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A NEW NUMERICAL TREATMENT FOR FRACTIONAL DIFFERENTIAL EQUATIONS BASED ON NON-DISCRETIZATION OF DATA USING LAGUERRE POLYNOMIALS

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Abstract

In this research work, we discuss an approximation techniques for boundary value problems (BVPs) of differential equations having fractional order (FODE). We avoid the method from discretization of data by applying polynomials of Laguerre and developed some matrices of operational types for the obtained numerical solution. By applying the operational matrices, the given problem is converted to some algebraic equation which on evaluation gives the required numerical results. These equations are of Sylvester types and can be solved by using matlab. We present some testing examples to ensure the correctness of the considered techniques.

Keywords: Boundary Value Problems; Laguerre Polynomials; Discretization of Data; Numerical Solution.

1. INTRODUCTION

For the last 30 years, fractional order differential equations (FODEs) have gained wide-ranging applications in different realms like electrical activity of heart,[1] tank reactors of continuous stirred,[2] plug flow reactors,[3] discussion of gravity and electromagnetism.[4,5] Likewise, researchers in fractional calculus have got popularity in applications in different fields such as heredity characters, medical engineering subjects, chemistry as well as mechanics, electrical networks, viscoelasticity, signaling, imaging and phenomenon of optics.[6–14] In addition, some other attractive applications in dynamical system, electrochemistry, culturing of microorganisms were studied in Refs. 15 and 16. These problems related to engineering and medical engineering are in terms of mathematical problems, which are modeled by FODEs discretized into problems of solving various systems.

Several techniques and methods in natural sciences, physical sciences, mechanics and fluid mechanics, optical and engineering technologies can be modified using partial differential equations. PDEs represent a large numbers of models like waves equation, wave production, propagation of long waves, heat equation for chemical reaction.[13,15,17,18] Fractional order differential equations have been investigated and studied from various aspects, like theory of existence and uniqueness of solution, checking solution that either it is stable or not and approximation of solutions. The theory of existence has been greatly gaining the interest of researchers and a lot of research work has been undertaken to explore this idea. In qualitative analysis, the exact solution is very difficult to be obtained. Therefore, the researchers have shown keen interest in numerical and series solution of fractional order partial differential equations (FOPDEs). Due to high applicability and much more importance of FODEs, the researchers have given the topic increased attention to develop numerous techniques to find numerical and optimal solutions for these equations. Generally, each and every FODE or FOPDE cannot be solved directly and exactly.

It is of fundamental interest to exploit various numerical schemes, efficient enough to avoid computational complexities or overcome the difficulty in obtaining explicit analytical solutions. The best option might be numerical solutions which are achieved through approximate solution methods or series-type solution, as found in Refs. 19–24. Several other well-known semi-numerical and numerical techniques are also developed, among them the "spectral method" needs discretization of data.[6] Further, scientists have introduced some matrix of operational types by using polynomials like Shifted, Legendre, Jacobi and various other polynomials forms.[25,26]

Haar and Legendre wavelet methods were also beneficial to solve linear FPDEs.[27,28] Mostly, the spectral methods based on matrices of operational types are applicable to find out the approximate solutions for both FODEs and FOPDEs, see Refs. 29–32. For few years, the Bernoulli wavelet method has been used to find numerical solutions to FDEs, see Ref. 33. Jacobi wavelet operational matrices and optimal homotopy asymptotic method were also used to find numerical solutions of FOPDEs, see Refs. 34 and 35. Similarly, we also use non-discretization of data, because it consumes less time and there is no need of extra memory, by obtaining the operational matrices using orthogonal polynomial named "Laguerre polynomials". These orthogonal polynomials help to convert the operational matrices to algebraic equations of the form

$MX + XN + L = 0$. After that, we take help from matlab or mathematica to solve the obtained algebraic equations for the unknown X. We now provide boundary value problems (BVPs) of FODE to find the approximate solutions by Laguerre polynomials in Case 1 of the main section.

$$\begin{aligned} &{}^cD^\gamma Z(t) + A_1{}^cD^\alpha Z(t) + A_2 Z(t) \\ &\quad = \phi(t), \quad 1 < \gamma \leq 2, \ 0 < \alpha \leq 1, \\ &Z(0) = Z_0, \quad Z(1) = Z_1, \ Z_0, \quad Z_1 \in \Re. \end{aligned}$$

Similarly, the coupled system of BVPs of FODEs is treated in Case 2 as follows:

$$\begin{cases} {}^cD^\gamma Y(t) + B_1{}^cD^\alpha Z(t) + B_2 Y(t) \\ \qquad + B_3 Z(t) = \varphi(t), \\ {}^cD^\gamma Z(t) + E_1^c D^\alpha Y(t) + E_2 Z(t) \\ \qquad + E_3 Y(t) = \psi(t), \end{cases}$$

$$\begin{aligned} &Z(0) = Z_0, \quad Y(0) = Y_0, \quad Z(1) = Z_1, \\ &Y(1) = Y_1, \ Z_0, \ Y_0, \ Z_1, \quad Y_1 \in R. \end{aligned}$$

2. PRELIMINARIES

In this section, requisite concepts and some basic results along with definitions are provided, stapled to the work of our discussion.

