(D) < 0$ there exists no closed manifold.

Proof [10]: The positivity of D and P exist for any $\upsilon \in (-\infty, \omega_-) \cup (\omega_-, a)$. We conclude that

$$Q(\omega_-) = 8c^{3/2}(\sqrt{c} - a) > 0, \quad Q(a) = -c^2 < 0, \tag{3.87}$$

means that inside the interval $(\omega_-, a), Q$ admits a simple zero. This implies a curvature singularity . This never happens for $\upsilon \in (-\infty, \omega_-)$ since $Q(\upsilon) > 0$. However ω_- is a zero of P together with $Q(\omega_-) > 0$ concluded by Lemma 2 [10].

3.3.4.3. The state $\Delta(D) > 0$

For $c < a^2$, it shows that [10]

$$D(v) = (a - v)(v^2 - {v_0}^2), \qquad v_0 = \sqrt{a^2 - c}\,,$$
$$P(v) = ((v - a)^2 - c)^2, \quad Q(v) = -P(v) + 4(a - v)D(v), \tag{3.88}$$

Through this set $(-\infty, 0) \cup \{0\} \cup (0, a^2)$, c takes its values.

Theorem 3[10]: The superintegrable systems ℓ_1 and ℓ_2 that given in (3.70) are globally defined on $\mathbb{S}^2$ if $c \in (-\infty, 0)$.

Proof [10]: Firstly we consider $P > 0$. Then $-v_0 < a < v_0$ is the ordering of the zeros of D. Means that two possible intervals that make its positivity be ensuring: either $v \in (-\infty, -v_0)$ or $v \in (a, v_0)$. The 1^{st} state is readily excluded since Qdecreases from $Q(-\infty) = +\infty$ to $Q(-v_0) = -P(-v_0) < 0$, thus it will be vanishing in the interval and will lead to a curvature singularity. So we consider $v \in (a, v_0)$.

Then $Q(a) = -P(a) = -c^2$ is non-positive, and due to the fact that Q is decreasing it will stay precisely non-positive anywhere on the interval. We conclude that [10]

$$G = \rho^2 \frac{dv^2}{(v-a)(1-v^2)} + 4\frac{(v-a)(1-v^2)}{(v^2-2av+1)^2} d\phi^2,\ v \in (a, 1),\ \emptyset \in \mathbb{S}^1, \tag{3.89}$$

where

$$a \in (-1,1), \quad -\rho = 1 + 4\frac{\left((v-a)\right)^2(1-v^2)}{(v^2-2av+1)^2}. \tag{3.90}$$

Both these end-points are obvious singularities due to the fact that

$$G(v \to 1-) \sim \frac{2}{1-a}(dr^2 + r^2 d\emptyset^2), \quad r = \sqrt{1-v}, \tag{3.91}$$

and

$$G(v \to a+) \sim \frac{4}{1-a^2}(dr^2 + r^2 d\emptyset^2), \quad r = \sqrt{v-a}. \tag{3.92}$$

Calculating the Euler characteristic. By resorting to Lemma 4, it will get [10]

$$\gamma(\upsilon)=\frac{(1-\upsilon^2)^2-4(\upsilon-a)^2}{Q(\upsilon)}\Rightarrow\chi(M)=\gamma(1)-\gamma(a)=2, \tag{3.93}$$

and this emphasis that the manifold is diffeomorphic to $\mathbb{S}^2$. Thus, the measure of this surface is given by

$$\mu_G\,(\mathbb{S}^2) \;=\; \frac{4\pi}{1+a}. \tag{3.94}$$

As a next step we discuss the global state of the integrals H, P_y, S_1, S_2. Using (3.88), so by referring to the theorem of Riemann uniformization, it ends up with [10]

$$H=\frac{1}{2}\left(\Pi^2+\;P\frac{P_\phi^2}{4D}\right)=\frac{1}{2\Omega^2}\left(P_\theta^2+\frac{P_\phi^2}{sin^2\theta}\right), \tag{3.95}$$

such that

$$t\equiv\tan\frac{\theta}{2}=\sqrt{\frac{(\upsilon-a)P}{(1-\upsilon^2)}}\;,\quad \Omega=\frac{(1-\upsilon^2)}{P}+\upsilon-a. \tag{3.96}$$

For all $\upsilon\in[a,1]$ the conformal factor is in fact C^∞. In addition we recall that [10]

$$L_1=-\sin\phi P_\phi-\frac{\cos\phi}{\tan\theta}P_\phi,\quad L_2=-\cos\phi P_\phi-\frac{\sin\phi}{\tan\theta}P_\phi,\quad L_3=P_\phi, \tag{3.97}$$

as well as the constrained coordinates, namely

$$x^1=\sin\theta\cos\phi,\quad x^2=\sin\theta\sin\phi,\;x^3=\cos\theta. \tag{3.98}$$

The relation $\Pi=-P_\theta/\Omega$ and formulas (3.64) and (3.65) produce [10]

$$S_1=-\,\frac{L_2}{\Omega}\left(\,\Pi^2-\mathrm{Q}\frac{P_\phi^2}{4D}\right)+x^2\,L_3\left(\mathrm{A}\,\Pi^2-\mathrm{B}\frac{P_\phi^2}{4D}\right), \tag{3.99}$$

such that the functions A,B of θ keep the forms as

$$A=\frac{\dot{D}-\sqrt{P}\cos\theta}{2\sin\theta\sqrt{D}}\;,\qquad B=\frac{W-Q\sqrt{P}\cos\theta}{2\sin\theta\sqrt{D}}. \tag{3.100}$$

P, Q and W are obviously globally defined, so long as the quantities Π^2 and $\frac{P_\phi^2}{4D}$ in the Hamiltonian. It is enough to check that the functions A and B are well-behaved near the poles. Starting with the north pole $(\upsilon\to a+)$ or $(\theta\to 0+)$ for which we obtain [10]

$$\begin{cases} A = \frac{\phi(a)}{2(1-a^2)} - \frac{sin^2\theta}{4(1-a^2)^2} + O(sin^4\theta), \\ B = -\frac{(1-a^2)}{2}\ \phi(a) + \frac{3}{4} sin^2\theta + O(sin^4\theta), \end{cases} \tag{3.101}$$

where $\phi(a) = a^4 - 2a^2 - 2a + 1$, whilst for the south pole ($\upsilon \to\ 1-$ or $\theta\ \to \pi-$) it will get

$$\begin{cases} A = \frac{\psi(a)}{2(1-a)} - \frac{(1-a)^4}{2} sin^2\theta + O(sin^4\theta), \\ B = -2(1-a)\ \psi(a) + 6(1-a)^6 sin^2\theta + O(sin^4\theta), \end{cases} \tag{3.102}$$

with $\psi(a) = 2a^2 - 2a + 1$. It notices that either $\psi(a)$ or $\phi(a)$ may be vanishing for some $a\ \in (0,1)$, but this does not endanger the result. In addition we have [10]

$$S_2 = \{P_\phi,\ S_1\} = \frac{L_1}{\Omega}\left(\Pi^2 - \mathrm{Q}\frac{P_\phi^2}{4D}\right) + x^1 L_3 \left(\mathrm{A}\,\Pi^2 - \mathrm{B}\ \frac{P_\phi^2}{4D}\right) \tag{3.103}$$

Proposition 9 [10]: There exists no closed manifold for $c\ =\ 0$.

Proof [10]: The above functions simplify as

$$D = -(\upsilon + a)(\upsilon - a)^2,\ \ P = (\upsilon - a)^4,$$
$$Q = (3\upsilon + 5a)(\upsilon - a)^3\ ,\quad a \neq 0. \tag{3.104}$$

When $a > 0$ the positivity of D implies $\upsilon \in I = (-\infty, -a)$, but since Q possess a simple zero $= -\frac{5}{3}a \in I$, according to the Lemma 1 there is no manifold construction. For $a < 0$ either $\upsilon \in (a, -a)$ or $\upsilon \in (-\infty, -a)$, emphasize the positivity of D. P vanishes for $\upsilon = a$, and the measure of the would-be manifold, $\mu_G = 12 \int \frac{(\upsilon-\upsilon_*)}{(\upsilon-a)^3} d\upsilon \int dy$ is divergent, ruling out a closed manifold. The residual state is $c \in (0, a^2)$. The debate relies robustly on the sign of a [1-].

Proposition 10 [10]: There exists no closed manifold for $c \in (0, a^2)$ together with $a < 0$.

Proof [10]: (D, P) are given by

$$D(v) = (a - v)(v^2 - v_0{}^2), \quad v_0 = \sqrt{a^2 - c},$$
$$P = (v - \omega_-)^2(v - \omega_+)^2, \quad \omega_{\pm} = a \pm \sqrt{c}\,, \tag{3.105}$$

with the ordering $\omega_- < a < \omega_+ < -v_0$. The positivity demands give three probable intervals [10]:

$$I_1 = (-\infty, \omega_-), \quad I_2 = (\omega_-, a), \quad I_3 = (-v_0, v_0). \tag{3.106}$$

When $\in I_1$, ω_- is a zero of P for which

$$Q(\omega_-) = 4(a - \omega_-)D(\omega_-) > 0, \tag{3.107}$$

and that concluded by Lemma 2 [10].

For $v \in I_2$ since $Q(\omega_-) > 0$ and $Q(a) = -P(a) < 0$, there exists a simple zero v_* of Q inside I_2, hence, by using the Lemma 1, no manifold structure can be founded. In the case when $v \in I_3$ it implies that $Q(-v_0) = -P(v_0) < 0$ and then Q decreases to $Q(v_0)$; it thus never vanishes and $P > 0$ in I_3, opening the probability of a manifold construction.

