Alexandria Engineering Journal (2022) 61, 2016–2028

Alexandria University

Alexandria Engineering Journal

www.elsevier.com/locate/aej
www.sciencedirect.com

A robust study on the listeriosis disease by adopting fractal-fractional operators

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Received 12 March 2021; revised 20 June 2021; accepted 8 July 2021
Available online 11 August 2021

KEYWORDS

Listeriosis model;
Mittag–Leffler kernel;
Power kernel;
Fractal-Fractional operators;
Stability analysis;
Disease-free equilibrium;
Endemic equilibrium

Abstract Listeriosis is one of the zoonotic diseases affecting most parts of the Sub-Saharan countries. The infection is often transmitted by eating and it can also pass by respiratory and direct contact. In this paper, a listeriosis mathematical model is formulated involving fractal-fractional orders in both Caputo and Atangana–Baleanu derivatives. Moreover, future behaviors of the disease are investigated by considering the fractal–fractional operators that are very effective in modeling the real-life phenomena by virtue of their memory effect. The basic properties and steady states are also obtained. The threshold parameter for determining the spread of the disease is computed. Numerical results are presented for each fractal-fractional-order operator. The results obtained in the paper show that the numerical schemes are effective for predicting and analyzing complex phenomena.

1. Introduction

Man has not done well in protecting the natural environment due to several reasons including population growth. The degrading of the immediate environment has so many negative consequences on mankind. Therefore, the spread of diseases has become inevitable in society including zoonotic ones such as Listeriosis [4,5]. This disease is acquired via two folds [4]. It is acquired after taking food contaminated with Listeria

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Peer review under responsibility of Faculty of Engineering, Alexandria University.

https://doi.org/10.1016/j.aej.2021.07.010

monocytogenes and also mothers who possess the disease can transmit it to the babies during delivery through their skin or respiratory track [9]. Listeria monocytogenes is known to be one of the easiest causes of bacterial meningitis in the class of neonates. Some of the factors which are known to be associated with this disease include weak immune suppression, cirrhosis, diabetes, and renal failure [6,8].

Researchers and biomedical scientists have continuously studied the qualitative properties and numerical solutions of fractional-order HIV epidemic models with stability analysis. Stability analysis of a system in epidemiology and immunology determines the behavior of the system in disease transmission. By stability analysis one knows when and where the disease spreads by calculating the most wanted quantity known as basic reproduction number denoted by R_0. It is the threshold quantity which shows whether infection spreads or dies in the susceptible population. In other words, we can say that the population is free from the disease if $R_0 < 1$ and disease spreads in the whole population if $R_0 > 1$.

Mathematical modeling in the last century has proven to be very effective tool in managing and controlling many diseases through vital qualitative information [7]. In absence of real data which sometimes very expensive mathematical models offer the alternative platform in this regard. Althought the fractional order mathematical modelling study of the listeriosis is totally very few in the literature, integer order models on listeriosis are not the new subject to researchers.

In [1], the authors constructed an integer model to examine the transmission dynamics of Listeriosis incorporating vaccination class. Authors in [2] formulated the Anthrax–Listeriosis coinfected model and obtained the steady states also undertook sensitivity analysis on the reproduction number. In [3] a deterministic model was developed on Listeriosis driven by cross contamination of ready-to-eat food products and suggested that reduction of contamination workers and removal of infected product would reduce the level of contamination of food product.

Fractional calculus possess memory effect and this property helps to predict accurately physical systems. Several fractional order derivatives have been formulated including, Riemann–Liouville-Caputo, Atangana-Baleanu, Caputo–Fabrizio [10–12]. Fractional calculus theory and its illustrative applications are attracting attention all over the world day by day. New fractional operators that have different features have been defined and have been used extensively to model real-life problems. The arise of the new operators in the literature can be considered as a result of the reproduction of new problems that model different types of real-life events. In this context, a number of effective problems have been solved by different types of fractional operators that contain different kernels and presented their illustrative numerical simulations in terms of the fractional operators. As suggestions one can follow these papers to have more opinion about the applications of FC [13–27]. Very recent, researchers have developed complex fractional operators consisting of more than single operator such fractal-fractional Caputo order derivative, fractal-fractional Caputo–Fabrizio and fractal-fractional Atangana–Baleanu derivative and conformable fractional order Caputo derivative [28,29]. These operators are adequate to describe highly complex phenomena that single operators cannot describe. Some interesting applications can be applied to engineering, biology, medicine and many other [31–50].

In this work, we have formulated the model in the context of fractal and fractional. Naturally, most objects in the biological point of view contain fractal. Therefore, modelling of epidemiology with respect to two dimensions presents a true reflection of the projection that comes out the analysis. In our model, the purpose of dealing with fractional-order systems is the memory and hereditary properties which are the complex behavioral patterns of biological systems gives us more realistic way to biological systems. In the fractional-order models, the memory property allows the integration of more information from the past which predicts and translates the models more accurately. Also, the hereditary property describes the genetic profile along with age and status of the immune system. Because of such properties fractional-order calculus have found wide applications to model dynamics processes in many well-known fields of science, engineering, biology, medicine and many others [51,52]. To the best of our knowledge no single mathematical Listeriosis model has been formulated in fractional dynamics and this study would lead to some vital qualitative information in the advancement of the Listeriosis. This study aims to use the newly-developed fractal-fractional derivatives in Caputo and Atangana-Baleanu sense to capture information on the dynamics of Listeriosis disease.

2. Preliminaries

In this section, we give the fundamental definitions that we use throughout the paper. These definitions generally explain the fractal-fractional derivative involving the power kernel and Mittag–Leffler kernel.

Definition 1. [30,53] Assuming that $\phi(t)$ is continuous on (a,b), and fractal-differentiable on (a,b) with order α, the fractal-fractional derivative of $\phi(t)$ in the Caputo sense involving power law-type kernel is given by

$$ {}_0{}^{FFC}D_t^{\alpha,\vartheta}\{\phi(t)\} = \frac{1}{\Gamma(k-\alpha)}\int_0^t \frac{d\phi(\tau)}{d\tau^{\vartheta}}(t-\tau)^{k-\alpha-1}d\tau, \tag{1}$$

where $k-1 < \alpha, \vartheta \leqslant k \in N$ and

$$\frac{d}{d\tau^{\vartheta}}\phi(\tau) = \lim_{t\to\tau}\frac{\phi(t)-\phi(\tau)}{t^{\vartheta}-\tau^{\vartheta}}. \tag{2}$$

If we consider the particular case of $\vartheta = 1$, Eq. (1) turns to well-known Caputo fractional derivative

$$ {}_0{}^{C}D_t^{\alpha}\{\phi(t)\} = \frac{1}{\Gamma(k-\alpha)}\int_0^t \frac{d^k}{d\tau^k}\phi(\tau)(t-\tau)^{k-\alpha-1}d\tau, \tag{3}$$

where $k-1 < \alpha \leqslant k \in N$.

Definition 2. [30,55] Assuming that $\phi(t)$ is continuous on (a,b), and fractal-differentiable on (a,b) with order α, the fractal-fractional derivative of $\phi(t)$ in the Caputo sense involving the generalized Mittag–Leffler type kernel is given by

$$ {}_0{}^{FFAB}D_t^{\alpha,\vartheta}\{\phi(t)\} = \frac{\Psi(\alpha)}{k-\alpha}\int_0^t \frac{d}{d\tau^{\vartheta}}\phi(\tau)E_{\alpha}\left(-\alpha\frac{(t-\tau)^{\alpha}}{k-\alpha}\right)d\tau, \tag{4}$$

where $k-1 < \alpha, \vartheta \leqslant k \in N, \Psi(\alpha)$ is an adjusting function and $\frac{d}{d\tau^{\vartheta}}\phi(\tau)$ is defined in Eq. (2). Considering the particular case of

$\vartheta = 1$, the Eq. (4) turns to well-known Atangana-Baleanu-Caputo fractional derivative

$$ {}_0^{ABC}D_t^{\alpha}\{\phi(t)\} = \frac{\Psi(\alpha)}{k-\alpha}\int_0^t \frac{d^k}{d\tau^k}\phi(\tau)E_{\alpha}\left(-\alpha\frac{(t-\tau)^{\alpha}}{k-\alpha}\right)d\tau, \tag{5} $$

where $k-1 < \alpha \leqslant k \in N$.

Definition 3. [53] The Caputo fractional integral of a given function $\phi(t)$ is defined as

$$ {}_0^{C}I_t^{\alpha}\{\phi(t)\} = \frac{1}{\Gamma(\alpha)}\int_0^t \phi(\tau)(t-\tau)^{\alpha-1}d\tau. \tag{6} $$

Definition 4. [54] The corresponding integral of the Atangana-Baleanu fractional derivative is defined as

$$ {}_0^{AB}I_t^{\alpha}\{\phi(t)\} = \frac{1-\alpha}{\Psi(\alpha)}\phi(t) + \frac{\alpha}{\Gamma(\alpha)\Psi(\alpha)} \times \int_0^t \phi(\tau)(t-\tau)^{\alpha-1}d\tau, \tag{7} $$

where $\Psi(\alpha)$ is defined in Eq. (5).

Definition 5. [30] The fractal-fractional integral of the given function $\phi(t)$ in the Caputo sense involving power-type kernel is defined as

$$ {}_0^{FFC}I_t^{\alpha,\vartheta}\{\phi(t)\} = \frac{\vartheta}{\Gamma(\alpha)}\int_0^t \tau^{\vartheta-1}\phi(\tau)(t-\tau)^{\alpha-1}d\tau. \tag{8} $$

which gives the Caputo fractional integral given in Eq. (6) in the particular case of $\vartheta = 1$.

