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# A hybrid analytical algorithm for thin film flow problem occurring in non-Newtonian fluid mechanics

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ABSTRACT

In this work, we investigate thin film flow of a third grade fluid down a inclined plane. The solution of a nonlinear boundary value problem (BVP) is derived by using an effective well organized computational scheme namely homotopy perturbation Elzaki transform method. Furthermore, this model is also resolved by Elzaki decomposition technique. The outcomes achieved by these two approaches are consistent with each other and because of that this technique may be regarded as an optional and effective scheme for determining results of linear and nonlinear BVP. Moreover, the homotopy perturbation Elzaki transform method leads over the Elzaki decomposition method since the nonlinear problems are solved without utilization of Adomian polynomials.



## 1. Introduction

Nonlinear differential equations (NDE) establish the non-linear case that emerge in several regions of engineering and scientific domains like solid state physics, fluid mechanics, biological models, plasma physics, financial models and social science models. Many difficulties that arise in scientific and engineering processes, mainly some fluid flow and heat transfer mathematical models are nonlinear, so few of these obstacles are resolved by numeric schemes and some are worked out by analytical perturbative technique [1–3]. In recent years several analytical and numerical techniques were introduced and developed to examine fluid flow problems [4–9]. In the analytical technique, accuracy and convergence are studied to keep away the different and unsuitable outcomes. For the scientific perturbation procedure, the small parameter is applied in the mathematical model [10]. The drawback in this technique that it includes the calculation of the small parameters (SP) and then use it into the mathematical model. The perturbation scheme is one of the common techniques that were considered by various investigators such as Bellman [11] and Cole [12]. In fact, both researchers had noticed to the mathematical features about the issue that incorporated a lack of physical inspection. This lack of physical inspection in this area was improved by Nayfeh [13] and Van Dyke [14].

Latterly, for the solution of BVPs, the scientific schemes possess ever amplifying the curiosity of the researchers, engineers and scientists. These methods are influenced by the perturbation approaches and have become demanding in natural challenges. The perturbation techniques, similar to other analytical schemes still have also their own drawbacks, such as all perturbation techniques have to introduce a SP in the mathematical model and solution of the mathematical model are presented in the series

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involving this parameter. Special skill is required for the selection of SP. That's why an computational scheme is used which does not need a SP in the mathematical model designing the problem.

As the drawbacks with this approach is that the condition of the general perturbation approach upon the presence of SP, so developing the technique for various uses is very tough. So, several types of new techniques are developed to remove the SP such as artificial parameter technique developed by Liu [15], the homotopy analysis technique [16–18], the variational iteration algorithm [19–23], the homotopy perturbation technique [24–30], the Adomian decomposition approach [31–33], the optimal homotopy asymptotic scheme [34,35], the homotopy perturbation sumudu transform methodology [36], the sumudu decomposition technique [37], homotopy perturbation Elzaki transform method (HPETM) [38], the Elzaki decomposition method (EDM) [39]. Recently, several mathematicians working in the field of nonlinear mathematical moelling have employed various techniques for handling physical problems [40–45].

Furthermore, we are applying the HPETM and the EDM to observe the solutions of nonlinear mathematical model controlling the thin film flow (TFF) of a third grade fluid (TGF) down a inclined plane, and numerical outcomes observed from these two methods are shown on graphical representation. The novelty of this work is that the HPETM and EDM are first time applied for examining the TFF of a TGF down a inclined plane. HPETM contributes the results in a quick convergent series which give rise to the results in a terminating series. The preference of these techniques is that it merges two powerful methods for achieving accurate and adjacent results for nonlinear differential mathematical models.