3.3.5. The global construct for $\epsilon \neq 0$

Proposition 11 [10]: The superintegrable system is never globally defined on a closed manifold if $\Delta(D) = 0$.

Proof [10]: Here it may have either

$$D(v) = (v_0 - v)^3 \quad or \quad D(v) = (v_0 - v)(v - v_1)^2, \tag{3.108}$$

with $v_0 \neq v_1$. The 1^{st} state is discussed as in Proposition 2 due to the fact that the metric G is of constant curvature. In the 2^{nd} state it has [10]

$$\begin{cases} P(v) = (v - v_1)^2 P(v), \quad P(v) = v^2 - 2(2v_0 + 3v_1)v + (2v_0 + v_1)^2. \\ Q(v) = 3(v - v_1)^3 (v - v_*), \qquad v_* = v_0 + \frac{v_0 - v_1}{3} \ . \end{cases} \tag{3.109}$$

First consider $v_0 < v_1$. Thus, D is positive iff $v \in I = (-\infty, v_0)$. If $\Delta(P) < 0$ then we conclude that $P > 0$ for all $v \in I$. Since $v_* < v_0$, a curvature singularity inside I will be there. When $\Delta(P)$ vanishes, it will get $P(v) = (v - \omega_0)^2$ and either $\omega_0 = v_0 < 0$ or $\omega_0 = 2v_0 < 0$. In the 1^{st} state at $v_* = \frac{4}{3}v_0 \in I$ there will be a curvature singularity whilst, in the 2^{nd} state, the positivity interval is $(-\infty, \omega_0)$. Since $v = \omega_0$ is a zero of P by utilizing the Lemma 2 [10]. Two real zeros exist and $P(v) = (v - \omega_-)(v - \omega_+)$ if $\Delta(P) < 0$. The interval of positivity can be written as $I = (-\infty, v_0) \cap (\omega_-, \omega_+)$. Therefore one of its end-points at least will coincide to a zero of P *as* concluded by Lemma 2. After that we discuss the state $v_0 > v_1$. D is positive if and only if $v \in I = (-\infty, v_0)$ [10]. When $\Delta(P) < 0$, then $P > 0$ for all $v \in I$, and we conclude the Lemma 3. If $\Delta(P) = 0$, it will obtain $P(v) = (v - \omega_0)^2$, and either $\omega_0 = v_0 > 0$ or $\omega_0 = 2v_0 > 0$. In the 1^{st} state remaining with $v \in (-\infty, v_0)$and end up with a conical singularity for $v \to \infty$, in the 2^{nd} state $v \in (\omega_0, v_0)$ where ω_0 is a zero of P, which turns out closedness by Lemma 2 [10]. Two real zeros are having and $P(v) = (v - \omega_-)(v - \omega_+)$ if $\Delta(P) > 0$. The interval of positivity becomes $I = (-\infty, v_0) \cap (\omega_-, \omega_+)$ and one of its end-points at least will coincide to a zero of P [10].

Proposition 12 [10]: If $\Delta(D) < 0$ the superintegrable systems are never globally defined on a closed manifold.

Proof [10]: We have

$$\begin{aligned} &D(v) = (v_0 - v)[(v - a)^2 + b^2], \quad b \neq 0, \\ &Q(v) = -P(v) + 4\ (v_0 + 2a - v)\ D(v), \end{aligned} \tag{3.110}$$

and

$$\begin{aligned} &P(v) = P(v)^2 - 16a[(v_0 + a)^2 + b^2]v, \\ &P(v) = v^2 - 2(v_0 + 2a)v - a^2 - b^2 - 2av_0, \end{aligned} \tag{3.111}$$

respectively. The proof of this theorem can be found in [10]

Theorem 4 [10]: If $\Delta(D) > 0$ it can put $D(v) = -(v - v_0)(v - v_1)(v - v_2)$ with $v_0 < v_1 < v_2$ the superintegrable systems ℓ_1 and ℓ_2 that given through (3.70) are in fact globally defined on $\mathbb{S}^2$ if and only if $v_0 + v_0 > 0$.

Proof: See [10].

3.3.6. Comparison with the outcomes of MSh

In [12] it was stated in Theorem (6.1) that $g = \frac{dx^2+dy^2}{h_x^2}$, $h_x = \frac{dh}{dx}$, where h is the solution of the differential equation (3.3) (b) with [10]

$$\mu = 1,\ A_0 = 1,\ A_1 = 0,\ A_3 = A_4 = A_e > 0 \quad and \quad A_2 \in \mathbb{R} \qquad (3.112)$$

is globally defined on S^2. Firstly, we write the metric in (v, ϕ) [10], namely

$$g = \frac{Q^2}{P^2}\frac{dv^2}{D} + \frac{4D}{P}d\phi^2, \qquad \phi \in \mathbb{S}^1, \qquad (3.113)$$

where from the knowledge of D the Q and P are yield by (3.73). We conclude that [10]

$$h = \frac{\dot{D}}{2\sqrt{2D}},\ \dot{D} = \frac{dD}{dv}, h_x = u = \sqrt{h^2 + v} = \sqrt{\frac{P}{8D}} > 0, \qquad (3.114)$$

and

$$y = \phi, \quad \frac{dx}{dv} = \frac{dx}{dh}\frac{dh}{dv} = \frac{1}{h_x}\frac{dh}{dv} = \frac{Q}{2D\sqrt{P}}. \qquad (3.115)$$

The conditions on the metric (3.113) to be Riemannian implies $D > 0$ and $P > 0$ and to use $v \to x$ it requires to be locally bijective and Q admits a fixed sign. Also we have $\epsilon = 1, a = -A_2, \lambda = 2A_e$, that implies [10]

$$D = -(v + A_2)(v^2 - A_2^2 + c) - 8A_e^2, \qquad (3.116)$$

which c denoting a constant of integration that is not appearing through the proof of the Theorem (6.1) in [12] and it can be arbitrarily chosen. The discriminant of D read as [10]

$$\Delta(D) = -27\xi^2 + 4A_2(8A_2^2 - 9c)\xi + 4c^2(A_2^2 - c), \quad \xi = 8A_e^2\,, \qquad (3.117)$$

but the discriminant sign is indefinite. The MSh Theorem (6.1) in [12] agrees with the Theorem 2 when $\Delta(D) > 0$, and in fact the metric is globally defined on $\mathbb{S}^2$ [10].

3.4. The state of affine

3.4.1. The metric

We start with [10]

$$h_x(h_x^2 + A_1 h + A_2) = A_3 x + A_4, \quad G = \frac{dx^2 + dy^2}{h_x^2}, \qquad (3.118)$$

respectively. We differentiate the equation for h and we conclude that

$$(3h_x^2 + A_1 h + A_2)h_{xx} + A_1 h_x^2 = A_3. \qquad (3.119)$$

Again regard that $u = h_x$ like a function of the new h, then it follows that [10]

$$u(3u^2 + A_1 h + A_2)\frac{du}{dh} = A_3 - A_1 u^2. \qquad (3.120)$$

Regarding $h(u)$ the inverse function we finish as a linear differential equation, namely [10]

$$(A_3 - A_1 u^2)\frac{dh}{du} - A_1 u h = u(3u^2 + A_2). \qquad (3.121)$$

Below we investigate two cases. When $A_1 = 0$ it implies that A_3 cannot vanish; assuming $\mu = \frac{u(3u^2 + A_2)}{A_3}$, the original x, and the metric G, are obtain as [10]

$$dx = \mu du, \ \mu = \frac{1}{u}\frac{dh}{du} \ \text{ which implies that } \ G = \frac{1}{u^2}(\mu^2\, du^2 + dy^2), \qquad (3.122)$$

The relations $h = h_0 + \frac{A_2}{2A_3}u^2 + \frac{3}{4}\frac{u^4}{A_3}$, $A_3x + A_4 = A_2u + u^3$ reveal that through expressing the variable x and the function h in terms of u we integrated the equation (3.118) [10].

Whether $A_1 \neq 0$ one can put $A_1 = 1$ and, through a change of h, it may set to $A_2 = 0$. To make these matters easy, we perform the rescaling: $G \to \frac{1}{4}G$ and $y \to 2y$. In this case we have $-2\mu = \frac{1}{u}\frac{dh}{du} \Rightarrow G = \frac{1}{u^2}(\mu^2\, du^2 + dy^2)$ and two possible solutions will be gotten for μ [10]:

$$\mu = 1 + \frac{C}{(u^2 - A_3)^{3/2}} \quad or \quad \mu = 1 + \frac{C}{(A_3 - u^2)^{3/2}}, \tag{3.123}$$

where C here is regard the real constant to the integration [10].

3.4.2. Global construction for vanishing A_1

It was already noticed that $\mu = \frac{3u^2 + A_2}{A_3}$. Two states are explained separately [10]:

a) State one: $A_2 = 0$, then we have $\mu = 2u^2$.

b) State two: $A_2 \neq 0$, thus $\mu = 1 + au^2$ [10].