Definition 2.1 (Refs. 13, 24, 36 and 37). The fractional integral of order $\gamma \in R^+$ of a function $Y : [0, \infty] \to R$ is defined as

$$I_t^\gamma Y(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\eta)^{\gamma-1} Y(\eta) d\eta.$$

Definition 2.2 (Ref. 6). The Caputo fractional derivative of fractional order of a function $Y \in \Re^+ \to R$ may be formulated as

$$\begin{aligned} {}^cD^\gamma Y(t) &= \frac{1}{\Gamma(m-\gamma)} \\ &\quad \times \int_0^t (t-\eta)^{m-\gamma-1} Y^m(\eta) d\eta. \end{aligned}$$

Lemma 2.1 (Ref. 38). *The solution of* ${}^cD^\gamma Y(t)$ *is given by* $Y(t) = d_0 + d_1 t + d_2 t^2 + \cdots + d_{m-1} t^{m-1}$ *such that* $d_p \in R, \ p = 0, 1, 2, \ldots, m-1$.

Lemma 2.2 (Ref. 39). *For* $\gamma > 0$, $I_t^\gamma[{}^cD^\gamma Y(t)] = Y(t) + d_0 + d_1 t + d_2 t^2 + \cdots + d_{m-1} t^{m-1}$, *where* $d_p \in \Re$, $p = 0, 1, 2, \ldots, m-1$.

Definition 2.3. The famous Laguerre polynomial "$L_p^\gamma(t)$" is defined as

$$\sum_{k=0}^{p} \frac{(-1)^k \Gamma(p+\gamma+1)}{\Gamma(k+1+\gamma)\Gamma(p+1-k)\Gamma(1+k)} y^k, \qquad p = 0, 1, 2, 3, \ldots.$$

Lemma 2.3. *Let us consider the function* φ, *which is continuous and differentiable up to* $n+1$ *and let*

$$Y = (L_{0,n}, L_{0,n}, \ldots, L_{0,n})$$

be Laguerre basis. If $L^t \ \Theta(t) \approx \varphi$ *in form of* Y *then the error will be*

$$\|\varphi - L^t \Theta(t)\| \leq \frac{\sqrt{(2)} \kappa S^{\frac{2n+3}{2}}}{\Gamma(n+2)\sqrt{2n+3}}.$$

Proof. For the proof, we refer to Ref. 40. □

If L_p^γ and L_q^γ are Laguerre polynomials, then their inner product is

$$\int_0^t L_p^\gamma(t) L_q^\gamma(t) X^\gamma(t) dt = \delta_{p,q} \mathfrak{Q}_k.$$

The weight function is

$$X^\gamma(t) = t^\gamma e^{-t},$$

with

$$\mathfrak{Q}_k = \begin{cases} \dfrac{\Gamma(1+\gamma+k)}{\Gamma(1+k)}, & p = q, \\ 0, & p \neq q. \end{cases}$$

We can express a function $Y(t)$ in terms of Laguerre polynomials as

$$\begin{aligned} Y(t) &= \sum_{k=0}^{M} d_k L_k^\gamma(t), \\ Y(t) &= d_0 L_0^\gamma(t) + d_1 L_1^\gamma(t) \\ &\quad + \cdots + d_M L_M^\gamma(t), \\ Y(t) &= [d_0 \ d_1 \ldots d_M] \begin{bmatrix} L_0^\gamma(t) \\ \vdots \\ L_n^\gamma(t), \end{bmatrix}, \\ Y(t) &= d^t H_M^T(t), \end{aligned} \tag{2.1}$$

where

$$d^t = [d_0 \ d_1 \ldots d_M],$$

and

$$H_M^T(t) = \begin{bmatrix} L_0^\gamma(t) \\ \vdots \\ L_n^\gamma(t) \end{bmatrix}.$$

Again, if

$$Y(t) = \sum_{k=0}^{m} d_p L_p^{\gamma}(t),$$

then the inner product is

$$\int_0^L Y(t)X^{\gamma}(t)L_q^{\gamma}(t)d(t) = \int_0^L \sum_{k=0}^{m} d_p L_p^{\gamma}(t)L_q^{\gamma}(t)X^{\gamma}(t)d(t),$$

which implies

$$\int_0^L Y(t)X^{\gamma}(t)L_q^{\gamma}(t)d(t) = \sum_{k=0}^{m} d_p \int_0^L L_p^{\gamma}(t)L_q^{\gamma}(t)X^{\gamma}(t)d(t).$$

We represent the left-hand side as

$$\int_0^L Y(t)X^{\gamma}(t)L_q^{\gamma}(t)d(t) = \sum_{k=0}^{m} d_p h_p,$$

where h_p is the general term of integration.