## 2. Elzaki transform

There are a number of integral transforms and broadly used in astronomy, engineering and also in physics. For solving differential models, the integral transforms are broadly used and consequently there are a number of works on the theory and use of integral transforms like the Fourier, Laplace, Hankel and Mellin to a name but a few. Successively, these transforms, in 2012, Elzaki et al. [46] have given a novel integral transform termed as the Elzaki transform (ET) and employed it to solve the various scientific problems. The ET is defined and expressed as [46]

$$E\,[\mathrm{f}(\mathrm{z})] \;= u\int_0^{\infty} \mathrm{f}\;(\mathrm{z})\;\mathrm{e}^{-\mathrm{z}/\mathrm{u}}\,\mathrm{dz},\; z>0. \tag{2.1}$$

The ET of the nth order derivative of a function $f(z)$ is expressed as [46]:

$$E[f^{(n)}(z)] = u^{-n}\bar{f}(u) - \sum_{k=0}^{n-1} u^{2-n+k} f^{(k)}(0). \tag{2.2}$$

## 3. Mathematical model

The mathematical model of thin film flow of a TGF down an inclined plane of inclination $\alpha \neq 0$ is expressed as [47,48]:

$$\frac{d^2w}{dz^2} + \frac{6(\beta_2+\beta_3)}{\mu}\left(\frac{dw}{dz}\right)^2\frac{d^2w}{dz^2} + \frac{\rho g \sin\alpha}{\mu} = 0, \tag{3.1}$$

$$w(0) = 0,\quad \frac{dw}{dz} = 0 \quad at\; z = \lambda. \tag{3.2}$$

Establishing the parameters

$$y = \lambda y_*,\quad w = \frac{\lambda^2 \rho g \sin\alpha}{\mu} w_*,$$

$$\beta_* = \frac{3\lambda^2\rho^2 g^2 \sin^2\alpha}{\mu^3}(\beta_2+\beta_3). \tag{3.3}$$

After excluding asterisks, Eqs. (3.1) and (3.2) takes the following form

$$\frac{d^2w}{dz^2} + 6\beta\left(\frac{dw}{dz}\right)^2\frac{d^2w}{dz^2} + 1 = 0, \tag{3.4}$$

$$w(0) = 0,\quad \frac{dw}{dz} = 0 \;\; at\; z = 1. \tag{3.5}$$

In Eqs. (3.1)–(3.5), $\mu$ represents dynamic viscosity, g indicates the gravity, $\rho$ denotes fluid density, $\beta_2, \beta_3$ indicate material constants and $\beta > 0$ gives material constant of a TGF. We observe that mathematical model (3.4) denotes an inhomogeneous and nonlinear differential mathematical model of second order with two boundary conditions (BCs). So, it is a well-defined problem.

On integrating Eq. (3.4), we find

$$\frac{dw}{dz} + 2\beta\left(\frac{dw}{dz}\right)^3 + z = C_1. \tag{3.6}$$

In Eq. (3.6) $C_1$ represents a constant. Applying the second BC of (3.5) in mathematical model (3.6), we get $C_1 = 1$. Therefore, Eqs. 3.4,3.5 can be represented as

$$\frac{dw}{dz} + 2\beta\left(\frac{dw}{dz}\right)^3 + (z-1) = 0, \tag{3.7}$$

$$w(0) = 0. \tag{3.8}$$

For $\beta = 0$, Eq. (3.8) relative to that of Newtonian fluid. The solution of Eq. (3.7) under the BCs (3.8) for $\beta = 0$ is expressed as below

$$w(z) = -\frac{1}{2}\left[(z-1)^2 - 1\right]. \tag{3.9}$$

In the next section, we derive the solution of the nonlinear model 3.7,3.8 with the aid of the HPETM and the EDM.

## 4. Solution by HPETM

### 4.1. Basic idea of HPETM

The HPETM is a joint form of HPM [24–30] and ET algorithm. To represent the fundamental plan of the HPETM, we take a nonlinear non-homogenous partial differential equation (NNHPDE) as shown below:

$$Lw(z) + Rw(z) + Nw(z) = g(z). \tag{4.1}$$

Here L represents linear differential operator (LDO) of highest order, R stands for the LDO of less order than L, N denotes nonlinear differential operator (NDO) and $g(z)$ indicates the term because of source.