3.4.2.1. The state $A_2 = 0$

In this case we have $G = dv^2 + \frac{dy^2}{v}$ which implies that $H = \frac{1}{2}\left(P_v^2 + vP_y^2\right)$, whilst the cubic integrals are written as [10]

$$S_1 = \frac{2}{3}P_v^3 + vP_y^2\left(vP_v + \frac{y}{2}P_y\right), \tag{3.124}$$

and

$$S_2 = yS_1 - \left(\frac{y^2}{4} + \frac{v^3}{9}\right)P_y^3 - \frac{2}{3}v^2 H\, P_y, \tag{3.125}$$

respectively. The superintegrable system is generated by (H, P_y, S_1). The Poisson brackets are written as [10]

$$\{P_y, S_1\} = \frac{1}{2}P_y^3, \quad \{P_y, S_2\} = S_1, \quad \{S_1, S_2\} = \frac{3}{2}S_2P_y^2\,. \tag{3.126}$$

Proposition 13 [10]: The superintegrable system (H, P_y, S_1) is not globally defined for $A_2 = 0$.

Proof [10]: The Riemannian character of the metric G requires $v > 0$ and $y \in \mathbb{R}$. The scalar curvature would be defined anywhere whether this metric G defined on a manifold. A simple calculation gives for result $R_G = -\frac{3}{2v^2}$ which is singular for $v \to 0^+$.

3.4.2.2. The state $A_2 \neq 0$

The corresponding Hamiltonian is given by [10]

$$2H = u^2 \left(\frac{P_u^2}{\mu^2} + P_y^2\right), \;\; u > 0, \;\; y \in \mathbb{R}, \;\; \mu = 1 + au^2, \;\; a \in \mathbb{R}, \qquad (3.127)$$

and S_1, S_2 are cubic integrals, namely

$$S_1 = \frac{2a}{3}\left(\frac{u}{\mu} P_u\right)^3 + P_y\left(uP_u P_y + y\, P_y^2\right) \qquad (3.128)$$

and

$$S_2 = \; yS_1 - \frac{1}{2}\left(y^2 + u^2\left(1 + \frac{au^2}{3}\right)^2\right) P_y^3 - \frac{a}{3} u^2 (2 + au^2) H P_y, \qquad (3.129)$$

respectively. Poisson brackets are written in this case as [10]

$$\{P_y, S_1\} = P_y^3, \;\; \{P_y, S_2\} = S_1, \;\; \{S_1, S_2\} = 3S_2 P_y^2 + 4P_y^3 H + \frac{16}{3} a P_y H^2. \quad (3.130)$$

Proposition 14 [10]: For $A_2 \neq 0$ the superintegrable system (H, P_y, S_1)

a) for $a < 0$ is not globally defined

b) for $a = 0$, is trivial

c) for $a > 0$ is globally defined on $\mathrm{M} \cong \mathbb{H}^2$.

Proof [10]: The scalar curvature has thee expression $R_G = -\frac{2}{\mu^3}(1 + 3au^2), u > 0,$ $y \in \mathbb{R}$. If $a < 0$ it is singular for $u_0 = |a|^{-1/2}$, and on a manifold the system cannot be defined. When $a = 0$, the metric G is reducing to the canonical metric

$G(\mathbb{H}^2, can) = \frac{du^2+dy^2}{u^2}$ of the hyperbolic plane $\mathbb{H}^2$. As a result of the theorem of Thompson [10] S_1 and S_2 are reducible. The set (H, P_y) stays as an integrable system but it is trivial, namely it is no longer superintegrable. We check the final state for which $a > 0$. The change of coordinates, namely [10]

$$t = u\left(1 + \frac{a}{3}u^2\right) \mapsto u = \frac{\xi^{1/3}}{a} - \xi^{-1/3}, \;\; \xi(t) = \frac{3}{2}a^2 t + \sqrt{a^3 + \frac{9}{4}a^4 t^2}, \qquad (3.131)$$

leads to the conclusion that $u(t)$ is C^∞ for all $t \geq 0$. The expression of the metric G is given below:

$$G = \Omega^2 \frac{dt^2+dy^2}{t^2} = \Omega^2 G(\mathbb{H}^2, can), \;\; \Omega(t) = 1 + \frac{a}{3}u^2(t), \;\; t > 0, \;\; y \in \mathbb{R} \quad (3.132)$$

and, since Ω does not vanishes, it is globally conformally related to the canonical metric of the hyperbolic plane, $\mathrm{M} \cong \mathbb{H}^2$. The Hamiltonian can be written as [10]

$$H = \frac{t^2}{2\Omega^2}\left(p_t^2 + p_y^2\right) = \frac{1}{2\Omega^2}(M_1^2 + M_2^2 - M_3^2). \qquad (3.133)$$

Taking into account that $P_y = M_2 + M_3$ and $t\,P_t = \frac{M_1 - x^1 P_1}{1+(x^1)^2}$ itreveals that [10]

$$S_1 = \frac{2a}{3}\left(\frac{t\,P_t}{\Omega}\right)^3 + P_y^2\left(\mu\frac{t\,P_t}{\Omega} + yP_y\right), \quad \mu(t) = 1 + au^2(t), \quad a > 0 \quad (3.134)$$

is globally defined on M. For S_2 it is also true [10].

3.4.3. The global structure for non-vanishing A_1

By exchanging $A_3 \to a$ in (3.123), there two states to be regarded based to $\epsilon = sign(u^2 - a)$ [10].

3.4.3.1. First state $\epsilon = +1$

By following the metric G and Hamiltonian, namely [10]

$$G = \frac{1}{u^2}(\mu^2 du^2 + dy^2), \;\; H = \frac{u^2}{2}\left(\frac{P_u^2}{\mu^2} + P_y^2\right), u^2 - a > 0, y \in \mathbb{R}, \qquad (3.135)$$

when $\mu = 1 + \frac{C}{(u^2-a)^{3/2}}$, then the cubic integrals are written as [10]

$$S_1 = (\frac{u}{\mu} P_u)^3 + u(u^2 - a)P_u P_y^2 - ay P_y^3 + 2y HP_y, \qquad (3.136)$$

$$S_2 = yS_1 - \frac{1}{2}\left(a(u^2+y^2) - \frac{2Cu^2}{\sqrt{u^2-a}} + \frac{C^2}{u^2-a}\right)P_y^3 - \left(u^2+y^2 - \frac{2C}{\sqrt{u^2-a}}\right)HP_y, \quad (3.137)$$

respectively. On $\mathbb{H}^2$ the state $C = 0$ coincides for the canonical metric, so the system will be trivial [10].

Proposition 15 [10]: The superintegrable system (H, P_y, S_1) is globally defined for $C = -1$ if and only if $a < 0$ and $|a| > 1$, in case of manifold is $\mathrm{M} \cong \mathbb{H}^2$.

Proof [10]: In this case the scalar curvature is written by

$$R_G = -\frac{2}{\mu^3}\left(1 + \frac{(2u^2+a)}{(u^2-a)^{5/2}}\right). \qquad (3.138)$$

When $a \geq 0$ it implies that $u > \sqrt{a}$ and R_G is be singular when $u_0 = \sqrt{a+1}$. For $a < 0$ it have $u > 0$. Thus, the curvature is singular when $u_0 = \sqrt{1-\rho}$ provided $\rho = |a| \leq 1$. When $\rho > 1$, the function μ is no longer vanishes, thus, the curvature stays continuous to $u \geq 0$. G has the following form, namely [10]

$$G = \frac{1}{u^2}(\mu^2 du^2 + dy^2), \qquad \mu = 1 - \frac{1}{(\rho+u^2)^{3/2}}, \; u > 0, \; y \in \mathbb{R}. \qquad (3.139)$$

Define the new variable $t = u\left(1 - \frac{1}{\rho\sqrt{\rho+u^2}}\right)$ such that $u \in [0, +\infty) \longmapsto t \in [0, +\infty)$. Since $\mu = \frac{dt}{\mathrm{du}}$ never vanishes, the inverse function $u(t)$ is $C^\infty\big([0, +\infty)\big)$ and the metric are given below [10]

$$G = \Omega^2 G(\mathbb{H}^2, can), \qquad \Omega(t) = 1 - \frac{1}{\rho\sqrt{\rho+u^2(t)}}, \; \rho > 1, \qquad (3.140)$$

where the conformal factor $\Omega(t)$ is C^∞ and never vanishes. Thus, the manifold is again $\mathrm{M} \cong \mathbb{H}^2$. The first cubic integral [10], namely

$$S_1 = (\tfrac{tP_t}{\Omega})^3 + \mu(t)\big(\rho + u^2(t)\big)(\tfrac{tP_t}{\Omega})\, P_y^2 + \rho y\, P_y^3 + 2y\, HP_y, \quad (3.141)$$

is globally defined and the equation (3.137) implies

$$S_2 = yS_1 + \tfrac{1}{2}\Big(-\rho(u^2 + y^2) + \tfrac{2u^2}{\sqrt{\rho+u^2}} + \tfrac{1}{\rho+u^2}\Big) P_y^3 - \Big(u^2 + y^2 + \tfrac{2}{\sqrt{\rho+u^2}}\Big) HP_y, \quad (3.142)$$

Proposition 16 [10]: When $C = +1$ the superintegrable system (H, P_y, S_1) is globally defined either if $a > 0$ and the manifold is $\mathrm{M} \cong \mathbb{R}^2$, or if $a < 0$ and $\mathrm{M} \cong \mathbb{H}^2$.