Then the coefficient d_p can be calculated as

$$d_p = \frac{1}{h_p}\int_0^L y(t)X^{\gamma}(t)L_q^{\gamma}(t)d(t).$$

Lemma 2.4. *Let $H_M^T(t)$ be a function vector, we can obtain the fractional order integral provided as*

$$I_t^{\gamma} H_M^T(t) = R_{M\times M}^{\gamma} H_M^T(t),$$

where the OM of fractional integral is represented in terms of $R_{M\times M}^{\gamma}$ given as follows:

$$\begin{bmatrix} U_{0,0,k,r}^{\gamma} & U_{0,1,k,r}^{\gamma} & \cdots & U_{0,q,k,r}^{\gamma} & \cdots & U_{0,m,k,r}^{\gamma} \\ U_{1,0,k,r}^{\gamma} & U_{1,p,k,r}^{\gamma} & \cdots & U_{1,q,k,r}^{\gamma} & \cdots & U_{1,m,k,r}^{\gamma} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ U_{p,0,k,r}^{\gamma} & U_{p,1,k,r}^{\gamma} & \cdots & U_{p,q,k,r}^{\gamma} & \vdots & U_{p,m,k,r}^{\gamma} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ U_{m,0,k,r}^{\gamma} & U_{m,1,k,r}^{\gamma} & \cdots & U_{m,q,k,r}^{\gamma} & \cdots & U_{m,m,k,r}^{\gamma} \end{bmatrix},$$

where

$$U_{p,q,k,r}^{\gamma} = \sum_{k=0}^{p}\sum_{r=0}^{p} \times \frac{(-1)^{k+r}\Gamma(q+1)\Gamma(p+\gamma+1) \times \Gamma(k+\gamma+\alpha+r+1)}{\Gamma(q-r+1)\Gamma(p-k+1)\Gamma(r+1) \times \Gamma(k+\gamma+1)\Gamma(k+\alpha+1) \times \Gamma(\gamma+r+1)}.$$

Proof. For the proof, we refer to Ref. 41. □

Lemma 2.5. *Let $H_M^T(t)$ represent a function vector, the fractional order derivative of this function, be given as*

$${}^{c}D^{\gamma} H_M^T(t) = G_{M\times M}^{\gamma} H_M^T(t),$$

such that $G_{M\times M}^{\gamma}$ is the OM of fractional order derivative and is equal to the matrix

$$\begin{bmatrix} 0 & 0 & & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ V_{\lceil\gamma\rceil,0,k,\alpha}^{\gamma} & V_{\lceil\gamma\rceil,1,k,\alpha}^{\gamma} & \cdots & V_{\lceil\gamma\rceil,q,k,\alpha}^{\gamma} & \cdots & V_{\lceil\gamma\rceil,n,k,\alpha}^{\gamma} \\ V_{p,0,k,\alpha}^{\gamma} & V_{p,1,k,\alpha}^{\gamma} & & V_{p,q,k,\alpha}^{\gamma} & \cdots & V_{p,n,k,\alpha}^{\gamma} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ V_{n,0,k,\alpha}^{\gamma} & V_{n,1,k,\alpha}^{\gamma} & & V_{n,q,k,\alpha}^{\gamma} & \cdots & V_{n,n,k,\alpha}^{\gamma} \end{bmatrix}, \tag{2.2}$$

where

$$V_{p,q,k,\alpha}^{\gamma} = \sum_{k=\gamma}^{p}\sum_{r=0}^{p} \times \frac{(-1)^{\gamma+k}\Gamma(q+1)\Gamma(p+\alpha+1) \times \Gamma(k+\alpha-r+\gamma+1)}{\Gamma(q-r+1)\Gamma(p-k+1) \times \Gamma(r+1)\Gamma(k+\alpha+1) \times \Gamma(k-\gamma+1)\Gamma(\alpha+\gamma+1)}.$$

Proof. For the proof, we refer to Ref. 42. □

Lemma 2.6. *We now obtain the OM for boundary conditions such that $X(t)$ be a function, with $L(t) = K_M H_M^T(t)$, then*

$$X(t)[I_t^{\gamma} L(t)] = K_M Q_{M\times M}^{\gamma} H_M^T(t),$$

where $Q_{M\times M}^{\gamma}$ is the operational matrix, given by

$$\begin{bmatrix} d_{0,0}, & d_{0,1} & \cdots & d_{0,q} & \cdots & d_{0,m} \\ d_{1,0} & d_{1,1} & \cdots & d_{1,q} & \cdots & d_{1,m} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ d_{p,0} & d_{p,1} & \cdots & d_{p,q} & \vdots & d_{p,m} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ d_{m,0} & d_{m,1} & \cdots & d_{m,q} & \cdots & d_{m,m} \end{bmatrix},$$

with

$$d_{p,q} = \frac{1}{h_p}\int_0^1 \Delta_{p,\gamma,k} X(t) L_q^{\gamma}(t)dt,$$

and

$$\Delta_{p,\gamma,k} = \sum_{k=0}^{p} \frac{(-1)^{p+1}\Gamma(p+1+\gamma)}{\Gamma(k+\gamma+1)\Gamma(1-k+p)\Gamma(k+\gamma)}.$$

