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A study of boundary value problem for generalized fractional differential inclusion via endpoint theory for weak contractions

Samiha Belmor[1], Fahd Jarad[2], Thabet Abdeljawad[3,4,5*] and Gülsen Kılınç[6]

*Correspondence:
tabdeljawad@psu.edu.sa
[3]Department of Mathematics and General Sciences, Prince Sultan University, P. O. Box 66833, Riyadh 11586, Saudi Arabia
[4]Department of Medical Research, China Medical University, Taichung 40402, Taiwan
Full list of author information is available at the end of the article

Abstract

This note is concerned with establishing the existence of solutions to a fractional differential inclusion of a ψ-Caputo-type with a nonlocal integral boundary condition. Using the concept of the endpoint theorem for φ-weak contractive maps, we investigate the existence of solutions to the proposed problem. An example is provided at the end to clarify the theoretical result.

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1 Introduction

On different time ranges, fractional calculus has had great impact due to a diversity of applications that have contributed to several fields of technical sciences and engineering [1–5]. One of the principal options behind the popularity of the area is that fractional-order differentiations and integrations are more beneficial tools in expressing real-world matters than the integer-order ones. Various studies in the literature, on distinct fractional operators such as the classical Riemann–Liouville, Caputo, Katugamploa, Hadamard, and Marchaud versions have shown versatility in modeling and control applications across various disciplines. However, such forms of fractional derivatives may not be able to explain the dynamic performance accurately, hence, many authors are found to be sorting out new fractional differentiations and integrations which have a kernel depending upon a function and this makes the range of definition expanded; see [6–8]. Furthermore, models based on these fractional operators provide excellent results to be compared with the integer-order differentiations [9–12].

Recently, the area of fractional-order differential inclusions has become mainly important as these equations were found to be of high importance in modeling stochastic and optimal controls problems [13]. By using techniques of nonlinear analysis the authors studied different aspects such as establishing the existence and the uniqueness of solutions, the upper and lower solutions, and stability. We refer the reader to [14–25] for various qualitative studies.

Details from the historical attitude and recent improvements in the area are detailed in the monograph of Ahmad *et al.* [26] and the survey of Agarwal *et al.* [27].

In this paper, we deal with the following ψ-fractional differential inclusions:

$$ {}^{c}\mathcal{D}_{\psi}^{\sigma} u(y) \in Z\bigl(y, u(y)\bigr), \quad y \in J = [1, T], 1 < \sigma \leq 2, \tag{1.1}$$

subject to ψ-boundary conditions of the form

$$ u(1) = \sum_{i=0}^{m} \lambda_i I_{\psi}^{\varrho} h\bigl(\eta_i, u(\eta_i)\bigr), \quad \delta_{\psi} u(T) = \delta_{\psi} u(1) = 0, \tag{1.2}$$

where ${}^{c}\mathcal{D}_{\psi}^{\sigma}$ is the ψ-Caputo fractional-order derivative, $Z : [1, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, I_{ψ}^{ϱ} is the ψ-Riemann–Liouville fractional integral of order $\varrho > 0$, $0 < \eta_i \leq T$, $\delta_{\psi} = \frac{1}{\psi'(y)} \frac{d}{dy}$, $h : [1, T] \times \mathbb{R} \to \mathbb{R}$ is given continuous function, and $\lambda_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, m$, are real constants such that

$$ -1 < \left(\theta \sum_{i=0}^{m} \lambda_i \frac{(\psi(\eta_i))^{\varrho}}{\Gamma(\varrho + 1)} \right) \leq 0. $$

θ will be determined later. We establish novel existence results of solutions for the above inclusion problem by using the endpoint theorem when the multivalued map is φ-weak contractive.

The result of the present paper unifies several classes of fractional differential inclusion with different boundary conditions. For example by taking $\psi(y) = y$ in (1.1)–(1.2) the results agree for the classical Caputo fractional inclusions [28] with a combination of classical nonlocal Riemann–Liouville fractional and Neumann boundary conditions of the form:

$$ u(1) = \sum_{i=0}^{m} \lambda_i I_{1^+}^{\varrho} h\bigl(\eta_i, u(\eta_i)\bigr), \quad u'(T) = u'(1) = 0, $$

when $\psi(y) = \ln(y)$, the results agree with the Caputo–Hadamard fractional inclusions [29] equipped with classical fractional integral boundary conditions of Hadamard type of the form

$$ u(1) = \sum_{i=0}^{m} \lambda_i {}^{H}I_{1^+}^{\varrho} h\bigl(\eta_i, u(\eta_i)\bigr), \quad \delta u(T) = \delta u(1) = 0, $$

while the results for generalized Caputo fractional inclusions [30] with nonlocal Katugampola type integral boundary conditions

$$ u(1) = \sum_{i=0}^{m} \lambda_i {}^{\rho}I_{1^+}^{\varrho} h\bigl(\eta_i, u(\eta_i)\bigr), \quad \delta_{\rho} u(T) = \delta_{\rho} u(1) = 0, $$

follow by taking $\psi(y) = y^{\rho}/\rho$.