Proof [10]: G is given by

$$G = \tfrac{1}{u^2}(\mu^2 du^2 + dy^2), \qquad \mu = 1 + \tfrac{1}{(u^2-a)^{3/2}}. \qquad (3.143)$$

We analyze the first the state $a > 0$ such that $u > \sqrt{a}$. We define the new coordinate $t = u\left(1 - \frac{1}{a\sqrt{u^2-a}}\right)$, $u \in \left(\sqrt{a}, +\infty\right) \mapsto t \in \mathbb{R}$. Because $\mu = \frac{dt}{\mathrm{du}}$ does not vanish, then $u(t)$ is $C^\infty(\mathbb{R})$, and the metric G [10]

$$G = \frac{dt^2+dy^2}{u^2(t)}, \quad t \in \mathbb{R}, \quad y \in \mathbb{R}\,, \qquad (3.144)$$

is globally conformally related to the flat metric (the manifold is therefore $\mathrm{M} \cong \mathbb{R}^2$). The corresponding cubic integral [10]

$$S_1 = (u(t)P_t)^3 + \mu(t)(u^2(t) - a)(u(t)P_t)P_y^2 - ayP_y^3 + 2yHP_y, \quad (3.145)$$

stays globally defined, and for S_2 the same holds true.

When $a = 0$ the function $\mu = 1 + \frac{1}{u^3}$ is no longer even, thus it ought to regard that $u \in \mathbb{R}$ and $R_G = 2u^6 \frac{(2-u^3)}{(1+u^3)^3}$ is not defined for $u = -1$, so no manifold construction exists. For $a < 0$ let $\rho = |a|$ and consider $u > 0$. We define the new coordinate as [10]

$$t = u\left(1 + \frac{1}{\rho\sqrt{\rho+u^2}}\right), u \in (0, +\infty) \mapsto t \in (0, +\infty). \qquad (3.146)$$

But $\mu = \frac{dt}{\mathrm{du}}$ never vanishes, thus the inverse function $u(t)$ is $C^{\infty}([0,+\infty))$. We conclude that the metric [10]

$$G = \Omega^2 \frac{dt^2+dy^2}{t^2}, \quad \Omega(t) = 1 + \frac{1}{\rho\sqrt{\rho+u^2}}, \ \rho > 0, t > 0, y \in \mathbb{R}, \qquad (3.147)$$

is globally conformally again that linked to the canonical metric on the manifold $\mathrm{M} \cong \mathbb{H}^2$ [10].

3.4.3.2. The 2^{nd} state$\epsilon = -1$

Let us start with the Hamiltonian and the metric as follows [10]

$$G = \frac{1}{u^2}(\mu^2 du^2 + dy^2), \ \ H = \frac{u^2}{2}\left(\frac{P_u^2}{\mu^2} + P_y^2\right), \quad a - u^2 > 0, y \in \mathbb{R}. \ (3.148)$$

For $\mu = 1 + \frac{C}{(a-u^2)^{3/2}}$, corresponding the scalar curvature is [10]

$$R_G = -\frac{2}{\mu^3}\left(1 + C\frac{(2u^2+a)}{(a-u^2)^{5/2}}\right) \qquad (3.149)$$

and the cubic integral S_1 is similar as in (3.136) but [10]

$$S_2 = yS_1 - \frac{1}{2}\left(a(u^2+y^2) - \frac{2cu^2}{\sqrt{a-u^2}} + \frac{c^2}{a-u^2}\right) P_y^3 - \left(u^2 + y^2 - \frac{2c}{\sqrt{a-u^2}}\right) HP_y. (3.150)$$

Proposition 17 [10]: The superintegrable system (H, P_y, S_1) is globally defined on the manifold, $\mathrm{M} \cong \mathbb{H}^2$ either for $C = +1$ or for $C = -1$ and $0 < a < 1$.

Proof [10]: It ought to have $a > 0$ to ensure $u \in (0, \sqrt{a}\,)$. For the case $C = -1$ the scalar curvature is singular as μ vanishes. This occurs for $u_0 = \sqrt{a-1}$ and $a \geq 1$. Thus, no manifold construction exists. Thus, for $0 < a < 1$ the function μ does not vanish, then it can be defined

$$t = -u\left(1 - \frac{1}{a\sqrt{a-u^2}}\right), \ \ u \in (0, \sqrt{a}\,) \longmapsto t \in (0, +\infty), \qquad (3.151)$$

so the inverse function $u(t)$ is $C^{\infty}([0,+\infty))$. This leads to the metric G [10]

$$G = \Omega^2 G(\mathbb{H}^2, can), \quad \Omega(t) = -1 + \frac{1}{a\sqrt{a - u^2(t)}}, \;\; 0 < a < 1, \quad (3.152)$$

where the conformal factor does not vanish, thus, again the manifold is $\mathrm{M} \cong \mathbb{H}^2$. We define [10]

$$t = u\left(1 + \frac{1}{a\sqrt{a - u^2}}\right), \; u \in (0, \sqrt{a}\,) \mapsto t \in (0, +\infty), \quad (3.153)$$

thus, the metric G becomes [10]

$$G = \Omega^2 G(\mathbb{H}^2, can), \;\; \Omega(t) = 1 - \frac{1}{a\sqrt{a - u^2(t)}}, \;\; a > 0, \quad (3.154)$$

where the conformal factor, Ω, does not vanish, thus, again the manifold is given by $\mathrm{M} \cong \mathbb{H}^2$. The proof that the cubic integral, S_1 and S_2 are also globally defined is the same as presented in Proposition 14 [10].

CHAPTER 4

FRACTIONAL KILLING-YANO TENSORS AND KILLING VECTORS

As it is well known, the Caputo definition is more convenient than Riemann-Liouville in all cases including the fractional differential geometry because the Caputo's derivative of the constant is zero whilst the derivative Riemann-Liouville is not zero [37,38].The Caputo differential operator has the form [39-46].

$$ {}_a\mathrm{D}_x^{\alpha} f(x) \equiv \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \\ x\int_a^x (x-u)^{n-\alpha-1} \dfrac{d^n f(u)}{du^n} du, \quad n-1<\alpha<n, \\ \dfrac{d^n}{dx^n} f(x), \qquad \alpha = n. \end{cases} \tag{4.1} $$

$\Gamma(.)$ refers to the Gamma function with $x > a$. Here $a = 0$ and $n-1 < \alpha \leq n$. If the $f(x) = x^p$ and $p \in R$ its Caputo fractional derivative gives [36]

$$ \mathrm{D}_x^{\alpha} x^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} \\ 0, \qquad p = 0,1,2,\ldots.,n-1\,. \end{cases} \tag{4.2} $$

Through the final decades the topic about the Killing and Killing-Yano (KY) tensors [36] was in connection to the geodesic motion of the superparticle and the particle in a curved [24,27,47-53]. They explain the hidden symmetries related to the fractional Killing vectors (KV) and KY tensors on curved spaces. We denote the fractional derivative by [36]

$$ {}_a D_x^{\alpha} f(x,y) \equiv \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \\ x\int_a^x (x-u)^{n-\alpha-1} \dfrac{\partial^n f(u,y)}{\partial u^n} du, \quad n-1<\alpha<n\,. \\ \dfrac{\partial^n}{\partial x^n} f(x,y), \qquad \alpha = n \end{cases} \tag{4.3} $$

We consider $a = 0$ and $n-1 < \alpha \leq n$ [36].

4.1. The one-dimensional case

We start with one-dimensional free Lagrangian [54].

$$L = \frac{1}{2}\dot{x}^2 + \dot{\lambda}_2\dot{x}. \tag{4.4}$$

(4.4.) can be reformulated as [36]

$$L = \frac{1}{2}g_{ij}\dot{u}^i\dot{u}^j, \tag{4.5}$$

where $g_{ij} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The related fractional Lagrangian has the form

$$L = \frac{1}{2}g_{ij}D_t^q u^i D_t^q u^j, \tag{4.6}$$

within the fractional Caputo derivative. The corresponding fractional Christofell symbols (ChS) for $n-1 < q < n$ are defined as [36]

$${}^q\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}\left(D^q_\mu g_{\alpha\beta} + D^q_\beta g_{\alpha\mu} - D^q_\gamma g_{\beta\mu}\right), \tag{4.7}$$

thus notice in the fractional state for order q the derivatives partial are defined. Here note that all the ChS are vanished and that is due to the metric is constant, namely [36]

$${}^q\Gamma^{\gamma}_{\mu\upsilon} = 0. \tag{4.8}$$

4.1.1. Fractional KV

From the generalized equations below the KV can be computed [36]

$$V^q_{\alpha;\beta} + V^q_{\beta;\alpha} = 0, \tag{4.9}$$

where $V^q_{\alpha;\beta}$ is the fractional covariant derivative denoted by

$$V^q_{\alpha;\beta} = D^q_\beta V_\alpha + g_{\alpha\mu}\,{}^q\Gamma^{\mu}_{\delta\beta}g^{\delta\lambda}V_\lambda. \tag{4.10}$$

Since all ChS are vanishes then it is simple to see that [36]

$$V^q_{1;1} = D^q_1 V_1 = 0,$$
$$V^q_{2;2} = D^q_2 V_2 = 0,$$
$$V^q_{1;2} + V^q_{2;1} = V^q_{1,2} + V^q_{2,1} = D^q_2 V_1 + D^q_1 V_2 = 0, \tag{4.11}$$

When $0 < q \leq 1$, the solution of (4.11) will be $V_1 = -cy^q$, $V_2 = cx^q$. Here c is constant. Whilst to $q > 1$ the corresponding solution becomes [36]

$$V_1 = -cy^q + \sum_{k=0}^{n-1}(a_k x^k + b_k y^k),$$
$$V_2 = cx^q + \sum_{k=0}^{n-1}(\grave{a}_k x^k + \grave{b}_k y^k), \tag{4.12}$$

where $c, a_k, b_k, \grave{a}_k, \grave{b}_k$ are constants.