Proof. We consider the general term of $H_M^T(t)$

$$I_t^\gamma L_p(t) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1-\eta)^{\gamma-1} L_p(\eta)d\eta,$$

$$I_t^\gamma L_p(t) = \frac{1}{\Gamma(\gamma)}\int_0^1 (1-\eta)^{\gamma-1} \sum_{k=0}^{p} (\eta)^k \times \frac{(-1)^k\Gamma(p+1+\gamma)}{\Gamma(-k+1+p)\Gamma(k+1+\gamma) \times \Gamma(1+k)} d\eta,$$

$$I_t^\gamma L_p(t) = \sum_{k=0}^{p} \frac{(-1)^k\Gamma(p+1+\gamma)}{\Gamma(\gamma)\Gamma(-k+1+p) \times \Gamma(k+1+\gamma)\Gamma(1+k)} \times \int_0^1 (1-\eta)^{\gamma-1}(\eta)^k d\eta. \tag{2.3}$$

Upon evaluating the integral

$$\int_0^1 (1-\eta)^{\gamma-1}\eta^k d\eta = \frac{\Gamma(\gamma)\Gamma(k+1)}{\Gamma(\gamma+k)},$$

(2.3) becomes

$$I_t^\gamma L_p(t) = \sum_{k=0}^{p} \frac{(-1)^k\Gamma(p+1+\gamma)}{\Gamma(\gamma)\Gamma(-k+1+p) \times \Gamma(k+1+\gamma)\Gamma(1+k)} \times \frac{\Gamma(\gamma)\Gamma(k+1)}{\Gamma(\gamma+k)}$$

$$\sum_{k=0}^{p} \frac{(-1)^k\Gamma(p+1+\gamma)}{\Gamma(-k+1+p)\Gamma(k+\gamma+1)\Gamma(1+k)} = \Delta_{p,\gamma,k}.$$

With the help of Laguerre polynomials we obtain

$$\Delta_{p,\gamma,k}X(t) = \sum_{q=0}^{m} d_{p,q}L_p(t),$$

where $d_{p,q}$ can be calculated by using orthogonality as

$$d_{p,q} = \frac{1}{hp}\int_0^1 \Delta_{p,\gamma,k}X(t)L_q^\gamma(t)dt. \tag{2.4}$$

For any result, we can get this relation with p and $q = 0, 1, \ldots, m$. □

3. MAIN RESULT

In this main section, we take the considered BVPs of FODE, and coupled system of BVPs of FODE, and construct the general procedure for their numerical solution such that in Case 1 we treat the BVP of FODE, while in Case 2 we treat the coupled system of BVPs of FODE.

Case I:

$${}^cD^\gamma Z(t) + A_1 {}^cD^\alpha Z(t) + A_2 Z(t) = \phi(t),$$
$$1 < \gamma \leq 2,\ \ 0 < \alpha \leq 1,$$
$$Z(0) = Z_0, \quad Z(1) = Z_1,\ Z_0,\ Z_1 \in \Re.$$

Assuming ${}^cD^\gamma Z(t) = L_M H_M^T(t)$, such that

$$Z(t) = L_M P_{M\times M}^\gamma H_M^T(t) - a_0 + a_1 t, \quad a_0, a_1 \in \Re.$$

Using the boundary conditions and then simplifying, we get

$$Z(t) = L_M P_{M\times M}^\gamma H_M^T(t) + Z_0 + t(Z_1 - Z_0) - tL_M P_{M\times M}^\gamma H_M^T(t). \tag{3.1}$$

We approximate

$$Z_0 + t(Z_1 - Z_0) \approx F_M^{(1)} H_M^T(t)$$

and

$$-L_M P_{M\times M}^\gamma H_M^T(t)|_{t=1} \approx L_M Q_{M\times M}^\gamma H_M^T(t).$$

Equation (3.1) implies that

$$Z(t) = L_M P_{M\times M}^\gamma H_M^T(t) + F_M^{(1)} H_M^T(t) + L_M Q_{M\times M}^\gamma H_M^T(t).$$

Applying ${}^cD^\alpha$ and using corollary

$${}^cD^\alpha[Z(t)] = {}^cD^\alpha[L_M P_{M\times M}^\gamma + F_M^{(1)} + L_M Q_{M\times M}^\gamma] H_M^T(t).$$

Approximation of $\phi(t) \approx F_M^{(2)} H_M^T$ and further calculation provides that

$$L_M H_M^T(t) + A_1[L_M P_{M\times M}^\gamma + F_M^{(1)} + L_M Q_{M\times M}^\gamma] G_{M\times M}^\alpha L_M^T(t) + A_2[L_M P_{M\times M}^\gamma + F_M^{(1)} + L_M Q_{M\times M}^\gamma] H_M^T(t) - F_M^2 H_M^T(t) = 0,$$

or

$$L_M + A_1[L_M P_{M\times M}^\gamma + F_M^{(1)} + L_M Q_{M\times M}^\gamma] G_{M\times M}^\alpha + A_2[L_M P_{M\times M}^\gamma + F_M^{(1)} + L_M Q_{M\times M}^\gamma] - F_M^2 = 0.$$

