Journal of Advanced Research 32 (2021) 149–160

Contents lists available at ScienceDirect

Journal of Advanced Research

journal homepage: www.elsevier.com/locate/jare

A hybrid fractional optimal control for a novel Coronavirus (2019-nCov) mathematical model

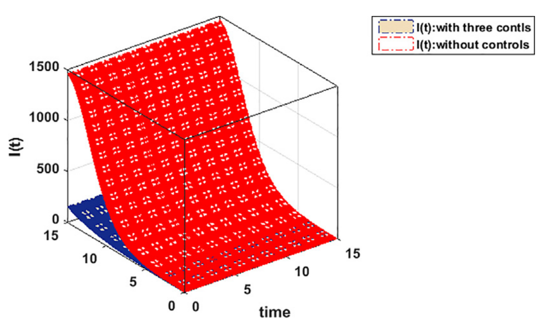
N.H. Sweilam [a,*], S.M. AL-Mekhlafi [b], D. Baleanu [c,d]

[a] Cairo University, Faculty of Science, Department of Mathematics, Giza, Egypt
[b] Sana'a University, Faculty of Education, Department of Mathematics, Yemen
[c] Cankaya University, Department of Mathematics, Turkey
[d] Institute of Space Sciences, Magurele-Bucharest, Romania

H I G H L I G H T S

- A novel mathematical model of Corona virus with new hybrid fractional operator derivative are presented.
- Three control variables are presented to minimize the number of infected population.
- Necessary control conditions are derived.
- Two numerical methods are constructed to study the behavior of the obtained fractional optimality system.
- The stability of the proposed methods are proved.
- Numerical simulations and comparative studies are given.

G R A P H I C A L A B S T R A C T



A R T I C L E I N F O

Article history:
Received 6 June 2020
Revised 23 July 2020
Accepted 10 August 2020
Available online 25 August 2020

Keywords:
Coronavirus diseases
A proportional derivative
Fractional order optimal control problems
Weighted average nonstandard finite difference method
Grünwald-Letnikov nonstandard finite difference method

A B S T R A C T

Introduction: Coronavirus COVID-19 pandemic is the defining global health crisis of our time and the greatest challenge we have faced since world war two. To describe this disease mathematically, we noted that COVID-19, due to uncertainties associated to the pandemic, ordinal derivatives and their associated integral operators show deficient. The fractional order differential equations models seem more consistent with this disease than the integer order models. This is due to the fact that fractional derivatives and integrals enable the description of the memory and hereditary properties inherent in various materials and processes. Hence there is a growing need to study and use the fractional order differential equations. Also, optimal control theory is very important topic to control the variables in mathematical models of infectious disease. Moreover, a hybrid fractional operator which may be expressed as a linear combination of the Caputo fractional derivative and the Riemann–Liouville fractional integral is recently introduced. This new operator is more general than the operator of Caputo's fractional derivative. Numerical techniques are very important tool in this area of research because most fractional order problems do not have exact analytic solutions.

Objectives: A novel fractional order Coronavirus (2019-nCov) mathematical model with modified parameters will be presented. Optimal control of the suggested model is the main objective of this work. Three control variables are presented in this model to minimize the number of infected populations. Necessary control conditions will be derived.

Peer review under responsibility of Cairo University.

* Corresponding author.
E-mail addresses: nsweilam@sci.cu.edu.eg (N.H. Sweilam), smdk100@gmail.com (S.M. AL-Mekhlafi), dumitru@cankaya.edu.tr (D. Baleanu).

https://doi.org/10.1016/j.jare.2020.08.006
2090-1232/

Methods: The numerical methods used to study the fractional optimality system are the weighted average nonstandard finite difference method and the Grünwald-Letnikov nonstandard finite difference method.
Results: The proposed model with a new fractional operator is presented. We have successfully applied a kind of Pontryagin's maximum principle and were able to reduce the number of infected people using the proposed numerical methods. The weighted average nonstandard finite difference method with the new operator derivative has the best results than Grünwald-Letnikov nonstandard finite difference method with the same operator. Moreover, the proposed methods with the new operator have the best results than the proposed methods with Caputo operator.
Conclusions: The combination of fractional order derivative and optimal control in the Coronavirus (2019-nCov) mathematical model improves the dynamics of the model. The new operator is more general and suitable to study the optimal control of the proposed model than the Caputo operator and could be more useful for the researchers and scientists.

Introduction

Coronavirus disease 2019 (COVID-19) is an infectious disease. In December 2019, the disease was first identified in China and rapidlied spread around that country and subsequently many others countries. It is reported that the virus might be bat origin, and the transmission of the virus might related to a seafood market (Huanan Seafood Wholesale Market) exposure. The genetic features and some clinical findings of the infection have been reported recently [10].

The spread of infectious diseases has serious effects on human society and healthy. The modeling study of infectious diseases is very useful in making strategies to control diseases [9]. Recently, many interesting papers in modeling the Coronavirus, see for example ([11–15]).

In general, mathematical models involved by the known ordinary differentiation could be used to capture dynamical systems of infectious disease, when only initial conditions are used to predict future behaviors of the spread. However, when the situation is unpredictable, which is the case of COVID-19, due to uncertainties associated to the pandemic, ordinal derivatives and their associated integral operators show deficient. The fractional order differential equations (FODEs) models seem more consistent with the real phenomena than the integer order models ([2–7]). This is due to the fact that fractional derivatives and integrals enable the description of the memory and hereditary properties inherent in various materials and processes. Hence there is a growing need to study and use the fractional order differential and integral equations. Moreover, the Caputo fractional derivative has been one of the most useful operators for modeling non-local behaviors by fractional differential equations [1].

Recently, Baleanu et. al., in [8] constructed a hybrid fractional operator which may be expressed as a linear combination of the Caputo fractional derivative and the Riemann–Liouville fractional integral. This new operator is more general than the operator of Caputo fractional derivative. In this work we will use this new derivative with an efficient nonstandard finite difference method (NSFDM) to study numerically the obtained fractional systems. The technique of the NSFDM was firstly proposed by Mickens [19]. Using this technique, some interesting real life applications are studied in ([16,17,20]).

Moreover, one of the new topics in mathematics is the fractional optimal control (FOC). FOC can be defined using varieties types of fractional derivatives definitions. Riemann–Liouville and Caputo fractional derivatives [20–23] can be considered the most important fractional derivatives definitions. Interesting numerical schemes for FOC are given in ([24–28]).

The main goal of this paper is to extend the mathematical model of Coronavirus given in [11] by using new hybrid fractional operator derivative. This operator can be written as a linear combination of a Riemann–Liouville integral with a Caputo derivative (CPC). We will introduce three control variables in order to minimize the number of the population of infected. Two numerical methods will be constructed to approximate the obtained fractional optimality system. These methods are: weighted average nonstandard finite difference method (WANFDM) and the Grünwald-Letnikov nonstandard finite difference method (GL-NSFDM). Stability analysis of the proposed methods will be proved. Comparative studies with Caputo derivative will be given.

To the best of our knowledge, a hybrid fractional optimal control for Coronavirus (2019-nCov) mathematical model has never been explored.

The organization of this article is as follows: The main mathematical formals will be given in Section 'Preliminaries and notations'. The proposed model with new fractional order derivatives and three controls are presented in Section 'Fractional order model of Coronavirus with control'. In Section 'The FOCPs', the formulation of the optimal control problem and the necessary optimality conditions are derived. In Section 'Numerical method for solving FOCPs', the numerical methods and there stability analysis are introduced. In Section 'Numerical experiments' numerical experiments with discussion are given. Finally, the conclusions are presented in Section 'Conclusions'.

Preliminaries and notations

In this section, we recall some important definitions of the fractional calculus used throughout the remaining sections of this paper.

- Let $0 < \alpha < 1, \Gamma$ be the Euler gamma function, then the Caputo fractional order derivative is defined as follows [1]:

$$ {}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} y'(s) ds, \tag{1}$$

- Let $y(t)$ be an integrable function, $0 < \alpha < 1$, then the Riemann–Liouville integral is defined as follows [1]:

$$ {}_0^{RL} D_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \tag{2}$$

- The new type of fractional operator is defined as a hybrid fractional operator from combining the proprotional and Caputo definition [8]:

$$ {}_0^{CP} D_t^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha} (K_1(\alpha,s) y(s) + K_0(\alpha,s) y'(s)) ds. \tag{3}$$

Table 1
The variables of system (6) [11].

The variable	Description
R	The class of recovery
H	The class of hospitalized
E	The class of exposed
I	The class of symptomatic and infectious
S	The class of susceptible
F	The class of fatality
P	The class of super-spreaders
A	The class of infectious but asymptomatic

Table 2
The parameters values for the Coronavirus model [11].