4.1.2. Fractional KY tensors

The fractional KY anti symmetric tensor denoted by ${}^q f_{\mu\gamma}$ fulfills [36]

$${}^q f_{\mu\upsilon;\lambda} + {}^q f_{\lambda\upsilon;\mu} = 0, \tag{4.13}$$

where ${}^q f_{\mu\gamma;\lambda}$ denotes the fractional covariant derivative of the KY tensor ${}^q f_{\mu\gamma}$ and it has the for

$${}^q f_{\mu\gamma;\lambda} = D^q_\lambda f_{\mu\gamma} - f_{\alpha\gamma}\, {}^q\Gamma^\alpha_{\lambda\mu} - f_{\mu\alpha}\, {}^q\Gamma^\alpha_{\lambda\gamma}. \tag{4.14}$$

By inspection we conclude that

$$D^q_\lambda f_{\mu\upsilon} = 0, \tag{4.15}$$

for every of μ, υ, λ.

Then the related solutions will be $f_{12} = c = -f_{21}$, $f_{11} = f_{22} = 0$ and c is constant for $0 < q \leq 1$. Whilst to $q > 1$, then the solution is

$$f_{12} = -f_{21} = \sum_{k=0}^{n-1}(a_k x^k + b_k y^k), \tag{4.16}$$

such that a_k, b_k are constants [36].

4.2. The two-dimensional case

Angular momentum is the first integral of motion in the case of two dimensional classical free Lagrangian, therefore the related Lagrangian becomes [54].

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \dot{\lambda}_3\,(x\dot{y} - y\dot{x}). \tag{4.17}$$

Then, we conclude that the fractional Lagrangian is given by

$$L = \frac{1}{2} g_{ij} D^{\alpha} q^{i} D^{\alpha} q^{j}, \tag{4.18}$$

which g_{ij} is the related matrix, namely [36]

$$g_{ij} = \begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{bmatrix}. \tag{4.19}$$

Generalized ChS given by [36]

$$ {}^{q}\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}\left(D^{q}_{\mu} g_{\alpha\beta} + D^{q}_{\beta} g_{\alpha\mu} - D^{q}_{\gamma} g_{\beta\mu}\right). \tag{4.20}$$

It can be seen as that such as $\gamma, \mu = 1,2,3$ [36]

$$ {}^{q}\Gamma^{\gamma}_{\mu\mu} = 0, \tag{4.21}$$

whilst

$$\begin{aligned} {}^{q}\Gamma^{\gamma}_{12} &= \frac{1}{2} g^{3\gamma}\left(D^{q}_{1} g_{32} + D^{q}_{2} g_{31}\right), \\ {}^{q}\Gamma^{\gamma}_{13} &= \frac{1}{2} g^{2\gamma}\left(D^{q}_{1} g_{32} + D^{q}_{2} g_{31}\right), \\ {}^{q}\Gamma^{\gamma}_{23} &= \frac{1}{2} g^{1\gamma}\left(D^{q}_{2} g_{13} + D^{q}_{1} g_{23}\right). \end{aligned} \tag{4.22}$$

4.2.1 Fractional KV

The equations of the fractional KV are [36]

$$V^q_{\alpha;\beta} + V^q_{\beta;\alpha} = 0, \tag{4.23}$$

such that $V^q_{\alpha;\beta}$ denotes the fractional covariant derivative, namely

$$V^q_{\alpha;\beta} = D^q_\beta V_\alpha + g_{\alpha\mu}\, {}^q\Gamma^\gamma_{\delta\beta} g^{\delta\lambda} V_\lambda. \tag{4.24}$$

By direct calculations we conclude that [36]

$$\begin{gathered}
V^q_{1;1} = D^q_1 V_1 = 0, \qquad V^q_{2;2} = D^q_2 V_2 = 0, \qquad V^q_{3;3} = D^q_3 V_3 = 0, \\
V^q_{1;2} + V^q_{2;1} = V^q_{1,2} + V^q_{2,1} = D^q_2 V_1 + D^q_1 V_2 = 0, \\
V^q_{1;3} + V^q_{3;1} = D^q_3 V_1 + D^q_1 V_3 + g^{2\lambda} V_\lambda D^q_2 g_{13} = 0, \\
V^q_{2;3} + V^q_{3;2} = D^q_3 V_2 + D^q_2 V_3 + g^{1\lambda} V_\lambda D^q_1 g_{23} = 0,
\end{gathered} \tag{4.25}$$

respectively. It is easy to find the solution of V_1 and V_2, therefore it is given below

$$\begin{gathered}
V_1 = c y^q + \sum_{k=0}^{n-1} (a_k x^k + b_k y^k), \\
V_2 = -c x^q + \sum_{k=0}^{n-1} (c_k x^k + d_k y^k),
\end{gathered} \tag{4.26}$$

which c, a_k, b_k, c_k, d_k are constants and $n - 1 < q \leq n$. It is difficult to get a solution for V_3 when $0 < q \leq 1$. When $n \geq 2$, which means that $q > 1$ the related equations be easy because [36]

$$D^q_2 g_{13} = D^q_1 g_{23} = 0. \tag{4.27}$$

In this case the common solution becomes

$$V_3 = \sum_{k=0}^{n-1} (\acute{a}_k x^k + \acute{b}_k y^k), \tag{4.28}$$

such that $\acute{a}_k, \acute{b}_k$ are constants [36].

4.2.2. Fractional KY tensors

The fractional KY tensors can be computed by solving [36]

$$ {}^{q}f_{\mu\upsilon;\lambda} + {}^{q}f_{\lambda\upsilon;\mu} = 0, \tag{4.29}$$

where ${}^{q}f_{\mu\upsilon;\lambda}$ defined the fractional covariant derivative of the KY tensors ${}^{q}f_{\mu\upsilon}$, namely [36]

$$ {}^{q}f_{\mu\upsilon;\lambda} = D_{\lambda}^{q} f_{\mu\upsilon} - f_{\alpha\upsilon}\,{}^{q}\Gamma_{\lambda\mu}^{\alpha} - f_{\mu\alpha}\,{}^{q}\Gamma_{\lambda\upsilon}^{\alpha}\,. \tag{4.30}$$

For $q > 1$, the ChS vanish, then we have [36]

$$ D_{\lambda}^{q} f_{\mu\upsilon} = 0, \tag{4.31}$$

for μ, υ, λ. Then the related solution is $f_{11} = f_{22} = f_{33} = 0$, where f_{12}, f_{13}, f_{23} are

$$\begin{aligned} f_{12} &= -f_{21} = \sum_{k=0}^{n-1} (a_k\, x^k + b_k y^k), \\ f_{13} &= -f_{31} = \textstyle\sum_{k=0}^{n-1} (\acute{a}_k\, x^k + \acute{b}_k y^k), \\ f_{23} &= -f_{32} = \sum_{k=0}^{n-1} (c_k\, x^k + d_k y^k)\,, \end{aligned} \tag{4.32}$$

which $a_k, b_k, \acute{a}_k, \acute{b}_k, c_k, d_k$ are constants. The fractional KY tensors can be produced new constant of motion [36].

CHAPTER 5

THE DUNKLE-COULOMB PROBLEM IN THE PLANE

Below we review the Dunkl-Coulomb (DC) system in the plan that is ruled by Hamiltonian

$$H = -\frac{1}{2}\nabla_{\mathcal{D}}^2 + \frac{\alpha}{r}. \tag{5.1}$$

Here $r^2 = x_1^2 + x_2^2$ and $\nabla_{\mathcal{D}}^2 = D_1^2 + D_1^2$. The Dunkl derivative terms it defined as [56]

$$D_i = \partial_{x_i} + \frac{\mu_i}{x_i}(1 - R_i), \qquad i = 1,2, \tag{5.2}$$

and R_i refers to the reflection operator $R_i f(x_i) = f(-x_i)$. By using (5.1) the model is maximally superintegrable and completely solvable. Using the realization of $so(2,1)$ with Dunkl operators, the spectrum of the Hamiltonian will be algebraically derived. The exact solutions of the model will be given with the help of the Laguerre polynomials and the Dunkl harmonic. The complete solutions of the model will be given on the circle [55].

5.1. Algebraic solution of the spectrum

The Hamiltonian spectrum (5.1) describing the DC system can be gotten algebraically by using $so(2,1)$ dynamical symmetry. First of all it explains the investigation of $so(2,1)$ presented in [57,58], then it is shown the connection between the DC problems and this realization [55].