Now using the ET on Eq. (4.1) and simplifying the resulting equation, we get

$$E\,[w(z)] = u^n\sum_{k=0}^{n-1} u^{2-n+k} w^{(k)}(0) + u^n E[g(z)] - u^n E[Rw(z) + Nw(z)]. \tag{4.2}$$

Now, using the inverse ET on (4.2), we have

$$w(z) = G(z) - E^{-1}[u^n E\,[Rw(z) + Nw(z)]]. \tag{4.3}$$

Here $G(z)$ indicates the term generated from the source term and the associated initial conditions (ICs).

Next, on making use of the HPM

$$w(z) = \sum_{m=0}^{\infty} p^m w_m(z) \tag{4.4}$$

and the nonlinear term can be expressed as

$$Nw(z) = \sum_{m=0}^{\infty} p^m H_m(w), \tag{4.5}$$

for some He's polynomials (HPs) [49,50] that are written as

$$H_m(w_0, w_1, \ldots, w_m) = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\left[N\left(\sum_{i=0}^{\infty} p^i w_i\right)\right]_{p=0}, m = 0, 1, \ldots . \tag{4.6}$$

The substitution of values from Eqs. (4.4) and (4.5) in Eq. (4.3), gives

$$\sum_{m=0}^{\infty} p^m w_m(z) = G(z) - p\left(E^{-1}\left[u^n E\left[R\sum_{m=0}^{\infty} p^m w_m(z) + \sum_{m=0}^{\infty} p^m H_m(w)\right]\right]\right), \tag{4.7}$$

which is the combined form of the ET and the HPM with the aid of HPs. Equating the coefficient of similar powers of $p$, we have

$$\begin{aligned}
p^0 &: \ w_0(z) = G(z), \\
p^1 &: \ w_1(z) = -E^{-1}[u^n E[R w_0(z) + H_0(w)]], \\
p^2 &: \ w_2(z) = -E^{-1}[u^n E[R w_1(z) + H_1(w)]], \\
p^3 &: \ w_3(z) = -E^{-1}[u^n E[R w_2(z) + H_2(w)]], \\
&\vdots \\
p^m &: \ w_m(z) = -E^{-1}[u^n E[R w_{m-1}(z) + H_{m-1}(w)]]].
\end{aligned} \tag{4.8}$$

Finally, The HPETM solution $w(z)$ is expressed as

$$w(z) = \lim_{N\to\infty} \sum_{m=0}^{N} w_m(z). \tag{4.9}$$

The above series solutions normally converge very speedily in a few terms. The convergence analysis of such kind of series is studied by Abbaoui and Cherruault [51].

*4.2. HPETM solution of the model*

$$E[w] = u^3 - u^4 - 2\beta u E\left[\left(\frac{dw}{dz}\right)^3\right]. \tag{4.10}$$

Applying the inverse ET, we have

$$w = z - \frac{z^2}{2} - 2\beta S^{-1}\left[uE\left[\left(\frac{dw}{dz}\right)^3\right]\right]. \tag{4.11}$$

Now on using the HPM, it gives

$$\sum_{m=0}^{\infty} p^m w_m(w) = z - \frac{z^2}{2} - 2\beta p\left(E^{-1}\left[uE\left[\sum_{m=0}^{\infty} p^m H_m(w)\right]\right]\right).$$

In the above expression $H_m(w)$ indicate HPs. The first some components of HPs are given below

$$\begin{aligned}
H_0 &= \left(\frac{dw_0}{dz}\right)^3, \\
H_1 &= 3\left(\frac{dw_0}{dz}\right)^2 \frac{dw_1}{dz}, \\
H_2 &= 3\frac{dw_0}{dz}\left(\frac{dw_1}{dz}\right)^2 + 3\left(\frac{dw_0}{dz}\right)^2 \frac{dw_2}{dz}, \\
H_3 &= \left(\frac{dw_1}{dz}\right)^3 + 6\frac{dw_0}{dz}\frac{dw_1}{dz}\frac{dw_2}{dz} + 3\left(\frac{dw_0}{dz}\right)^2 \frac{dw_3}{dz}, \\
&\vdots
\end{aligned} \tag{4.12}$$