Parameter	Description	Value (per $day^{-\alpha}$)
β^α	Transmission coefficient from infected individuals	2.55^α
L	Relative transmissibility of hospitalized patients	1.56 dimensionless
β_1^α	Transmission coefficient due to super-spreaders	7.65^α
K^α	Rate at which exposed become infectious	0.25^α
ρ_1	Rate at which exposed people become infected I	0.580 dimensionless
ρ_2	Rate at which exposed people become super-spreaders	0.001 dimensionless
γ_a^α	Rate of being hospitalized	0.94^α
γ_i^α	Recovery rate without being hospitalized	0.27^α
γ_r^α	Recovery rate of hospitalized patients	0.5^α
δ_i^α	Disease induced death rate due to infected class	3.5^α
δ_p^α	Disease induced death rate due to super-spreaders	1^α
δ_h^α	Disease induced death rate due to hospitalized class	0.3^α

Let the kernels are given as follows: $K_0(\alpha,t)=\alpha C^{2\alpha}t^{(1-\alpha)}$, $K_1(\alpha,t)=(1-\alpha)t^\alpha$,

where $0<\alpha<1, C$ is constant. In the special case when K_0 and K_1 are independent of t, the new operators are given as follows:
Definition 2.1. The proprotial-Caputo hybrid operator is defined either as general way [8]:

$$\begin{aligned}{}^{CP}_0D_t^\alpha y(t) &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{-\alpha}(K_1(\alpha,s)y(s)+K_0(\alpha,s)y'(s))ds,\\ &= (K_1(\alpha,t)y(t)+K_0(\alpha,t)y'(t))\times\left(\tfrac{t^\alpha}{\Gamma(1-\alpha)}\right).\end{aligned} \tag{4}$$

Or as the following simple expression [8]:

$$\begin{aligned}{}^{CPC}_0D_t^\alpha y(t) &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{-\alpha}(K_1(\alpha)y(s)+K_0(\alpha)y'(s))ds\\ &= K_1(\alpha)\ {}^{RL}_0I_t^{1-\alpha}y(t)+K_0(\alpha)\ {}^{C}_0D_t^\alpha y(t),\end{aligned} \tag{5}$$

where, $K_1(\alpha), K_0(\alpha)$ are constants with respect to t and depending only on α. Also, in this paper we consider the kernels as follows: $K_0(\alpha)=\alpha C^{2\alpha}Q^{(1-\alpha)}$, $K_1(\alpha)=(1-\alpha)Q^\alpha$, where Q is constant and $C=1$.

Fractional order model of Coronavirus with control

Herein, we consider the recent Coronavirus spreading model given in [11] using a new hybrid fractional order derivative. This model consists of eight nonlinear differential equations. We change the order of the equations to α, the dimension of the left-hand side would be $(time)^{-\alpha}$. In order to have the dimensions match we should change the dimensions of the parameters. Also, when $\alpha\to 1$ the fractional order system reduces to classical one. Three controls, u_I, u_P, u_h are added in order to health care such as isolating patients in private health rooms and providing respirators and give them treatments soothing regularly. Let us assume that $\delta_i^\alpha=\delta_p^\alpha=\delta_h^\alpha=0$. The description of all the variables given in Table 1. Also, Table 2 describes the parameters. The CPC-modified model is then represented as follows:

$$\begin{aligned}
&{}^{CPC}_0D_t^\alpha S = -\beta^\alpha\tfrac{IS}{N} - L\beta^\alpha\tfrac{HS}{N} - \beta_1^\alpha\tfrac{PS}{N},\\
&{}^{CPC}_0D_t^\alpha E = \beta^\alpha\tfrac{IS}{N} + L\beta^\alpha\tfrac{HS}{N} + \beta_1^\alpha\tfrac{PS}{N} - K^\alpha E,\\
&{}^{CPC}_0D_t^\alpha I = K^\alpha\rho_1 E - (\gamma_a^\alpha+\gamma_i^\alpha)I - \delta_i^\alpha I - \nu u_I I,\\
&{}^{CPC}_0D_t^\alpha P = K^\alpha\rho_2 E - (\gamma_a^\alpha+\gamma_i^\alpha)P - \delta_p^\alpha P - \nu u_P P,\\
&{}^{CPC}_0D_t^\alpha A = K^\alpha(1-\rho_1-\rho_2)E,\\
&{}^{CPC}_0D_t^\alpha H = \gamma_a^\alpha(I+P) - \gamma_r^\alpha H - \delta_h^\alpha H - \nu u_h H + 0.5\nu u_I I + 0.5\nu u_P P,\\
&{}^{CPC}_0D_t^\alpha R = \gamma_i^\alpha(I+P) + \gamma_r^\alpha H + 0.5\nu u_I I + 0.5\nu u_P P + \nu u_h H,\\
&{}^{CPC}_0D_t^\alpha F = \delta_i^\alpha I + \delta_p^\alpha P + \delta_h^\alpha H,
\end{aligned} \tag{6}$$

where, $0<\nu\leqslant 1$. The existence and uniqueness of the solutions of (6) follow from the results given in [29]. The basic reproduction number of the proposed model (6) is given as follows [11]:

$$R_0 = \frac{\beta^\alpha\rho_1(\gamma_a^\alpha L+\chi_h)}{\chi_i\chi_h} + \frac{\beta^\alpha\rho_2\gamma_a^\alpha L+\rho_2\beta_1^\alpha\chi_h}{\chi_p\chi_h}. \tag{7}$$

where, $\chi_i=\gamma_a^\alpha+\gamma_i^\alpha+\delta_i^\alpha, \chi_p=\gamma_a^\alpha+\gamma_i^\alpha+\delta_p^\alpha$ and $\chi_h=\gamma_r^\alpha+\delta_h^\alpha$. The endemic threshold is given at $R_0=1$ and indicates the minimal transmission potential that sustains endemic disease, that is, when $R_0<1$, the disease will die out and for $R_0>1$ the disease may become endemic [30]. In this work we consider $R_0>1$.

The FOCPs

Consider the system (6) in $\mathbb{R}^8$, let

$$\Omega=\{(u_I(.),u_P(.),u_h(.))|u_I(.),u_P(.),u_h \textit{ are Lebsegue measurable on } [0,1],$$

$$0\leqslant u_I(.),u_P(.),u_h(.)\le 1, \forall t\in[0,T_f]\},$$

be the admissible control set. We will define the objective functional as follows:

$$J(u_I,u_P,u_h)=\int_0^{T_f}\left(I(t)+H(t)+B_1u_I^2(t)+B_2u_P^2(t)+B_3u_h^2(t)\right)dt. \tag{8}$$

The aim now is to find $u_I(t), u_P(t)$ and $u_h(t)$ such that the following cost functional is minimum:

$$J(u_I,u_P,u_H)=\int_0^{T_f}\eta(t,S,E,I,P,A,H,R,F,u_I,u_P,u_h)dt, \tag{9}$$

subject to the constraints

$${}^{CPC}_aD_t^\alpha\Psi_j=\xi_i. \tag{10}$$

Where

$$\begin{aligned}&\xi_i=\xi_i(t,S,E,I,P,A,H,R,F,u_I,u_P,u_h),\quad i,j=1,\ldots,8,\\ &\Psi_j=\{S,E,I,P,A,H,R,F\},\end{aligned}$$

$\Psi_1(0)=S_0, \Psi_2(0)=E_0, \Psi_3(0)=I_0, \Psi_4(0)=P_0, \Psi_5(0)=A_0, \Psi_6(0)=H_0, \Psi_6(0)=R_0, \Psi_6(0)=F_0$.

We will use a kind of Pontryagin's maximum principle in fractional order case, this idea is given by Agrwal in [23]:

Consider a modified cost functional as follows [25]:

$$\widetilde{J} = \int_0^{T_f}\Big[H(t,S,E,I,P,A,H,R,F,u_I,u_P,u_h) - \sum_{i=1}^{8}\lambda_i\xi_i(t,S,E,I,P,A,H,R,F,u_I,u_P,u_h)\Big]dt. \tag{11}$$

The Hamiltonian is define as follows:

$$\begin{aligned} &H(t,S,A,P,I,E,H,R,F,u_I,u_P,u_h,\lambda_i)\\ &= \eta(t,S,A,P,I,E,H,R,F,u_I,u_P,u_h,\lambda_i)\\ &+\sum_{i=1}^{8}\lambda_i\xi_i(t,S,E,I,P,A,H,R,F,u_I,u_P,u_h). \end{aligned} \tag{12}$$

From (11) and (12), we have:

$${}_t^{CPC}D_{t_f}^{\alpha}\lambda_\imath = \frac{\partial H}{\partial \vartheta_\imath}, \quad \imath = 1,\dots,8, \tag{13}$$

where,

$$\vartheta_\imath = \{t,S,E,I,P,A,H,R,F,u_I,u_P,u_h, \quad \imath = 1,\dots,8\},$$

$$0 = \frac{\partial H}{\partial u_k}, \quad k = I,P,h, \tag{14}$$

$${}_0^{CPC}D_t^{\alpha}\vartheta_\imath = \frac{\partial H}{\partial \lambda_\kappa}, \quad \imath = 1,\dots,8, \tag{15}$$

and it is also required that the Lagrange multipliers satisfies:

$$\lambda_\imath(T_f) = 0, \quad \imath = 1,2,\dots,8. \tag{16}$$

Theorem 4.1. *There exists optimal control variables u_I^*, u_P^*, u_h^* with the corresponding solutions $S^*, E^*, I^*, P^*, A^*, H^*, R_p^*, F^*$, that minimizes $J(u_I,u_p,u_h)$ over Ω. Furthermore, there exists adjoint variables λ_i, $i = 1,2,3,\dots,8$, satisfy the following:*