Definition 6. [30] The fractal-fractional integral of the given function $\phi(t)$ in the Caputo sense involving the generalized Mittag–Leffler type kernel is defined as

$$ {}_0^{FFAB}I_t^{\alpha,\vartheta}\{\phi(t)\} = \frac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)}\phi(t) + \frac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)} \times \int_0^t \tau^{\vartheta-1}\phi(\tau)(t-\tau)^{\alpha-1}d\tau, \tag{9} $$

which gives the Atangana-Baleanu fractional integral given in Eq. (7) in the particular case of $\vartheta = 1$.

3. Mathematical Model Formulation

The model sub-divides the entire related human population at time t, denoted by $\Upsilon(t)$, into the following sub-populations of susceptible individuals who have not infected with listeriosis yet $S(t)$, those that have been infected with listeriosis $I(t)$, those recovered from the infection $R(t)$, and $C(t)$ is the population of carcasses of animals in the soil that may have died of listeriosis. Since the carcasses of animals that may have not been properly disposed of have the tendency of generating pathogens. Thus, the total human population is $\Upsilon = S + I + R + C$. The concentration of carcasses and ingestion rate are depicted by η and ξ, respectively. And listeriosis related death rate is λ, waning immunity rate is ω, natural death rate is given by μ, carcasses (bacteria) mortality rate is μ_d. Human recruitment rate is Θ, listeriosis waning immunity rate is ω, listeriosis recovery rate is δ and listeriosis contribution to environment is depicted by σ.

Based on the interrelationship with the compartments the following nonlinear ordinary differential equations are obtained [2].

$$ \begin{aligned} &\tfrac{dS}{dt} = \Theta + \omega R(t) - \tfrac{\eta C(t)}{\xi + C(t)}S(t) - \mu S(t),\\ &\tfrac{dI}{dt} = \tfrac{\eta C(t)}{\xi + C(t)}S(t) - (\delta + \lambda + \mu)I(t),\\ &\tfrac{dR}{dt} = \delta I(t) - (\omega + \mu)R(t),\\ &\tfrac{dC}{dt} = \sigma I(t) - \mu_d C(t),\\ &\Upsilon(t) = S(t) + I(t) + R(t) + C(t). \end{aligned} \tag{10} $$

The above ordinary differential model (10) is further modified to a fractional-order system of order α. The purpose of considering the fractional-order case is the significant uniqueness of these varieties of fractional-order systems with non-local characteristics (memory) and hereditary properties that have not been seen with the integer-order differential operators which widely exists in biology. Non-integer modeling possesses the previous and current in order to make prediction for the future. The value α, is the fractional order and ϑ constitutes the fractal dimension. In reality, objects are made up fractal and including fractal perspective would lead to better understanding of phenomena. Thus, our proposed fractional-order model for listeriosis disease transmission has the following form

$$ \begin{aligned} &{}_0^{*}D_t^{\alpha,\vartheta}S(t) = \Theta + \omega R(t) - \tfrac{\eta C(t)}{\xi + C(t)}S(t) - \mu S(t),\\ &{}_0^{*}D_t^{\alpha,\vartheta}I(t) = \tfrac{\eta C(t)}{\xi + C(t)}S(t) - (\delta + \lambda + \mu)I(t),\\ &{}_0^{*}D_t^{\alpha,\vartheta}R(t) = \delta I(t) - (\omega + \mu)R(t),\\ &{}_0^{*}D_t^{\alpha,\vartheta}C(t) = \sigma I(t) - \mu_d C(t). \end{aligned} \tag{11} $$

with the initial values

$$ S_0(t) = S(0), I_0(t) = I(0), R_0(t) = R(0), C_0(t) = C(0), \tag{12} $$

where $0 < \alpha \leqslant 1$ shows the fractional order of the system and ${}_0^{*}D_t^{\alpha,\vartheta}$ represents the corresponding fractional or fractal-fractional operator, i.e., Caputo- Fractional, Atangana-Baleanu-Fractional, Caputo-Fractal-Fractional or Atangana-Baleanu-Fractal-Fractional operator. If we take into account $\alpha = 1$ in the system (11), then it reduces to the system (10) which gives the integer order one. Moreover, one can reduce the fractal-fractional system (11) to the fractional system by taking $\vartheta = 1$.

4. Investigation of the Dynamics of the Model

Stability analysis of a system in epidemiology and immunology determines the behavior of the system in disease status. By stability analysis one knows when and where the disease spread by calculating the most wanted quantity known as basic reproduction number denoted by Φ_0. It is the threshold quantity which shows whether infection spreads or not in the susceptible population. In this section, the positivity and boundedness of the solution for the proposed model given by Eq. (11) are given, after that the basic reproduction number is pointed out. Finally, the existence conditions and the stability results for both disease-free equilibrium and endemic equilibrium are provided.

4.1. Positivity and Boundedness

By positivity means the population survives and boundedness refers as a natural restriction to growth because of limited resources. In order to show the positivity of the solution, let us depict $R_+^4 = \{\Lambda(t) \in R^4 : \Lambda(t) \geqslant 0\}$, and let $\Lambda(t) = [S(t), I(t), R(t), C(t)]^T$. Hence, the following theorem arises:

Theorem 1. *The region R_+^4 is positively invariant and the solutions remain bounded throughout the region.*

Proof 1. By considering the results in [58], we can determine the solution on the positive axes by solving the fractional system (11) taking into account the initial values given by (12). Afterwards, we need to explain the positive region R_+^4 remains positively invariant. Then we have

$$\begin{aligned} {}_0{}^*D_t^{\alpha,\vartheta} S(t)\big|_{S=0} &= \Theta + \tfrac{\eta C(t)}{\xi + C(t)} R(t) \geqslant 0, \\ {}_0{}^*D_t^{\alpha,\vartheta} I(t)\big|_{I=0} &= \tfrac{\eta C(t)}{\xi + C(t)} S(t) \geqslant 0, \end{aligned} \tag{13}$$

$$\begin{aligned} {}_0{}^*D_t^{\alpha,\vartheta} R(t)\big|_{R=0} &= \delta I(t) \geqslant 0, \\ {}_0{}^*D_t^{\alpha,\vartheta} C(t)\big|_{C=0} &= \sigma I(t) \geqslant 0. \end{aligned} \tag{14}$$

On each hyperplane bounding the non-negative orthant, the vector field falls into the region R_+^4. This means that the region is a positively invariant set. Moreover, from Eq. (11) we get

$$\begin{aligned} {}_0{}^*D_t^{\alpha,\vartheta} \Upsilon(t) &= \Theta - \mu_d C(t) - \mu(S(t) + I(t) + R(t)) + (\sigma - \lambda) I(t) \\ &\leqslant \Theta - \mu \Upsilon(t) - (\lambda - \sigma) I(t) \\ &\leqslant \Theta - \mu \Upsilon(t). \end{aligned} \tag{15}$$

Applying the Laplace transform of Eq. (15), we have

$$s^{\alpha} \tilde{\Upsilon}(s) - s^{\alpha-1} \Upsilon(0) \leqslant \frac{\Theta}{s} - \mu \tilde{\Upsilon}(s),$$

which implies

$$\tilde{\Upsilon}(s) \leqslant \frac{s^{-1}\Theta(t)}{s^{\alpha} + \mu} + \frac{s^{\alpha-1}}{s^{\alpha} + \mu} \Upsilon(0).$$

If we apply the inverse Laplace transform to the last equation and taking into account that

$$\Upsilon(0) = (S(0), I(0), R(0), C(0)) \in R_+,$$

we get

$$\begin{aligned} \Upsilon(t) &\leqslant \Theta t^{\alpha} E_{\alpha,\alpha+1}(-\mu t^{\alpha}) + E_{\alpha}(-\mu t^{\alpha}) \Upsilon(0) \\ &= \tfrac{\Theta}{\mu} (\mu t^{\alpha} E_{\alpha,\alpha+1}(-\mu t^{\alpha}) + E_{\alpha}(-\mu t^{\alpha})) \\ &= \tfrac{\Theta}{\mu}. \end{aligned} \tag{16}$$

Hence, the biologically feasible region for the system (11) is given by

$$R_f^+ = \left\{ (S, I, R, C) \in R_+^4 : 0 < S + I + R + C \leqslant \frac{\Theta}{\mu} \right\}. \tag{17}$$

4.2. Determining the Equilibria and Their Stabilities

The equilibrium points are obtained by equating to zero the right-hand side of system (11) as

$$\begin{aligned} &\Theta + \omega R(t) - \tfrac{\eta C(t)}{\xi + C(t)} S(t) - \mu S(t) = 0, \\ &\tfrac{\eta C(t)}{\xi + C(t)} S(t) - (\delta + \lambda + \mu) I(t) = 0, \\ &\delta I(t) - (\omega + \mu) R(t) = 0, \\ &\sigma I(t) - \mu_d C(t) = 0. \end{aligned} \tag{18}$$

Simplifying the system (18) we obtain the disease free equilibrium point $P_{DF}^0 = (S^0, I^0, R^0, C^0) = \left(\frac{\Theta}{\mu}, 0, 0, 0\right)$, which is defined as the point in which the population remains in the lack of disease [59].

