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Research Article

A New Approach to Increase the Flexibility of Curves and Regular Surfaces Produced by 4-Point Ternary Subdivision Scheme

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In this article, we present a new subdivision scheme by using an interpolatory subdivision scheme and an approximating subdivision scheme. The construction of the subdivision scheme is based on translation of points of the 4-point interpolatory subdivision scheme to the new position according to three displacement vectors containing two shape parameters. We first study the characteristics of the new subdivision scheme analytically and then present numerical experiments to justify these analytical characteristics geometrically. We also extend the new derived scheme into its bivariate/tensor product version. This bivariate scheme is applicable on quadrilateral meshes to produce smooth limiting surfaces up to C^3 continuity.

1. Introduction

CAGD is considered as an emerging research field of computational mathematics, which has been fast growing in the last two decades due to a vast range of applications in a number of scientific fields and in real life. It has been extended into new directions owing to several generalizations and applications. The field is concerned with modeling and designing of different complex objects with the help of elegant mathematical algorithms. In CAGD, subdivision schemes have become one of the most important, efficient, and emerging modeling tools for designing and modeling of objects. It defines a smooth curve after applying a sequence of successive refinements. The subdivision schemes are main approaches used to create a curve from an initial control polygon or a surface from an initial control mesh by subdividing them according to the refining rules. These refining rules take the initial control polygon or mesh to produce a sequence of finer polygons or meshes converging to a smooth limiting curve or surface.

Subdivision schemes are classified into interpolatory subdivision schemes and approximating subdivision schemes. Interpolatory subdivision schemes produce the limit curves that pass through all the initial points, whereas the approximating subdivision schemes generate the limit curves that do not pass through the initial control points. The combined subdivision schemes produce the limit curves that may or may not pass through the initial control points. So, their construction has become a new and important trend in CAGD. Different variants of a method to construct the ternary combined subdivision schemes from ternary approximating subdivision schemes have been discussed in [1–3]. Hameed and Mustafa [4] discussed a recursive process for constructing the family of combined binary subdivision schemes. Han and Jia [5] analyzed the approximation and smoothness properties of fundamental and refinable functions that arise from interpolatory subdivision schemes in multidimensional spaces. The push-back operators have been used by [6–9] for the construction of new subdivision schemes. In this article, we give a new method to construct a Modified Combined Ternary Subdivision Scheme (MCTSS). We start from two schemes, one of which is interpolatory with good approximation order and the other one is approximating with good continuity. Hence the MCTSS gives good approximation order and continuity. The motivation for the construction of a new combined subdivision scheme with shape parameters is explained in the following.

1.1. Motivation. We construct a new subdivision scheme with shape parameters by using interpolatory and approximating subdivision schemes so that shape parameters allow the limit curves to move outside the interpolatory curve, inside the approximating curve, or in between the interpolatory and approximating curves. This can be seen in Figure 1. In this figure, red bullets are the initial control points. Blue and green lines show the curves generated by schemes (10) and (11), respectively. These schemes are also the special cases of the MCTSS. Black lines show the curves generated by the MCTSS for $(\alpha,\beta)=(0.15,0.3)$, $(\alpha,\beta)=(-0.15,-0.3)$, $(\alpha,\beta)=(-0.5,-1)$, and $(\alpha,\beta)=(-1.5,-3)$ from outside to inside, respectively.

The remainder of this article is organized as follows. In Section 2, we present basic notations and results. In Section 3, the framework for the construction of MCTSS is presented. In Section 4, we study the properties of the MCTSS analytically. Comparison with existing schemes is given in Section 5. We give numerical examples of the MCTSS in Section 6. In Section 7, we extend the MCTSS into one of its bivariate versions. Conclusions are given in Section 8.

2. Basic Notations and Results

A univariate linear ternary subdivision scheme S_a is based on repeated application of the refinement rules, which are used to map a polygon $P^k=\{P_i^k\}_{i\in\mathbb{Z}}\in l(\mathbb{Z})$ to a refined polygon $P^{k+1}=\{P_i^{k+1}\}_{i\in\mathbb{Z}}\in l(\mathbb{Z})$. The general compact form of these refinement rules is defined as

$$p_{3i+\xi}^{k+1}=\sum_{j\in\mathbb{Z}}a_{3j+\xi}p_{i+j}^{k},\quad \xi=0,1,2, \tag{1}$$

where $l(\mathbb{Z})$ denotes the space of scaler-valued sequences. The sequence $a=\{a_j\}_{j\in\mathbb{Z}}$ is called the refinement mask. The polynomial that uses this mask as coefficient is called the Laurent polynomial. Therefore, the Laurent polynomial corresponding to subdivision scheme (1) is

$$a(z)=\sum_{j\in\mathbb{Z}}a_{3j}z^{3j}+\sum_{j\in\mathbb{Z}}a_{3j+1}z^{3j+1}+\sum_{j\in\mathbb{Z}}a_{3j+2}z^{3j+2}. \tag{2}$$

The necessary condition for the convergence of a ternary subdivision scheme is

$$\sum_{j\in\mathbb{Z}}a_{3j}=1,\sum_{j\in\mathbb{Z}}a_{3j+1}=1,\sum_{j\in\mathbb{Z}}a_{3j+2}=1, \tag{3}$$

which is equivalent to the following relation:

$$a(1)=3\text{ and }a\left(e^{(2\pi i/3)j}\right)=0,\quad \text{where, } j=1,2\ \&\ i=\sqrt{-1}, \tag{4}$$

and (4) is also called the basic sum rule of the ternary subdivision scheme (1).

Definition 1 (see [2]). A combined ternary subdivision scheme is characterized by a parameter-dependent Laurent polynomial $a(z)$, $z\in\mathbb{C}\backslash\{0\}$, which satisfies the odd-symmetry property $a(z)=a(z^{-1})$ for all choices of the shape parameters and the interpolation property $\sum_{j=0}^{2}a(e^{(2\pi i/3)j}z)=3$ only for some special choices of the shape parameters.

Theorem 1 (see [10]). *A convergent subdivision scheme S_a corresponding to the Laurent polynomial,*

$$a(z)=\left(\frac{1+z+z^2}{3z^2}\right)^n b(z), \tag{5}$$

is C^n-continuous if and only if the subdivision scheme S_b corresponding to the Laurent polynomial $b(z)$ is convergent.

Theorem 2 (see [10]). *ie scheme S_a corresponding to the Laurent polynomial $a(z)$ converges, if and only if the scheme S_b corresponding to the Laurent polynomial $b(z)$ is contractive, and the scheme S_b is contractive, if $\|b^\ell\|_\infty<1$, for some $\ell>0$, with*

$$\left\|b^\ell\right\|_\infty=\max\left\{\sum_{i\in\mathbb{Z}}\left|b_{k-3^{\ell_i}}^{\ell}\right|:\ 0\le k<3^\ell\right\}, \tag{6}$$

where b_i^ℓ is the mask of the scheme S_b^ℓ with Laurent polynomial $b^\ell(z)=b(z)b(z^3),\ldots,b(z^{3^{\ell-1}})$.

Let ρ_d be the space of polynomials of degree d and $p\in\rho_d$. A subdivision operator S_a is said to generate polynomials of degree d if $S_a(p)=g$, where $g\in\rho_d$. A subdivision operator is said to reproduce polynomials of degree d if $S_a(p)=p$.

Theorem 3 (see [11]). *A convergent ternary subdivision scheme generates polynomials of degree d if and only if it satisfies the following condition:*

$$a^{(t)}\left(e^{(2\pi i/3)j}\right)=0,\quad \text{where, } t=0,1,\ldots,d \text{ and } j=1,2, \tag{7}$$

where $a^{(t)}$ denotes the t-th derivative of $a(z)$ with respect to z.

Theorem 4 (see [11]). *A convergent subdivision scheme reproduces polynomials of degree d if and only if it generates polynomials of degree d and satisfies the following condition:*

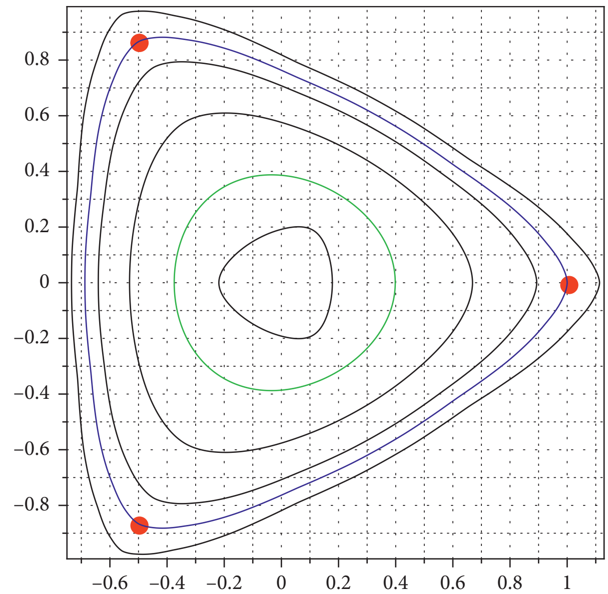


Figure 1: Curves produced by the MCTSS for different values of shape parameters.

$$a^{(t)}(1) = 3\prod_{l=0}^{t-1}(\tau - l), \quad \text{where } \tau = \frac{a'(1)}{3}, \; t = 0, 1, 2, \ldots, d. \tag{8}$$

Convexity is an important shape property. The applications of convexity are in the following:

(i) Designing of telecommunication system
(ii) Nonlinear programming
(iii) Engineering optimization theory
(iv) Approximation theory, and many other fields

In order to analyze this property for our subdivision scheme, we use the following notations and results.

Definition 2 (see [12]). The mask/coefficient of an n-th degree polynomial $a(z) = \sum_{i=0}^{n} a_i z^i$ is said to be bell-shaped if it satisfies

$$\begin{cases} \operatorname{Supp}(a) = [0, n], \\ a_i > 0, \quad i \in [0, n], \\ a_i = a_{n-i}, \quad i \in [0, n], \\ a_i < a_{i+1}, \quad i \in \left[0, \dfrac{n-1}{2}\right], \end{cases} \tag{9}$$

where $a = \{a_0, a_1, \ldots, a_n\}$ is the set of masks/coefficients.

A subdivision scheme is said to be monotonicity-preserving if it preserves the monotonicity of the starting sequence. That is, at any refinement/subdivision step $k > 0$, the difference sequence ΔP^k is positive/negative whenever the difference sequence ΔP^0 is positive/negative, respectively. We use the following result of [12] to analyze this property for our subdivision scheme.

Theorem 5 (see [12]). *Any ternary subdivision scheme associated with a bell-shaped mask (9) satisfying the basic sum rule (4) is monotonicity-preserving.*

A subdivision scheme preserves convexity if it preserves the convexity of the starting sequence. That is, at any refinement step $k > 0$, the difference sequence $\Delta^2 P^k$ is positive/negative whenever the difference sequence $\Delta^2 P^0$ is positive/negative, respectively. We use the following result of [12] to analyze convexity-preserving property for our subdivision scheme.