The comparison of the coefficients of equal powers of $p$, enables us to get

$$\begin{aligned}
p^0 &: w_0(z) = z - \frac{z^2}{2}, \\
p^1 &: w_1(z) = \frac{\beta}{2}\left[(z-1)^4 - 1\right], \\
p^2 &: w_2(z) = -2\beta^2\left[(z-1)^6 - 1\right], \\
p^3 &: w_3(z) = 12\beta^3\left[(z-1)^8 - 1\right], \\
p^4 &: w_4(z) = -88\beta^4\left[(z-1)^{10} - 1\right], \\
p^5 &: w_5(z) = 632\beta^5\left[(z-1)^{12} - 1\right], \\
&\vdots
\end{aligned} \tag{4.13}$$

Thus, the HPETM solution is presented as

$$\begin{aligned}
w(z) = z - \frac{z^2}{2} &+ \frac{\beta}{2}\left[(z-1)^4 - 1\right] - 2\beta^2\left[(z-1)^6 - 1\right] \\
&+ 12\beta^3\left[(z-1)^8 - 1\right] - 88\beta^4\left[(z-1)^{10} - 1\right] \\
&+ 632\beta^5\left[(z-1)^{12} - 1\right] + \ldots
\end{aligned} \tag{4.14}$$

In Eq. (4.14) the terms which include the terms of $\beta$ provide the input of the non-Newtonian fluid. It is very important to see that by putting $\beta = 0$ in the expression (4.14), we get the solution of the Newtonian fluid problem. Therefore the first iteration of the nonlinear model 3.7,3.8 derived with the aid of the HPETM is similar to the exact solutions of the linear mathematical model. This reveals that the HPETM can be similarly used in handling linear models. The influence of the non-Newtonian parameter $\beta$ on the velocity profile presented in Eq. (4.14) is demonstrated in Fig. 1. It can be seen that as we reduce the value of $\beta$ the velocity profile enhances and converges to the Newtonian case.

## 5. Solution by EDM

*5.1. Basic idea of EDM*

The Elzaki decomposition method (EDM) is a combination of the ET and ADM. Let us consider a NNHPDE:

$$Lw(z) + Rw(z) + Nw(z) = g(z), \tag{5.1}$$

On applying the ET on Eq. (5.1), we have

$$E[w(z)] = u^n \sum_{k=0}^{n-1} u^{2-n+k} w^{(k)}(0) + u^n E[g(z)] - u^n E[Rw(z) + Nw(z)]. \tag{5.2}$$

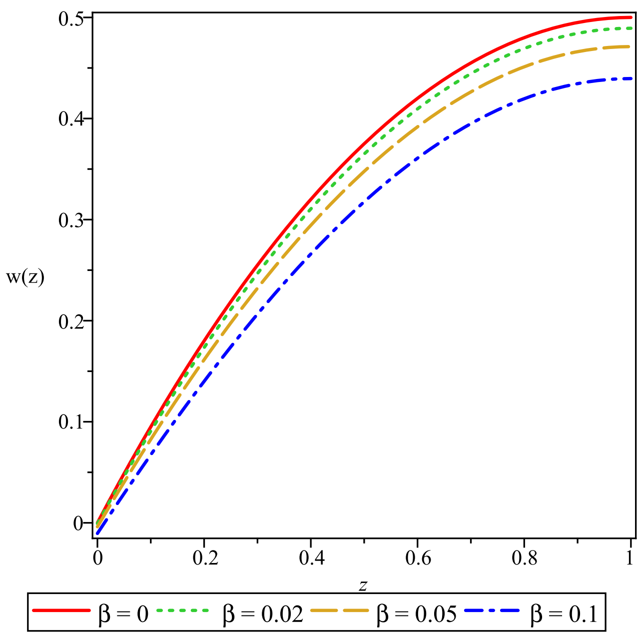


**Fig. 1.** Plots of velocity profile with respect to $z$ for distinct values of $\beta$.

Now, using the inverse ET on Eq. (5.2), we get

$$U(z) = G(z) - E^{-1}[u^n E[Rw(z) + Nw(z)]], \tag{5.3}$$

here $G(z)$ comprises the term which arise from the source term and the associated ICs.