Now, in order to investigate the local stability of the disease-free equilibrium, we need to evaluate the basic reproduction number that can be regarded as a measure to predict the future trend of the disease. It is considered as the mean number of newly infected individuals procured by a single infection individual. We use next generation matrix method [59] in order to obtain the basic reproduction number and we evaluate the matrices at the disease-free equilibrium point P_{DF}^0 for the new infection and transfer terms as

$$A = \begin{pmatrix} 0 & 0 & \frac{\Theta\eta}{\mu\xi} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} \delta + \lambda + \mu & 0 & 0 \\ -\delta & \omega + \mu & 0 \\ -\sigma & 0 & \mu_d \end{pmatrix}.$$

Thus, the basic reproduction number is given by the following equation

$$\Phi_0 = \rho(AB^{-1}) = \frac{\Theta\sigma\eta}{\mu\mu_d\xi(\delta + \lambda + \mu)}, \tag{19}$$

where ρ represents the spectral radius of the matrix AB^{-1}.

Theorem 2. *The system (11) has always a unique endemic equilibrium if $\Phi_0 > 1$.*

Proof 2. By equating to zero the right-hand side of system (11), we get the endemic equilibrium point $P_{EE}^0 = (S^{\otimes}, I^{\otimes}, R^{\otimes}, C^{\otimes})$, where

$$\begin{aligned} S^{\otimes}(t) &= \tfrac{1}{-\mu^2\rho - \mu\rho\omega} \begin{pmatrix} -\Theta\mu\rho - \Theta\rho\omega - (\delta\mu + \lambda\mu + \mu^2 + \lambda\omega + \mu\omega) \\ \times \frac{(\mu+\omega)\left(-\eta\Theta\rho + \delta\mu\xi\mu_d + \lambda\mu\xi\mu_d + \mu^2\xi\mu_d\right)}{\delta\eta\mu + \eta\lambda\mu + \delta\mu^2 + \eta\mu^2 + \lambda\mu^2 + \mu^3 + \eta\lambda\omega + \delta\mu\omega + \eta\mu\omega + \lambda\mu\omega + \mu^2\omega} \end{pmatrix}, \\ I^{\otimes}(t) &= \tfrac{(\mu+\omega)\left(\eta\Theta\rho - \delta\mu\xi\mu_d - \lambda\mu\xi\mu_d - \mu^2\xi\mu_d\right)}{\rho(\delta\eta\mu + \eta\lambda\mu + \delta\mu^2 + \eta\mu^2 + \lambda\mu^2 + \mu^3 + \eta\lambda\omega + \delta\mu\omega + \eta\mu\omega + \lambda\mu\omega + \mu^2\omega)} \\ &= \tfrac{B_1(\mu+\omega)}{\rho B_2}(\Phi_0 - 1), \\ R^{\otimes}(t) &= \tfrac{\delta\left(\eta\Theta\rho - \delta\mu\xi\mu_d - \lambda\mu\xi\mu_d - \mu^2\xi\mu_d\right)}{\rho(\delta\eta\mu + \eta\lambda\mu + \delta\mu^2 + \eta\mu^2 + \lambda\mu^2 + \mu^3 + \eta\lambda\omega + \delta\mu\omega + \eta\mu\omega + \lambda\mu\omega + \mu^2\omega)} \\ &= \tfrac{B_1\delta}{\rho B_2}(\Phi_0 - 1), \\ C^{\otimes}(t) &= \tfrac{(\mu+\omega)\left(\eta\Theta\rho - \delta\mu\xi\mu_d - \lambda\mu\xi\mu_d - \mu^2\xi\mu_d\right)}{(\delta\eta\mu + \eta\lambda\mu + \delta\mu^2 + \eta\mu^2 + \lambda\mu^2 + \mu^3 + \eta\lambda\omega + \delta\mu\omega + \eta\mu\omega + \lambda\mu\omega + \mu^2\omega)\mu_d} \\ &= \tfrac{B_1(\mu+\omega)}{\mu_d B_2}(\Phi_0 - 1), \end{aligned} \tag{20}$$

where $B_1 = \eta\Theta\rho - \delta\mu\xi\mu_d - \lambda\mu\xi\mu_d - \mu^2\xi\mu_d$, and $B_2 = \delta\eta\mu + \eta\lambda\mu + \delta\mu^2 + \eta\mu^2 + \lambda\mu^2 + \mu^3 + \eta\lambda\omega + \delta\mu\omega + \eta\mu\omega + \lambda\mu\omega + \mu^2\omega$. It is clear that if $\Phi_0 > 1$, then there is always a unique endemic equilibrium of the system. Hence the proof.

Therefore, the proposed nonlinear fractional-order listeriosis epidemic model has at most two equilibria namely disease-free equilibrium point $P^0_{DF} = \left(\frac{\Theta}{\mu}, 0, 0, 0\right)$, and the endemic equilibrium point $P^0_{EE} = (S^{\otimes}, I^{\otimes}, R^{\otimes}, C^{\otimes})$. At this stage, for the stabilities of the equilibria the next theorem arises:

Theorem 3. *The disease-free equilibrium point P^0_{DF} is locally asymptotically stable if $\Phi_0 < 1$ and otherwise unstable.*

Proof 3. By solving the system (11) at the disease-free equilibrium (DFE), we get the following general Jakobian matrix:

$$J|_{P^0_{DF}} = \begin{pmatrix} -\mu & 0 & \omega & -\frac{\Theta\eta}{\mu\xi} \\ 0 & -\delta-\lambda-\mu & 0 & \frac{\Theta\eta}{\mu\xi} \\ 0 & \delta & -\omega-\mu & 0 \\ 0 & \sigma & 0 & -\mu_d \end{pmatrix}.$$

Hence, the DFE is locally asymptotically stable if all the eigenvalues $\varphi_i, i = 1, 2, 3, 4$ of the matrix $J|_{P^0_{DF}}$ satisfy the following condition:

$$\left|\arg\left(eig\left(J|_{P^0_{DF}}\right)\right)\right| = |\arg(\varphi_i)| > \alpha\frac{\pi}{2}, i = 1, 2, 3, 4. \tag{21}$$

Thus, the corresponding characteristic equation is given as

$$\left|J|_{P^0_{DF}} - \varphi I\right| = 0,$$

which gives

$$(\mu+\varphi)(\mu+\omega+\varphi)\left(\varphi^2 + A\varphi + B\right) = 0, \tag{22}$$

where $A = \delta+\lambda+\mu+\mu_d$, $B = (1-\Phi_0)(\delta+\lambda+\mu)\mu_d$. Clearly, $\varphi_1 = -\mu$and $\varphi_2 = -\mu-\omega$ are negative. Moreover, we have for other eigenvalues that $\varphi_{3,4} = -(\delta+\lambda+\mu+\mu_d) \pm \sqrt{(\delta+\lambda+\mu+\mu_d)^2 - 4(1-\Phi_0)(\delta+\lambda+\mu)\mu_d}$, which gives that when $\Phi_0 < 1$ all eigenvalues of Eq. (22) have roots whose real parts are negative. Otherwise, at least one of the eigenvalues has positive real part. Hence the proof.

Further, it can be seen that when $\Phi_0 < 1$, then the disease does not spread in the population and the infection dies. On the other hand, if $\Phi_0 > 1$, then the disease persists in the whole population. This represents, that the basic reproduction number Φ_0 determines the next behaviour of the model.

Theorem 4. *The endemic equilibrium point P^0_{EE} is locally asymptotically stable if $\Phi_0 > 1$ and otherwise unstable.*

Proof 4. To establish the local stability of the endemic equilibrium P^0_{EE}, the Jacobian matrix of the system (11) is given by

$$J|_{P^0_{EE}} = \begin{pmatrix} -\mu-\frac{C^{\otimes}\eta}{C^{\otimes}+\xi} & 0 & \omega & -\frac{S^{\otimes}\xi\eta}{(C^{\otimes}+\xi)^2} \\ \frac{C^{\otimes}\eta}{C^{\otimes}+\xi} & -\delta-\lambda-\mu & 0 & \frac{S^{\otimes}\xi\eta}{(C^{\otimes}+\xi)^2} \\ 0 & \delta & -\omega-\mu & 0 \\ 0 & \sigma & 0 & -\mu_d \end{pmatrix}.$$

The characteristic equation of $J|_{P^0_{EE}}$ is obtained by $\left|J|_{P^0_{EE}} - \varphi I\right| = 0$, which is expressed as the following polynomial

$$P(\varphi) = \varphi^4 + x_1\varphi^3 + x_2\varphi^2 + x_3\varphi + x_4 = 0, \tag{23}$$

where,

$$\begin{aligned}
x_1 =\ & \delta+\lambda+3\mu+\mu_d+\omega+\tfrac{C^{\otimes}\eta}{C^{\otimes}+\xi}, \\
x_2 =\ & 3\mu^2+\delta\mu_d+\delta\omega+3\mu\mu_d+\lambda\mu_d+\lambda\omega+\mu_d\omega+2\delta\mu+2\lambda\mu+2\mu\omega \\
& +\tfrac{C^{\otimes}\eta}{\xi+C^{\otimes}}(2\mu+\lambda+\delta+\mu_d+\omega)-\tfrac{S^{\otimes}\xi\rho\eta}{(\xi+C^{\otimes})^2}, \\
x_3 =\ & \tfrac{C^{\otimes}\eta}{\xi+C^{\otimes}}(\mu^2+\lambda\mu+\lambda\mu_d+2\mu\mu_d+\lambda\omega+\mu\omega+\mu_d\omega+\delta\mu+\delta\mu_d) \\
& +\mu^3+\delta\mu^2+\lambda\mu^2+3\mu^2\mu_d+\mu^2\omega+2\lambda\mu\mu_d+\delta\mu\omega+\delta\mu_d\omega+\lambda\mu\omega \\
& +\lambda\mu_d\omega+2\mu\mu_d\omega+2\delta\mu\mu_d-\tfrac{S^{\otimes}\xi\rho\eta(2\mu+\omega)}{(\xi+C^{\otimes})^2}, \\
x_4 =\ & \delta\mu\mu_d\omega+\lambda\mu\mu_d\omega+\mu^3\mu_d+\delta\mu^2\mu_d+\lambda\mu^2\mu_d+\mu^2\mu_d\omega \\
& +\tfrac{C^{\otimes}\eta}{\xi+C^{\otimes}}(\lambda\mu\mu_d+\lambda\mu_d\omega+\mu\mu_d\omega+\delta\mu\mu_d+\mu^2\mu_d)-\tfrac{S^{\otimes}\xi\rho\eta\mu(\mu+\omega)}{(\xi+C^{\otimes})^2}.
\end{aligned}$$