We obtain the matrix equation as

$$\begin{aligned} &L_M + L_M[A_1 P^{\gamma}_{M\times M} + A_1 Q^{\gamma}_{M\times M} G^{\alpha}_{M\times M} \\ &\quad + A_2 P^{\gamma}_{M\times M} + A_2 Q^{\gamma}_{M\times M}] \\ &\quad + A_1 F^{(1)}_M G^{\alpha}_{M\times M} + A_2 F^{(1)}_M - F^{(2)}_M = 0. \end{aligned}$$

Solving the above "matrix equation" by matlab, we get the coefficient matrix L_M.

Case II: Similarly, the FDE with the boundary conditions are given as

$$\begin{cases} {}^cD^{\gamma} Y(t) + B_1 {}^cD^{\alpha} Z(t) \\ \qquad + B_2 Y(t) + B_3 Z(t) = \varphi(t), \\ {}^cD^{\gamma} Z(t) + E_1 {}^cD^{\alpha} Y(t) \\ \qquad + E_2 Z(t) + E_3 Y(t) = \psi(t), \end{cases} \tag{3.2}$$

$$Z(0) = Z_0, \quad Y(0) = Y_0,$$
$$Z(1) = Z_1, \quad Y(1) = Y_1.$$

Let us assume

$$\begin{cases} {}^cD^{\gamma} Y(t) = L_M H^T_M(t), \\ {}^cD^{\gamma} Z(t) = N_M H^T_M(t), \end{cases} \tag{3.3}$$

$$\begin{cases} Y(t) = e_0 + e_1(t) + L_M P^{\gamma}_{M\times M} H^T_M(t), \\ Z(t) = d_0 + d_1(t) + N_M P^{\gamma}_{M\times M} H^T_M(t). \end{cases} \tag{3.4}$$

Using the boundary conditions

$$\begin{aligned} Y(t) &= L_M P^{\gamma}_{M\times M} H^T_M(t) + Y_0 + t(Y_1 - Y_0) \\ &\quad - t L_M P^{\gamma}_{M\times M} H^T_M(t)|_{t=1}, \\ Z(t) &= N_M P^{\gamma}_{M\times M} H^T_M(t) + Z_0 + t(Z_1 - Z_0) \\ &\quad - t N_M P^{\gamma}_{M\times M} H^T_M(t)|_{t=1}. \end{aligned}$$

Upon Approximation

$$\begin{aligned} Y_0 + t(Y_1 - Y_0) &\approx F^1_M H^T_M(t), \\ Z_0 + t(Z_1 - Z_0) &\approx F^2_M H^T_M(t), \\ -t L_M P^{\gamma}_{M\times M} H^T_M(t) &= L_M Q^{\gamma,t}_{M\times M} H^T_M(t), \\ -t N_M P^{\gamma}_{M\times M} H^T_M(t) &= N_M Q^{\gamma,t}_{M\times M} H^T_M(t). \end{aligned}$$

Equation (3.2) is then written as

$$\begin{cases} Y(t) = L_M P^{\gamma}_{M\times M} H^T_M(t) \\ \qquad + L_M Q^{\gamma,t}_{M\times M} H^T_M(t) + F^1_M H^T_M(t), \\ Z(t) = N_M P^{\gamma}_{M\times M} H^T_M(t) \\ \qquad + N_M Q^{\gamma,t}_{M\times M} H^T_M(t) + F^2_M H^T_M(t). \end{cases} \tag{3.5}$$

The source functions $\varphi(t)$ and $\psi(t)$ are approximated such that

$$\begin{cases} \varphi(t) = F^3_M H^T_M(t), \\ \psi(t) = F^4_M H^T_M(t). \end{cases} \tag{3.6}$$

Due to application of ${}^cD^{\alpha}$ to $Z(t)$ and ${}^cD^{\alpha}$ to $Z(t)$

$$\begin{cases} {}^cD^{\alpha *} Y(t) = [L_M P^{\gamma}_{M\times M} + L_M Q^{\gamma,t}_{M\times M} + F^1_M] \\ \qquad \times G^{\alpha}_{M\times M} H^T_M(t), \\ {}^cD^{\alpha} Z(t) = [N_M P^{\gamma}_{M\times M} + N_M Q^{\gamma,t}_{M\times M} + F^2_M] \\ \qquad \times G^{\alpha}_{M\times M} H^T_M(t), \end{cases} \tag{3.7}$$