(i) adjoint equations:

$$\begin{aligned} {}_t^{CPC}D_{t_f}^{\alpha}\lambda_1 &= \lambda_1\Big(-\beta^{\alpha}\frac{I^*}{N} - L\beta^{\alpha}\frac{H^*}{N} - \beta_1^{\alpha}\frac{P^*}{N}\Big) + \lambda_2\Big(\beta^{\alpha}\frac{I^*}{N} + L\beta^{\alpha}\frac{H^*}{N} + \beta_1^{\alpha}\frac{P^*}{N}\Big),\\ {}_t^{CPC}D_{t_f}^{\alpha}\lambda_2 &= -K^{\alpha}\lambda_2 + \lambda_3K^{\alpha}\rho_1 + K^{\alpha}\rho_2\lambda_4 + K^{\alpha}(1-\rho_1-\rho_2)\lambda_5,\\ {}_t^{CPC}D_{t_f}^{\alpha}\lambda_3 &= -\lambda_1\beta^{\alpha}\tfrac{S^*}{N} + \lambda_2\beta^{\alpha}\tfrac{S^*}{N} - \lambda_3(\gamma_a + \gamma_i^{\alpha} + \delta_i^{\alpha} + \nu u_I^*) + \lambda_6(\gamma_a^{\alpha} + 0.5\nu u_I^*) + (\gamma_i^{\alpha} + 0.5\nu u_I^*)\lambda_7 + \delta_i^{\alpha}\lambda_8,\\ {}_t^{CPC}D_{t_f}^{\alpha}\lambda_4 &= -\lambda_1\beta_1^{\alpha}\tfrac{S^*}{N} + \lambda_2\beta_1^{\alpha}\tfrac{S^*}{N} - \lambda_4\big(\gamma_a^{\alpha} + \gamma_i^{\alpha} + \delta_p^{\alpha} + \nu u_P^*\big) + \lambda_6(\gamma_a^{\alpha} + 0.5\nu u_P^*) + (\gamma_i^{\alpha} + 0.5\nu u_P^*)\lambda_7 + \delta_p^{\alpha}\lambda_8,\\ {}_t^{CPC}D_{t_f}^{\alpha}\lambda_6 &= -\lambda_1\beta^{\alpha}L\tfrac{S^*}{N} + \lambda_2\beta^{\alpha}L\tfrac{S^*}{N} - \lambda_6(\gamma_r^{\alpha} + \delta_h^{\alpha} + (\gamma_a^{\alpha} + \nu u_h^*)) + \lambda_7(\gamma_r^{\alpha} + \nu u_P^*) + \delta_h^{\alpha}\lambda_8, \end{aligned} \tag{17}$$

where

$${}_t^{CPC}D_{t_f}^{\alpha}\lambda_5 = {}_t^{CPC}D_{t_f}^{\alpha}\lambda_7 = {}_t^{CPC}D_{t_f}^{\alpha}\lambda_8 = 0.$$

(ii) The transversality conditions

$$\lambda_\imath(T_f) = 0, \quad \imath = 1,2,\dots,8. \tag{18}$$

(iii) Optimality conditions:

$$\begin{aligned} &H(S,E,I,P,A,H,R,F,u_I,u_P,u_h,\lambda,t) =\\ &\min_{0\leqslant u_I,u_P,u_h\leqslant 1} H(S,E,I,P,A,H,R,F,u_I,u_P,u_h,\lambda,t). \end{aligned} \tag{19}$$

Moreover:

$$u_I^* = \min\Big\{1, \max\Big\{0, \frac{\nu I^*(\lambda_3 - 0.5\lambda_6 - 0.5\lambda_7)}{B_1}\Big\}\Big\}, \tag{20}$$

$$u_P^* = \min\Big\{1, \max\Big\{0, \frac{\nu P^*(\lambda_4 - 0.5\lambda_6 - 0.5\lambda_7)}{B_2}\Big\}\Big\}, \tag{21}$$

$$u_h^* = \min\Big\{1, \max\Big\{0, \frac{\nu H^*(\lambda_6 - \lambda_7)}{B_3}\Big\}\Big\}. \tag{22}$$

Proof. Eq. (17) can be obtained from (13, where:

$$\begin{aligned} H &= \lambda_1{}_0^{CPC}D_t^{\alpha}S^* + \lambda_2{}_0^{CPC}D_t^{\alpha}E^* + \lambda_3{}_0^{CPC}D_t^{\alpha}I^*\\ &+\lambda_4{}_0^{CPC}D_t^{\alpha}P^* + \lambda_5{}_0^{CPC}D_t^{\alpha}A^* + \lambda_6{}_0^{CPC}D_t^{\alpha}H^*\\ &+\lambda_7{}_0^{CPC}D_t^{\alpha}R^* + \lambda_8{}_0^{CPC}D_t^{\alpha}F^*\\ &+P^* + H^* + I^* + B_1u_I^* + B_2u_P^* + B_3u_h^*, \end{aligned} \tag{23}$$

is the Hamiltonian. $\lambda_\kappa(T_f) = 0$, $\kappa = 1,\dots,8$, are hold. Eqs. (20)–(22) can be obtained from (19). □

Now, by substituting u_I^*, u_P^*, u_h^* in (6):

$$\begin{aligned} &{}_0^{CPC}D_t^{\alpha}S^* = -\beta^{\alpha}\tfrac{I^*S^*}{N} - L\beta^{\alpha}\tfrac{H^*S^*}{N} - \beta_1^{\alpha}\tfrac{P^*S^*}{N},\\ &{}_0^{CPC}D_t^{\alpha}E^* = \beta^{\alpha}\tfrac{I^*S^*}{N} + L\beta^{\alpha}\tfrac{H^*S^*}{N} + \beta_1^{\alpha}\tfrac{P^*S^*}{N} - K^{\alpha}E^*,\\ &{}_0^{CPC}D_t^{\alpha}I^* = K^{\alpha}\rho_1E^* - (\gamma_a^{\alpha} + \gamma_i^{\alpha})I^* - \delta_i^{\alpha}I^* - \nu u_I^*I^*,\\ &{}_0^{CPC}D_t^{\alpha}P^* = K^{\alpha}\rho_2E^* - (\gamma_a^{\alpha} + \gamma_i^{\alpha})P^* - \delta_p^{\alpha}P^* - \nu u_P^*P^*,\\ &{}_0^{CPC}D_t^{\alpha}A^* = K(1-\rho_1-\rho_2)E^*,\\ &{}_0^{CPC}D_t^{\alpha}H^* = \gamma_a^{\alpha}(I^* + P^*) - \gamma_r^{\alpha}H^* - \delta_h^{\alpha}H^* - \nu u_hH^* + 0.5\nu u_I^*I^* + 0.5\nu u_P^*P^*,\\ &{}_0^{CPC}D_t^{\alpha}R^* = \gamma_i^{\alpha}(I^* + P^*) + \gamma_r^{\alpha}H^* + \nu u_h^*H^* + 0.5\nu u_I^*I^* + 0.5\nu u_P^*P^*,\\ &{}_0^{CPC}D_t^{\alpha}F^* = \delta_i^{\alpha}I^* + \delta_p^{\alpha}P^* + \delta_h^{\alpha}H^*. \end{aligned} \tag{24}$$

Numerical method for solving FOCPs

NWAFDM

Let us consider the following fractional order differential equation with the hybrid fractional operator:

$${}_0^{CP}D_t^{\alpha}y(t) = \xi(t,y(t)), \quad 0 < \alpha \leqslant 1, \quad y(0) = y_0. \tag{25}$$

We can discretize (25) by using definition (3) as follows:

$$\begin{aligned} &\frac{1}{\phi(\tau)^{\alpha-1}\Gamma(2-\alpha)}\sum_{i=0}^{n}\Big((1-\alpha)t_i^{\alpha}y_{n-i+1} + \alpha C^{2\alpha}t_i^{(1-\alpha)}\frac{y_{n-i+1}-y_{n-i}}{\phi(\tau)}\Big)\\ &\times\Big[(i+1)^{(1-\alpha)} - (i)^{(1-\alpha)}\Big] = \Theta\xi(t_{n+1},y(t_{n+1})) + (1-\Theta)\xi(t_n,y(t_n)), \end{aligned} \tag{26}$$

where,

$$\phi(\tau) = \tau + O(\tau^2), \quad 0 < \phi(\tau) < 1, \quad \tau \longrightarrow 0.$$

Also, we can discretize (25) by using definition (5) and using GL-approximation to approximate the Caputo fractional derivatives:

$$\begin{aligned}&\frac{Q^{\alpha}(1-\alpha)}{\phi(\tau)^{\alpha-1}\Gamma(2-\alpha)}\sum_{i=0}^{n+1} y_{n-i+1}\left[(i+1)^{(1-\alpha)} - (i)^{(1-\alpha)}\right]\\ &+\frac{\alpha C^{2\alpha} Q^{(1-\alpha)}}{\phi(\tau)^{\alpha}}\left(y_{n+1} - \sum_{i=1}^{n+1}\mu_i y_{n+1-i} - q_{n+1}y_0\right)\\ &= \Theta\xi(t_{n+1}, y(t_{n+1})) + (1-\Theta)\xi(t_n, y(t_n)),\end{aligned} \tag{27}$$

where, $K_0(\alpha) = \alpha C^{2\alpha} Q^{(1-\alpha)}$, $K_1(\alpha) = (1-\alpha)Q^{\alpha}, \omega_0 = 1, \omega_i = (1 - \frac{\alpha}{i})\omega_{i-1}$, $t^n = n\tau$, $\tau = \frac{T_f}{N_n}$, $N_n \in \mathbb{N}$. $\mu_i = (-1)^{i-1}\binom{\alpha}{i}$, $\mu_1 = \alpha$, $q_i = \frac{i^{\alpha}}{\Gamma(1-\alpha)}$ and $i = 1, 2, \ldots, n+1$. Additionally, consider ([18,24]):

$$0 < \mu_{i+1} < \mu_i < \ldots < \mu_1 = \alpha < 1,$$

$$0 < q_{i+1} < q_i < \ldots < q_1 = \frac{1}{\Gamma(1-\alpha)}.$$

The main advantage of this method is it can be explicit i. e., $(\Theta = 0)$ or implicit i. e., $(0 < \Theta < 1)$ or fully implicit i. e., $(\Theta = 1)$, the advantage of implicit case is it has large stability regions by using the idea of the weighed step introduced by the nonstandard finite difference method. In this article we will use the method given in (27).