Then, according to the Routh-Hurwitz criteria [60], we can say that all the roots of the fourth degree characteristic polynomial are negative or have negative real part if and only if the following conditions are satisfied:

$$\begin{aligned}
&(\mathrm{i}) x_1 > 0, \\
&(\mathrm{ii}) x_3 > 0, \\
&(\mathrm{iii}) x_4 > 0, \\
&(\mathrm{iv}) x_1x_2x_3 > x_3^2 + x_1^2x_4.
\end{aligned} \tag{24}$$

Since all the parameters are positive, clearly, we have $x_1 = \delta+\lambda+3\mu+\mu_d+\omega+\frac{C^{\otimes}\eta}{C^{\otimes}+\xi} > 0$. For the condition (ii), since we can the following inequality,

$$\begin{aligned}
& \tfrac{C^{\otimes}\eta}{\xi+C^{\otimes}}(\mu^2+\lambda\mu+\lambda\mu_d+2\mu\mu_d+\lambda\omega+\mu\omega+\mu_d\omega+\delta\mu+\delta\mu_d) \\
& +\mu^3+\delta\mu^2+\lambda\mu^2+3\mu^2\mu_d+\mu^2\omega+2\lambda\mu\mu_d+\delta\mu\omega+\delta\mu_d\omega+\lambda\mu\omega \\
& +\lambda\mu_d\omega+2\mu\mu_d\omega+2\delta\mu\mu_d > \tfrac{S^{\otimes}\xi\rho\eta(2\mu+\omega)}{(\xi+C^{\otimes})^2},
\end{aligned}$$

which gives that $x_3 > 0$. Considering the same procedure and taking into account the variables given in Table 1, it can be used to show that other two conditions hold. Therefore, all the roots of the characteristic polynomial in Eq. (23) have negative real parts and satisfy the condition

Table 1 Parameters and variables with their values for fractional order listeriosis epidemic model.

Parameters	*Meaning*	*Values/Unit*
Θ	Human recruitment rate	0.001 (assumed)
ω	Waning immunity rate	0.001 (assumed)
δ	Listeriosis recovery rate	0.002 (assumed)
μ	Natural death rate	0.2[56]
σ	Listeriosis contribution to environment	0.65 (assumed)
μ_d	Carcasses (bacteria) mortality rate	0.0025 (assumed)
λ	Listeriosis related death rate	0.2 [2]
η	Concentration of carcasses	10000 [57]
ξ	Bacteria Ingestion rate	0.5 [57]
$S(t)$	Susceptible individuals	Variable
$I(t)$	Infected individuals	Variable
$R(t)$	Recovered Individuals	Variable
$C(t)$	Population of animal carcasses	Variable

$|\arg(\varphi_i)| > \alpha\frac{\pi}{2}, i = 1,2,3,4,$ when $\Phi_0 > 1$. This proves the theorem.

5. Existence and Uniqueness of the Solution Under ABC-Fractal-Fractional Derivative

In this section we show the existence and uniqueness of the ordinary differential equations constructed by the Atangana-Baleanu operator in the frame of fractal-fractional. For this we consider the following general Cauchy problem:

$${}_0^{FF}D_t^{\alpha}\{\phi(t)\} = \varpi(t,\phi(t)). \tag{25}$$

Then we can get the following by the corresponding definition

$$\frac{\Psi(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t \varpi(\varsigma,\phi(\varsigma))E_{\alpha}\left(-\alpha\frac{(t-\tau)^{\alpha}}{1-\alpha}\right)d\tau. \tag{26}$$

Hence, considering the fact that the integral is differentiable, we may arrange it as

$$\frac{\Psi(\alpha)}{1-\alpha}\vartheta^{-1}t^{1-\vartheta}\frac{d}{dt}\int_0^t \varpi(\varsigma,\phi(\varsigma))E_{\alpha}\left(-\alpha\frac{(t-\vartheta)^{\alpha}}{1-\alpha}\right)d\vartheta. \tag{27}$$

This allows us to write the main problem in Eq. (25) as

$$\frac{\Psi(\alpha)}{1-\alpha}\frac{d}{dt}\int_0^t \varpi(\varsigma,\phi(\varsigma))E_{\alpha}\left(-\alpha\frac{(t-\vartheta)^{\alpha}}{1-\alpha}\right)d\vartheta = \vartheta t^{\vartheta-1}\varpi(t,\phi(t)). \tag{28}$$

Taking the Atangana-Baleanu fractional integral of the right hand side of Eq. (28) we get

$$\phi(t) = \phi(0) + \frac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)}\varpi(t,\phi(t)) \tag{29}$$

$$+\frac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}\int_0^t \varsigma^{\vartheta-1}\varpi(\varsigma,\phi(\varsigma))(t-\vartheta)^{\alpha-1}d\vartheta. \tag{30}$$

Considering such as Picard-Lindelöf theorem [61] we take

$$\Pi_p^q = \nabla_m(t_m)\times\overline{\Omega_0(\phi_0)}, \tag{31}$$

where $\overline{\nabla_m(t_m)} = [t_{m-p}, t_{m+p}], \overline{\Omega_0(\phi_0)} = [t_0 - q, t_0 + q]$. Now we proceed by letting

$$W = \sup_{t\in\Pi_p^q}\|\varpi\|, \tag{32}$$

and by defining the norm

$$\|\varpi(t)\|_{\infty} = \sup_{t\in\Pi_p^q}\|\varpi(t)\|. \tag{33}$$

Thus, the following assignment can be developed

$$\mho.C[\nabla_m(t_m),\Omega_q(t_m)] \Rightarrow C(\nabla_m(q),\Omega_q(t_m)), \tag{34}$$

defined by

$$\begin{aligned}\mho\varepsilon(t) = {} & \phi_0 + \tfrac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)}\varpi(t,\varepsilon(t)) \\ & + \tfrac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}\int_0^t \varsigma^{\vartheta-1}\varpi(\varsigma,\varepsilon(\varsigma))(t-\vartheta)^{\alpha-1}d\vartheta.\end{aligned} \tag{35}$$

The main goal is to indicate that the operator we have defined maps to Q which is complete norm empty metric space Q into itself, and moreover it is a contraction mapping. Then the first thing is to demonstrate that

$$\begin{aligned}\|\mho\varepsilon(t) - \phi_0\| \leqslant {} & q, \\ \leqslant {} & \tfrac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)}\|\varpi(t,\varepsilon(t))\|_{\infty} \\ & + \tfrac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}\int_0^t \varsigma^{\vartheta-1}\|\varpi(\varsigma,\varepsilon(\varsigma))\|(t-\vartheta)^{\alpha-1}d\vartheta, \\ \leqslant {} & \frac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)}W + \frac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}W\int_0^t \varsigma^{\vartheta-1}(t-\vartheta)^{\alpha-1}d\vartheta.\end{aligned} \tag{36}$$

Taking $\varsigma = ty$, we can rewrite Eq. (36) as

$$\|\mho\varepsilon - \phi_0\| \leqslant W\left(\frac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)} + \frac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}t^{\vartheta+\alpha-3}Z(\vartheta,\alpha)\right), \tag{37}$$

which yields the following equation by considering the inequality in Eq. (36)

$$W < \frac{q}{\frac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)} + \frac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}t^{\vartheta+\alpha-3}Z(\vartheta,\alpha)}. \tag{38}$$

Theorem 5. *The operator $\mho$has the unique solution.*

Proof 5. To demonstrate that the operator $\mho$has the unique solution, we take ε_1 and $\varepsilon_2$such that $\varepsilon_i \in C[\nabla_m(t_m),\Omega_q(t_m)], i = 1,2$ by using the Banach fixed point theorem to have the following result:

$$\|\mho\varepsilon_1 - \mho\varepsilon_2\| \leqslant M\|\varepsilon_1 - \varepsilon_2\|_{\infty}, \tag{39}$$

where $M < 1$.

$$\begin{aligned}&\|\mho\varepsilon_1 - \mho\varepsilon_2\| \leqslant \tfrac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)}\|\varpi(t,\varepsilon_1) - \varpi(t,\varepsilon_2)\| \\ &+\tfrac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}\int_0^t \varsigma^{\vartheta-1}\|\varpi(\varsigma,\varepsilon_1) - \varpi(\varsigma,\varepsilon_2)\|(t-\vartheta)^{\alpha-1}d\vartheta,\end{aligned} \tag{40}$$

where ϖbeing contraction, we get

$$\begin{aligned}&\|\mho\varepsilon_1 - \mho\varepsilon_2\| \leqslant \tfrac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)}M\|\varepsilon_1 - \varepsilon_2\|_{\infty} \\ &+\tfrac{\alpha\vartheta M}{\Gamma(\alpha)\Psi(\alpha)}\|\varepsilon_1 - \varepsilon_2\|_{\infty}\int_0^t \varsigma^{\vartheta-1}(t-\vartheta)^{\alpha-1}d\vartheta, \\ &\leqslant \tfrac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)}M\|\varepsilon_1 - \varepsilon_2\|_{\infty} + \tfrac{\alpha\vartheta M}{\Gamma(\alpha)\Psi(\alpha)}\|\varepsilon_1 - \varepsilon_2\|_{\infty}t^{\vartheta+\alpha-3}Z(\vartheta,\alpha).\end{aligned} \tag{41}$$