Theorem 6 (see [12]). *Any ternary subdivision scheme associated with a bell-shaped mask (9) such that its Laurent polynomial $a(z)$ has a factor $(1 + z + z^2)^2$ is convexity-preserving.*

3. Framework for the Construction of the MCTSS

We construct a new combined approximating and interpolating subdivision scheme with two shape parameters. The method that we adopt for the construction of the new subdivision scheme is described here. Here, we provide the subdivision rules of MCTSS in a vector approach. For this, firstly we take the ternary 4-point interpolating subdivision scheme presented in [13]:

$$\begin{pmatrix} \overline{P}_{3i}^{k+1} \\ \overline{P}_{3i+1}^{k+1} \\ \overline{P}_{3i+2}^{k+1} \end{pmatrix} = \frac{1}{81}\begin{pmatrix} 0 & 81 & 0 & 0 \\ -5 & 60 & 30 & -4 \\ -4 & 30 & 60 & -5 \end{pmatrix}\begin{pmatrix} P_{i-1}^{k} \\ P_{i}^{k} \\ P_{i+1}^{k} \\ P_{i+2}^{k} \end{pmatrix}. \tag{10}$$

Now, we take the following ternary 4-point approximating B-spline scheme of degree 4:

$$\begin{pmatrix} Q_{3i}^{k+1} \\ Q_{3i+1}^{k+1} \\ Q_{3i+2}^{k+1} \end{pmatrix} = \frac{1}{81} \begin{pmatrix} 15 & 51 & 15 & 0 \\ 5 & 45 & 30 & 1 \\ 1 & 30 & 45 & 5 \end{pmatrix} \begin{pmatrix} P_{i-1}^{k} \\ P_{i}^{k} \\ P_{i+1}^{k} \\ P_{i+2}^{k} \end{pmatrix}. \tag{11}$$

Now differences between the points of (11) and the points of (10) give three displacement vectors, which are defined as follows:

$$\begin{pmatrix} D_{3i}^{k+1} \\ D_{3i+1}^{k+1} \\ D_{3i+2}^{k+1} \end{pmatrix} = \frac{1}{81} \begin{pmatrix} -15 & 30 & -15 & 0 \\ -10 & 15 & 0 & -5 \\ -5 & 0 & 15 & -10 \end{pmatrix} \begin{pmatrix} P_{i-1}^{k} \\ P_{i}^{k} \\ P_{i+1}^{k} \\ P_{i+2}^{k} \end{pmatrix}. \tag{12}$$

Now we obtain a new combined ternary subdivision scheme with two shape parameters by moving the points $\overline{P}_{3i}^{k+1}$, $\overline{P}_{3i+1}^{k+1}$, and $\overline{P}_{3i+2}^{k+1}$ of (10) to the new position according to the displacement vectors αD_{3i}^{k+1}, $(\beta-\alpha)D_{3i+1}^{k+1}$, and $(\beta-\alpha)D_{3i+2}^{k+1}$, respectively, where α and β are the shape parameters. The shape parameters are chosen in such a way that the interpolating and approximating behaviors of the new subdivision scheme depend on the shape parameter α, while β is used only for providing tension in the curves. Mathematically,

$$\begin{pmatrix} P_{3i}^{k+1} \\ P_{3i+1}^{k+1} \\ P_{3i+2}^{k+1} \end{pmatrix} = \begin{pmatrix} \overline{P}_{3i}^{k+1} \\ \overline{P}_{3i+1}^{k+1} \\ \overline{P}_{3i+2}^{k+1} \end{pmatrix} + \begin{pmatrix} \alpha & 0 & 0 \\ 0 & (\beta-\alpha) & 0 \\ 0 & 0 & (\beta-\alpha) \end{pmatrix} \begin{pmatrix} D_{3i}^{k+1} \\ D_{3i+1}^{k+1} \\ D_{3i+2}^{k+1} \end{pmatrix}. \tag{13}$$

Hence, we get the following combined subdivision scheme with two shape parameters:

$$\begin{pmatrix} P_{3i}^{k+1} \\ P_{3i+1}^{k+1} \\ P_{3i+2}^{k+1} \end{pmatrix} = \frac{1}{81} \begin{pmatrix} -15\alpha & 81+30\alpha & -15\alpha & 0 \\ -5-10(\beta-\alpha) & 60+15(\beta-\alpha) & 30 & -4-5(\beta-\alpha) \\ -4-5(\beta-\alpha) & 30 & 60+15(\beta-\alpha) & -5-10(\beta-\alpha) \end{pmatrix} \begin{pmatrix} P_{i-1}^{k} \\ P_{i}^{k} \\ P_{i+1}^{k} \\ P_{i+2}^{k} \end{pmatrix}. \tag{14}$$

Hence, the mask of scheme (14) is

$$a_{\alpha,\beta} = \frac{1}{81} \begin{pmatrix} -15\alpha & 81+30\alpha & -15\alpha & 0 \\ -5-10(\beta-\alpha) & 60+15(\beta-\alpha) & 30 & -4-5(\beta-\alpha) \\ -4-5(\beta-\alpha) & 30 & 60+15(\beta-\alpha) & \beta-\alpha(\beta-\alpha) \end{pmatrix}, \tag{15}$$

and, equivalently,

$$a_{\alpha,\beta} = \begin{pmatrix} \gamma_{-3,\alpha,\beta} & \gamma_{0,\alpha,\beta} & \gamma_{3,\alpha,\beta} & 0 \\ \gamma_{-4,\alpha,\beta} & \gamma_{-1,\alpha,\beta} & \gamma_{2,\alpha,\beta} & \gamma_{5,\alpha,\beta} \\ \gamma_{-5,\alpha,\beta} & \gamma_{-2,\alpha,\beta} & \gamma_{1,\alpha,\beta} & \gamma_{4,\alpha,\beta} \end{pmatrix}, \tag{16}$$

where

$$\begin{cases} \gamma_{-5,\alpha,\beta} = \frac{1}{81}\{-4-5(\beta-\alpha)\}, \\ \gamma_{-4,\alpha,\beta} = \frac{1}{81}\{-5-10(\beta-\alpha)\}, \\ \gamma_{-3,\alpha,\beta} = \frac{-15\alpha}{81}, \\ \gamma_{-2,\alpha,\beta} = \frac{30}{81}, \\ \gamma_{-1,\alpha,\beta} = \frac{1}{81}\{60+15(\beta-\alpha)\}, \\ \gamma_{-1,\alpha,\beta}\frac{1}{81}\{(81+30\alpha)\}, \\ \gamma_{1,\alpha,\beta} = \frac{1}{81}\{60+15(\beta-\alpha)\}, \\ \gamma_{2,\alpha,\beta} = \frac{30}{81}, \\ \gamma_{3,\alpha,\beta} = \frac{-15\alpha}{81}, \\ \gamma_{4,\alpha,\beta} = \frac{1}{81}\{-5-10(\beta-\alpha)\}, \\ \gamma_{5,\alpha,\beta} = \frac{1}{81}\{-4-5(\beta-\alpha)\}. \end{cases} \tag{17}$$

The Laurent polynomial corresponding to scheme (14) is

$$a_{\alpha,\beta}(z) = \sum_{i=-5}^{5} \gamma_{i,\alpha,\beta} z^i, \tag{18}$$

where $\gamma_{i,\alpha,\beta} : i = -5, \ldots, 5$ are defined in (17).

Remark 1. $S_{a_{\alpha,\beta}}$ is the subdivision scheme, corresponding to the Laurent polynomial $a_{\alpha,\beta}(z)$. We obtain several subschemes from scheme (14) for special choices of shape parameters. That is, $S_{a_{0,\beta}}$, $S_{a_{\alpha,0}}$, $S_{a_{\alpha,2\alpha}}$, and $S_{a_{(\beta/2),\beta}}$ denote the subdivision schemes obtained from scheme (14) for $(\alpha,\beta) = (0,\beta)$, $(\alpha,\beta) = (\alpha,0)$, $(\alpha,\beta) = (\alpha,2\alpha)$, and $(\alpha,\beta) = ((\beta/2),\beta)$, respectively.

Remark 2. For $(\alpha,\beta) = (0,0)$, the scheme $S_{a_{\alpha,\beta}}$ approaches the interpolatory scheme defined in (10), whereas, for $(\alpha,\beta) = (-1,-2)$, this scheme approaches the approximating scheme defined in (11).

4. Properties of the MCTSS

In this section, we analyze the behavior of the MCTSS. A detailed analysis of the scheme is presented here by discussing the important features of the scheme such as continuity, degree of polynomial generation, and degree of polynomial reproduction. We use the Laurent polynomial method to check the continuity of the MCTSS for different values of shape parameters. We also show that the MCTSS has a bell-shaped mask for the specific ranges of parameters. Moreover, we show that the MCTSS preserves monotonicity and convexity for the specific ranges of shape parameters.

4.1. Smoothness Analysis. In this part of the paper, we discuss the level of continuity to which the MCTSS can produce smooth limiting curves or 2D models. It is well known that a continuous subdivision scheme must be convergent. So, we derive the following lemma.

Lemma 1. *The ternary subdivision scheme, which is defined in (14), satisfies the necessary conditions for the convergence.*

Proof 1. The Laurent polynomial of scheme (14), which is defined in (17) and (18), can be written as

$$\begin{aligned} a_{\alpha,\beta}(z) = {} & \frac{1}{81}\{-4-5(\beta-\alpha)\}z^{-5} + \frac{1}{81}\{-5-10(\beta-\alpha)\}z^{-4} - \frac{15\alpha}{81}z^{-3} + \frac{30}{81}z^{-2} + \frac{1}{81} \\ & \times\{60+15(\beta-\alpha)\}z^{-1} + \frac{1}{81}\{(81+30\alpha)\} + \frac{1}{81}\{60+15(\beta-\alpha)\}z + \frac{30}{81}z^2 - \\ & \frac{15\alpha}{81}z^3 + \frac{1}{81}\{-5-10(\beta-\alpha)\}z^4 + \frac{1}{81}\{-4-5(\beta-\alpha)\}z^5. \end{aligned} \tag{19}$$

By factorizing (19), we get

$$a_{\alpha,\beta}(z) = \frac{z^{-5}}{81}\left(1+z+z^2\right)^2\left[(5\alpha-5\beta-4)z^0+(3)z^1+(-30\alpha+15\beta+6)z^2+(50\alpha-20\beta+17)z^3+(-30\alpha+15\beta+6)z^4+(3)z^5 + (5\alpha-5\beta-4)z^6\right]. \tag{20}$$

The necessary conditions for the convergence of subdivision scheme (14) are

$$a_{\alpha,\beta}(1) = 3 \text{ and } a_{\alpha,\beta}\left(e^{(2\pi i/3)j}\right) = 0, \quad \text{where } j = 1,2. \tag{21}$$

From (20), we can easily calculate $a_{\alpha,\beta}(1) = (243/81) = 3$.