The paper is organized as follows. Section 2 recalls some basic and fundamental definitions and lemmas. In Sect. 3, we prove the existence of a solution to the proposed problem (1.1)–(1.2). An example is provided to demonstrate the main results in Sect. 4.

2 Preliminary results

Let $\mathcal{W} = C([1,T],\mathbb{R})$ be the set of all continuous functions u from $[1,T]$ into $\mathbb{R}$ with the uniform norm

$$\|u\| = \sup_{y\in[1,T]} \bigl|u(y)\bigr|.$$

$L^1([1,T],\mathbb{R})$ be the Banach space of measurable functions $u:[1,T]\to\mathbb{R}$ with the norm

$$\|u\|_1 = \int_1^T \bigl|u(y)\bigr|\,dy.$$

We define $\mathrm{AC}^m_\psi([1,T],\mathbb{R})$ by

$$\mathrm{AC}^m_\psi\bigl([1,T],\mathbb{R}\bigr) = \biggl\{u:[1,T]\to\mathbb{R};\bigl(\delta_\psi^{m-1}u\bigr)(y)\in \mathrm{AC}\bigl([1,T],\mathbb{R}\bigr), \delta_\psi = \frac{1}{\psi'(y)}\frac{d}{dy}\biggr\},$$

which is supplied with the norm described by

$$\|u\|_{C^m_\psi} = \sum_{j=0}^{m-1}\bigl\|\delta_\psi^j u(y)\bigr\|_\infty,$$

where $\psi\in C^m([1,T],\mathbb{R})$, with $\psi'(y)>0$ on $[1,T]$, and

$$\delta_\psi^j = \underbrace{\delta_\psi\delta_\psi\ldots\delta_\psi}_{j\text{ times}},$$

and $\mathrm{AC}([1,T],\mathbb{R})$ is the space of absolutely continuous functions from $[1,T]$ into $\mathbb{R}$.

Now we introduce some notations and definitions of fractional calculus with respect to another function and give preliminary results that we will need in our proofs later.

Definition 2.1 ([28]) The ψ-fractional integration operator in the Riemann–Liouville sense of order $\sigma>0$ with lower limit 1 for an integrable function g is defined by

$$I^\sigma_\psi g(y) = \Gamma(\sigma)^{-1}\int_1^y \psi'(\xi)\bigl(\psi(y)-\psi(\xi)\bigr)^{\sigma-1}g(\xi)\,d\xi, \tag{2.1}$$

provided the integral exists.

Definition 2.2 ([7]) The ψ-fractional differentiation operator in the Riemann–Liouville sense of order $\sigma > 0$ of a function $g \in \mathrm{AC}_{\psi}^{m}([1,T],\mathbb{R})$ is defined by

$$\begin{aligned}\mathcal{D}_{\psi}^{\sigma}g(y) &= I_{\psi}^{m-\sigma}\big(\delta_{\psi}^{m}g\big)(y) + \sum_{j=0}^{m-1}\frac{(\delta_{\psi}^{j}g)(1)}{\Gamma(j-\sigma+1)}\big(\psi(y)-\psi(1)\big)^{j-\sigma} \\ &= \Gamma(m-\sigma)^{-1}\int_{1}^{y}\psi'(\xi)\big(\psi(y)-\psi(\xi)\big)^{m-\sigma-1}\delta_{\psi}^{m}g(\xi)\,d\xi \\ &\quad + \sum_{j=0}^{m-1}\frac{(\delta_{\psi}^{j}g)(1)}{\Gamma(j-\sigma+1)}\big(\psi(y)-\psi(1)\big)^{j-\sigma},\end{aligned} \tag{2.2}$$

provided the integral exists, where $m = [\sigma]+1$, and Γ is the Gamma Euler function.