Remark 1. In (27), if we put $K_0(\alpha) = 1$ and $K_1(\alpha) = 0$, we obtained the discretization of the Caputo fractional derivative as follows:

$$\begin{aligned}&\frac{1}{\phi(\tau)^{\alpha}}\left(y_{n+1} - \sum_{i=1}^{n+1}\mu_i y_{n+1-i} - q_{n+1}y_0\right)\\ &= \Theta\xi(t_{n+1}, y(t_{n+1})) + (1-\Theta)\xi(t_n, y(t_n)),\end{aligned} \tag{28}$$

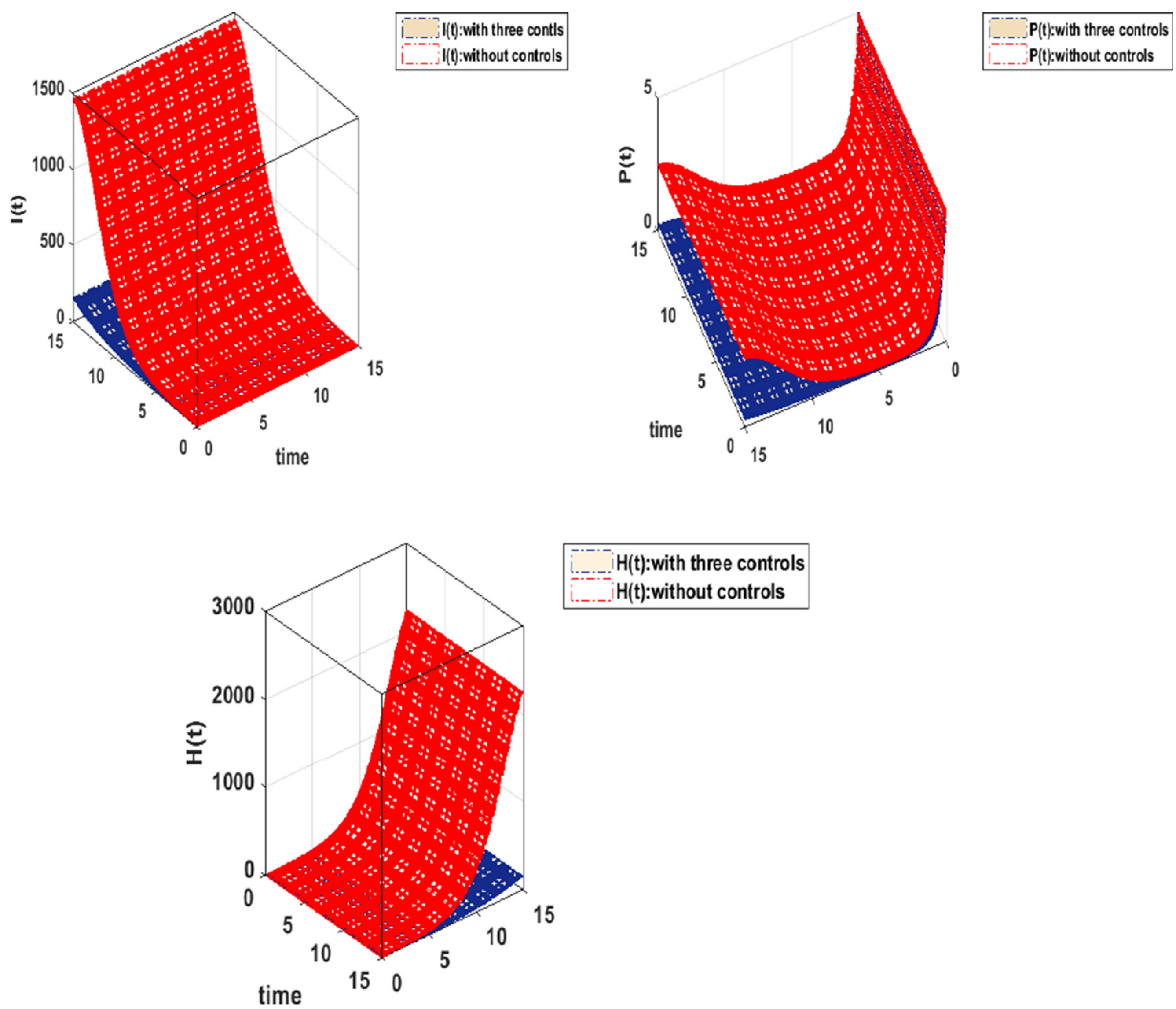


Fig. 1. Numerical simulations of the variables I, P and H with and without controls at $\alpha = 0.95, T_f = 15$ and $\Theta = 0.5$ using scheme (27).

GL-NSFDM

We can rewrite the relation (5) in another way as follows:

$$\begin{aligned} {}^{CPC}_{0}D^{\alpha}_{t}y(t) &= \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{-\alpha}(K_1(\alpha)y(s)+K_0(\alpha)y'(s))ds, \\ &= K_1(\alpha){}^{RL}_{0}I^{1-\alpha}_{t}y(t)+K_0(\alpha){}^{C}_{0}D^{\alpha}_{t}y(t), \\ &= K_1(\alpha){}^{RL}_{0}D^{\alpha-1}_{t}y(t)+K_0(\alpha){}^{C}_{0}D^{\alpha}_{t}y(t), \end{aligned} \tag{29}$$

where, $K_1(\alpha), K_0(\alpha)$ are constant with respect to t and depending only on α. Using GL-approximation and NSFDM, we can discretize (29) as follows:

$$\begin{aligned} {}^{CPC}_{0}D^{\alpha}_{t}y(t)|_{t=t^n} = \frac{K_1(\alpha)}{\phi(\tau)^{\alpha-1}}\left(y_{n+1}+\sum_{i=1}^{n+1}\omega_i y_{n+1-i}\right) \\ +\frac{K_0(\alpha)}{\phi(\tau)^{\alpha}}\left(y_{n+1}-\sum_{i=1}^{n+1}\mu_i y_{n+1-i}-q_{n+1}y_0\right), \end{aligned} \tag{30}$$

$$\begin{aligned} &\tfrac{K_1(\alpha)}{\phi(\tau)^{\alpha-1}}\left(y_{n+1}+\sum_{i=1}^{n+1}\omega_i y_{n+1-i}\right) \\ &+\tfrac{K_0(\alpha)}{\phi(\tau)^{\alpha}}\left(y_{n+1}-\sum_{i=1}^{n+1}\mu_i y_{n+1-i}-q_{n+1}y_0\right)=\xi(y(t_n),t_n), \end{aligned} \tag{31}$$

where,

$\phi(\tau)=\tau+O(\tau^2),\quad 0<\phi(\tau)<1,\quad \triangle(t)\longrightarrow 0,$

$\omega_0=1, \omega_i=(1-\frac{\alpha}{i})\omega_{i-1},\ t^n=n\tau,\quad \tau=\frac{T_f}{N_n},\quad N_n$ is a natural number.

$\mu_i=(-1)^{i-1}\binom{\alpha}{i},\quad \mu_1=\alpha,\quad q_i=\frac{i^{\alpha}}{\Gamma(1-\alpha)}$ and $i=1,2,\ldots,n+1.$

Additionally, let us assume that [18]:

$0<\mu_{i+1}<\mu_i<\ldots<\mu_1=\alpha<1,$

$0<q_{i+1}<q_i<\ldots<q_1=\dfrac{1}{\Gamma(1-\alpha)}.$

Stability of NWAFDM

In the following we will show that the NWAFDM in case $0<\Theta\leqslant 1$ (implicit case) is unconditionally stable. In order to investigate the stability of the proposed method when($\Theta\neq 0$), consider the following test problem of linear fractional differential equation:

$$({}^{CPC}_{0}D^{\alpha}_{t})y(t)=Ay(t),\quad t>0,\quad 0<\alpha\leqslant 1,\quad A<0. \tag{32}$$

Let $y(t_n)=y_n$ is the approximate solution of this equation then using GL-NFDM with (29) we rewrite Eq. (32) in the following form:

$$\begin{aligned} &=\tfrac{Q^{\alpha}(1-\alpha)}{\phi(\tau)^{\alpha-1}\Gamma(2-\alpha)}\sum_{i=0}^{n+1}y_{n-i+1}\left[(i+1)^{(1-\alpha)}-(i)^{(1-\alpha)}\right] \\ &\quad+\tfrac{C^{2\alpha}\alpha Q^{(1-\alpha)}}{(\phi(\tau)^{(\alpha)})}\left(y_{n+1}-\sum_{i=1}^{n+1}\mu_i y_{n+1-i}-q_{n+1}y_0\right) \\ &=\Theta Ay_{n+1}+(1-\Theta)Ay_n, \end{aligned} \tag{33}$$

put,

$$g_1=\frac{Q^{\alpha}(1-\alpha)}{\phi(\tau)^{\alpha-1}\Gamma(2-\alpha)},\quad W^{\alpha}=\left[(i+1)^{(1-\alpha)}-(i)^{(1-\alpha)}\right], g_2=\frac{C^{2\alpha}\alpha Q^{(1-\alpha)}}{\left(\phi(\tau)^{(\alpha)}\right)}.$$

We can write (33) as follows:

$$\begin{aligned} &g_1\sum_{i=0}^{n+1}y_{n-i+1}W^{\alpha}+g_2\left(y_{n+1}-\sum_{i=1}^{n+1}\mu_i y_{n+1-i}-q_{n+1}y_0\right) \\ &=\Theta Ay_{n+1}+(1-\Theta)Ay_n. \end{aligned} \tag{34}$$

Then,

$$\begin{aligned} &g_1 y_{n+1}+g_1\sum_{i=1}^{n}y_{n-i+1}W^{\alpha}+g_2\left(y_{n+1}-\sum_{i=1}^{n+1}\mu_i y_{n+1-i}-q_{n+1}y_0\right) \\ &=\Theta Ay_{n+1}+(1-\Theta)Ay_n, \end{aligned} \tag{35}$$

Table 3
Comparison between the values of objective functional with and without controls, for $T_f=60$, using scheme (27) and $\Theta=1$.