The second step in EDM is that we express result as an infinite series presented as

$$w(z) = \sum_{m=0}^{\infty} w_m(z), \tag{5.4}$$

and the nonlinear term can be break down as

$$Nw(z) = \sum_{m=0}^{\infty} A_m, \tag{5.5}$$

where $A_m$ are Adomian's polynomials (APs) [52] of $w_0, w_1, w_2, \ldots, w_m$ and it can be determined by using the below given formula

$$A_m = \frac{1}{n!}\frac{d^m}{d\lambda^m}\left[N\left(\sum_{i=0}^{\infty}\lambda^i w_i\right)\right]_{\lambda=0}, \quad m = 0, 1, 2, \ldots \tag{5.6}$$

Using Eqs. (5.4) and (5.5) in Eq. (5.3), we get

$$\sum_{m=0}^{\infty} w_m(z) = G(z) - E^{-1}\left[u^n E\left[R\sum_{m=0}^{\infty} w_m(z) + \sum_{m=0}^{\infty} A_m\right]\right]. \tag{5.7}$$

On equating both sides of the Eq. (5.7), we have

$$w_0(z) = G(z), \tag{5.8}$$

$$w_1(z) = -E^{-1}[u^n E[Rw_0(z) + A_0]], \tag{5.9}$$

$$w_2(z) = -E^{-1}[u^n E[Rw_1(z) + A_1]]. \tag{5.10}$$

In general, the recursive formula is stated below

$$w_{m+1}(z) = -E^{-1}[u^n E[Rw_m(z) + A_m]], \; m \geqslant 1. \tag{5.11}$$

Now by solving Eq. (5.11), we get the values of $w_1, w_2, \ldots, w_m$.

### 5.2. Solution of the problem

On applying the ET on Eq. (3.7), we have

$$E[w] = u^3 - u^4 - 2\beta u E\left[\left(\frac{dw}{dz}\right)^3\right]. \tag{5.12}$$

The inverse Elzaki transform implies that

$$w = z - \frac{z^2}{2} - 2\beta E^{-1}\left[uE\left[\left(\frac{dw}{dz}\right)^3\right]\right]. \tag{5.13}$$

On using the scheme (ADM), if we suppose a solution expressed in the form of Eq. (5.4), we get

$$\sum_{m=0}^{\infty} w_m(z) = z - \frac{z^2}{2} - 2\beta E^{-1}\left[uE\left[\sum_{m=0}^{\infty} A_m(w)\right]\right], \tag{5.14}$$

where $A_m(w)$ are APs. The first few components of APs are as follows

$$\begin{aligned}
A_0 &= \left(\frac{dw_0}{dz}\right)^3, \\
A_1 &= 3\left(\frac{dw_0}{dz}\right)^2 \frac{dw_1}{dz}, \\
A_2 &= 3\frac{dw_0}{dz}\left(\frac{dw_1}{dz}\right)^2 + 3\left(\frac{dw_0}{dz}\right)^2 \frac{dw_2}{dz}, \\
A_3 &= \left(\frac{dw_1}{dz}\right)^3 + 6\frac{dw_0}{dz}\frac{dw_1}{dz}\frac{dw_2}{dz} + 3\left(\frac{dw_0}{dz}\right)^2 \frac{dw_3}{dz}, \\
&\vdots
\end{aligned} \tag{5.15}$$

The recursive relation is given below

$$\begin{aligned}
w_0(z) &= z - \frac{z^2}{2}, \\
w_1(z) &= -2\beta E^{-1}[uE[A_0(w)]], \\
w_{m+1}(z) &= -2\beta E^{-1}[uE[A_m(w)]].
\end{aligned} \tag{5.16}$$