Therefore, we can write

$$\begin{aligned}\|\mho\varepsilon_1 - \mho\varepsilon_2\| \leqslant {} & \left(\frac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)} + \frac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}t^{\vartheta+\alpha-3}Z(\vartheta,\alpha)\right) \\ & M\|\varepsilon_1 - \varepsilon_2\|_{\infty},\end{aligned} \tag{42}$$

which gives

$$\begin{aligned}&M\left(\tfrac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)} + \tfrac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}t^{\vartheta+\alpha-3}Z(\vartheta,\alpha)\right) \\ &< M\left(\tfrac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)} + \tfrac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)}p^{\vartheta+\alpha-3}Z(\vartheta,\alpha)\right).\end{aligned} \tag{43}$$

Moreover, $\mho$ is the contraction if

$$\|\mho\varepsilon_1 - \mho\varepsilon_2\| \leqslant \mho\|\varepsilon_1 - \varepsilon_2\|.$$

Then

$$M < \frac{1}{\frac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)} + \frac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)} t^{\vartheta+\alpha-3} Z(\vartheta,\alpha)}, \tag{45}$$

and

$$W < \frac{q}{\frac{(1-\alpha)\vartheta t^{\vartheta-1}}{\Psi(\alpha)} + \frac{\alpha\vartheta}{\Gamma(\alpha)\Psi(\alpha)} p^{\vartheta+\alpha-3} Z(\vartheta,\alpha)}. \tag{46}$$

This proves that the operator $\mho$ has the unique solution.

6. Existence and Uniqueness of the Solution Under Caputo-Fractal-Fractional Derivative

In this section we show the existence and uniqueness of the solution in the frame of Caputo fractal-fractional operator [29]. For this, we consider the same equation in (25), then we have

$$\phi(t) = \phi(0) + \frac{\vartheta\alpha}{\Gamma(\alpha)} \int_0^t \varsigma^{\vartheta-1} \varpi(\varsigma, \phi(\varsigma)) d\varsigma. \tag{47}$$

Then we have the following assignment

$$\mho\varepsilon(t) = \phi_0 + \frac{\vartheta\alpha}{\Gamma(\alpha)} \int_0^t \varsigma^{\vartheta-1} \varpi(\varsigma, \varepsilon(\varsigma)) d\varsigma, \tag{48}$$

which implies

$$\|\mho\varepsilon(t) - \phi_0\| < q \Rightarrow W, \tag{49}$$

where $W = \sup_{t\in\Pi_p^q} \|\varpi\|$, and $W < \frac{q\Gamma(\alpha)}{\alpha\vartheta p^{\vartheta+\alpha-3} Z(\vartheta,\alpha)}$. Following the similar steps in the previous section we take $\varepsilon_i \in C\left[\nabla_m(t_m), \Omega_q(t_m)\right], i = 1, 2$. Then we have

$$\|\mho\varepsilon_1 - \mho\varepsilon_2\| < \frac{\alpha M\vartheta}{\Gamma(\alpha)} p^{\vartheta+\alpha-3} Z(\vartheta,\alpha). \tag{50}$$

Therefore, the contraction property is given if the inequality holds:

$$M < \frac{\Gamma(\alpha)}{\alpha\vartheta p^{\vartheta+\alpha-3} Z(\vartheta,\alpha)}, \tag{51}$$

which implies

$$W < \frac{q\Gamma(\alpha)}{\alpha\vartheta p^{\vartheta+\alpha-3} Z(\vartheta,\alpha)}. \tag{52}$$

Thus, the mentioned problem has a unique solution and moreover this result proves the existence and uniqueness condition under the Caputo-fractal-fractional operator.

7. Numerical Scheme for the Fractal-Fractional Caputo Listeriosis Model

In this section,

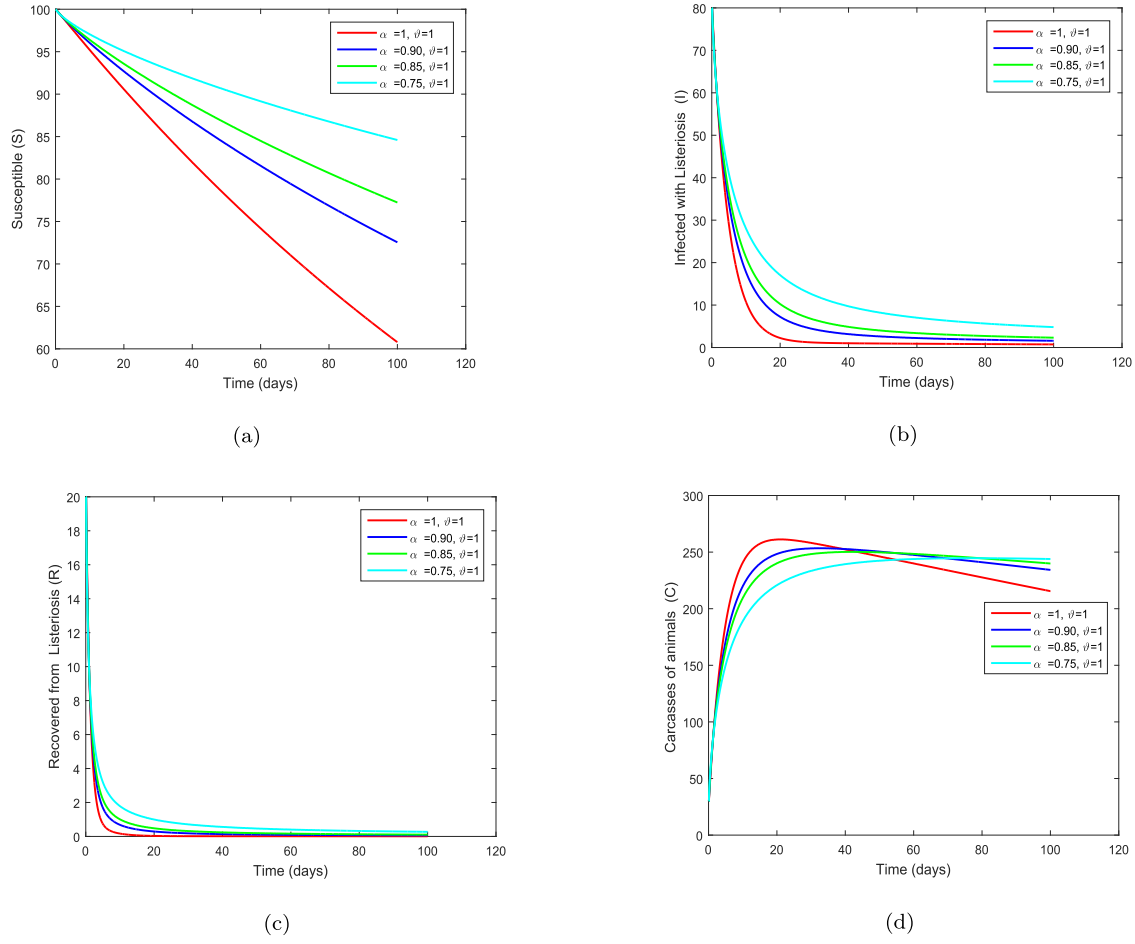

Fig. 1 Simulation of the fractal–fractional Caputo derivative model (11), when $\vartheta = 1, \vartheta = 0.90, \vartheta = 0.85, \vartheta = 0.75$ and $\alpha = 1$.

$$
\begin{aligned}
{}_0^{FF-C}D_t^{\alpha,\vartheta}S(t) &= \Theta + \omega R(t) + \frac{\eta C(t)S(t)}{\xi + C(t)} - \mu S(t),\\
{}_0^{FF-C}D_t^{\alpha,\vartheta}I(t) &= \frac{\eta C(t)S(t)}{\xi + C(t)} - (\delta + \lambda + \mu)I(t),\\
{}_0^{FF-C}D_t^{\alpha,\vartheta}R(t) &= \delta I(t) - (\omega + \mu)R(t),\\
{}_0^{FF-C}D_t^{\alpha,\vartheta}C(t) &= \sigma I(t) - \mu_d C(t),
\end{aligned} \tag{53}
$$

with the corresponding initial conditions

$$S_0 = S(0), I_0 = I(0), R_0 = R(0), C_0 = C(0).$$

In this regard we investigate the following:

$$
\begin{aligned}
\Phi_1(S(t), I(t), R(t), C(t)) &= \Theta + \omega R(t) + \frac{\eta C(t)S(t)}{\xi + C(t)} - \mu S(t),\\
\Phi_2(S(t), I(t), R(t), C(t)) &= \frac{\eta C(t)S(t)}{\xi + C(t)} - (\delta + \lambda + \mu)I(t),\\
\Phi_3(S(t), I(t), R(t), C(t)) &= \delta I(t) - (\omega + \mu)R(t),\\
\Phi_4(S(t), I(t), R(t), C(t)) &= \sigma I(t) - \mu_d C(t).
\end{aligned} \tag{54}
$$