5.1.1. A realization of $so(2,1)$ within Dunkl operators

The dilation operators is defined by $E = x_1\partial_{x_1} + x_2\partial_{x_2} + (\mu_1 + \mu_2 + \frac{1}{2})$. Besides we have [55]

$$L_0 = -r\nabla_{\mathcal{D}}^2 + \frac{r}{4}, L_\pm = -r\nabla_{\mathcal{D}}^2 - \frac{r}{4} \pm E, \tag{5.3}$$

respectively. The operators $L_0, L_\pm$ fulfills the $so(2,1)$ algebra, namely

$$[L_0,\, L_\pm] = \pm L_\pm, [L_+,\, L_-] = -2L_0.$$

$so(2,1)$ is the Casimir operator (CO) and it has the for

$$C = L_0^2 - \frac{1}{2}(L_+L_- + L_-L_+)$$

or

$$C = J_3^2 + (\mu_1 R_1 + \mu_2 R_2)^2 - \frac{1}{4}. \tag{5.4}$$

J^3 denotes the generator of Dunkl angular momentum, namely [55]

$$J^3 = -i(x_1 D_2 - x_2 D_1). \tag{5.5}$$

It can be seen that in irreducible representation of the non-negative- discrete series of $so(2,1)$, CO acts as multiple of the identity

$$C = \gamma(\gamma - 1), \gamma > 0, \tag{5.6}$$

so, the eigenvalue of the generator L_0, induced by $\lambda_{L_0}(\zeta)$ becomes [52]

$$\lambda_{L_0}(\zeta) = \zeta + \gamma. \tag{5.7}$$

Here ζ is positive integer [59,60,61,62]. It divides into two sectors coinciding to the probable eigenvalues of the operator R_1R_2 that commutes with equation (5.5). When $R_1R_2 = +1$, the corresponding eigenvalues of equation (5.5) defined by $\lambda_{J_3}^{+}(m)$ are [55]

$$\lambda_{J_3}^{+}(m) = \pm 2\sqrt{m(m + \mu_1 + \mu_2)}. \tag{4.8}$$

Here$m$ denotes a positive integer. For $R_1 R_2 = -1$, the related eigenvalues denoted by $\lambda_{J_3}^{-}(k)$ and corresponding to (5.5) are [55]

$$\lambda_{J_3}^{-}(k) = \pm 2\sqrt{(k+\mu_1)(k+\mu_2)}. \tag{5.9}$$

Here k denotes a non-negative half-integer, that means $k \in \left\{\frac{1}{2}, \frac{3}{2}, \dots\right\}$. Observable corresponding to the eigenvalues such that $\lambda_{J_3}^{+}(0) = 0$ is regarded as non-degenerate. Combining (5.8) and (5.9), one can get that the eigenvalues of CO (5.4) in the representation (5.3) are of the type (5.6) with $\gamma(n) = 2n + \mu_1 + \mu_2 + \frac{1}{2}$. Thus n is a positive integer for $R_1 R_2 = +1$ and a non-negative half-integer for $R_1 R_2 = -1$. The irreducible $so(2,1)$ representations contained in (5.3) that has the form [55]

$$\lambda_{L_0}(\zeta, n) = \zeta + 2n + \mu_1 + \mu_2 + \frac{1}{2}, \tag{5.10}$$

for the spectrum of L_0 [55].

5.1.2. Connection with the DC problem

Let us start with the equation of Schrödinger for the Hamiltonian (5.1) of the DC system, namely

$$\left(-\frac{1}{2}\nabla_{\mathcal{D}}^2 + \frac{\alpha}{r}\right)\Psi_\varepsilon = \varepsilon\Psi_\varepsilon.$$

For bound cases, we get $\left(-\frac{r}{2}\nabla_{\mathcal{D}}^2 - \varepsilon r\right)\Psi_\varepsilon = -\alpha\,\Psi_\varepsilon$. For rescaling the coordinates based to $x_i \rightarrow \frac{x_i}{\sqrt{-8\varepsilon}}$, the equation of eigenvalue is transformed to $L_0\Psi_\varepsilon = -\frac{2\alpha}{\sqrt{-8\varepsilon}}\Psi_\varepsilon$. The energy spectrum of the Hamiltonian (5.1) for the DC system possess the expression, namely [55]

$$\varepsilon(\zeta, n) = \frac{-\alpha^2}{2(\zeta + 2n + \mu_1 + \mu_2 + \frac{1}{2})^2}. \tag{5.11}$$

Here ζ is a positive integer and n is a positive integer in the sector $R_1 R_2 = +1$ and a non-negative half-integer in the sector $R_1 R_2 = -1$. Observe that Hamiltonian and

R_1R_2 can be diagonalized at the same time because of the commutation between Hamiltonian in (5.1) with both reflection operators. If $\mu_1 = \mu_2 = 0$, the system of DC reduces to the standard Kepler–Coulomb problem [55].

5.2. The invariance algebra and superintegrability

Two algebraically independent operators commuting with the Hamiltonian are required. Assume A_1, A_2 be the operators and defined them as [55]

$$A_1 = \frac{x_1}{r} - \frac{\mu_1}{\alpha} D_1 R_1 - \frac{1}{2\alpha}\{\tau, D_2\}, \quad A_2 = \frac{x_2}{r} - \frac{\mu_2}{\alpha} D_2 R_2 - \frac{1}{2\alpha}\{\tau, D_1\}, \tag{5.12}$$

such that
$$\tau = (x_1 D_2 - x_2 D_1) = iJ_3, \tag{5.13}$$
and $\{x, y\} = xy + yx$ stands for the anticommutator. A direct calculation reveals that A_1, A_2 and τ are first integrals of motion for the DC system, means, namely [55]

$$[H, \tau] = [H, A_1] = [H, A_1] = 0, \tag{5.14}$$

Since the commutation between (5.1) with the R_1, R_2, these operators are the symmetries of the DC Hamiltonian. The first integrals of motion A_1 and A_2 are similar to the ingredients of the Lenz–Runge vector for the system of standard Kepler–Coulomb in two dimensions, namely

$$\begin{gathered}[A_1, A_2] = -\frac{2}{\alpha^2} H\tau, \\ [A_1, \tau] = A_2(1 + 2\mu_1 R_1), \\ [\tau, A_2] = A_1(1 + 2\mu_2 R_2).\end{gathered} \tag{5.15}$$

The commutation relations including the reflections are expressed below [55]

$$\begin{gathered}\{\tau, R_1\} = 0, \{\tau, R_2\} = 0, \\ \{A_1, R_1\} = 0, [A_1, R_2] = 0, \\ \{A_2, R_2\} = 0, [A_2, R_1] = 0,\end{gathered} \tag{5.16}$$

with $[R_1, R_2] = 0$. The CO is written as (5.15)

$$Q = A_1^2 + A_2^2 + \frac{2H}{\alpha^2}\tau^2 - \frac{2H}{\alpha^2}(\mu_1 R_1 + \mu_2 R_2 + 2\mu_1 \mu_2 R_1 R_2), \tag{5.17}$$

that commutes with all symmetries. A direct calculation reveals that the Q is given by [55]

$$Q = \frac{H}{\alpha^2}\left(2\mu_1^2 + 2\mu_2^2 + \frac{1}{2}\right) + 1\,. \tag{5.18}$$

We introduce the renormalized operators with a given value of the energy ε, $J_1 = \sqrt{\frac{\alpha^2}{-2H}}A_1$, $J_2 = \sqrt{\frac{\alpha^2}{-2H}}A_2$ [55] and we have

$$\begin{aligned} [J_1, J_2] &= iJ_3, \\ [J_2, J_3] &= iJ_1(1 + 2\mu_2 R_2), \\ [J_3, J_1] &= iJ_2(1 + 2\mu_1 \mathrm{R}_1), \end{aligned} \tag{5.19}$$

with

$$\{J_1, \mathrm{R}_1\} = \{J_2, \mathrm{R}_2\} = \{J_3, \mathrm{R}_1\} = \{J_3, \mathrm{R}_2\} = 0,$$
$$[J_1, \mathrm{R}_2] = [J_2, \mathrm{R}_1][\mathrm{R}_1, \mathrm{R}_2] = 0.$$

So the CO becomes [55]

$$Q = J_1^2 + J_2^2 + J_3^2 + \mu_1 R_1 + \mu_2 R_2 + 2\mu_1 \mu_2 R_1 R_2.$$

5.3. Exact solutions

The corresponding Schrödinger equations is given by

$$H\Psi = \varepsilon\Psi. \tag{5.20}$$

In polar coordinates, by using separation of variables, it can be exactly solved. Considering $x_1 = r cos\,\theta$ and $x_2 = r\sin\theta$, then R_1, R_2 have these actions $R_1\, f(r,\phi) = f(r, \pi - \phi)$, $R_2\, f(r,\phi) = f(r, -\phi)$, since the commutation between H with both R_1 and R_2. Thus, it is suitable to search for the joint eigen functions of Hamiltonian and reflections R_1, R_2. The Hamiltonian (5.1) in polar coordinates, becomes $H = A_r + \frac{1}{r^2}B_\phi$, where A_r is [55]

$$A_r = \frac{1}{2}\partial_r^2 - \frac{1}{2r}(1 + 2\mu_1 + 2\mu_2)\partial_r + \frac{\alpha}{r},$$

and

$$B_\phi = \frac{1}{2}\partial_\phi^2 + (\mu_1\, tg\phi - \mu_2\, ctg\phi)\partial_\phi + \frac{\mu_1(1-R_1)}{2cos^2\phi} + \frac{\mu_2(1-R_2)}{2sin^2\phi},$$

respectively. By taking $\Psi = R(r)\Phi(\phi)$, it can discovered that (5.20) becomes

$$\left(A_r - \varepsilon + \frac{m^2}{2\rho^2}\right)R(r) = 0, \tag{5.21a}$$

$$(B_\phi - \frac{m^2}{2})\Phi(\phi) = 0, \tag{5.21b}$$

where the separation constant reads as $\frac{m^2}{2}$ [55]. So, (5.21b) is the same to the one that arises in the study of the two dimensional system of Dunkl harmonic oscillator [60]. The solutions are described by (e_1, e_2) associatedto the eigenvalues which are $(1-2e_1, 1-2e_2)$ of the (R_1, R_2) where $(e_i \in \{0,1\})$. Thus, we conclude that

$$\Phi_n^{(e_1,e_2)}(\phi) = \eta_n^{(e_1,e_2)} \times cos^{e_1}\phi\; sin^{e_2}\phi\; P_{n-\frac{e_1}{2}-\frac{e_2}{2}}^{(\mu_1-\frac{1}{2}+e_1,\mu_2-\frac{1}{2}+e_2)}(-cos2\phi), \tag{5.22}$$

with $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomials [63]. We have that

$$\eta_n^{(e_1,e_2)} = \sqrt{(\frac{2n+\mu_1+\mu_2}{2})(n-\frac{e_1+e_2}{2})!} \times \sqrt{\frac{\Gamma(n+\mu_1+\mu_2+\frac{e_1+e_2}{2})}{\Gamma(n+\mu_1+\frac{1+e_1-e_2}{2})\Gamma(n+\mu_2+\frac{1+e_2-e_1}{2})}}. \tag{5.23}$$