α	$J(u_l^*,u_p^*,u_h^*)$ without control	$J(u_l^*,u_p^*,u_h^*)$ with 3 controls $\phi(\tau)=(1-e^{-\tau})$	$J(u_l^*,u_p^*,u_h^*)$ with 3 controls $\phi(\tau)=0.1(1-e^{-\tau})$
1	5.9739×10^4	3.2372×10^4	2.0261×10^3
0.97	4.9343×10^4	2.6898×10^4	3.0983×10^3
0.92	3.4850×10^4	1.9303×10^4	4.7571×10^3
0.85	2.0857×10^4	1.1886×10^4	7.2660×10^3
0.70	2.5082×10^3	630.4559	373.6541

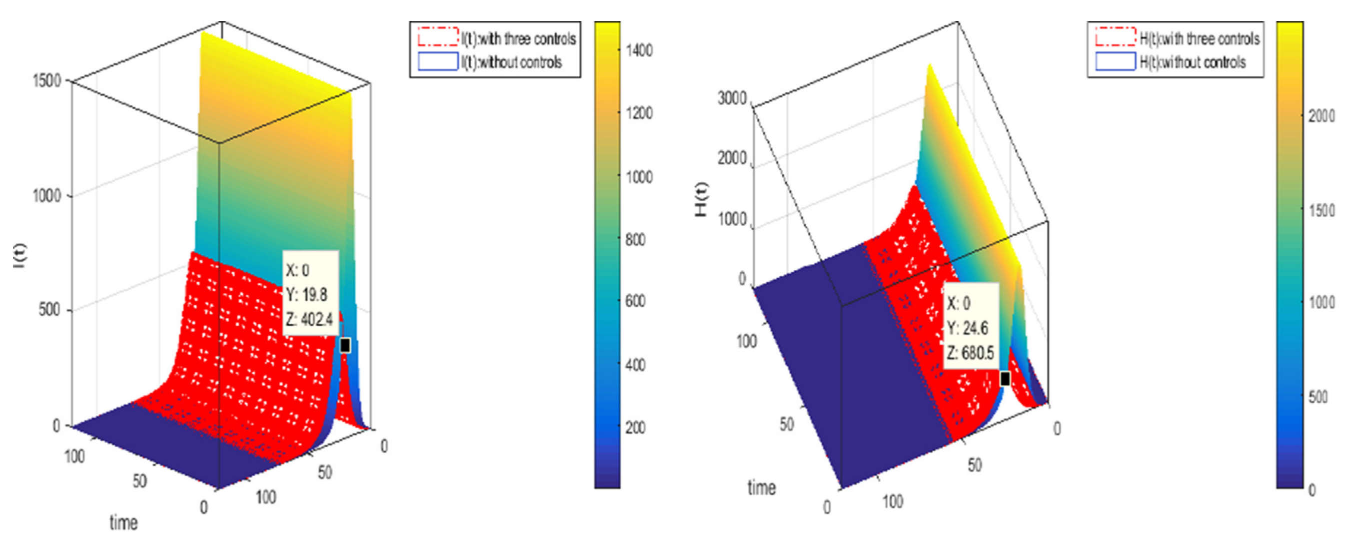


Fig. 2. Numerical simulations of the variables I and H with and without controls at $\alpha=0.95$ and $T_f=100$ and $\Theta=0.5$ using scheme (27).

Table 4
Comparison between the values of objective functional with three controls, for $T_f = 90$, using WANFDM (27), $\Theta = 0$.

The operator of fractional	α	$J(u_I^*, u_p^*, u_h^*)$ with 3 controls
CPC (27)	1	3.2651×10^4
CPC (31)		3.2651×10^4
Caputo(28)		3.2651×10^4
CPC (27)	0.99	3.0977×10^4
CPC (31)		3.0911×10^4
Caputo(28)		3.3816×10^4
CPC (27)	0.80	6.7811×10^3
CPC (31)		9.1050×10^3
Caputo(28)		3.9292×10^4
CPC (27)	0.75	4.1461×10^3
CPC (31)		4.3852×10^3
Caputo(28)		3.0509×10^4
CPC (27)	0.7	816.10564
CPC (31)		2.0535×10^3
Caputo(28)		1.7234×10^4

$$y_{n+1} = \frac{\left((1-\Theta)Ay_i - g_1\sum_{i=1}^{n} y_{n-i+1}W^{\alpha} + g_2\left(\sum_{i=1}^{n+1}\mu_i y_{n+1-i} + q_{n+1}y_0\right)\right)}{(g_1+g_2-\Theta A)}, \quad (36)$$

we have $\frac{1}{(g_1+g_2-\Theta A))} < 1$, hence

$y_1 \leqslant y_0,$

$y_0 \geqslant y_1 \geqslant \ldots \geqslant y_{n-1} \geqslant y_n \geqslant y_{n+1}.$

So the proposed implicit scheme is stable.

Stability of GL-NSFDM

In order to investigate the stability of the proposed method 37 consider the test problem of linear fractional differential Eq. (32). Using GL-approximation and NSFDM (29) we can discretize (32) as follows:

$$\frac{K_1(\alpha)}{\phi(\tau)^{\alpha-1}}\left(y_{n+1} + \sum_{i=1}^{n+1}\omega_i y_{n+1-i}\right) + \frac{K_0(\alpha)}{\phi(\tau)^{\alpha}}\left(y_{n+1} - \sum_{i=1}^{n+1}\mu_i y_{n+1-i} - q_{n+1}y_0\right) = Ay_n, \quad (37)$$

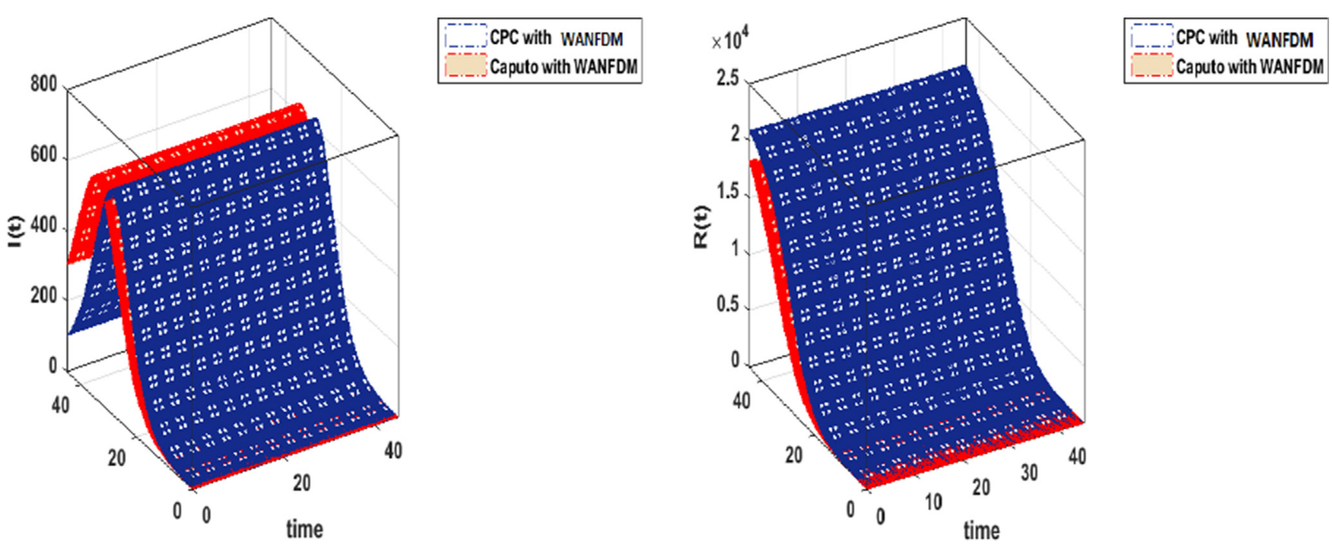


Fig. 3. Numerical simulations of I and R with control case at $\alpha = 0.98$ and $T_f = 45$ and $\Theta = 1$ using schemes (27) and (28).

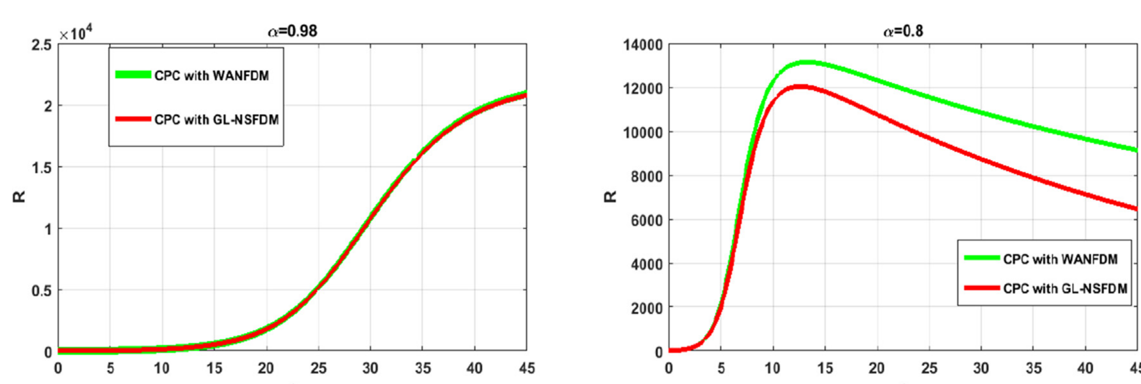


Fig. 4. Numerical simulations of $R(t)$ with control case using schemes (31) and (27).

put $C = \frac{K_1(\alpha)}{\phi(\tau)^{\alpha-1}}, \quad B = \frac{K_0(\alpha)}{\phi(\tau)^{\alpha}}$. Then, we have:

$$y_{n+1} = \frac{1}{C+B}\left(Ay_n - C\sum_{i=1}^{n+1}\omega_i y_{n+1-i} + B\left(\sum_{i=1}^{n+1}\mu_i y_{n+1-i} + q_{n+1}y_0\right)\right), \tag{38}$$

since, $C + B > 1$ then we have: $y_1 < y_0$ and $y_0 \geqslant y_1 \geqslant \ldots \geqslant y_{n-1} \geqslant y_n \geqslant y_{n+1}$. So the proposed scheme is stable.