Eq. (53) is transformed into Volterra integrals, and the following numerical scheme given at t_{n+1} are obtained:

$$
\begin{aligned}
S_{n+1} &= S(0) + \frac{\vartheta}{\Gamma(\alpha)}\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}\theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Phi_1(S(\theta), I(\theta), R(\theta), C(\theta))d\theta,\\
I_{n+1} &= I(0) + \frac{\vartheta}{\Gamma(\alpha)}\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}\theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Phi_2(S(\theta), I(\theta), R(\theta), C(\theta))d\theta,\\
R_{n+1} &= R(0) + \frac{\vartheta}{\Gamma(\alpha)}\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}\theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Phi_3(S(\theta), I(\theta), R(\theta), C(\theta))d\theta,\\
C_{n+1} &= C(0) + \frac{\vartheta}{\Gamma(\alpha)}\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}\theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Phi_4(S(\theta), I(\theta), R(\theta), C(\theta))d\theta.
\end{aligned} \tag{55}
$$

Here the purpose is to approximate the function $\theta^{\vartheta-1}\Phi_i(S, I, R, C, \theta)\forall i = 1, 2, 3, 4$ making use of Lagrangian piece-wise interpolation principle so that we have

$$
\begin{aligned}
\Psi_v(s) = &\frac{\theta - t_{v-1}}{t_v - t_{v-1}}t_v^{\vartheta-1}\Phi_i(S_v, I_v, R_v, C_v, t_v)\\
&- \frac{\theta - t_v}{t_v - t_{v-1}}\Phi_i(S_{v-1}, I_{v-1}, R_{v-1}, C_{v-1}, t_{v-1})
\end{aligned} \tag{56}
$$

Therefore we arrive at the following:

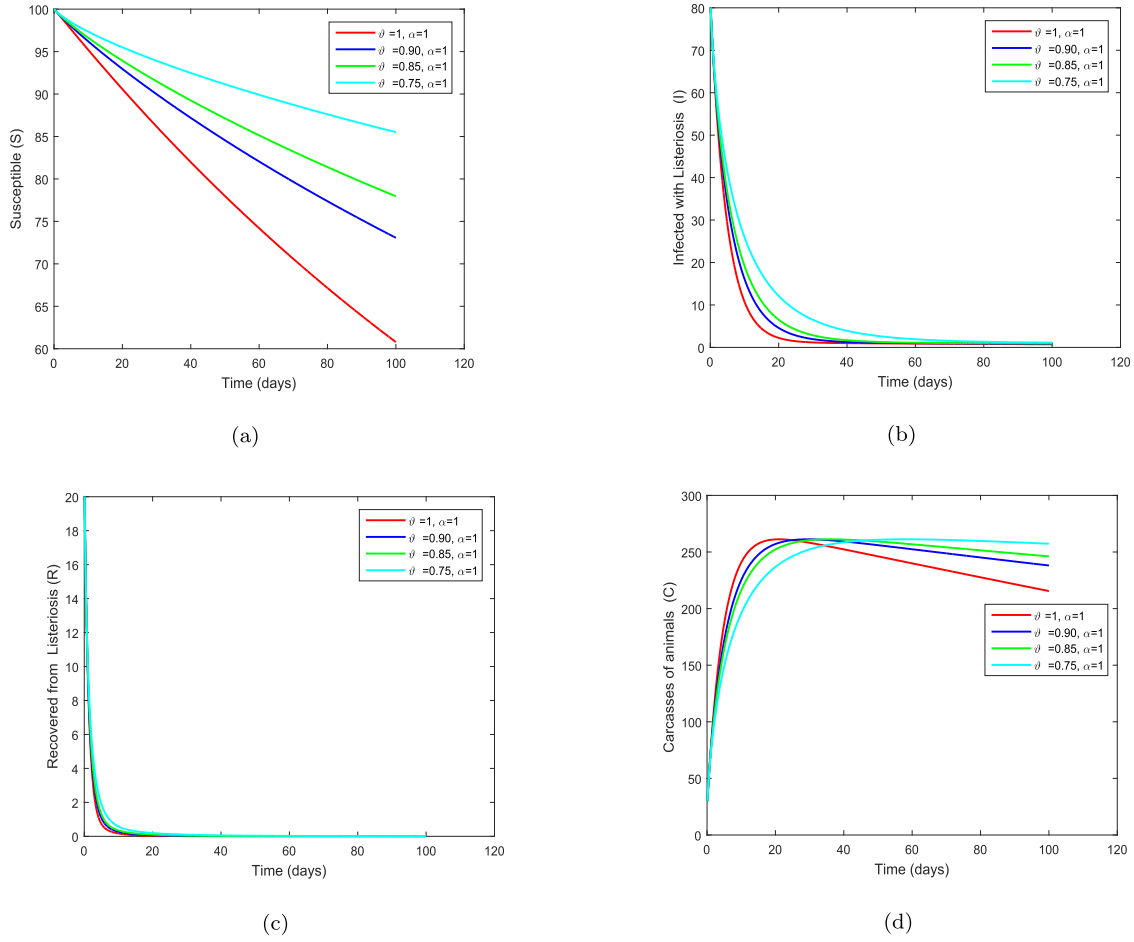

Fig. 2 Simulation of the fractal–fractional Caputo derivative model (11), when $\alpha = 1, \alpha = 0.90, \alpha = 0.85, \alpha = 0.75$ and $\vartheta = 1$.

$$
\begin{aligned}
S_{n+1} &= S(0) + \frac{\vartheta}{\Gamma(\alpha)} \sum_{v=0}^{n} \int_{t_v}^{t_{v+1}} \theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Psi_v(\theta)d\theta,\\
I_{n+1} &= I(0) + \frac{\vartheta}{\Gamma(\alpha)} \sum_{v=0}^{n} \int_{t_v}^{t_{v+1}} \theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Psi_v(\theta)d\theta,\\
R_{n+1} &= R(0) + \frac{\vartheta}{\Gamma(\alpha)} \sum_{v=0}^{n} \int_{t_v}^{t_{v+1}} \theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Psi_v(\theta)d\theta,\\
C_{n+1} &= C(0) + \frac{\vartheta}{\Gamma(\alpha)} \sum_{v=0}^{n} \int_{t_v}^{t_{v+1}} \theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Psi_v(\theta)d\theta.
\end{aligned} \tag{57}
$$

Obtaining solution of the right -hand side of the integral leads to the following numerical scheme:

$$
\begin{aligned}
S_{n+1} &= S(0) + \frac{\vartheta(\Delta t)^{\alpha}}{\Gamma(\alpha+2)} \sum_{v=0}^{n} \Phi_1(S_v, I_v, R_v, C_v, t_v)((1-v+n)^{\alpha}(-v+n+2+\alpha)\\
&\quad -(-v+n)^{\alpha}(-v+n+2\alpha+2))\\
&\quad -t_{v-1}^{\vartheta-1}\Phi_1(S_{v-1}, I_{v-1}, R_{v-1}, C_{v-1}, t_{v-1})\Big((1-v+n)^{\alpha+1}-(-v+n)^{\alpha}(-v+n+\alpha+1)\Big)\Big],\\
I_{n+1} &= I(0) + \frac{\vartheta(\Delta t)^{\alpha}}{\Gamma(\alpha+2)} \sum_{v=0}^{n} \Phi_2(S_v, I_v, R_v, C_v, t_v)((1-v+n)^{\alpha}(-v+n+2+\alpha)\\
&\quad -(-v+n)^{\alpha}(-v+n+2\alpha+2))\\
&\quad -t_{v-1}^{\vartheta-1}\Phi_2(S_{v-1}, I_{v-1}, R_{v-1}, C_{v-1}, t_{v-1})\Big((1-v+n)^{\alpha+1}-(-v+n)^{\alpha}(-v+n+\alpha+1)\Big)\Big],\\
R_{n+1} &= R(0) + \frac{\vartheta(\Delta t)^{\alpha}}{\Gamma(\alpha+2)} \sum_{v=0}^{n} \Phi_3(S_v, I_v, R_v, C_v, t_v)((1-v+n)^{\alpha}(-v+n+2+\alpha)\\
&\quad -(-v+n)^{\alpha}(-v+n+2\alpha+2))\\
&\quad -t_{v-1}^{\vartheta-1}\Phi_3(S_{v-1}, I_{v-1}, R_{v-1}, C_{v-1}, t_{v-1})\Big((1-v+n)^{\alpha+1}-(-v+n)^{\alpha}(-v+n+\alpha+1)\Big)\Big],\\
C_{n+1} &= C(0) + \frac{\vartheta(\Delta t)^{\alpha}}{\Gamma(\alpha+2)} \sum_{v=0}^{n} \Phi_4(S_v, I_v, R_v, C_v, t_v)((1-v+n)^{\alpha}(-v+n+2+\alpha)\\
&\quad -(-v+n)^{\alpha}(-v+n+2\alpha+2))\\
&\quad -t_{v-1}^{\vartheta-1}\Phi_4(S_{v-1}, I_{v-1}, R_{v-1}, C_{v-1}, t_{v-1})\Big((1-v+n)^{\alpha+1}-(-v+n)^{\alpha}(-v+n+\alpha+1)\Big)\Big].
\end{aligned} \tag{58}
$$