Here $\Gamma(x)$ refers to the classical Gamma function [64], and this emphasis that the orthogonality relation is satisfied for the wave functions, namely

$$\int_0^{2\pi} \Phi_n^{(e_1,e_2)}(\phi)\Phi_{\acute{n}}^{(\acute{e}_1,\acute{e}_2)}(\phi)|\cos\phi|^{2\mu_1}\,|\sin\phi|^{2\mu_2}\,d\phi = \delta_{n\acute{n}}\delta_{e_1\acute{e}_1}\delta_{e_2\acute{e}_2}. \tag{5.24}$$

We report that B_ϕ is linked to J_3, namely

$$J_3^2 = 2B_\phi + 2\mu_1\mu_2(1 - R_1R_2)$$

Nevertheless, it is easily shown that the wave functions (5.22) coincide to that named Dunkl harmonics oscillators on the circle [65]. The separation constant to the solutions (5.22) has the formula $m^2 = 4n(n + \mu_1 + \mu_2)$. One finds that [55]

$$R_{\zeta,n}(r) = \xi_{\zeta,n} \times e^{\frac{-\beta r}{2}}(\beta r)^{2n} L_\zeta^{(4n+2\mu_1+2\mu_2)}(\beta r). \tag{5.25}$$

Thus, the Laguerre polynomials are $L_n^{(\alpha)}(x)$ [66]. We have $\beta = \sqrt{-8\varepsilon(\zeta, n)}$, so $\varepsilon(\zeta, n)$ is taken by (5.11). The normalization factor reads as

$$\xi_{\zeta,n} = \sqrt{\frac{\zeta!}{\Gamma(\zeta+4n+2\mu_1+2\mu_2+1)}} \times \sqrt{\frac{\beta^{2\mu_1+2\mu_2+2}}{(2\zeta+4n+2\mu_1+2\mu_2+1)}}, \tag{5.26}$$

thus

$$\int_0^\infty R_{\zeta,n}(r) R_{\dot{\zeta},n}(r) r^{2\mu_1+2\mu_2+1}\, dr = \delta_{\zeta\dot{\zeta}}\ . \tag{5.27}$$

According the results that above, $\Psi_{\zeta,n}(r,\phi)$ which is the eigenfunctions of the DC Hamiltonian (5.1) coinciding to the $\varepsilon(\zeta, n)$ which is energy values given by (4.64), namely [55]

$$\Psi_{\zeta,n}(r,\phi) = \mathrm{R}_{n,\zeta}(r)\ \Phi_n^{(e_1,e_2)}(\phi). \tag{5.28}$$

Thus, the radial part is given by (5.22) while the angular parts is given by (5.25). The wave functions under the scalar product are perpendicular taking into account [55]

$$< f, g >= \int_0^\infty \int_0^{2\pi} f^*(r,\emptyset) g(r,\emptyset) |r\cos\phi|^{2\mu_1}\ |\mathrm{r}\sin\phi\,|^{2\mu_2}\, r dr d\emptyset. \tag{5.29}$$

The Laplace–Dunkl operator is self-adjoint [60]. By taking $\mu_1 = \mu_2 = 0$, the regular wave functions of the two dimensional Kepler–Coulomb system are returned from (5.28). The equation of Schrödinger is linked to the DC system does not appear to allow the separation of the variables in any another coordinate system. This is in contrast with the classical two dimensional Kepler–Coulomb Hamiltonian [55,66,67].

CHAPTER 6

CONCLUSION

We recall that the superintegrable systems have huge applications in science and engineering. In my thesis, I presented a review on the superintegrability concepts and some of their applications.

At first I reviewed some basic definitions of the Hamiltonian approach and integrable systems. Particularly, the integrability with action angle variable was explained in detail.

After that the preliminaries of the two-dimensional cubically superintegrable systems were reviewed. We presented the analyses of the trigonometric state by integrating (3.3) (a) to obtain an explicit local formula of the metric together with the cubic integrals. The global issues are reviewed and it was shown that there is no closed manifold on which the superintegrable model can be defined. It was shown that the trigonometric state never leads to superintegrable systems defined on a closed manifold. Then, we showed the investigation of the hyperbolic state (purely imaginary eigenvalues) and we recall the integration of the differential equation (3.3)(b) providing an explicit formula for the related metric and the cubic integrals as well. The analysis of the affine state was reviewed.

After that, I reviewed the existence of fractional Killing vector and Killing-Yano tensors for the geometry induced by fractionalizing the classical free Lagrangian. I reviewed the result of one and two dimensional curved spaces. The expressions of the fractional ChS within Caputo derivative and the explicit solution to the fractional KY tensors and KV are displayed.

Finally, I reviewed the Dunkl-Coulomb (DC) system in the plane. I concluded that this model is exactly solvable and also superintegrable. The first integrals of motion were gained and the symmetry algebra they satisfy was presented and the separated solutions were showed explicitly in polar coordinates.

I hope my master thesis will be very useful for students and young researchers willing to study the new trends in superintegrable systems and their properties.

REFERENCES

1. **Miller W. Jr., Post S., Winternitz P., (2013),** "*Classical and Quantum Superintegrability with Applications*", Journal of Physics A: Mathematical and Theoretical, vol.46, no.42, pp.1-17.

2. **Defterli Ö., (2004),** "*Mathematical Aspects of Superintegrable Systems in Two Dimensions*", M.Sc. Thesis, Department of Mathematics and Computer Science, Cankaya University, Turkey.

3. **Baleanu D., Defterli Ö., (2004),** "*Killing-Yano Tensors and Superintegrable Systems*", Czechoslovak Journal of Physics, vol.54, no.11, pp.1215 –1222.

4. **Baleanu D., Defterli Ö., (2007),** "*Hidden Symmetries of Two Dimensional Superintegrable Systems*", In Mathematical Methods in Engineering, pp.159-166, Springer, The Netherlands.

5. **Beisert N., (2013),** "*Integrability in QFT and AdS/CFT*", Lecture Notes, Ecole Polytechnique Fédérale de Lausanne (EPF), Switzerland.

6. **Frolov V., (2012),** "*Applications of Hidden Symmetries to Black Hole Physics*", Journal of Physics: Conference Series, IOP Publishing, vol.283, no.1, pp.1-3.

7. **Dunajski M., (2012),** "*Integrable Systems*", Department of Applied Mathematics and Theoretical Physics, University of Cambridge, UK, pp.20-35.

8. **Dunajski, M., (2009),** "*Solitons, Instantons and Twistor* ", vol.19, OUP Oxford.

9. **Arnold V. I., (1989),** "*Mathematical Methods of Classical Mechanics Second Edition*", Springer, vol.60, p.271, New York.

10. **Valent G., Duval C., Shevchishin V., (2014),** "*Explicit Metrics for a Class of Two-Dimensional Cubically Superintegrable Systems*", Journal of Geometry and Physics, vol.87, pp.461-481.

11. **Koenigs G., Darboux G., (1972),** Note in "*Leçons Sur la Théorie Générale des Surfaces*", Chelsea Publishing, vol.4, pp.368.

12. **Matveev V.S., Shevchishin V.V., (2011),** "*Two Dimensional Super-integrable Metrics with One Liner and One Cubic Integral*", Journal of Geometry and Physics, vol. 61, no.8, pp.1353-1377.

13. **Valent G., (2010),** "*On a Class of Integrable Systems with a Cubic First Integral*", Communications in Mathematical Physics, vol.299, no.3, pp.631-649.

14. **Valent G., (2013),** "*On a class of Integrable Systems with a Quartic First Integral*", Regular and Chaotic Dynamics, vol. 18, no.4, pp.394-424.

15. **Thompson G., (1986),** "*Killing Tensors in Spaces of Constant Curvature*", Journal of Mathematical Physics, vol.27, no.11, pp.2693-2699.

16. **Crasmareanu M., Baleanu D., (2008),** "*New Aspects in Killing Tensors of Rank Two*", Romanian Journal Physics, vol.54, no.3-4, pp.275-280.

17. **Kerr R.P., (1963),** "*Gravitional Filed of a Spinning Mass as an Example of Algebraically Special Metrics*", Physical Review Letters, vol.11, no.5, pp.237.

18. **Carter B., (1968),** "*Hamilton-Jacobi and Schrödinger Separable Solutions of Einstein's Equations*", Communications Mathematical Physics, vol.10, no.4, pp.280-310.