Now, to show that $a_{\alpha,\beta}(e^{(2\pi i/3)j}) = 0$ for $j = 1,2$, it is sufficient to show that

$$\left(1+z+z^2\right)^2 = 0, \quad \text{for } z = \left(e^{(2\pi i/3)}\right) \text{ and } z = \left(e^{(4\pi i/3)}\right). \tag{22}$$

Since $e^{(2\pi i/3)} = -12 + i2\sqrt{3} = (e^{(4\pi i/3)})^2$ and $(e^{(2\pi i/3)})^2 = -(1/2) - (i/2)\sqrt{3} = e^{(4\pi i/3)}$, the result in (22) is proved. Hence, the necessary conditions for the convergence of scheme (14) are satisfied. This completes the proof. □

Theorem 7. *The scheme $S_{a_{0,\beta}}$ is C^0-continuous for the parametric interval $-(16/5) < \beta < (5/2)$ and it is C^1-continuous for the parametric interval $-(1/5) < \beta < (6/5)$.*

Proof 2. The Laurent polynomial corresponding to the scheme $S_{a_{0,\beta}}$ is given by

$$a_{0,\beta}(z) = -\frac{z^{-5}}{81}\left(1+z+z^2\right)^2\left[(4+5\beta)+(-3)z+(-6-15\beta)z^2+(-17+20\beta)z^3+(-6-15\beta)z^4+(-3)z^5+(4+5\beta)z^6\right]. \tag{23}$$

Now, we check the C^0-continuity of the scheme $S_{a_{0,\beta}}$. For this, we write (23) as

$$a_{0,\beta}(z) = \left(\frac{1+z+z^2}{3z^2}\right)^0 b_0(z), \tag{24}$$

where

$$b_0(z) = -\frac{z^{-5}}{81}\left(1+z+z^2\right)^2\left[(4+5\beta)+(-3)z+(-6-15\beta)z^2+(-17+20\beta)z^3+(-6-15\beta)z^4+(-3)z^5+(4+5\beta)z^6\right], \tag{25}$$

which can also be written as

$$b_0(z) = \left(1+z+z^2\right)c_0(z), \tag{26}$$

where

$$c_0(z) = z^{-5}\left[\left(-\frac{4}{81}-\frac{5\beta}{81}\right)z^0+\left(-\frac{5\beta}{81}-\frac{1}{81}\right)z^1+\left(\frac{5}{81}+\frac{10\beta}{81}\right)z^2+\left(\frac{26}{81}-\frac{5\beta}{81}\right)z^3+\left(\frac{29}{81}+\frac{10\beta}{81}\right)z^4+\left(\frac{26}{81}-\frac{5\beta}{81}\right)z^5+\left(\frac{5}{81}+\frac{10\beta}{81}\right)z^6 + \left(-\frac{1}{81}-\frac{5\beta}{81}\right)z^7+\left(-\frac{4}{81}-\frac{5\beta}{81}\right)z^8\right], \tag{27}$$

and, for C^0-continuity of the scheme $S_{a_{0,\beta}}$ corresponding to Laurent polynomial $a_{0,\beta}(z)$, we have to show that the scheme S_{b_0} is convergent. For this purpose, we develop a difference scheme S_{c_0} corresponding to the Laurent polynomial $c_0(z)$. Now, we have to show that the scheme S_{c_0} is contractive. For this, we use Theorem 2 to calculate $\|c_0\|_\infty$; that is,

$$\|c_0\|_\infty = \max\left\{\left|-\frac{4}{81}-\frac{5\beta}{81}\right|+\left|\frac{26}{81}-\frac{5\beta}{81}\right|+\left|\frac{5}{81}+\frac{10\beta}{81}\right|,\left|-\frac{5\beta}{81}-\frac{1}{81}\right|+\left|\frac{29}{81}+\frac{10\beta}{81}\right|+\left|-\frac{1}{81}-\frac{5\beta}{81}\right|,\left|\frac{5}{81}+\frac{10\beta}{81}\right|+\left|\frac{26}{81}-\frac{5\beta}{81}\right|+\left|-\frac{4}{81}-\frac{5\beta}{81}\right|\right\}, \tag{28}$$

and, from above, we can easily calculate that $\|c_0\|_\infty < 1$ for $-(16/5) < \beta < (5/2)$.

It follows that scheme S_{c_0} is contractive, S_{b_0} is convergent, and $S_{a_{0,\beta}}$ is C^0-continuous. So, the scheme $S_{a_{0,\beta}}$ is C^0-continuous for $-(16/5) < \beta < (5/2)$. Now we find C^1-continuity of the scheme $S_{a_{0,\beta}}$. From (23), we get

$$a_{0,\beta}(z) = \left(\frac{1+z+z^2}{3z^2}\right)^1 b_1(z), \tag{29}$$

where

$$b_1(z) = \frac{z^{-3}}{27}\left(1+z+z^2\right)^1\left[(4+5\beta)+(-3)z+(-6-15\beta)z^2+(-17+20\beta)z^3+(-6-15\beta)z^4+(-3)z^5+(4+5\beta)z^6\right], \tag{30}$$

which can be written as

$$b_1(z) = \left(1+z+z^2\right)c_1(z), \tag{31}$$

where

$$c_1(z) = z^{-3}\left[\left(-\frac{4}{27}-\frac{5\beta}{27}\right)z+\left(\frac{2}{9}+\frac{5\beta}{9}\right)z^2+\left(\frac{17}{27}-\frac{20\beta}{27}\right)z^3+\left(\frac{2}{9}+\frac{5\beta}{9}\right)z^4+\left(\frac{1}{9}\right)z^5+\left(-\frac{4}{27}-\frac{5\beta}{27}\right)z^6\right]. \tag{32}$$

To find C^1-continuity of the scheme $S_{a_{0,\beta}}$, we have to show that the scheme corresponding to the Laurent polynomial (32) is contractive. To check the contractiveness of the scheme S_{c_1}, we use Theorem 2 and calculate the following expression:

$$\|c_1\|_\infty = \max\left\{\left|-\frac{4}{27}-\frac{5\beta}{27}\right|+\left|\frac{17}{27}-\frac{20\beta}{27}\right|+\left|-\frac{4}{27}-\frac{5\beta}{27}\right|,\left|\frac{3}{27}\right|+\left|\frac{6}{27}+\frac{15\beta}{27}\right|,\left|\frac{6}{27}+\frac{15\beta}{27}\right|+\left|\frac{3}{27}\right|\right\}. \tag{33}$$

It is easy to calculate that $\|c_1\|_\infty < 1$ for $-(1/5) < \beta < (6/5)$ and $-2 < \beta < (6/5)$. The common interval for which $\|c_1\|_\infty < 1$ is $-(1/5) < \beta < (6/5)$.

Hence, the scheme $S_{a_{0,\beta}}$ is C^1-continuous for $-(1/5) < \beta < (6/5)$. This completes the proof.

To increase the range of continuity for the shape parameter β, we apply Theorem 2 for $\ell = 2$ and derive the following result. □

Corollary 1. *The scheme* $S_{a_{0,\beta}}$ *produces* C^0*-continuous curves for the parametric interval* $(43/25) - (3\sqrt{1961}/25) < \beta < (17/5)$, *and it produces* C^1*-continuous curves for the parametric interval* $(41/5) - (3\sqrt{205}/5) < \beta < (6/5)$.

Remark 3. The subdivision scheme $S_{a_{0,\beta}}$ is an interpolatory subdivision scheme. This scheme is a special case of scheme (14). In other words, MCTSS (14) is interpolatory for $\alpha = 0$ and $\forall\beta$.

The proofs of the rest of the theorems are similar to the proof of Theorem 7.

Theorem 8. *The scheme* $S_{a_{\alpha,0}}$ *is* C^0*-continuous for the parametric interval* $-(49/25) < \alpha < (32/25)$. *Moreover, it is* C^1*-continuous for the parametric interval* $-(3/5) < \alpha < (1/20)$.

Remark 4. To increase the range of continuity for the shape parameter α, we apply Theorem 2 for $\ell = 2$ and get the result that the scheme $S_{a_{\alpha,0}}$ generates C^0-continuous curves for the parametric interval $(3/25)-(\sqrt{3034}/25)<\alpha<-(5/11)+(9\sqrt{115}/55)$. It also generates C^1-continuous curves for the parametric interval $-(3/5)<\alpha<-(11/17)+(3\sqrt{491}/85)$.

Theorem 9. *The scheme $S_{a_{\alpha,2\alpha}}$ is C^0-continuous for the parametric interval $-(53/10)<\alpha<(5/2)$. Moreover, it is C^1-continuous for the parametric interval $-(13/5)<\alpha<(1/10)$ and C^2-continuous for the parametric interval $-2<\alpha<-(1/5)$. Furthermore, it is C^3-continuous for the parametric interval $-(7/5)<\alpha<-(4/5)$.*

Remark 5. To increase the range of continuity for the shape parameter α, we apply Theorem 2 for $\ell=2$ and seek the result that the scheme $S_{a_{\alpha,2\alpha}}$ generates C^0-continuous curves for the parametric interval $-(7/5)-(3\sqrt{91}/5)<\alpha<-(23/10)+(3\sqrt{373}/10)$. Moreover, it generates C^1-continuous curves for the parametric interval $-(7/5)-(3\sqrt{10}/5)<\alpha<-(7/5)+(3\sqrt{10}/5)$ and C^2-continuous curves for the parametric interval $-2<\alpha<-(1/5)$. Furthermore, it generates C^3-continuous curves for the parametric interval $-(7/5)<\alpha<-(4/5)$.

Theorem 10. *The scheme $S_{a_{(\beta/2),\beta}}$ is C^0-continuous for the parametric interval $-(53/5)<\beta<5$. Also it is C^1-continuous for the parametric interval $-(26/5)<\beta<(1/5)$, C^2-continuous for the parametric interval $-4<\beta<-(2/5)$, and C^3-continuous for the parametric interval $-(14/5)<\beta<-(8/5)$.*

Remark 6. In order to increase the range of continuity for the shape parameter β, we apply Theorem 2 for $\ell=2$ and get the result that the scheme $S_{a_{(\beta/2),\beta}}$ generates C^0-continuous curves for the parametric interval $-(14/5)-(6\sqrt{91}/5)<\beta<-(23/5)+(3\sqrt{373}/5)$. Moreover it generates C^1-continuous curves for the parametric interval $-(14/5)-(6\sqrt{10}/5)<\beta<-(14/5)+(6\sqrt{10}/5)$, C^2-continuous curves for the parametric interval $-4<\beta<-(2/5)$, and C^3-continuous curves for the parametric interval $-(14/5)<\beta<-(8/5)$.