Numerical experiments

In the following, numerical simulations for the models (17) and (24) without and with optimal control are presented. Two schemes (27), (31) are presented to solve the proposed model with the following initial conditions [11]: $S(0) = N - 6, R(0) = 0, A(0) = 0, F(0) = 0, E(0) = 0, P(0) = 5, I(0) = 1, H(0) = 0$. Then by using the nonstandard implicit finite difference method [27] we will solve

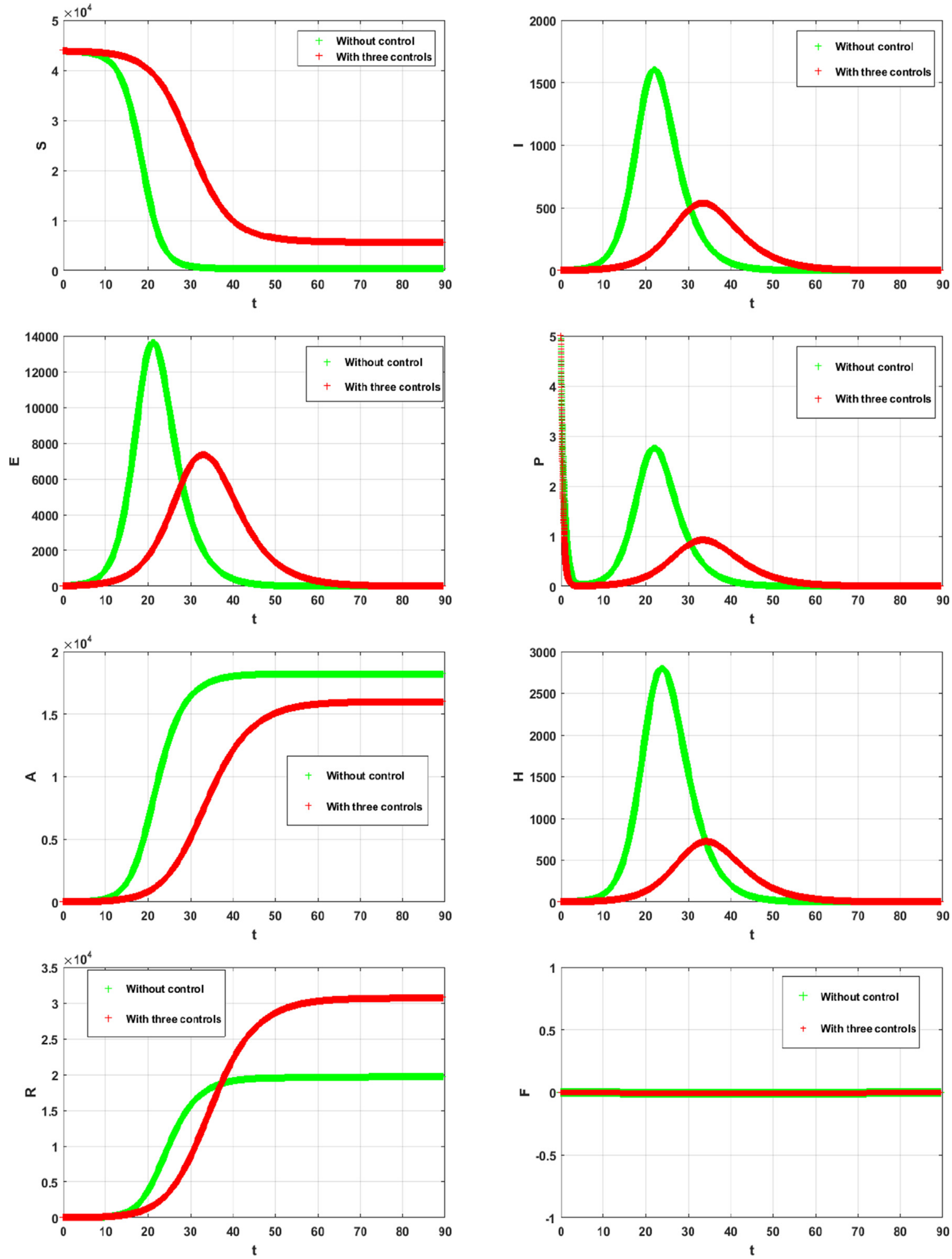


Fig. 5. Numerical simulations of the all variables of system (6) with and without controls at $\alpha = 0.99$ using scheme (31).

the co-state Eq. (17) with the transversality conditions (18). The controls are updated by using a convex combination of the previous controls and the value from the characterizations of u_I^*, u_P^* and u_h^* . This process is reiterated and the iteration is ended if the current state, the adjoint state, and the control values converge sufficiently. In this case we use different values of $0 < \alpha \leqslant 1$ with $B_1 = 100, B_2 = 50$, and $B_3 = 100, \nu = 1$. In the numerical simulations the time level is chosen in days.

Fig. 1 demonstrate the effective of three controls case for the proposed model (6) using the scheme WANFDM (27) at final time equal 15 and $\Theta = 0.5$. We noted that in uncontrolled case, the number of the population of I, P, H are increasing, while the number of these population are decreasing in controlled case in the same interval. Moreover, when the final time equal 100, as we can see in Fig. 2 the population number of I, P, H are increasing in interval $(0, 25)$, in uncontrolled case while the number of

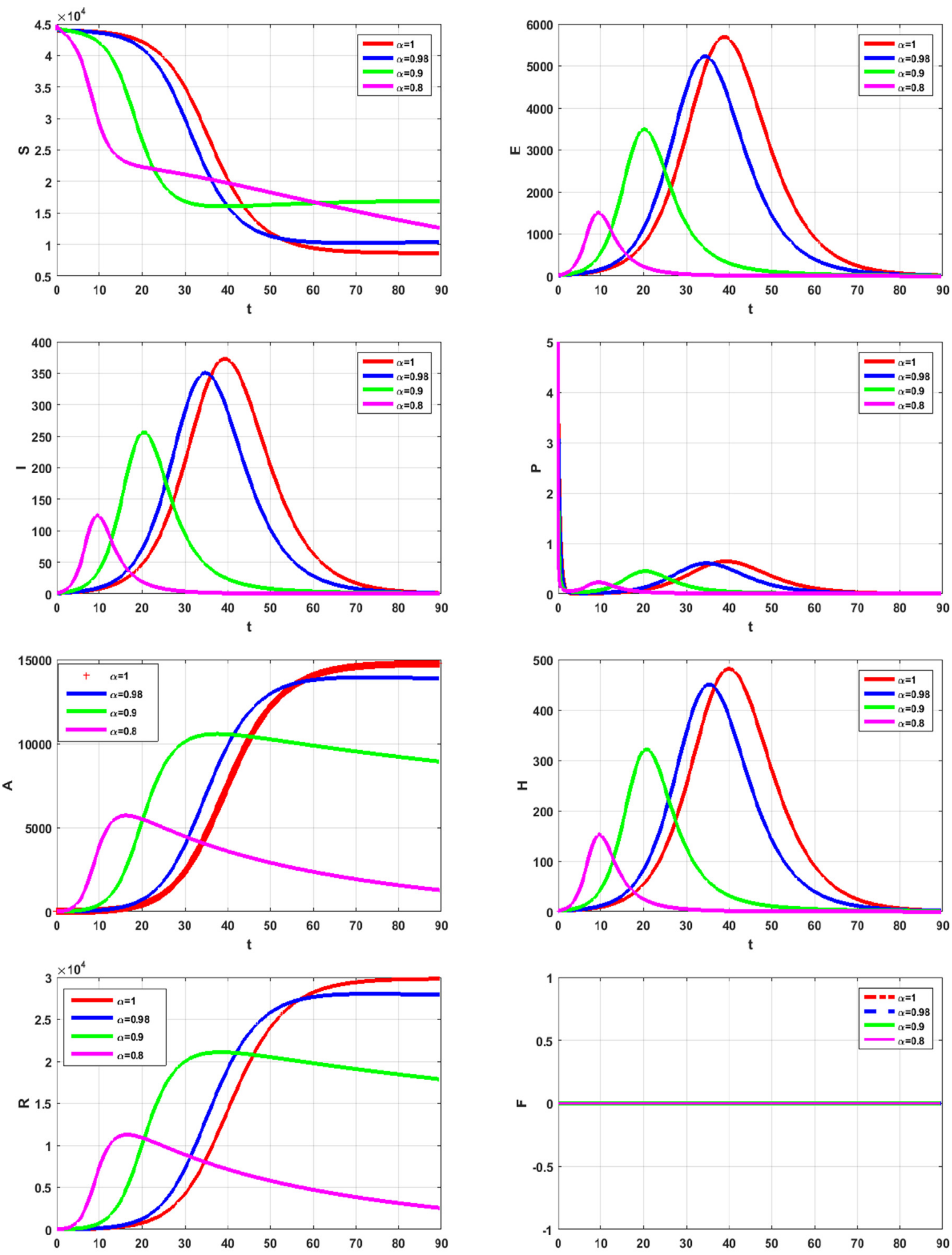


Fig. 6. Numerical simulations of the all variables of system (6) at different α and three controls case using scheme (31).

these population are decreasing in controlled case in the same interval.