19. **Walkerand M., Penrose R., (1970),** "*On Quadratic First Integrals of the Geodesic Equations for Type {22} Spacetimes*", Communications Mathematical Physics, vol.18, no.4, pp.265-274.

20. **Yano K., (1952),** "*Some Remarks on Tensor Fields and Curvature*", Annals of Mathematical, vol.55, no.2, pp.328-347.

21. **Penrose R., (1973),** "*Naked Singularities*", Annals of the New York Academy of Sciences, vol. 224, no.1, pp.125–134.

22. **Floyd R., (1973),** "*The Dynamics of Kerr Fields*", PhD. Thesis, University of London, England.

23. **Cotaescu I. I., Visinescu M., (2008),** "*Superalgebra of Dirac-Type Operators of the Euclidean Taub-NUT Space*", Fortschritte der Physik, vol.56, no.4-5, pp.400-405.

24. **Visinescu M., (1994),** "*The Geodesic Motion in Taub-NUT Spinning Space*", Classical Quantum Gravity, vol.11, no.7, pp.1867-1879.

25. **Cotaescu I.I., Visinescu M., (2000),** "*Schrödinger Quantum Modes on the Taub-NUT Background*", Modern Physics Letters A, vol.15, no.2, pp.145–157.

26. **Baleanu D., Başkal S., (2002),** "*Killing-Yano Symmetry for a Class of Spacetimes Admitting Parallel Null 1-Planes*", Nuovo Cimento, vol.117 B, no.5.

27. **Baleanu D., Başkal S., (2002),** "*Dual Metrics and Non-Generic Supersymmetries for a Class of Siklos Spacetimes*", International Journal of Modern Physics A, vol.17, no.26, pp.3737-3747.

28. **Frolov V. P., (2008),** "*Hidden Symmetries of Higher-Dimensional Black Hole Spacetimes*", Progress of Theoretical Physics Supplement, vol.172, pp.210-219.

29. **Teukolsky S. A., (1973),** "*Perturbations of a Rotating Black Hole .I. Fundamental Equations for Gravitational, Electromagnetic and Neutrino-Field Perturbations*", Astrophysical Journal, vol.185, pp.635-674.

30. **Unruh W., (1973),** "*Separability of the Neutrino Equations in a Kerr Background*", Physics Review Letters, vol.31, no.20, pp.1265.

31. **Chandrasekhar S., (1976),** "*The Solution of Dirac's Equation in Kerr Geometry*", Proceedings of the Royal Society London A: Mathematical, Physical and Engineering Sciences, vol.349, no.1659, pp.571-575.

32. **Page D. N., (1976),** "*Dirac Equation Around a Charged, Rotating Black Hole*", Physics Review D, vol.14, no.6, pp.1509.

33. **Boccaletti D., Pucacco G., (1997),** "*Killing Equations in Classical Mechanics*", Nuovo Cimento B, vol.112, pp.181–212.

34. **Baker A., (2003),** "*Matrix Groups. An Introduction to Lie Group Theory*", Springer Undergraduate Mathematics Series, 2nd Edition, Springer-Verlag London, Ltd., P.161-162.

35. **Pettini M., (2007),** "*Geometry and Topology to Hamiltonian Dynamics and Statistical Mechanics*", Springer, vol.33, P.141.

36. **Malkawi E., Baleanu D., (2014),** "*Fractional Killing-Yano Tensors and Killing Vectors Using the Caputo Derivative in Some One and Two-Dimensional Curved Space*", Abstract and Applied Analysis, vol.2014, pp.4.

37. **Baleanu D., Vacaru S.I., (2011),** "*Fedosov Quantization of Fractional Lagrange Spaces*", International Journal of Theoretical Physics.,vol.50, no.1, pp.233–243.

38. **Baleanu D., Vacaru S.I., (2011),** "*Fractional Almost Kähler-Lagrange Geometry*", Nonlinear Dynamics, vol.64, no.4, pp.365–373.

39. **Kilbas A.A., Srivastava H.M., Trujillo J.J., (2006),** "*Theory and Applications of Fractional Differential Equations*", North-Holland Mathematics Studies, vol. 204.

40. **Lakshmikantham V., Vatsala A.S., (2009),** "*Basic Theory of Fractional Differential Equations*", Nonlinear Analysis: Theory, Methods & Applications, vol.69, no.8, pp.2677–2682.

41. **Podlubny I., (1999),** "*Fractional Differential Equations*", Academic Press, San Diego, California, USA, vol.198.

42. **Sabatier J., Agrawal O.P., Machado J. A. T., (2007),** "*Advances in Fractional Calculus*: *Theoretical Developments and Applications in Physics and Engineering* ", Springer, Dordrecht.

43. **Miller K. S., Ross B., (1993),** "*An Introduction to the Fractional Calculus and Fractional Differential Equations*", John Wiley & Sons.

44. **Oldhamand K. B., Spanier J., (1974),** "*The Fractional Calculus; Theory and Applications of Differentiation and Integration to Arbitrary Order*" Mathematics in Science and Engineering, V, Academic Press, vol.111.

45. **Herrmann R., (2011),** "*Fractional Calculus: An Introduction for Physicists*", World Scientific, Singapore.

46. **Samko S. G., Kilbas A. A., Marichev O. I., (1993),** "*Fractional Integral and Derivatives Theory and Applications*", Gordon and Breach Science Publishers, Yverdon, Switzerland, pp.517-397.

47. **Van Holten J. W., (1995),**"*Supersymmetry and the Geometry of Taub-NUT*", Physics Letters, vol. 342, no.1–4, pp.47–52.

48. **Baleanu D., Defterli Ö., (2004),** "*Killing-Yano Tensors and Angular Momentum*", Czechoslovak Journal of Physics, vol.54, no.2, pp.157–165.

49. **Garfinkle D., Glass E.N., (2010),** "*Killing Tensors and Symmetries*", Classical and Quantum Gravity, vol. 27, no.9, Article ID 095004.

50. **Tachibana Sh., (1969),** "*On Conformal Killing Tensor in a Riemannian Space*", Tohoku Mathematical Journal, 2nd Series, vol.21, no.2, pp.56–64.

51. **Houri T., Yamamoto K., (2013),** "*Killing-Yano Symmetry of Kaluza-Klein Black Holes in Five Dimensions*", Classical and Quantum Gravity, vol.30, no. 7, Article ID075013.

52. **Baleanu D., Karasu A., (1999),** "*Lax Tensors, Killing Tensors and Geometric Duality*" , Modern Physics Letters A, vol.14, no.37, pp.2587–2594.

53. **Baleanu D., Başkal S., (2000),** "*Geometrization of the Lax Pair Tensors*", Modern Physics Letters A, vol. 15, no. 24, pp. 1503–1510.

54. **Güler Y., Baleanu D., Cenk M., (2003),** "*Surface Terms, Angular Momentum and Hamilton-Jacobi Formalism*", Nuovo Cimento, vol.118 B, no.3, pp.293–306.

55. **Genest V. X., Lapointe A., Vinet L.,(2015),** "*The Dunkl–Coulomb Problem in the Plane*", Physics Letters A, vol.379, no.12, pp.923-927.

56. **Dunkl C. F., (1989),** "*Differential-Difference Operators Associated to Reflection Groups*", Transactions of the American Mathematical Society, vol.311, no.1, pp.167-183.

57. **DeBie H., Ørsted B., Somberg P., Souĉek V., (2012),** "*Dunkl Operators and a Family of Realizations of Osp (1|2)*", Transactions of the American Mathematical Society, vol.364, no.7, pp.3875– 3902.

58. **Ben Saïd S., Kobayshi T., Ørsted B., (2012),** "*Laguerre Semigroup and Dunkl Operators*", Compositio Mathematica, vol.148, no.4, pp.1265-1336.

59. **Vilenkin N. J., Klimyk A. U., (1991),** "*Representation of Lie Groups and Special Functions*", vol.1, vol 72 of Mathematics and its Applications (Soviet Series).

60. **Genest V. X., Ismail M. E. H., Vinet L., Zhedanov A., (2013)**, "*The Dunkl Oscillator in the Plane: I, Superintegrability, Separated Wave Functions and Overlap Coefficients*", Journal of Physics A: Mathematical and Theoretical, vol.46, no.14, pp.145201.

61. **Genest V. X., Ismail M. E. H., Vinet L., Zhedanov A., (2014),** "*The Dunkl Oscillator in the Plane II: Representations of the Symmetry Algebra",* Communications in Mathematical Physics, vol.329, no.3, pp.999-1029.

62. **Lapointe A., (2014),** "*Le Problème de Dunkl–Coulomb Dans le Plan*", M.Sc. Thesis, Universitè de Montrèal.

63. **Koekoek R., Lesky P. A., Swarttouw R. F., (2010),** "*Hypergeometric Orthogonal Polynomials and Their q-Analogues*", 1st edition, Springer.

64. **Arfken G. B., Weber H., Harris F. E., (2012),** "*Mathematical Methods for Physicists*", 7th edition, Academic Press.

65. **Dunkl C. F., Y. Xu., (2001),** "*Orthogonal Polynomials of Several Variables*", 2nd edition, no.155, Cambridge University Press.

66. **Boiteux M., (1982),** "*Theory of Nonbijective Canonical Transformations in Mechanics: Application to the Coulomb Problem*", Journal of Mathematical Physics, vol.23, no.7, pp.1311-1314.

67. **Levi-Civita T., (1920)**, "*Sur la Régularisation du Problème des Trois Corps*", Acta Mathematica, vol.42, no.1, pp.99-144.

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