4.2. Order of MCTSS for Generating and Reproducing Polynomials. Polynomial generation and polynomial reproduction are the important properties of the subdivision schemes. In this section, we discuss the order/degree of polynomial generation and degree of polynomial reproduction of the scheme defined in (14). Generation is the highest degree of polynomials that are generated by the scheme. Any subdivision scheme that reproduces polynomials of degree d also generates polynomials of degree d. By [14], if a subdivision scheme reproduces polynomials of degree d, then it is said to have an approximation order $d+1$. In the next part of paper, we check the capacity of the MCTSS (14) for generating and reproducing polynomials.

Theorem 11. *The scheme associated with the Laurent polynomial $a_{0,\beta}(z)$ generally generates polynomials of degree 1 and for $\beta=0$ it generates polynomials of degree 3.*

Proof 3. We have

$$\begin{aligned} a^{(0)}\left(e^{(2\pi i/3)}\right)&=0,\\ a^{(0)}\left(e^{(4\pi i/3)}\right)&=0,\\ a^{(1)}\left(e^{(2\pi i/3)}\right)&=0,\\ a^{(1)}\left(e^{(4\pi i/3)}\right)&=0, \end{aligned}\tag{34}$$

$$\begin{aligned} a^{(2)}\left(e^{(2\pi i/3)}\right)&=\frac{5}{6}\beta(1+i\sqrt{3})^2,\\ a^{(2)}\left(e^{(4\pi i/3)}\right)&=\frac{5}{6}\beta(-1+i\sqrt{3})^2,\\ a^{(3)}\left(e^{(2\pi i/3)}\right)&=-10\beta,\\ a^{(3)}\left(e^{(4\pi i/3)}\right)&=-10\beta,\\ a^{(4)}\left(e^{(2\pi i/3)}\right)&=\frac{-20}{3}(1+i\sqrt{3})(3+8\beta),\\ a^{(4)}\left(e^{(4\pi i/3)}\right)&=\frac{20}{3}(-1+i\sqrt{3})(3+8\beta). \end{aligned}\tag{35}$$

It is easy to see that the first four relations in (34) return zero when $\beta=0$. This completes our proof by Theorem 3. □

Theorem 12. *The scheme associated with the Laurent polynomial $a_{0,\beta}(z)$ generally reproduces polynomials of degree 1 and for specific value of β (i.e., $\beta=0$) it reproduces polynomials of degree 3.*

Proof 4. By ([11], Corollary 1), an interpolatory subdivision scheme that generates polynomials up to degree d also reproduces polynomials up to degree d. Thus, the result is proved. □

Remark 7. In a similar way, by using Theorems 3 and 4, the following results can be proved:

(i) The scheme $S_{a_{\alpha,0}}$ generally generates polynomials of degree 1 and for $\alpha=0$, it generates polynomials of degree 3
(ii) The scheme $S_{a_{\alpha,0}}$ generally reproduces polynomials of degree 1 and for $\alpha=0$, it reproduces polynomials of degree 3.
(iii) The scheme associated with the Laurent polynomial $a_{\alpha,2\alpha}(z)$ generally generates polynomials of degree 3.
(iv) The scheme associated with the Laurent polynomial $a_{\alpha,2\alpha}(z)$ generally reproduces polynomials of degree 1 and for $\alpha=0$, it reproduces polynomials of degree 3.
(v) The scheme associated with the Laurent polynomial $a_{(\beta/2),\beta}(z)$ generally generates polynomials of degree 3.

(vi) The scheme associated with the Laurent polynomial $a_{(\beta/2),\beta}(z)$ generally reproduces polynomials of degree 1. For $\beta = 0$, it reproduces polynomials of degree 3.

4.3. Shape Preservation of MCTSS. Here, we check the range of shape parameter for which MCTSS (14) preserves monotonicity and convexity. For this purpose, we use Theorems 5 and 6.

Lemma 2. *The scheme $S_{a_{\alpha,2\alpha}}$ associated with the Laurent polynomial $a_{\alpha,2\alpha}(z)$ has a bell-shaped mask for $-(7/5) < \alpha < -(4/5)$.*

Proof 5. If we put $\beta = 2\alpha$ in (18), we get the Laurent polynomial corresponding to the scheme $S_{a_{\alpha,2\alpha}}$; that is,

$$\begin{aligned} a_{\alpha,2\alpha}(z) = \frac{1}{81}\big[&(-4-5\alpha)z^{-5} + (-5-10\alpha)z^{-4} + (-15\alpha)z^{-3} + (30)z^{-2} + (60+15\alpha)z^{-1} + (81+30\alpha)z^{0} + (60+15\alpha)z^{1} + (30)z^{2} \\ &+ (-15\alpha)z^{3} + (-5-10\alpha)z^{4} + (-4-5\alpha)z^{5}\big] \\ = &\sum_{i=-5}^{5} \gamma_{i,\alpha,2\alpha} z^{i}. \end{aligned} \tag{36}$$

It is easy to see from (36) that $\operatorname{Supp}(a_{a,2a}) = [-5,5]$. Also, from (36), $\gamma_{i,\alpha,2\alpha} > 0$: $i = -5,\dots,5$ for special interval of α, which is calculated as

$$\begin{aligned} &\gamma_{-5,\alpha,2\alpha} > 0, \quad \text{for } \alpha < \frac{4}{5}, \\ &\gamma_{-4,\alpha,2\alpha} > 0, \quad \text{for } \alpha < \frac{1}{2}, \\ &\gamma_{-3,\alpha,2\alpha} > 0, \quad \text{for } \alpha < 0, \\ &\gamma_{-2,\alpha,2\alpha} > 0, \quad \text{for } \alpha > -4, \\ &\gamma_{-1,\alpha,2\alpha} > 0, \quad \text{for } \alpha > -4, \\ &\gamma_{0,\alpha,2\alpha} > 0, \quad \text{for } \alpha > -\frac{27}{10}, \\ &\gamma_{1,\alpha,2\alpha} > 0, \quad \text{for } \alpha > -4 \\ &\gamma_{2,\alpha,2\alpha} > 0, \quad \text{for } \alpha < 0, \\ &\gamma_{3,\alpha,2\alpha} > 0, \quad \text{for } \alpha < 0, \\ &\gamma_{4,\alpha,2\alpha} > 0, \quad \text{for } \alpha < -\frac{1}{2} \text{ and } \gamma_{5,\alpha,2\alpha} > 0, \quad \text{for } \alpha < -\frac{4}{5}. \end{aligned} \tag{37}$$

Hence, the interval for which $\gamma_{i,\alpha,2\alpha} > 0$: $i = -5,\dots,5$ is

$$\frac{-27}{10} < \alpha < \frac{-4}{5}. \tag{38}$$

We also see that $\gamma_{-5,\alpha,2\alpha} = \gamma_{5,\alpha,2\alpha}, \gamma_{-4,\alpha,2\alpha} = \gamma_{4,\alpha,2\alpha}, \gamma_{-3,\alpha,2\alpha} = \gamma_{3,\alpha,2\alpha}, \gamma_{-2,\alpha,2\alpha}\backslash\backslash = \gamma_{2,\alpha,2\alpha}, \gamma_{-1,\alpha,2\alpha} = \gamma_{1,\alpha,2\alpha}$, and $\gamma_{0,\alpha,2\alpha} = \gamma_{0,\alpha,2\alpha}$.

Now, we have to prove that $\gamma_{i,\alpha,2\alpha} < \gamma_{i+1,\alpha,2\alpha}$ for $i \in [-5,-1]$. It is also observed that

$$\begin{aligned} &\gamma_{-5,\alpha,2\alpha} < \gamma_{-4,\alpha,2\alpha}, \quad \text{for } \alpha < -\frac{1}{5}, \\ &\gamma_{-4,\alpha,2\alpha} < \gamma_{-3,\alpha,2\alpha}, \quad \text{for } \alpha < 1, \\ &\gamma_{-3,\alpha,2\alpha} < \gamma_{-2,\alpha,2\alpha}, \quad \text{for } \alpha < -2, \\ &\gamma_{-2,\alpha,2\alpha} < \gamma_{-1,\alpha,2\alpha}, \quad \text{for } \alpha > -2 \text{ and } \gamma_{-1,\alpha,2\alpha} < \gamma_{0,\alpha,2\alpha}, \quad \text{for } \alpha > -\frac{7}{5}. \end{aligned} \tag{39}$$

So, the interval of α for which $\gamma_{i,\alpha,2\alpha} < \gamma_{i+1,\alpha,2\alpha}$ is

$$\frac{-7}{5} < \alpha < \frac{-1}{5}. \tag{40}$$

Now, by combining the two intervals given in (38) and (40), we get a common interval; that is, $(-7/5) < \alpha < (-4/5)$.

Hence, the interval of α for which scheme $S_{a_{\alpha,2\alpha}}$ has a bell-shaped mask is $(-7/5) < \alpha < (-4/5)$. This completes the proof. □

Remark 8. Similarly, the following results can be proved:

(i) The scheme $S_{a_{(\beta/2),\beta}}$ associated with the Laurent polynomial $a_{(\beta/2),\beta}(z)$ has a bell-shaped mask for the interval $-(14/5) < \beta < -(8/5)$

(ii) The scheme $S_{a_{\alpha,0}}$ corresponding to the Laurent polynomial $a_{\alpha,0}$ does not contain bell-shaped mask for all α

(iii) The scheme $S_{a_{0,\beta}}$ corresponding to the Laurent polynomial $a_{0,\beta}$ does not contain bell-shaped mask for all β

Theorem 13. *The scheme $S_{a_{\alpha,2\alpha}}$ associated with the Laurent polynomial $a_{\alpha,2\alpha}(z)$ preserves monotonicity for $-(7/5) < \alpha < -(4/5)$.*

Proof 6. Since by Lemma 1, the MCTSS satisfies the basic sum rule and by Lemma 2, it has a bell-shaped mask for $-(7/5) < \alpha < -(4/5)$, and by Theorem 5, it preserves the monotonicity for $-(7/5) < \alpha < -(4/5)$. □

Theorem 14. *The scheme $S_{a_{\alpha,2\alpha}}$ associated with the Laurent polynomial $a_{\alpha,2\alpha}(z)$ preserves convexity for $-(7/5) < \alpha < -(4/5)$.*

Proof 7. By (20), the MCTSS has a factor $(1+z+z^2)^2$ and by Lemma 2, it has a bell-shaped mask for $-(7/5) < \alpha < -(4/5)$. So, by Theorem 6, it preserves the convexity for $-(7/5) < \alpha < -(4/5)$. □

Remark 9. The scheme $S_{a_{(\beta/2),\beta}}$ associated with the Laurent polynomial $a_{\beta/2,\beta}(z)$ preserves monotonicity and convexity for $-(14/5) < \beta < -(8/5)$.