using the above equations in (3.2) we get

$$\begin{cases} L_M H^T_M(t) + B_1[N_M P^{\gamma}_{M\times M} \\ \quad + N_M Q^{\gamma,t}_{M\times M} + F^2_M] G^{\alpha}_{M\times M} H^T_M(t) \\ \quad + B_2[L_M P^{\gamma}_{M\times M} + L_M Q^{\gamma,t}_{M\times M} \\ \quad + F^1_M] H^T_M(t) + B_3[N_M P^{\gamma *}_{M\times M} \\ \quad + N_M Q^{\gamma,t}_{M\times M} + F^2_M] H^T_M(t) - F^3_M H^T_M(t) = 0, \\ N_M H^T_M(t) + E_1[L_M P^{\gamma}_{M\times M} + L_M Q^{\gamma,t}_{M\times M} + F^1_M] \\ \quad \times G^{\alpha}_{M\times M} H^T_M(t) + E_2[N_M P^{\gamma}_{M\times M} \\ \quad + N_M Q^{\gamma,t}_{M\times M} + F^2_M] H^T_M(t) \\ \quad + E_3[L_M P^{\gamma}_{M\times M} + L_M Q^{\gamma,t}_{M\times M} + F^1_M] \\ \quad \times H^T_M(t) - F^4_M H^T_M(t) = 0. \end{cases} \tag{3.8}$$

Now, let us assume that

$$\begin{cases} A^{\gamma}_{M\times M} = B_1[P^{\gamma}_{M\times M} + Q^{\gamma,t}_{M\times M}] G^{\alpha}_{M\times M} \\ \qquad + B_3[P^{\gamma}_{M\times M} + Q^{\gamma,t}_{M\times M}], \\ B^{\gamma}_{M\times M} = E_1[P^{\gamma}_{M\times M} + Q^{\gamma,t}_{M\times M}] G^{\alpha}_{M\times M} \\ \qquad + E_3[P^{\gamma}_{M\times M} + Q^{\gamma,t}_{M\times M}], \\ O^{\gamma}_{M\times M} = B_2[P^{\gamma}_{M\times M} + Q^{\gamma,t}_{M\times M}], \\ R^{\gamma}_{M\times M} = E_2[P^{\gamma}_{M\times M} + Q^{\gamma,t}_{M\times M}], \end{cases} \tag{3.9}$$

with

$$\begin{cases} S_M = B_1 F^2_M G^{\alpha}_{M\times M} + B_2 F^1_M + B_3 F^2_M - F^3_M, \\ T_M = E_1 F^1_M G^{\alpha}_{M\times M} + E_2 F^2_M + E_3 F^1_M - F^4_M. \end{cases} \tag{3.10}$$

The matrix equation is obtained from (3.8) as

$$\begin{cases} L_M H_M^T(t) + N_M A_{M\times M}^{\gamma} H_M^T(t) \\ \quad + L_M O_{M\times M}^{\gamma} H_M^T(t) + S_M H_M^T(t) = 0, \\ N_M H_M^T(t) + L_M B_{M\times M}^{\gamma} H_M^T(t) \\ \quad + N_M R_{M\times M}^{\gamma} H_M^T(t) + T_M H_M^T(t) = 0, \end{cases} \tag{3.11}$$

which by simplification yields

$$[L_M \ N_M] + [L_M \ N_M] \times \begin{bmatrix} O_{M\times M}^{\gamma} & B_{M\times M}^{\gamma} \\ A_{M\times M}^{\gamma} & R_{M\times M}^{\gamma} \end{bmatrix} + [S_M \ T_M] = 0.$$

4. TEST PROBLEMS

In this section, we provide some test problems to elaborate the above analysis.

Example 4.1. Consider the following fractional order problem with boundary conditions:

$$\begin{aligned} {}^cD^{\gamma}Z(t) + 5\,{}^cD^{\alpha}Z(t) + 6Z(t) &= \frac{720t^{6-\gamma}}{\Gamma(7-\gamma)} + \frac{120t^{5-\gamma}}{\Gamma(6-\gamma)} \\ &\quad - \frac{24t^{4-\gamma}}{\Gamma(3-\gamma)} + \frac{2t^{2-\gamma}}{\Gamma(3-\gamma)} \\ &\quad + \frac{3600t^{6-\alpha}}{\Gamma(7-\alpha)} + \frac{600t^{5-\alpha}}{\Gamma(5-\alpha)} \\ &\quad - \frac{120t^{4-\alpha}}{\Gamma(5-\alpha)} + \frac{10t^{2-\alpha}}{\Gamma(3-\alpha)} \\ &\quad + 6t^6 + 6t^5 - 6t^4 + 6t^2 + 6, \end{aligned}$$

$$Z(0) = 1, \quad Z(1) = 3,$$

where $1 < \gamma \leq 2,\ 0 < \alpha \leq 1$. At $\gamma = 2,\ \alpha = 1$, the solution at integer order is given by $Z(t) = t^6 + t^5 - t^4 + t^2 + 1$. We investigate the approximate solutions at various fractional orders by using the aforementioned proposed method corresponding to different fractional orders. Also, we provide the absolute error graphs. From Figs. 1 and 2, we observe that the spectral method under the operational matrices by using Laguerre polynomials provides very good solutions. As the scale level enlarges the approximate solution is tending to converge the exact solution at integer order. But on the other hand when $(\gamma, \alpha) \to (2, 1)$, the numerical solution is tending to the integer order solution.