Table 3 reports the cost functional values for the scheme (27) at fully implicit case with and without controls at different α and $\phi(\tau)$. We have the best results in controlled case at $\phi(\tau) = 0.1(1 - e^{-\tau})$.

A comparison between cost functional values derived by three schemes (27), (28) and (31) with three controls at $T_f = 90$, is given in Table 4, where the scheme (28) is a special case for the schemes (27) and (31) when we put $K_0 = 1, K_1 = 0$. We concluded that when $\alpha = 1$, all schemes give the same result of the objective functional, also in interval $0 < \alpha < 0.8$ the difference of the schemes are very small and almost the scheme (31) gives the best results, while at interval $0.8 < \alpha < 0.7$ the scheme (27) gives the best results. This mean that the new operator derivative CPC is more general and suitable to study the optimal control of the biological phenomena than the Caputo operator derivative.

Fig. 3 shows how the behavior of I and R are change when we use the general scheme (27) with the new operator derivative CPC and the Caputo derivative. We noted that the results which obtained by (27) are the best, because the number of I which obtained by (27) is less than the number of I which obtained by (28), also, the number of R which obtained by (27) is bigger than the number of R which obtained by (28). This mean that, the new operator CPC is more suitable to describe the biological phenomena than the Caputo operator.

Fig. 4 shows comparesion between the results obtained by the two schemes (27) and (31) at $\alpha = 0.98$ and $\alpha = 0.8$. We noted that at $\alpha = 0.8$, the number of R which obtained by scheme (27) is bigger than the number of R which obtained by (31), this mean that, the scheme (27) is the best to study the optimal control problems.

Fig. 5 shows the behavior of the solutions for the proposed model (6) using (31) in controlled and uncontrolled cases. Fig. 6 shows how the behavior of the solutions in control case are changing by using different values of α and $T_f = 90$ using (31). Fig. 7 shows how the behavior of the solutions I, P and H in control case are changing by using different values of α and $T_f = 300$ using (27).

Fig. 8 shows the evolution of the approximate solutions for the control variables with several values of α. We noted that the range of the solutions remain between zero and one.

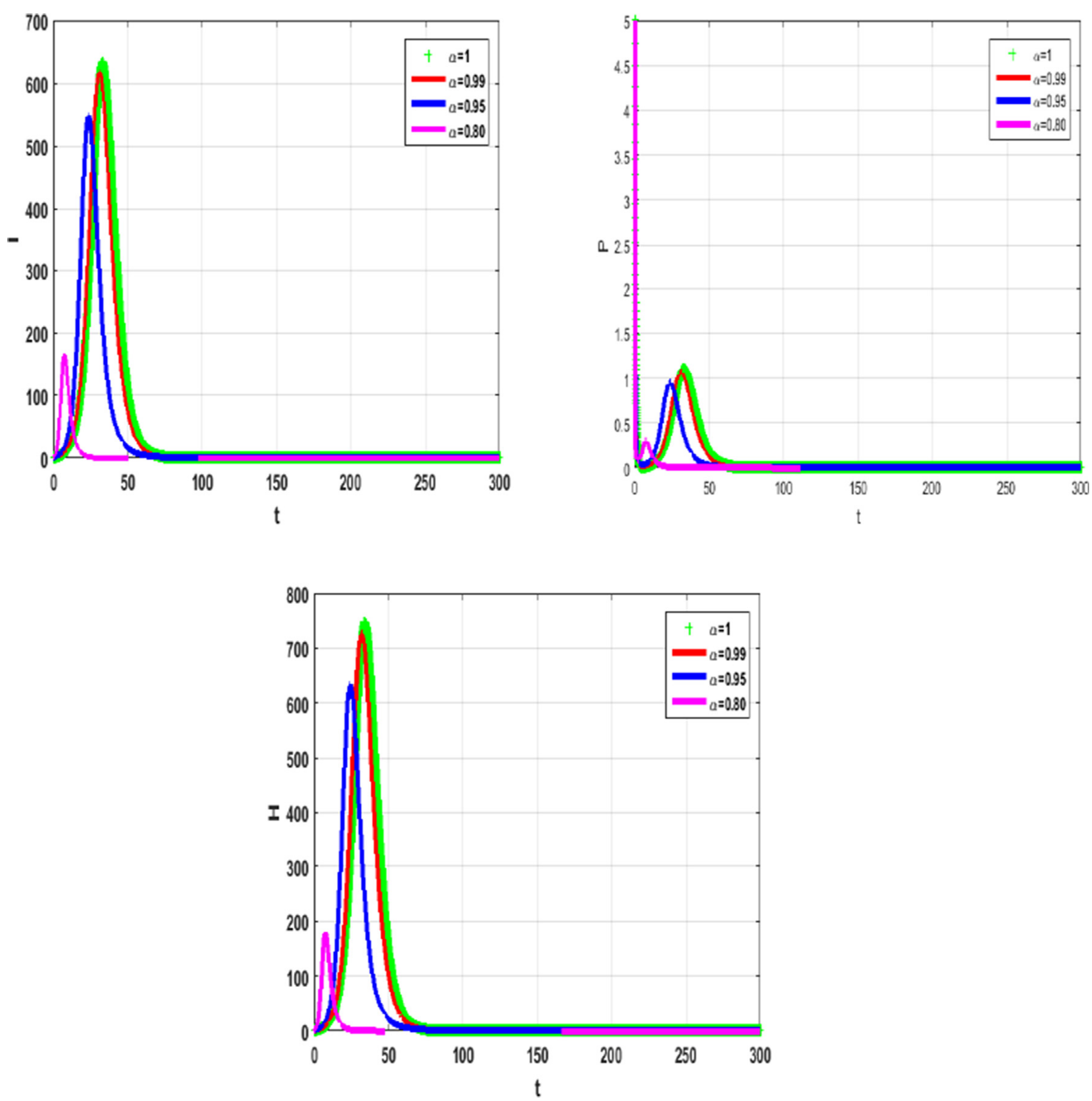


Fig. 7. Numerical simulations of I, P and H at different α and three controls case using scheme (27) and $\Theta = 1$.

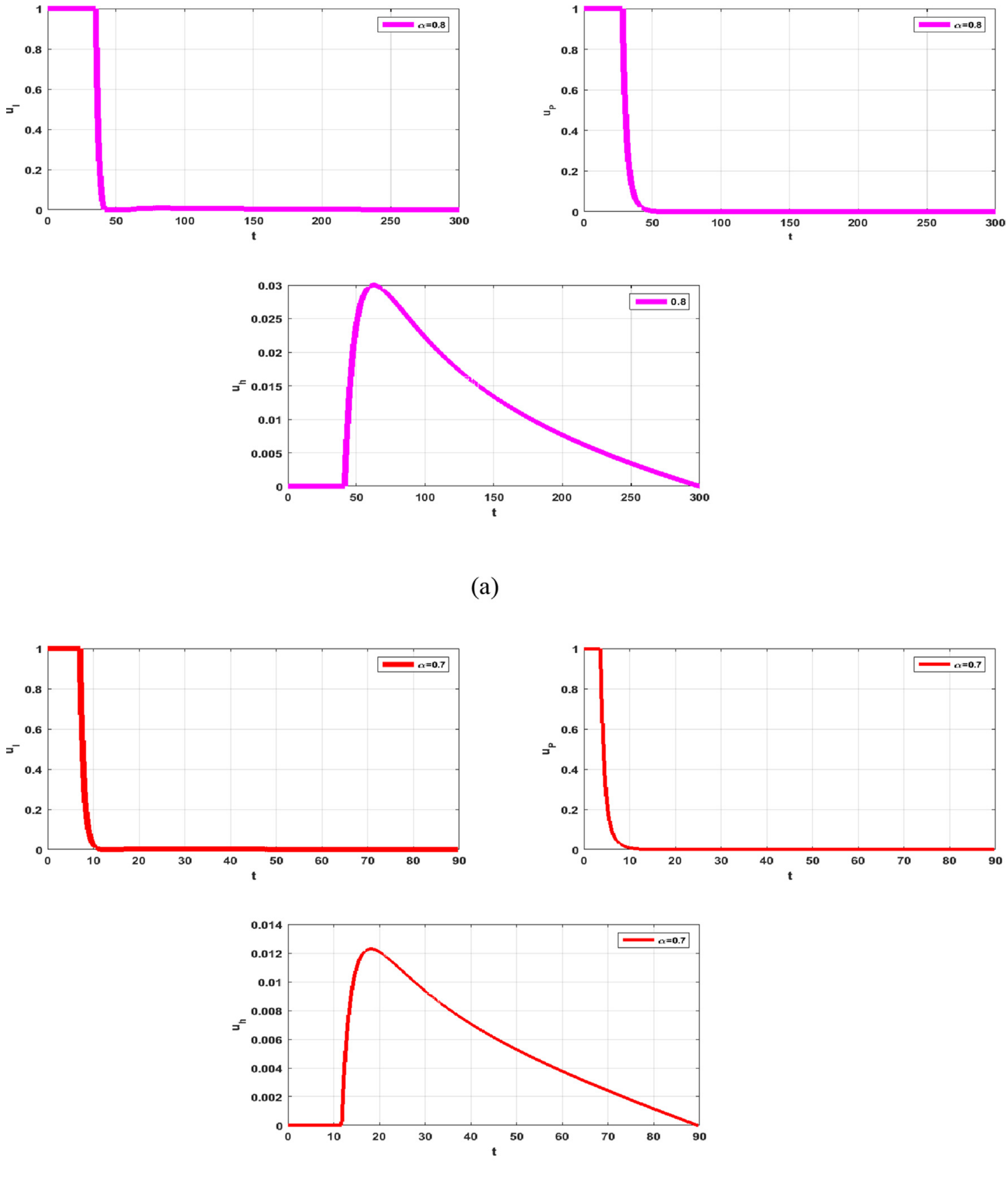


Fig. 8. Numerical simulations of u_I, u_P and u_h for the system (6) at (a)$\alpha = 0.8$ and $T_f = 300$ and (b) $\alpha = 0.7$ and $T_f = 90$ using scheme (27) and $\Theta = 1$.