Remark 10. The support of the MCTSS is $[-2.5, 2.5]$, which is calculated by ([3], Theorem 1).

5. Comparison with Existing Schemes

Here we give comparison of the MCTSS with the existing ternary schemes that produce limiting curves up to C^3 smoothness and summarize the results in Table 1. This table shows that MCTSS keeps detailed features better than existing schemes. In this table, G-D and R-D denote the degree of polynomials generation and the degree of polynomial reproduction of the ternary subdivision schemes, respectively. Moreover, scheme of [12] is a special case of the MCTSS for $(\alpha, \beta) = (-1, -2)$.

6. Numerical Experiments by MCTSS

Here, we show the behavior of the MCTSS (14) by presenting different models and show how it controls the shape of limiting curves. We develop the initial polygons by using functional and nonfunctional initial data. We also compare the results of the MCTSS (14) at different values of shape parameters. In the figures of these experiments, red solid circles and red lines represent initial points and initial polygons, respectively.

Experiment 1. In this experiment, we draw the initial control model by using initial control points $(12, 10)$, $(14, 7)$, $(14, 5)$, $(19, 5)$, $(19, 7)$, $(15, 13)$, $(20, 14)$, $(25, 13)$, $(25, 11)$, $(22, 7)$, $(22, 5)$, $(27, 5)$, $(29, 11)$, $(32, 13)$, $(35, 9)$, $(35, 5)$, $(40, 5)$, $(40, 7)$, $(37, 12)$, $(40, 16)$, $(44, 7)$, $(47, 5)$, $(49, 7)$, $(47, 9)$, $(47, 12)$, $(50, 16)$, $(50, 21)$, $(45, 26)$, $(36, 26)$, $(33, 25)$, $(30, 26)$, $(26, 27)$, $(14, 27)$, $(10, 25)$, $(5, 15)$, $(10, 21)$, $(10, 16)$, $(6, 13)$, $(6, 9)$, $(7, 8)$, $(7, 5)$, $(12, 5)$, and $(12, 10)$. Figure 2(a) shows initial control model, and Figures 2(b)–2(d) show the models generated by the MCTSS (14) for $(\alpha, \beta) = (-1.8, -3.6)$, $(\alpha, \beta) = (-0.25, -0.5)$, and $(\alpha, \beta) = (0.24, 0.48)$, respectively.

Experiment 2. In this experiment, we draw the initial control model by using the initial control points $(1, -3)$, $(5, -4)$, $(4, -3)$, $(9, 1)$, $(7, 2)$, $(8, 5)$, $(5, 4)$, $(5, 5)$, $(3, 4)$, $(4, 9)$, $(2, 7)$, $(0, 10)$, $(-2, 7)$, $(-4, 8)$, $(-3, 3)$, $(-5, 6)$, $(-5, 4)$, $(-8, 5)$, $(-7, 2)$, $(-9, 1)$, $(-4, -3)$, $(-5, -4)$, $(0, -3)$, $(2, -7)$, $(2, -6)$, and $(1, -3)$. Figure 3(a) shows initial control model, and Figures 3(b)–3(d) show models generated by the MCTSS (14) for $(\alpha, \beta) = (-1.4, -2.8)$, $(\alpha, \beta) = (-0.25, -0.5)$ and $(\alpha, \beta) = (0.24, 0.48)$, respectively.

Experiment 3. In this experiment, we draw the initial control model by using the initial control points $(12, 1)$, $(18, 1)$, $(23, 5)$, $(26, 8)$, $(28, 11)$, $(28, 15)$, $(26, 18)$, $(23, 20)$, $(20, 22)$, $(18, 22)$, $(16, 20)$, $(16, 22)$, $(17, 28)$, $(15, 28)$, $(14, 22)$, $(14, 20)$, $(12, 22)$, $(10, 22)$, $(7, 20)$, $(4, 18)$, $(1, 15)$, $(1, 11)$, $(4, 8)$, $(7, 5)$, and $(12, 1)$. Figure 4(a) shows the initial control model, and Figures 4(b)–4(d) show the models generated by the MCTSS for $(\alpha, \beta) = (-1.4, -2.8)$, $(\alpha, \beta) = (-0.25, -0.5)$ and $(\alpha, \beta) = (0.24, 0.48)$, respectively.

Experiment 4. Here, we take the monotone data from the monotonic function $f_1(x) = \sqrt{x}$ and show that MCTSS (14) preserves monotonicity for $(\alpha, \beta) = (-1, -2)$. The graphical results obtained by MCTSS (14) are shown in Figure 5(b), while monotone data are shown in Figure 5(a).

Experiment 5. Here, we show the convexity preservation of scheme (14) graphically. We take the convex data from the convex function $f_2(x) = e^x + 6$ and show that scheme (14) preserves convexity for $(\alpha, \beta) = (-(6/5), (-12/5))$. Figure 6(b) shows the graphical results obtained by the MCTSS (14) and Figure 6(a) shows the convex data.

Experiment 6. Here, to show the generation and reproduction degree of scheme (14), we take the data from the linear polynomial function $f_3(x) = x + 2$, quadratic polynomial function $f_4(x) = x^2 - 2x + 8$, and cubic polynomial function $f_5(x) = x^3 - 4x$, respectively. Then, we show that, for $(\alpha, \beta) = (0, 0)$, scheme (14) generates polynomials of degree 3 and also reproduces polynomials of degree 3. Figures 7(a)–7(c) show the graphical results obtained by the MCTSS (14).

7. Tensor Product Version of the MCTSS

In this section, we extend the MCTSS into its tensor product version. The tensor product scheme is designed for quadrilateral meshes. Since this scheme is the tensor product version of the MCTSS, it consists of nine refinement rules. One rule is for vertex, four rules are for edges, and four rules are for faces. Hence, at each subdivision step, the bivariate MCTSS splits each mesh into 9 meshes. The outer lawyer or boundary of the mesh which consists of different faces enclosed by edges constitutes the surface. Nine rules of the bivariate subdivision scheme are used for fitting the surface by initial quadrilateral mesh. The bivariate MCTSS, the construction of which is explained in Appendix A, is given by

Table 1: Comparison of the MCTSS with existing ternary C^3 schemes.

Scheme	Parameters	Type	Support	G-D	R-D	Mask type
Scheme [15]	0	Approximating	[−2.75, 2.75]	1	3	Bell-shaped
Scheme [12]	0	Approximating	[−2.5, 2.5]	3	1	Bell-shaped
Scheme [2]	3	Combined	[−2.5, 2.5]	Up to 3	1	Not bell-shaped
MCTSS	2	Combined	[−2.5, 2.5]	Up to 3	Up to 3	Bell-shaped

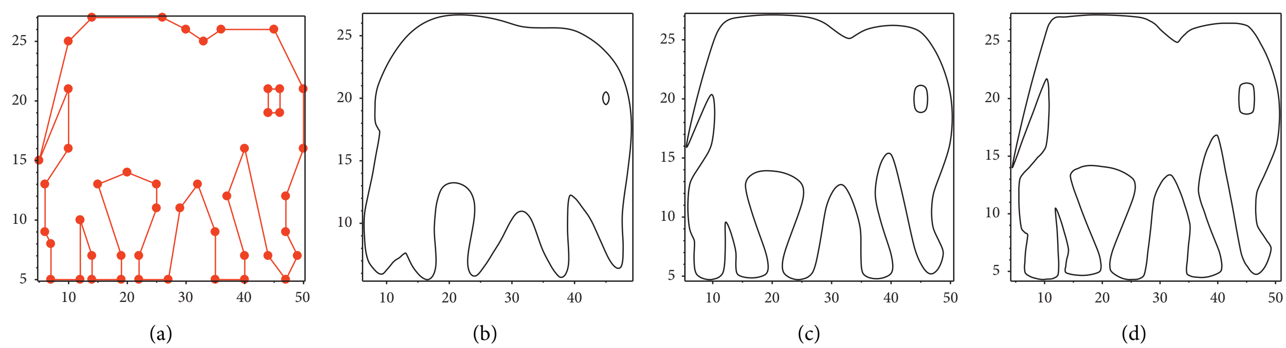


Figure 2: Black lines represent models generated by the MCTSS after four subdivision steps.

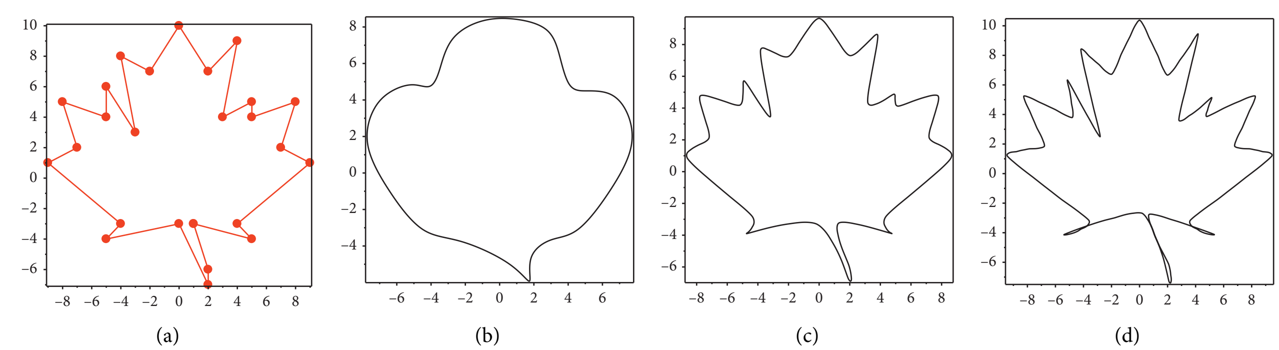


Figure 3: Black lines represent models generated by the MCTSS after four subdivision steps.