Example 4.2. Consider the given coupled system as

$$\begin{cases} {}^cD^{\gamma}Y(t) + 10\,{}^cD^{\alpha}Z(t) + 40Y(t) \\ \quad - 5Z(t) = 49\cos t - 5\sin t, \\ {}^cD^{\gamma}Z(t) - 10\,{}^cD^{\alpha}y(t) + 20Z(t) \\ \quad + 6Y(t) = 29\sin t + 6\cos t, \end{cases} \tag{4.1}$$

under initial and boundary conditions as $Z(0) = 1,\ Y(0) = 0,\ Z(1) = \cos 1,\ Y(1) = \sin 1$.

Let the solution at integer orders $\gamma = 2, \alpha = 1$ be $Y(t) = \cos t,\ Z(t) = \sin t$. We approximate the solution through the proposed method for the given coupled system of fractional order. We see

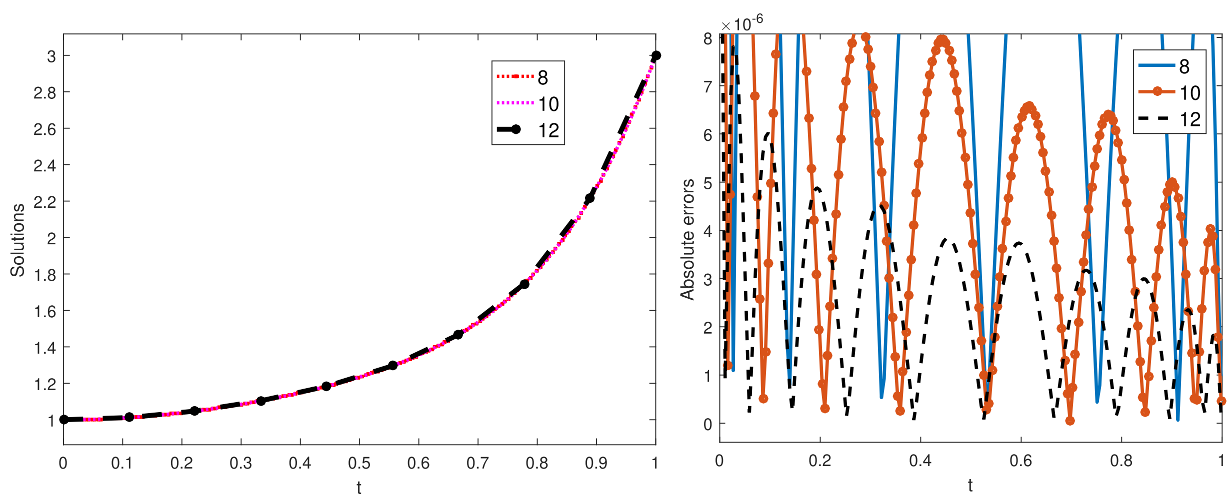

Fig. 1 Graphical representation at various values of M and taking $\gamma = 1.9,\ \alpha = 0.9$ and the absolute errors.

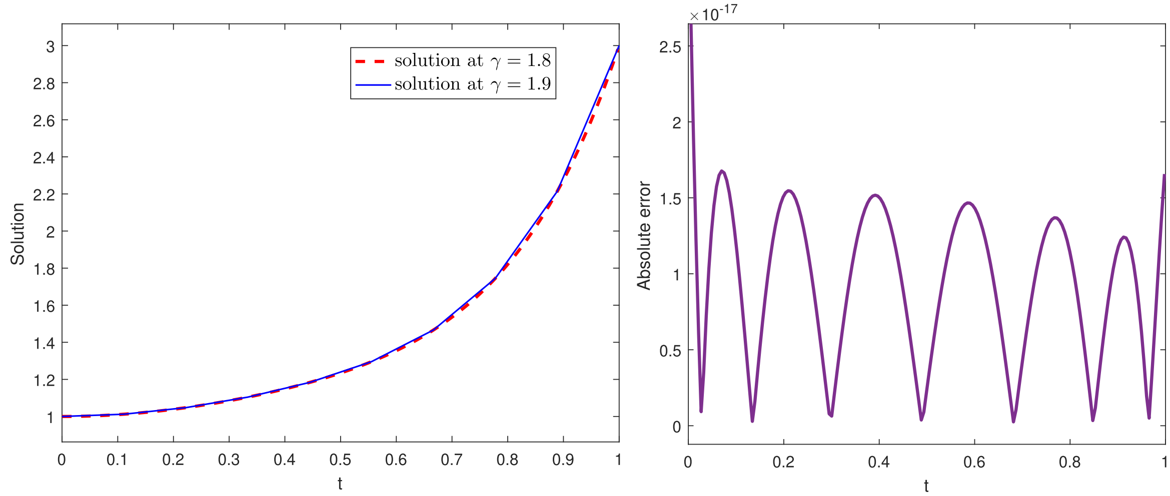

Fig. 2 Numerical solution at $M = 12$ and $\alpha = 0.9$ and their absolute errors.