Table 5
Comparison between the values of I, P and H with and without controls, $T_f = 30$, using WANFDM (27), $\Theta = 0.5$.

The Controls	α	I	P	H
With	1	489.5754	5.1367×10^3	633.4476
Without		581.1148	1.6621×10^4	1.3685×10^3
With	0.98	448.6273	3.7270×10^3	474.3764
Without		636.5596	1.5491×10^4	1.6026×10^3
With	0.90	127.7800	1.0190×10^3	127.7851
Without		1.1239×10^3	1.0617×10^3	1.9801×10^3
With	0.8	28.0352	244.5881	27.5343
Without		85.8616	4.3370×10^3	1.0642×10^3

Table 6
CPU time in seconds for the solution of optimality systems at different values of α and $\Theta = 0$.

α	CPU time of CPC (27)	CPU time of CPC (31)	CPU time of Caputo (28)
1	2.457034	2.445327	2.561605
0.98	4.661529	2.191416	4.807198
0.90	4.936312	2.176197	4.983086
0.8	4.984231	2.130180	4.965147

Table 5 reports the values of the objective functional obtained by the scheme (27) with and without controls at different values of α, $\Theta = 0.5$. Table 6 shows the CPU time for the optimality systems using NWAFDM (27) and GL-NSFDM(31) with CPC definition and NWAFDM (28) with Caputo definition at different values of α. We noted that the second method GL-NSFDM is faster than the first and third methods.

Conclusions

In the present work, the optimal control of Coronavirus model with new fractional operator is presented. This operator can be written as a linear combination of a Riemann–Liouville integral with a Caputo derivative. This dynamical model is more suitable to describe the biological phenomena with memory than the integer order model. Three control variables, $u_l(t), u_p(t)$ and $u_h(t)$ are introduced in order to health care such as isolating patients in private health rooms and providing respirators and give them treatments soothing regularly. These have been implemented to minimize the number of infected population. Necessary optimality conditions are derived. Also, the combination of fractional order derivative and optimal control in the model improves the dynamics and increases complexity of the model. Two schemes are constructed to study the behavior of the proposed problems. We can conclude from the obtained numerical results that the new operator derivative CPC is more general and suitable to study the optimal control of the biological phenomena than the Caputo operator derivative. Moreover, the WANFDM (27) is depending on the values of the factor Θ, it can be explicit or implicit with large stability regions. This scheme is the best for solve the obtained fractional optimality system. Numerical simulations are presented to support our theoretical findings. Moreover, the CPC fractional derivative provides best results and could be more useful for the researchers and scientists.

Declaration of Competing Interest

The authors have declared no conflict of interest.

Compliance with Ethics Requirements

This article does not contain any studies with human or animal subjects.

References

[1] Podlubny I. Fractional differential equations. New York: Academic Press; 1999.
[2] Carvalho ARM, Pinto CMA. Non-integer order analysis of the impact of diabetes and resistant strains in a model for TB infection. Commun Nonlinear Sci Numer Simulat 2018;61:104–26.
[3] Sweilam NH, AL-Mekhlafi SM, Hassan AN. Numerical treatment for solving the fractional two-Group Influenza model. Progr Fract Differ Appl 2018;4:1–15.
[4] Kumar S, Ghosh S, Lotayif MSM, Samet B. A model for describing the velocity of a particle in Brownian motion by Robotnov function based fractional operator. Alexandria Eng J 2020. doi: https://doi.org/10.1016/j.aej.2020.04.019.
[5] Rihan FA, Baleanu D, Lakshmanan S, Rakkiyappan R. On fractional SIRC model with Salmonella bacterial infection. Abstract Appl Anal 2014:1–9.
[6] Machado JAT. Fractional-order derivative approximations in discrete-time control systems. Syst Anal Model Simul 1999;34:419–34.
[7] Dehghan M, Hamedi E, Khosravian-Arab H. A numerical scheme for the solution of a class of fractional variational and optimal control problems using the modified Jacobi polynomials. J Vib Control 2016;22:1547–59.
[8] Baleanu D, Fernandez A, Akgül A. On a fractional operator combining proportional and classical differintegrals, mathematics. 2020;8(3). doi:10.3390/math8030360
[9] Brauer F, Driessche P, Wu J. Mathematical epidemiology. Springer; 2008.
[10] World Health Organization. Coronavirus. World Health Organization, cited January 19, 2020. Available: https://www.who.int/health-topics/coronavirus.
[11] Ndäırou F, Area I, Nieto JJ, Torres DFM. Mathematical modeling of COVID-19 transmission dynamics with a case study of Wuhan, Chaos. Solit Fractals 2020. doi: https://doi.org/10.1016/j.chaos.2020.109846.
[12] Khan MA, Atangana A. Modeling the dynamics of novel Coronavirus (2019-nCov) with fractional derivative. Alexandria Eng J 2020. doi: https://doi.org/10.1016/j.aej.2020.02.033.
[13] Chen T-M, Rui J, Wang Q-P, Zhao Z-Y, Cui J-A, Yin L. A mathematical model for simulating the phase-based transmissibility of a novel coronavirus. 2020: 9–24 ,https://doi.org/10.1186/s40249-020-00640-3.
[14] Atangana A. Modelling the spread of COVID-19 with new fractal-fractional operators: Can the lockdown save mankind before vaccination? Chaos, Solit Fractals 2020;136:109860.
[15] Ivorra B, Ferrández MR, Vela-Pérez M, Ramos AM. Mathematical modeling of the spread of the corona virus disease 2019 (COVID-19) taking into account the undetected infections. The Case of China 2020. http://www.doi.org/10.13140/RG.2.2.21543.29604.
[16] Arenas AJ, Gonzàlez-Parra G, Chen-Charpentierc BM. Construction of nonstandard finite difference schemes for the SI and SIR epidemic models of fractional order. Mathe Comput Simulat 2016;121:48–63.
[17] Iqbal Z, Ahmed N, Baleanu D, Adel W, Rafiq M, Rehman MA, Alshomrani AS. Positivity and boundedness preserving numerical algorithm for the solution of fractional nonlinear epidemic model of HIV/AIDS transmission. Chaos, Solit Fractals 2020;134:109706.
[18] Scherer R, Kalla S, Tang Y, Huang J. The Grünwald-Letnikov method for fractional differential equations. Comput Math Appl 2011;62:902–17.
[19] Mickens R. Nonstandard finite difference models of differential equations. Singapore: World Scientific; 1994.
[20] Hajipour M, Jajarmi A, Baleanu D. An efficient nonstandard finite difference scheme for a class of fractional chaotic systems. J Comput Nonlinear Dyn 2018;13:1–9. doi: https://doi.org/10.1115/1.4038444.
[21] Zaky MA, Tenreiro Machado J. On the formulation and numerical simulation of distributed-order fractional optimal control problems. Commun Nonlinear Sci Numer Simul 2017;52:177–89.
[22] Agrawal OP. A formulation and numerical scheme for fractional optimal control problems. IFAC Proceedings Vol. 2006;39:68–72.
[23] Agrawal OP, Defterli O, Baleanu D. Fractional optimal control problems with several state and control variables. J Vib Control 2010;16:1967–76.
[24] Sweilam NH, Abou Hasan MM, Baleanu D. New studies for general fractional financial models of awareness and trial advertising decisions. Chaos, Solit Fractals 2017;104:772–84.
[25] Sweilam NH, AL-Mekhlafi SM, Baleanu D. Efficient numerical treatments for a fractional optimal control nonlinear tuberculosis model. Int J Biomathemat 2018;11:1–31.
[26] Sweilam NH, AL-Mekhlafi SM. Optimal control for a nonlinear mathematical model of tumor under immune suppression: A numerical approach. Optim Control Appl Meth 2018;39:1581–96.
[27] Sweilam NH, AL-Mekhlafi SM, Alshomrani AS, Baleanu D. Comparative study for optimal control nonlinear variable-order fractional tumor model. Chaos, Solitons Fractals 2020;136. doi: https://doi.org/10.1016/j.chaos.2020.109810.
[28] Sweilam NH, AL-Mekhlafi SM. Optimal control for a time delay multi-strain tuberculosis fractional model: a numerical approach. IMA J Mathe Control Informat 2019;36:317–40.
[29] Lin W. Global existence theory and chaos control of fractional differential equations. JMAA 2007;332:709–26.
[30] Van den Driessche P, Watmough J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math Biosci 2002;180:29–48.

Nasser H. Sweilam, Professor of numerical analysis at the Department of Mathematics, Faculty of Science, Cairo University. He earned his M.Sc. in Mathematis, Faculty of Science, Cairo University. He was a channel system Ph.D. student between Cairo University, Egypt, and TU-Munich, Germany. He received his Ph.D. in "Optimal Control of Variational Inequalities, the Dam Problem". He was the Head of the Department of Mathematics, Faculty of Science, Cairo University (May 2012–May 2018). Prof. Sweilam published more than 140 publications and supervised more than 33 thesis' for Ph.D., M.Sc. He is referee and editor of several international journals, in the frame of pure and applied Mathematics. His main research interests are numerical analysis, optimal control of differential equations, fractional and variable order calculus, bio-informatics and cluster computing, ill-posed problems.