8. Numerical Scheme for the Fractal-Fractional ABC Listeriosis Model

This aspect considers the Listeriosis model in fractal-fractional ABC operator in Caputo derivative sense and given by:

$$
\begin{aligned}
{}_0^{FF-ABC}D_t^{\alpha,\vartheta} S(t) &= \Theta + \omega R(t) + \frac{\eta C(t)S(t)}{\xi + C(t)} - \mu S(t),\\
{}_0^{FF-ABC}D_t^{\alpha,\vartheta} I(t) &= \frac{\eta C(t)S(t)}{\xi + C(t)} - (\delta + \lambda + \mu)I(t),\\
{}_0^{FF-ABC}D_t^{\alpha,\vartheta} R(t) &= \delta I(t) - (\omega + \mu)R(t),\\
{}_0^{FF-ABC}D_t^{\alpha,\vartheta} C(t) &= \sigma I(t) - \mu_d C(t).
\end{aligned} \tag{59}
$$

The associated initial condition is given by:

$$S_0 = S(0), I_0 = I(0), R_0 = R(0), C_0 = C(0).$$

Here we examine:

$$
\begin{aligned}
\Phi_1(S(t), I(t), R(t), C(t)) &= \Theta + \omega R(t) + \frac{\eta C(t)S(t)}{\xi + C(t)} - \mu S(t),\\
\Phi_2(S(t), I(t), R(t), C(t)) &= \frac{\eta C(t)S(t)}{\xi + C(t)} - (\delta + \lambda + \mu)I(t),\\
\Phi_3(S(t), I(t), R(t), C(t)) &= \delta I(t) - (\omega + \mu)R(t),\\
\Phi_4(S(t), I(t), R(t), C(t)) &= \sigma I(t) - \mu_d C(t).
\end{aligned} \tag{60}
$$

Making use of ABC integral at t_{n+1} in Eq. (60) gives the following:

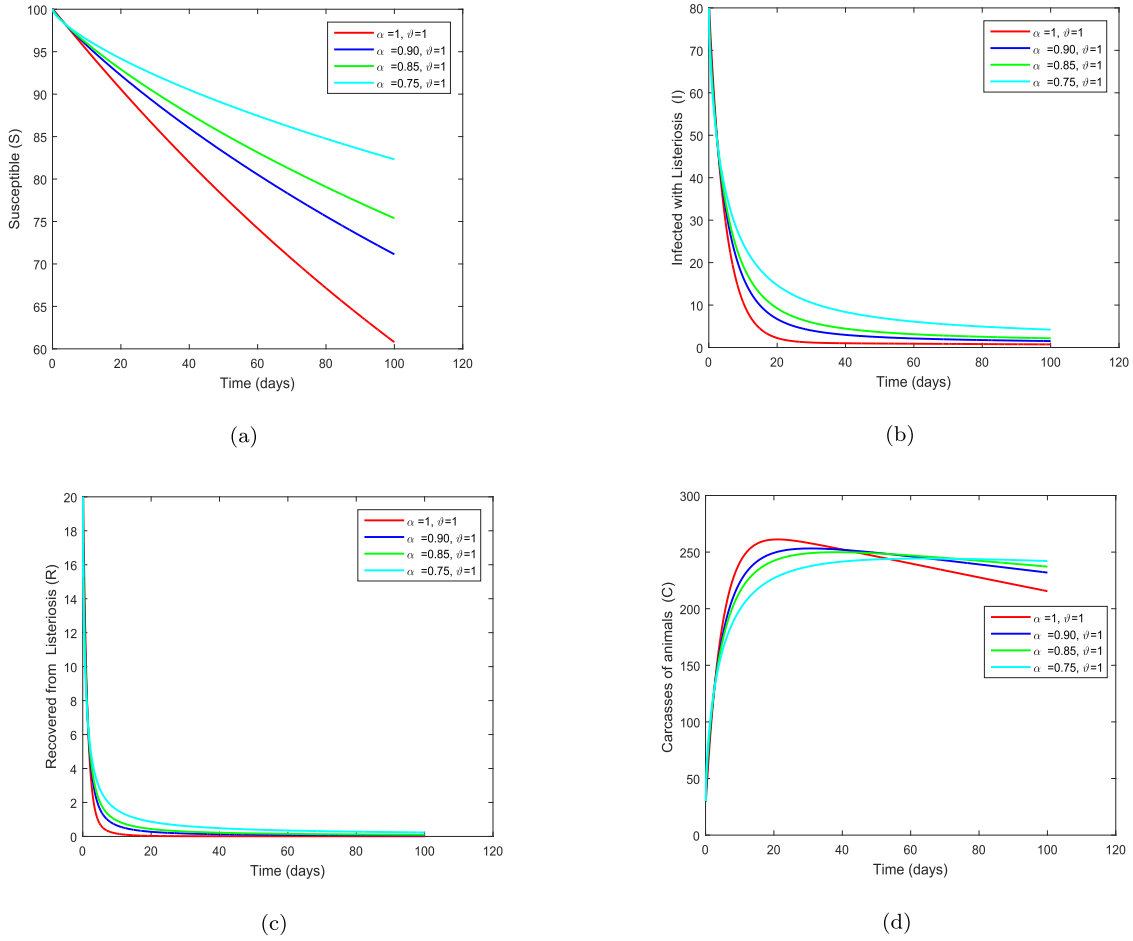

Fig. 3 Simulation of the fractal–fractional ABC derivative of the model (11), when $\vartheta = 1, \vartheta = 0.90, \vartheta = 0.85, \vartheta = 0.75$ and $\alpha = 1$.

$$
\begin{aligned}
S_{n+1} &= S(0) + \frac{\vartheta t_n^{\vartheta-1}(1-\alpha)}{M(\alpha)}\Phi_1\big(S(t)_n, I(t)_n, R(t)_n, C(t)_n, t_n\big) \\
&+ \frac{\alpha\vartheta}{M(\alpha)\Gamma(\alpha)}\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}\theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Phi_1(S(\theta), I(\theta), R(\theta), C(\theta), \theta)d\theta, \\
I_{n+1} &= I(0) + \frac{\vartheta t_n^{\vartheta-1}(1-\alpha)}{M(\alpha)}\Phi_2\big(S(t)_n, I(t)_n, R(t)_n, C(t)_n, t_n\big) \\
&+ \frac{\alpha\vartheta}{M(\alpha)\Gamma(\alpha)}\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}\theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Phi_2(S(\theta), I(\theta), R(\theta), C(\theta), \theta)d\theta, \\
R_{n+1} &= R(0) + \frac{\vartheta t_n^{\vartheta-1}(1-\alpha)}{M(\alpha)}\Phi_3\big(S(t)_n, I(t)_n, R(t)_n, C(t)_n, t_n\big) \\
&+ \frac{\alpha\vartheta}{M(\alpha)\Gamma(\alpha)}\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}\theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Phi_3(S(\theta), I(\theta), R(\theta), C(\theta), \theta)d\theta, \\
C_{n+1} &= C(0) + \frac{\vartheta t_n^{\vartheta-1}(1-\alpha)}{M(\alpha)}\Phi_4\big(S(t)_n, I(t)_n, R(t)_n, C(t)_n, t_n\big) \\
&+ \frac{\alpha\vartheta}{M(\alpha)\Gamma(\alpha)}\sum_{v=0}^{n}\int_{t_v}^{t_{v+1}}\theta^{\vartheta-1}(t_{n+1}-\theta)^{\alpha-1}\Phi_4(S(\theta), I(\theta), R(\theta), C(\theta), \theta)d\theta.
\end{aligned}
\tag{61}
$$

Introducing approximation of $\theta^{\vartheta_i-1}\Phi_i(S(t), I(t), R(t), C(t), \theta)$ $\forall i = 1, 2, 3, 4[t_v, int_{v+1}]$ the following numerical scheme is obtained:

$$
\begin{aligned}
S_{n+1} &= S(0) + \tfrac{\vartheta t_n^{\vartheta-1}(1-\alpha)}{M(\alpha)}\Phi_1\big(S(t)_n, I(t)_n, R(t)_n, C(t)_n, t_n\big) + \tfrac{\vartheta(\Delta t)^{\alpha}}{M(\alpha)\Gamma(\alpha+2)} \\
&\times \sum_{v=0}^{n}\big[\Phi_1\big(S(t)_v, IS(t)_v, RS(t)_v, CS(t)_v, t_v\big)\big((1-v+n)^{\alpha}(-v+n+2+\alpha) - (-v+n)^{\alpha}(-v+n+2\alpha+2)\big) \\
&- t_{v-1}^{\vartheta-1}\Phi_1\big(S(t)_{v-1}, I(t)_{v-1}, R(t)_{v-1}, C(t)_{v-1}, t_{v-1}\big)\big(v-n+1\big)^{\alpha+1} - (-v+n)^{\alpha}(-v+n+\alpha+1)\big], \\
I_{n+1} &= I(0) + \tfrac{\vartheta t_n^{\vartheta-1}(1-\alpha)}{M(\alpha)}\Phi_2\big(S(t)_n, I(t)_n, R(t)_n, C(t)_n, t_n\big) + \tfrac{\vartheta(\Delta t)^{\alpha}}{M(\alpha)\Gamma(\alpha+2)} \\
&\times \sum_{v=0}^{n}\big[\Phi_2\big(S(t)_v, IS(t)_v, R(t)_v, C(t)_v, t_v\big)\big((1-v+n)^{\alpha}(-v+n+2+\alpha) - (-v+n)^{\alpha}(-v+n+2\alpha+2)\big) \\
&- t_{v-1}^{\vartheta-1}\Phi_2\big(S(t)_{v-1}, I(t)_{v-1}, R(t)_{v-1}, C(t)_{v-1}, t_{v-1}\big)\big(v-n+1\big)^{\alpha+1} - (-v+n)^{\alpha}(-v+n+\alpha+1)\big], \\
R_{n+1} &= R(0) + \tfrac{\vartheta t_n^{\vartheta-1}(1-\alpha)}{M(\alpha)}\Phi_3\big(S(t)_n, I(t)_n, R(t)_n, C(t)_n, t_n\big) + \tfrac{\vartheta(\Delta t)^{\alpha}}{M(\alpha)\Gamma(\alpha+2)} \\
&\times \sum_{v=0}^{n}\big[\Phi_3\big(S(t)_v, I(t)_v, R(t)_v, C(t)_v, t_v\big)\big((1-v+n)^{\alpha}(-v+n+2+\alpha) - (-v+n)^{\alpha}(-v+n+2\alpha+2)\big) \\
&- t_{v-1}^{\vartheta-1}\Phi_3\big(S(t)_{v-1}, I(t)_{v-1}, R(t)_{v-1}, C(t)_{v-1}, t_{v-1}\big)\big(v-n+1\big)^{\alpha+1} - (-v+n)^{\alpha}(-v+n+\alpha+1)\big], \\
C_{n+1} &= C(0) + \tfrac{\vartheta t_n^{\vartheta-1}(1-\alpha)}{M(\alpha)}\Phi_4\big(S(t)_n, I(t)_n, R(t)_n, C(t)_n, t_n\big) + \tfrac{\vartheta(\Delta t)^{\alpha}}{M(\alpha)\Gamma(\alpha+2)} \\
&\times \sum_{v=0}^{n}\big[\Phi_4\big(S(t)_v, I(t)_v, R(t)_v, C(t)_v, t_v\big)\big((1-v+n)^{\alpha}(-v+n+2+\alpha) - (-v+n)^{\alpha}(-v+n+2\alpha+2)\big) \\
&- t_{v-1}^{\vartheta-1}\Phi_4\big(S(t)_{v-1}, I(t)_{v-1}, R(t)_{v-1}, C(t)_{v-1}, t_{v-1}\big)\big(v-n+1\big)^{\alpha+1} - (-v+n)^{\alpha}(-v+n+\alpha+1)\big].
\end{aligned}
\tag{62}
$$