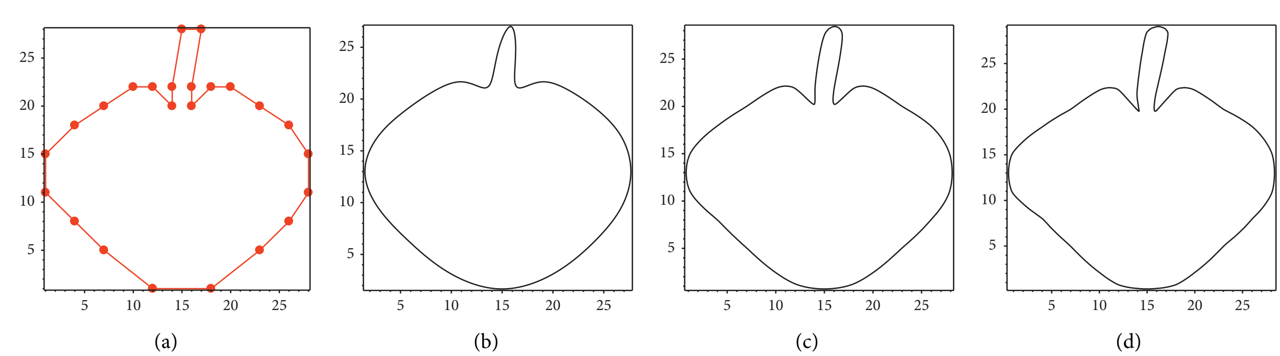
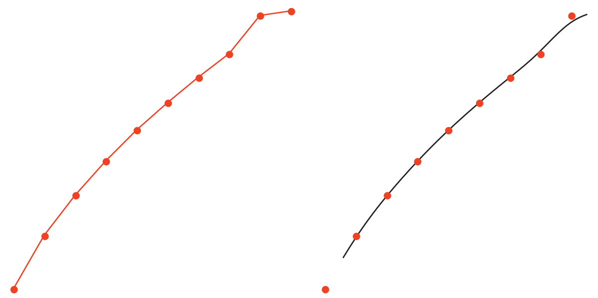
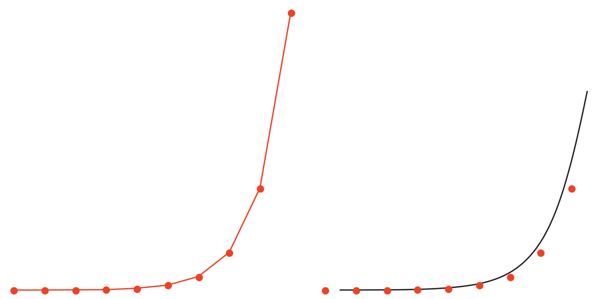


Figure 4: Black lines (b–d) represent models generated by the MCTSS after four subdivision steps.



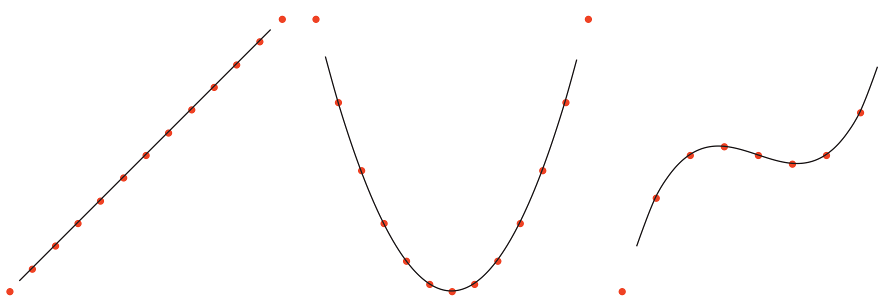
(a) (b)

Figure 5: Black solid line represents curve fitted by our subdivision scheme.



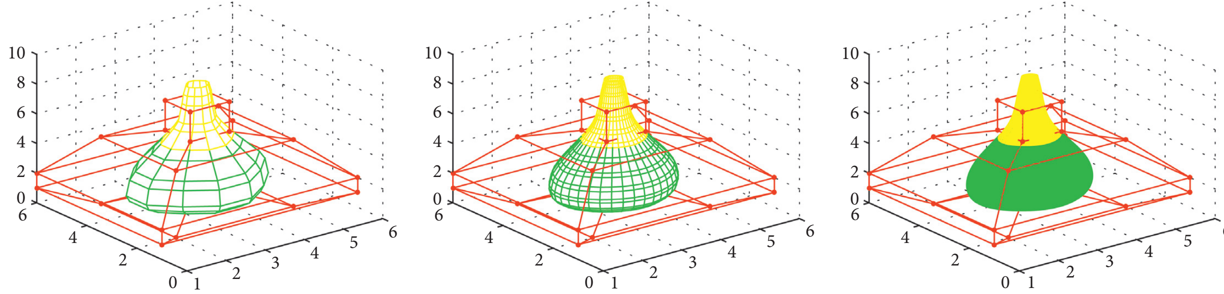
(a) (b)

Figure 6: Black solid line represents curve fitted by our subdivision scheme.



(a) (b) (c)

Figure 7: Black solid lines represent curves fitted by our subdivision scheme. (a–c) The curves fitted by the MCTSS when initial data is taken from linear, quadratic, and cubic functions, respectively.



(a) (b) (c)

Figure 8: Meshes and surface produced by the MCTSS (41) with $(\alpha, \beta) = (-(3/2), -3)$ after one, two, and four subdivision levels, respectively.

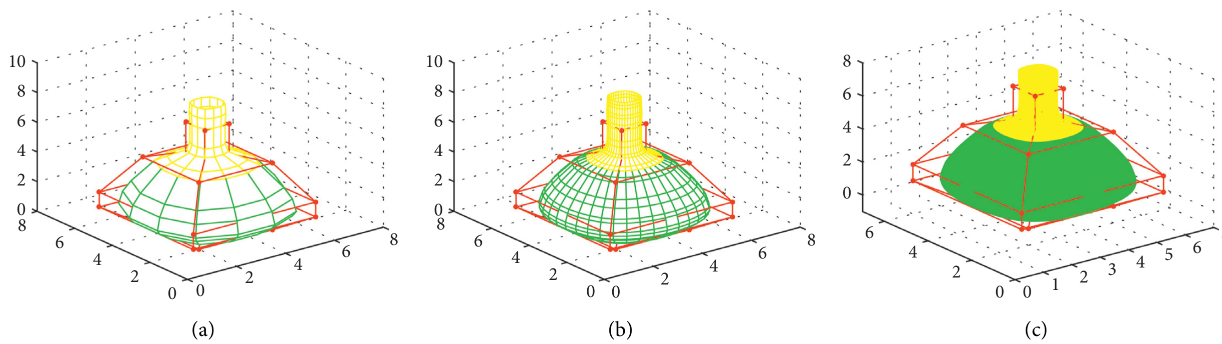


FIGURE 9: Meshes and surface produced by the MCTSS (41) with $(\alpha, \beta) = (-(1/2), -1)$ after one, two, and four subdivision levels, respectively.

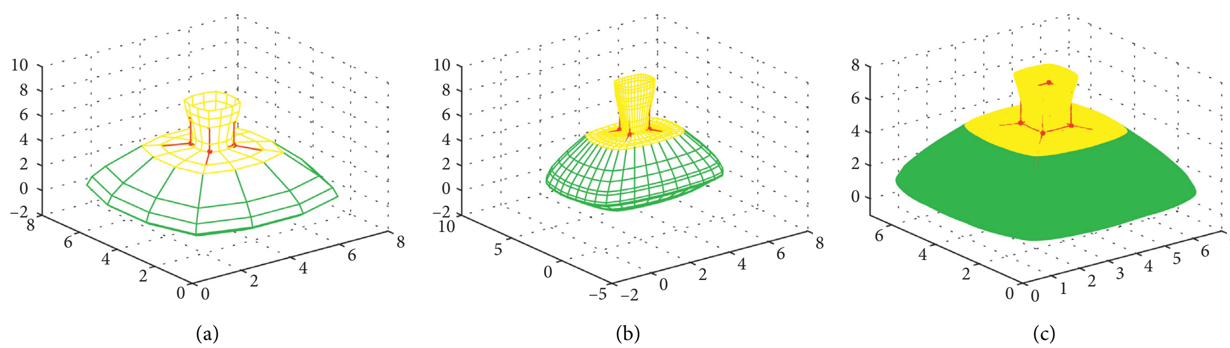


FIGURE 10: Meshes and surface produced by the bivariate MCTSS (41) with $(\alpha, \beta) = ((1/4), (1/2))$ after one, two, and four subdivision levels, respectively.

$$\begin{cases} P^{k+1}_{3i,3j} = \sum_{m=-1}^{1}\sum_{n=-1}^{1} A_{m+2,n+2} P^{k}_{i+m,j+n}, \\ P^{k+1}_{3i+1,3j} = \sum_{m=-1}^{2}\sum_{n=-1}^{1} B_{m+2,n+2} P^{k}_{i+m,j+n}, \\ P^{k+1}_{3i+2,3j} = \sum_{m=-1}^{2}\sum_{n=-1}^{1} C_{m+2,n+2} P^{k}_{i+m,j+n}, \\ P^{k+1}_{3i,3j+1} = \sum_{m=-1}^{1}\sum_{n=-1}^{2} D_{m+2,n+2} P^{k}_{i+m,j+n}, \\ P^{k+1}_{3i+1,3j+1} = \sum_{m=-1}^{2}\sum_{n=-1}^{2} E_{m+2,n+2} P^{k}_{i+m,j+n}, \\ P^{k+1}_{3i+2,3j+1} = \sum_{m=-1}^{2}\sum_{n=-1}^{2} F_{m+2,n+2} P^{k}_{i+m,j+n}, \\ P^{k+1}_{3i,3j+2} = \sum_{m=-1}^{1}\sum_{n=-1}^{2} G_{m+2,n+2} P^{k}_{i+m,j+n}, \\ P^{k+1}_{3i+1,3j+2} = \sum_{m=-1}^{2}\sum_{n=-1}^{2} H_{m+2,n+2} P^{k}_{i+m,j+n}, \\ P^{k+1}_{3i+2,3j+2} = \sum_{m=-1}^{2}\sum_{n=-1}^{2} I_{m+2,n+2} P^{k}_{i+m,j+n}, \end{cases} \quad (41)$$

where

$$\begin{gathered} A_{m+2,n+2}\colon m = -1..1; n = -1..1, \\ B_{m+2,n+2} := m = -1..2; n \\ C_{m+2,n+2}\colon m = -1..2; n = -1..1, \\ D_{m+2,n+2}\colon m = -1..1; n = -1..2, \\ E_{m+2,n+2}\colon m = -1..2; n = -1..2, \\ F_{m+2,n+2}\colon m = -1..2; n = -1..2, \\ G_{m+2,n+2}\colon m = -1..1; n = -1..2, \\ H_{m+2,n+2}\colon m = -1..2; n = -1..2, \\ I_{m+2,n+2}\colon m = -1..2; n = -1..2, \end{gathered} \quad (42)$$

are defined in (A.1)–(A.15) of Appendix A, respectively.

Figures 8–10 show the surfaces produced by the bivariate MCTSS at different values of the shape parameters α and β. In these figures, red bullets and red lines represent the initial points and initial meshes, respectively.

8. Conclusion

In this paper, we have introduced a new method to construct a combined ternary subdivision scheme with two shape parameters. We have analyzed the properties of the MCTSS for different ranges of shape parameters. We have also showed that the MCTSS produces smooth 2D and 3D models at specific choices of shape parameters. Moreover, we have shown that the graphical results of the MCTSS verify the analytical results of the scheme. Furthermore, we have derived the bivariate subdivision scheme with nine refinement rules. This scheme is used to produce smooth surface when all the initial meshes are quadrilateral.