The rest of terms of the EDM solution can be computed with the aid of above iterative scheme as follows

$$\begin{aligned}
w_1(z) &= \frac{\beta}{2}\left[(z-1)^4 - 1\right], \\
w_2(z) &= -2\beta^2\left[(z-1)^6 - 1\right], \\
w_3(z) &= 12\beta^3\left[(z-1)^8 - 1\right], \\
w_4(z) &= -88\beta^4\left[(z-1)^{10} - 1\right], \\
w_5(z) &= 632\beta^5\left[(z-1)^{12} - 1\right], \\
&\vdots
\end{aligned} \tag{5.17}$$

Adding all above terms, we can present the solution as follows

$$\begin{aligned}
w(z) = z - \frac{z^2}{2} &+ \frac{\beta}{2}\left[(z-1)^4 - 1\right] - 2\beta^2\left[(z-1)^6 - 1\right] \\
&+ 12\beta^3\left[(z-1)^8 - 1\right] - 88\beta^4\left[(z-1)^{10} - 1\right] \\
&+ 632\beta^5\left[(z-1)^{12} - 1\right] + \ldots
\end{aligned} \tag{5.18}$$

which is the same solution as derived with the aid of HPETM. As earlier by taking $\beta = 0$ in Eq. (5.18), we can get the exact solution

**Table 1**
Comparison of solutions using HPETM and EDM for $\beta = 0.04$.

| $z$ | HPETM | EDM |
|---|---|---|
| 0.0 | 0.0000000000 | 0.0000000000 |
| 0.2 | 0.1655347200 | 0.1655347200 |
| 0.4 | 0.2999347200 | 0.2999347200 |
| 0.6 | 0.3978547200 | 0.3978547200 |
| 0.8 | 0.4573747200 | 0.4573747200 |
| 1.0 | 0.4773427200 | 0.4773427200 |

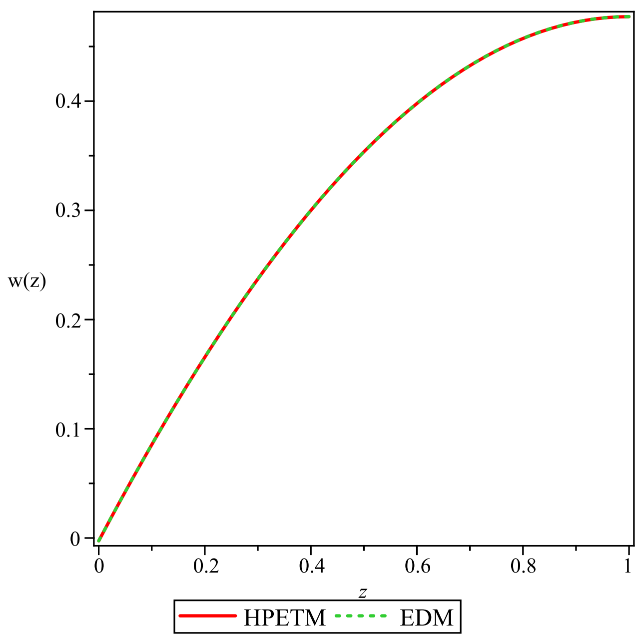


**Fig. 2.** Plots of the HPETM and the EDM solutions at $\beta$ = 0.04.

for the Newtonian fluid model. In Table 1, a comparative analysis between HPETM and EDM is presented. It can be noticed a the outcomes of both of the schemes are in a great agreement.

## 6. Conclusions

In this work, we have investigated the thin flow problem with a TGF and achieved its results by utilizing the homotopy perturbation Elzaki transform technique and the Elzaki decomposition technique. The comparison among the fifth order iterative solution of the HPETM and the six terms of the EDM is demonstrated in the form of Fig. 2. It is noticed that for $\beta = 0.04$, that two suggested computational techniques are in good agreement with each other. That's why, both approaches are very powerful and well organized algorithms for handling linear and nonlinear mathematical models which arise in distinct domains of science, finance and engineering. Moreover, the HPETM is advantageous over the EDM is that the nonlinear problems are solved without making use of the APs in this technique. To conclude, the HPETM and EDM can be considered as a great improvement in existing numerical schemes and find the vast range of utilities in handling nonlinear mathematical models.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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