9. Numerical Simulations and Discussion

Fractional order influences the dynamics of a phenomenon. However, researches have established that the realistic frac-

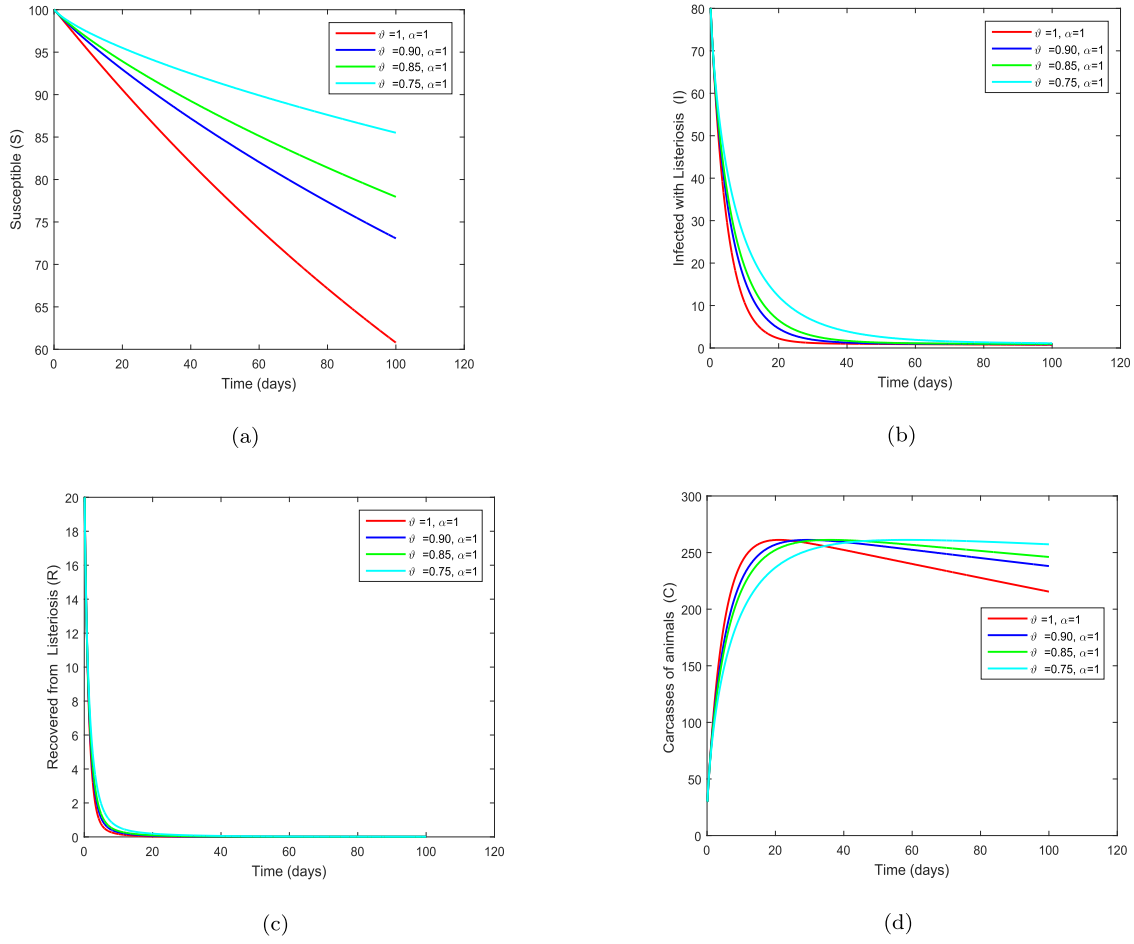

Fig. 4 Simulation of the fractal–fractional ABC derivative of the model (11), when $\alpha = 1, \alpha = 0.90, \alpha = 0.85, \alpha = 0.75$ and $\vartheta = 1$.

tional order in nonlinear models are from 0.5 to 0.99. This is the reason why almost all fractional models have the orders ranging from these values. When both fractal and fractional orders are the same it means that the fractional dimension coincides with the fractional order and depending on the model would determine the physical model to be involved. The step size employed in this work was 0.01 with the time interval [0,30] taking into account the following initial conditions $S(0) = 100, I(0) = 3, R(0) = 40, C(0) = 30$. The parameter values used for this study are indicated in Table 1 as follows $\Theta = 0.001, \delta = 0.002, \mu = 0.2, \sigma = 0.65, \mu_d = 0.0025, \lambda = 0.2, \eta = 10000, \xi = 0.5, \omega = 0.001$.

Fig. 1 (a-d) represent the numerical simulation results based on the fractal–fractional Caputo derivative model as fractal order α is varied and fractional order ϑ is kept constant. In Figs. 1 (a-c) as the α values are increased the number of individuals in these compartments turn to be asymptotically to the x- axis. Thus, the number of individuals in these respective classes reduces as fractal order α values increase. This indicates that naturally most biological objects are influenced by fractal properties. Integer order derivatives do not have such properties. However, in Fig. 1 (d) the number of carcasses of animals increases as the α values increase. This means that the fractal order influences on the control of the carcasses of animals. So, in order to reduce this infection the fractal order must be reduced. Figs. 2 (a-d) are the numerical simulation results hinged on fractal–fractional Caputo derivative as ϑ values are varied and α is kept constant. In Figs. 2 (a-c), the number of individuals decreases as fractional order ϑ values go up and a sharp contrast can be seen in Fig. 2 (d) as the ϑ values increase the number of carcasses of animals in the class also appreciate in values. This implies that the fractional order directly affects the number of carcasses of animals at any given time. The number of carcasses of animals can be kept at minimal by equally reducing the fractional order derivative and keeping the fractal order constant. This indicates that the fractional order derivative can be employed to obtain a clear qualitative information on this disease. Figs. 3 (a-d) are the numerical results based on the fractal–fractional ABC derivative in Caputo sense. Figs. 3 (a-c) indicate that as the fractal order α values are increased the number of individuals in each respective classes reduces. This indicates that the fractal order affects the dynamics of listeriosis. In Fig. 3 (d) as α values are appreciated the number of carcasses of animals increases and similarly in order to reduce this disease the fractal order must be reduced. Figs. 4 (a-d) represent the numerical simulation results based on the fractal–fractional ABC derivative in Caputo sense as the fractional order ϑ values are varied. It can be seen in Figs. 4 (a-c) that as ϑ values increase the number of individuals in the respective classes reduces, however, in Fig. 4 (d) the number of carcasses of animals rises up as the fractional order ϑ values increase. Thus, it is important to control the number of carcasses of animals which can be done by reducing fractional order values so that the spread of the disease can be controlled.

10. Conclusions

In this paper, a listeriosis model has been formulated involving fractal-fractional order derivatives in Caputo and Atangana–Baleanu sense. Moreover, future behaviors of the disease are investigated by considering the fractal–fractional operators that are very effective in modeling the real-life phenomena by virtue of their memory effect. The reproduction number of the listeriosis model has been computed and the existence and uniqueness of solutions to both operators involving fractal-fractional derivatives have been proved, also numerical schemes for each fractal-fractional derivative operator have been derived. The numerical results have indicated that the fractal-fractional Atangana–Baleanu derivative has performed better than the fractal-fractional Caputo derivative operator. These newly defined operators can be used to model other complex dynamics in the scientific environment, too.

11. Declarations:

Availability of data and materials: Not applicaple in this article.

Competing interests: The authors state that there is no conflict of interest in the publication of this article.

Funding: Not applicable.

Author's Contributions: All authors have made equal contributions to the writing of this article. The final manuscript read and approved by all authors.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

Mehmet Yavuz was supported by TUBITAK (The Scientific and Technological Research Council of Turkey).

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