Appendix

A. Construction of the Bivariate MCTSS

The Laurent polynomial of three refinement rules P_{3i}^{k+1}, P_{3i+1}^{k+1}, and P_{3i+2}^{k+1} of the MCTSS can be written as

$$\begin{aligned} a_{\alpha,\beta,3i}(z) &= \sum_{j=-1}^{1} \gamma_{3j,\alpha,\beta} z^{3j}, \\ a_{\alpha,\beta,3i+1}(z) &= \sum_{j=-2}^{1} \gamma_{3j+2,\alpha,\beta} z^{3j+2}, \\ a_{\alpha,\beta,3i+2}(z) &= \sum_{j=-2}^{1} \gamma_{3j+1,\alpha,\beta} z^{3j+1}, \end{aligned} \quad (A.1)$$

where

$$a_{\alpha,\beta}(z) = a_{\alpha,\beta,3i}(z) + a_{\alpha,\beta,3i+1}(z) + a_{\alpha,\beta,3i+2}(z). \quad (A.2)$$

We get the Laurent polynomial of 9 refinement rules of tensor product version of MCTSS by

$$a_{\alpha,\beta,3i+m_1,3j+m_2}(z_1, z_2) = a_{\alpha,\beta,3i+m_1}(z_1) \times a_{\alpha,\beta,3j+m_2}(z_2), \quad (A.3)$$

where $(m_1, m_2) = (0,0)$, $(1,0)$, $(2,0)$, $(0,1)$, $(1,1)$, $(2,1)$, $(0,2)$, $(1,2)$, $(2,2)$.

Hence, for $(m_1, m_2) = (0,0)$, we get

$$\begin{aligned} a_{\alpha,\beta,3i,3j}(z_1, z_2) = {} & A_{1,1} z_1^{-3} z_2^{-3} + A_{1,2} z_1^{-3} z_2^{0} + A_{1,3} z_1^{-3} z_2^{3} \\ & + A_{2,1} z_1^{0} z_2^{-3} + A_{2,2} z_1^{0} z_2^{0} + A_{2,3} z_1^{0} z_2^{3} \\ & + A_{3,1} z_1^{3} z_2^{-3} + A_{3,2} z_1^{3} z_2^{0} + A_{3,3} z_1^{3} z_2^{3}, \end{aligned} \quad (A.4)$$

where

$$\begin{aligned} A_{1,1} &= \gamma_{-3,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ A_{1,2} &= \gamma_{-3,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ A_{1,3} &= \gamma_{-3,\alpha,\beta}\gamma_{3,\alpha,\beta}, \\ A_{2,1} &= \gamma_{0,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ A_{2,2} &= \gamma_{0,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ A_{2,3} &= \gamma_{0,\alpha,\beta}\gamma_{3,\alpha,\beta}, \\ A_{3,1} &= \gamma_{3,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ A_{3,2} &= \gamma_{3,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ A_{3,3} &= \gamma_{3,\alpha,\beta}\gamma_{3,\alpha,\beta}. \end{aligned} \quad (A.5)$$

Now, for $(m_1, m_2) = (1,0)$, we get

$$\begin{aligned} a_{\alpha,\beta,3i+1,3j}(z_1, z_2) = {} & B_{1,1} z_1^{-4} z_2^{-3} + B_{1,2} z_1^{-4} z_2^{0} + B_{1,3} z_1^{-4} z_2^{3} \\ & + B_{2,1} z_1^{-1} z_2^{-3} + B_{2,2} z_1^{-1} z_2^{0} + B_{2,3} z_1^{-1} z_2^{3} \\ & + B_{3,1} z_1^{2} z_2^{-3} + B_{3,2} z_1^{2} z_2^{0} + B_{3,3} z_1^{2} z_2^{3} \\ & + B_{4,1} z_1^{5} z_2^{-3} + B_{4,2} z_1^{5} z_2^{0} + B_{4,3} z_1^{5} z_2^{3}, \end{aligned} \quad (A.6)$$

where

$$\begin{aligned} B_{1,1} &= \gamma_{-4,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ B_{1,2} &= \gamma_{-4,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ B_{1,3} &= \gamma_{-4,\alpha,\beta}\gamma_{3,\alpha,\beta}, \\ B_{2,1} &= \gamma_{-1,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ B_{2,2} &= \gamma_{-1,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ B_{2,3} &= \gamma_{-1,\alpha,\beta}\gamma_{3,\alpha,\beta}, \\ B_{3,1} &= \gamma_{2,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ B_{3,2} &= \gamma_{2,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ B_{3,3} &= \gamma_{2,\alpha,\beta}\gamma_{3,\alpha,\beta}, \\ B_{4,1} &= \gamma_{5,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ B_{4,2} &= \gamma_{5,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ B_{4,3} &= \gamma_{5,\alpha,\beta}\gamma_{3,\alpha,\beta}. \end{aligned} \quad (A.7)$$

Now, for $(m_1, m_2) = (2,0)$, we get

$$\begin{aligned} a_{\alpha,\beta,3i+2,3j}(z_1, z_2) = {} & C_{1,1} z_1^{-5} z_2^{-3} + C_{1,2} z_1^{-5} z_2^{0} + C_{1,3} z_1^{-5} z_2^{3} \\ & + C_{2,1} z_1^{-2} z_2^{-3} + C_{2,2} z_1^{-2} z_2^{0} + C_{2,3} z_1^{-2} z_2^{3} \\ & + C_{3,1} z_1^{1} z_2^{-3} + C_{3,2} z_1^{1} z_2^{0} + C_{3,3} z_1^{1} z_2^{3} \\ & + C_{4,1} z_1^{4} z_2^{-3} + C_{4,2} z_1^{4} z_2^{0} + C_{4,3} z_1^{4} z_2^{3}, \end{aligned} \quad (A.8)$$

where

$$\begin{aligned} C_{1,1} &= \gamma_{-5,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ C_{1,2} &= \gamma_{-5,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ C_{1,3} &= \gamma_{-5,\alpha,\beta}\gamma_{3,\alpha,\beta}, \\ C_{2,1} &= \gamma_{-2,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ C_{2,2} &= \gamma_{-2,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ C_{2,3} &= \gamma_{-2,\alpha,\beta}\gamma_{3,\alpha,\beta}, \\ C_{3,1} &= \gamma_{1,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ C_{3,2} &= \gamma_{1,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ C_{3,3} &= \gamma_{1,\alpha,\beta}\gamma_{3,\alpha,\beta}, \\ C_{4,1} &= \gamma_{4,\alpha,\beta}\gamma_{-3,\alpha,\beta}, \\ C_{4,2} &= \gamma_{4,\alpha,\beta}\gamma_{0,\alpha,\beta}, \\ C_{4,3} &= \gamma_{4,\alpha,\beta}\gamma_{3,\alpha,\beta}. \end{aligned} \quad (A.9)$$

Similarly, for $(m_1, m_2) = (0,1)$, we get

$$\begin{aligned} a_{\alpha,\beta,3i,3j+1}(z_1,z_2) = {} & D_{1,1} z_1^{-3} z_2^{-4} + D_{1,2} z_1^{-3} z_2^{-1} + D_{1,3} z_1^{-3} z_2^{2} \\ & + D_{1,4} z_1^{-3} z_2^{5} + D_{2,1} z_1^{0} z_2^{-4} + D_{2,2} z_1^{0} z_2^{-1} \\ & + D_{2,3} z_1^{0} z_2^{2} + D_{2,4} z_1^{0} z_2^{5} + D_{3,1} z_1^{3} z_2^{-4} \\ & + D_{3,2} z_1^{3} z_2^{-1} + D_{3,3} z_1^{3} z_2^{2} + D_{3,4} z_1^{3} z_2^{5}, \end{aligned} \tag{A.10}$$

where

$$\begin{aligned} D_{1,1} &= \gamma_{-3,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ D_{1,2} &= \gamma_{-3,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ D_{1,3} &= \gamma_{-3,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ D_{1,4} &= \gamma_{-3,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ D_{2,1} &= \gamma_{0,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ D_{2,2} &= \gamma_{0,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ D_{2,3} &= \gamma_{0,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ D_{2,4} &= \gamma_{0,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ D_{3,1} &= \gamma_{3,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ D_{3,2} &= \gamma_{3,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ D_{3,3} &= \gamma_{3,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ D_{3,4} &= \gamma_{3,\alpha,\beta}\gamma_{2,\alpha,\beta}. \end{aligned} \tag{A.11}$$

Again, for $(m_1, m_2) = (1,1)$, we get

$$\begin{aligned} a_{\alpha,\beta,3i+1,3j+1}(z_1,z_2) = {} & E_{1,1} z_1^{-4} z_2^{-4} + E_{1,2} z_1^{-4} z_2^{-1} + E_{1,3} z_1^{-4} z_2^{2} \\ & + E_{1,4} z_1^{-4} z_2^{5} + E_{2,1} z_1^{-1} z_2^{-4} + E_{2,2} z_1^{-1} z_2^{-1} \\ & + E_{2,3} z_1^{-1} z_2^{2} + E_{2,4} z_1^{-1} z_2^{5} + E_{3,1} z_2^{3} z_2^{-4} \\ & + E_{3,2} z_1^{2} z_2^{-1} + E_{3,3} z_1^{2} z_2^{2} + E_{3,4} z_1^{2} z_2^{5} \\ & + E_{4,1} z_2^{3} z_5^{-4} + E_{4,2} z_1^{5} z_2^{-1} + E_{4,3} z_1^{5} z_2^{2} \\ & + E_{4,4} z_1^{5} z_2^{5}, \end{aligned} \tag{A.12}$$

where

$$\begin{aligned} E_{1,1} &= \gamma_{-4,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ E_{1,2} &= \gamma_{-4,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ E_{1,3} &= \gamma_{-4,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ E_{1,4} &= \gamma_{-4,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ E_{2,1} &= \gamma_{-1,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ E_{2,2} &= \gamma_{-1,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ E_{2,3} &= \gamma_{-1,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ E_{2,4} &= \gamma_{-1,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ E_{3,1} &= \gamma_{2,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ E_{3,2} &= \gamma_{3,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ E_{3,3} &= \gamma_{2,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ E_{3,4} &= \gamma_{2,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ E_{4,1} &= \gamma_{5,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ E_{4,2} &= \gamma_{5,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ E_{4,3} &= \gamma_{5,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ E_{4,4} &= \gamma_{5,\alpha,\beta}\gamma_{5,\alpha,\beta}. \end{aligned} \tag{A.13}$$