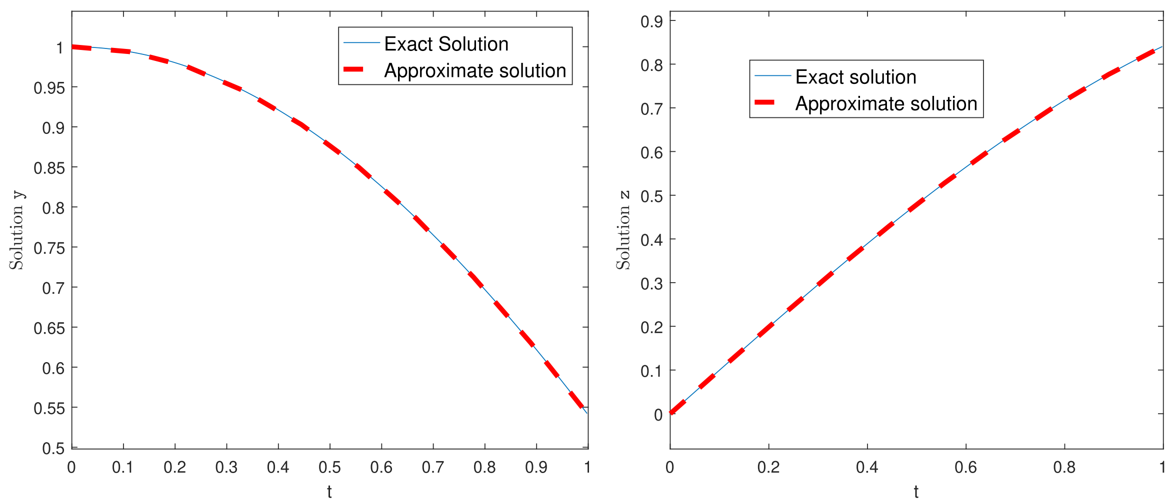

Fig. 3 Comparison between exact and approximate solutions at $M = 10$ and taking $\gamma = 1.9,\ \alpha = 0.9$.

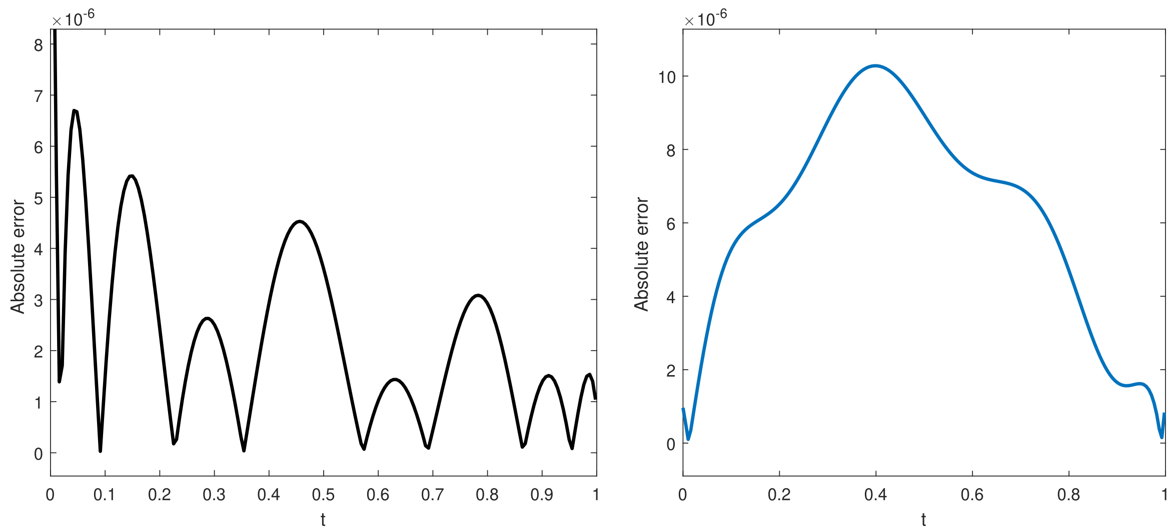

Fig. 4 Absolute errors at $M = 10$ and $\gamma = 1.9,\ \alpha = 0.9$.

from Figs. 3 and 4 that the adopted spectral method produces very accurate solution at the given values scales and fractional orders.

5. CONCLUSION

We have successfully used Laguerre polynomials to obtain the operational matrix without discretization of data. Based on these matrices, we have converted some FODEs and their systems to Sylvester-type algebraic equations which then solved for numerical solutions. In this regard, a new operational matrix "Q" has been obtained for the BVPs. From examples and their analysis, we observe that the considered polynomials also provide excellent numerical results for FODEs.

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