Now, for $(m_1, m_2) = (2,1)$, we get

$$\begin{aligned} a_{\alpha,\beta,3i+2,3j+1}(z_1,z_2) = {} & F_{1,1} z_1^{-5} z_2^{-4} + F_{1,2} z_1^{-5} z_2^{-1} + F_{1,3} z_1^{-5} z_2^{2} + F_{2,1} z_1^{-5} z_2^{5} + F_{2,2} z_1^{-2} z_2^{-4} + F_{2,3} z_1^{-2} z_2^{-1} + F_{3,1} z_1^{-2} z_2^{2} + F_{3,2} z_1^{-2} z_2^{5} \\ & + F_{3,3} z_1^{1} z_2^{-4} + F_{4,1} z_1^{1} z_2^{-1} \\ & + F_{4,2} z_1^{1} z_2^{2} + F_{4,3} z_1^{1} z_2^{5} + F_{3,3} z_1^{4} z_2^{-4} + F_{5,1} z_1^{4} z_2^{-1} + F_{5,2} z_1^{4} z_2^{2} + F_{5,3} z_1^{4} z_2^{5}, \end{aligned} \tag{A.14}$$

where

$$\begin{aligned} F_{1,1} &= \gamma_{-5,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ F_{1,2} &= \gamma_{-5,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ F_{1,3} &= \gamma_{-5,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ F_{1,4} &= \gamma_{-5,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ F_{2,1} &= \gamma_{-2,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ F_{2,2} &= \gamma_{-2,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ F_{2,3} &= \gamma_{-2,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ F_{2,4} &= \gamma_{-2,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ F_{3,1} &= \gamma_{1,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ F_{3,2} &= \gamma_{1,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ F_{3,3} &= \gamma_{1,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ F_{3,4} &= \gamma_{1,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ F_{4,1} &= \gamma_{4,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ F_{4,2} &= \gamma_{4,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ F_{4,3} &= \gamma_{4,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ F_{4,4} &= \gamma_{4,\alpha,\beta}\gamma_{5,\alpha,\beta}. \end{aligned} \tag{A.15}$$

Furthermore, for $(m_1, m_2) = (0, 2)$, we get

$$\begin{aligned} a_{\alpha,\beta,3i,3j+2}(z_1, z_2) = {} & G_{1,1}z_1^{-3}z_2^{-5} + G_{1,2}z_1^{-3}z_2^{-2} + G_{1,3}z_1^{-3}z_2^{1} \\ & + G_{1,4}z_1^{-3}z_2^{4} + G_{2,1}z_1^{0}z_2^{-5} + G_{2,2}z_1^{0}z_2^{-2} \\ & + G_{2,3}z_1^{0}z_2^{1} + G_{2,4}z_1^{0}z_2^{4} \\ & + G_{3,1}z_1^{3}z_2^{-5} + G_{3,2}z_1^{3}z_2^{-2} + G_{3,3}z_1^{3}z_2^{1} \\ & + G_{3,4}z_1^{3}z_2^{4}, \end{aligned} \tag{A.16}$$

where

$$\begin{aligned} G_{1,1} &= \gamma_{-3,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ G_{1,2} &= \gamma_{-3,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ G_{1,3} &= \gamma_{-3,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ G_{1,4} &= \gamma_{-3,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ G_{2,1} &= \gamma_{0,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ G_{2,2} &= \gamma_{0,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ G_{2,3} &= \gamma_{0,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ G_{2,4} &= \gamma_{0,\alpha,\beta}\gamma_{5,\alpha,\beta}, \\ G_{3,1} &= \gamma_{3,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ G_{3,2} &= \gamma_{3,\alpha,\beta}\gamma_{-4,\alpha,\beta}, \\ G_{3,3} &= \gamma_{3,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ G_{3,4} &= \gamma_{3,\alpha,\beta}\gamma_{2,\alpha,\beta}. \end{aligned} \tag{A.17}$$

Now, for $(m_1, m_2) = (1, 2)$, we get

$$\begin{aligned} a_{\alpha,\beta,3i+1,3j+2}(z_1, z_2) = {} & H_{1,1}z_1^{-4}z_2^{-5} + H_{1,2}z_1^{-4}z_2^{-2} + H_{1,3}z_1^{-4}z_2^{1} \\ & + H_{1,4}z_1^{-4}z_2^{4} + H_{2,1}z_1^{-1}z_2^{-5} \\ & + H_{2,2}z_1^{-1}z_2^{-2} + H_{2,3}z_1^{-1}z_2^{1} + H_{2,4}z_1^{-1}z_2^{4} \\ & + H_{3,1}z_2^{3}z_2^{-5} + H_{3,2}z_1^{2}z_2^{-2} + H_{3,3}z_1^{2}z_1^{1} \\ & + H_{3,4}z_1^{2}z_2^{4} + H_{4,1}z_2^{3}z_5^{-5} + H_{4,2}z_1^{5}z_2^{-2} \\ & + H_{4,3}z_1^{5}z_2^{1} + H_{4,4}z_1^{5}z_2^{4}, \end{aligned} \tag{A.18}$$

where

$$\begin{aligned} H_{1,1} &= \gamma_{-4,\alpha,\beta}\gamma_{-5,\alpha,\beta}, \\ H_{1,2} &= \gamma_{-4,\alpha,\beta}\gamma_{-2,\alpha,\beta}, \\ H_{1,3} &= \gamma_{-4,\alpha,\beta}\gamma_{1,\alpha,\beta}, \\ H_{1,4} &= \gamma_{-4,\alpha,\beta}\gamma_{4,\alpha,\beta}, \\ H_{2,1} &= \gamma_{-1,\alpha,\beta}\gamma_{-5,\alpha,\beta}, \\ H_{2,2} &= \gamma_{-2,\alpha,\beta}\gamma_{-1,\alpha,\beta}, \\ H_{2,3} &= \gamma_{1,\alpha,\beta}\gamma_{2,\alpha,\beta}, \\ H_{2,4} &= \gamma_{-1,\alpha,\beta}\gamma_{4,\alpha,\beta}, \\ H_{3,1} &= \gamma_{2,\alpha,\beta}\gamma_{-5,\alpha,\beta}, \\ H_{3,2} &= \gamma_{3,\alpha,\beta}\gamma_{-2,\alpha,\beta}, \\ H_{3,3} &= \gamma_{2,\alpha,\beta}\gamma_{1,\alpha,\beta}, \\ H_{3,4} &= \gamma_{2,\alpha,\beta}\gamma_{4,\alpha,\beta}, \\ H_{4,1} &= \gamma_{5,\alpha,\beta}\gamma_{-5,\alpha,\beta}, \\ H_{4,2} &= \gamma_{5,\alpha,\beta}\gamma_{-2,\alpha,\beta}, \\ H_{4,3} &= \gamma_{5,\alpha,\beta}\gamma_{1,\alpha,\beta}, \\ H_{4,4} &= \gamma_{5,\alpha,\beta}\gamma_{4,\alpha,\beta}. \end{aligned} \tag{A.19}$$

Finally, for $(m_1, m_2) = (2, 2)$, we get

$$\begin{aligned} a_{\alpha,\beta,3i+2,3j+2}(z_1, z_2) = {} & I_{1,1}z_1^{-5}z_2^{-5} + I_{1,2}z_1^{-5}z_2^{-2} + I_{1,3}z_1^{-5}z_2^{1} \\ & + I_{1,4}z_1^{-5}z_2^{4} + I_{2,1}z_1^{-2}z_2^{-5} \\ & + I_{2,2}z_1^{-2}z_2^{-2} + I_{2,3}z_1^{-2}z_2^{1} + I_{2,4}z_1^{-2}z_2^{4} \\ & + I_{3,1}z_2^{1}z_2^{-5} + I_{3,2}z_1^{1}z_2^{-2} + I_{3,3}z_1^{1}z_2^{1} \\ & + I_{3,4}z_1^{1}z_2^{4} + I_{4,1}z_2^{4}z_5^{-5} + I_{4,2}z_1^{4}z_2^{-2} \\ & + I_{4,3}z_1^{4}z_2^{1} + I_{4,4}z_1^{4}z_2^{4}, \end{aligned} \tag{A.20}$$

where

$$
\begin{aligned}
I_{1,1} &= \gamma_{-5,\alpha,\beta}\gamma_{-5,\alpha,\beta},\\
I_{1,2} &= \gamma_{-5,\alpha,\beta}\gamma_{-2,\alpha,\beta},\\
I_{1,3} &= \gamma_{-5,\alpha,\beta}\gamma_{1,\alpha,\beta},\\
I_{1,4} &= \gamma_{-5,\alpha,\beta}\gamma_{4,\alpha,\beta},\\
I_{2,1} &= \gamma_{-2,\alpha,\beta}\gamma_{-5,\alpha,\beta},\\
I_{2,2} &= \gamma_{-2,\alpha,\beta}\gamma_{-1,\alpha,\beta},\\
I_{2,3} &= \gamma_{-2,\alpha,\beta}\gamma_{2,\alpha,\beta},\\
I_{2,4} &= \gamma_{-2,\alpha,\beta}\gamma_{4,\alpha,\beta},\\
I_{3,1} &= \gamma_{1,\alpha,\beta}\gamma_{-5,\alpha,\beta},\\
I_{3,2} &= \gamma_{1,\alpha,\beta}\gamma_{-2,\alpha,\beta},\\
I_{3,3} &= \gamma_{1,\alpha,\beta}\gamma_{1,\alpha,\beta},\\
I_{3,4} &= \gamma_{1,\alpha,\beta}\gamma_{4,\alpha,\beta},\\
I_{4,1} &= \gamma_{4,\alpha,\beta}\gamma_{-5,\alpha,\beta},\\
I_{4,2} &= \gamma_{4,\alpha,\beta}\gamma_{-2,\alpha,\beta},\\
I_{4,3} &= \gamma_{4,\alpha,\beta}\gamma_{1,\alpha,\beta},\\
I_{4,4} &= \gamma_{4,\alpha,\beta}\gamma_{4,\alpha,\beta}.
\end{aligned}
\tag{A.21}
$$

$\gamma_{i,\alpha,\beta}$: $i = -5, \ldots, 5$ used in (A.1)–(A 15) are defined by (17). Hence, the Laurent polynomial of the bivariate MCTSS is

$$a_{\alpha,\beta}(z_1, z_2) = \sum_{m_1=0}^{2}\sum_{m_2=0}^{2} a_{\alpha,\beta,3i+m_1,3j+m_2}(z_1, z_2). \tag{A.22}$$

Data Availability

The data for implementation of the result are included in the paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Ghulam Mustafa, Rabia Hameed, and Amina Liaqat contributed to conceptualization and wrote the original draft; Dumitru Baleanu, Maysaa M. Al-Qurashi, and Faheem Khan contributed to formal analysis; Yu-Ming Chu and Faheem Khan contributed to methodology; Rabia Hameed and Ghulam Mustafa supervised the paper; Dumitru Baleanu and Yu-Ming Chu contributed to writing, reviewing, and editing the paper.

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