Definition 2.3 ([6, 7]) The ψ-Caputo differentiation operator of fractional-order $\sigma > 0$ for a given $g \in \mathrm{AC}_{\psi}^{m}([1,T],\mathbb{R})$ is given by

$$\begin{aligned}{}^{c}\mathcal{D}_{\psi}^{\sigma}g(y) &= I_{\psi}^{m-\sigma}\big(\delta_{\psi}^{m}g\big)(y) \\ &= \Gamma(m-1)^{-1}\int_{1}^{y}\psi'(\xi)\big(\psi(y)-\psi(\xi)\big)^{m-\sigma-1}\big(\delta_{\psi}^{m}g\big)(\xi)\,d\xi, \quad m=[\sigma]+1,\end{aligned}$$

provided the integral exists. If $\sigma = m \in \mathbb{N}$ we have

$${}^{c}\mathcal{D}_{\psi}^{\sigma}g(y) = \big(\delta_{\psi}^{m}g\big)(y).$$

Lemma 2.4 ([7]) *For $\sigma > 0$ and a given function $g \in \mathrm{AC}_{\psi}^{m}([1,T],\mathbb{R})$, we have*

$$I_{\psi}^{\sigma}\,{}^{c}\mathcal{D}_{\psi}^{\sigma}g(y) = g(y) - \sum_{j=0}^{m-1}\frac{(\delta_{\psi}^{j}g)(1)}{j!}\big(\psi(y)-\psi(1)\big)^{j}. \tag{2.3}$$

Particularly, for $0 < \sigma < 1$, we obtain

$$I_{\psi}^{\sigma}\,{}^{c}\mathcal{D}_{\psi}^{\sigma}g(y) = g(y) - g(1).$$

We will investigate the existence of solutions to the problem (1.1)–(1.2) with the help of the following lemma.

Lemma 2.5 *Let $\phi : [1,T] \to \mathbb{R}$ be a continuous function, and $1 < \sigma \leq 2$. Then the ψ-fractional problem*

$${}^{c}\mathcal{D}_{\psi}^{\sigma}u(y) = \phi(y), \quad y \in [1,T], \tag{2.4}$$

$$u(1) = \sum_{i=0}^{m}\lambda_{i}I_{\psi}^{\varrho}h\big(\eta_{i},u(\eta_{i})\big), \quad \delta_{\psi}u(T) = \delta_{\psi}u(1) = 0, \tag{2.5}$$

is solvable, and its solution is given by

$$u(y) = I_{\psi}^{\sigma}\phi(y) + \sum_{i=0}^{m}\lambda_{i}I_{\psi}^{\varrho}h\big(\eta_{i},u(\eta_{i})\big). \tag{2.6}$$

Proof Performing the ψ-Riemann–Liouville fractional integration I_{ψ}^{σ} to both sides of (2.4) and making use of Lemma 2.4, we derive

$$u(y) = \lambda_1 + \lambda_2\bigl(\psi(y) - \psi(1)\bigr) + I_{\psi}^{\sigma}\phi(y), \tag{2.7}$$

where λ_1, λ_2 are real constants. Applying the δ_{ψ}-differentiation in (2.7) the following equation is formulated:

$$(\delta_{\psi} u)(y) = \lambda_2 + I_{\psi}^{\sigma-1}\phi(y). \tag{2.8}$$

Using the boundary conditions $(\delta_{\psi} u)(T) = (\delta_{\psi} u)(1) = 0$ in (2.8), we get $\lambda_2 = 0$, then using the condition $u(1) = \sum_{i=0}^{m} \lambda_i I_{\psi}^{\varrho} h(\eta_i, u(\eta_i))$ in (2.7), after inserting $\lambda_2 = 0$, gives us

$$\lambda_1 = \sum_{i=0}^{m} \lambda_i I_{\psi}^{\varrho} h\bigl(\eta_i, u(\eta_i)\bigr). \tag{2.9}$$

Thus by substituting values of λ_1 and λ_2 in (2.7), we get the solution (2.6). □

3 Main results

We introduce in this section the function class Ψ of all mappings $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi^{-1}(0) = \{0\}$, and $\varphi(z) < z$ for all $z > 0$, $\varphi(z_n) \to 0$ when $z_n \to 0$

Definition 3.1 ([31]) Let $\mathcal{W}$ be a complete space endowed with a metric ρ. A multivalued operator $S : \mathcal{W} \to \mathcal{P}_{\mathrm{cl,bd}}(\mathcal{W})$ is said to be a φ-weak contraction if there exists a function $\varphi \in \Psi$, such that

$$H_{\rho}(Sz, Sw) \le \rho(z, w) - \varphi\bigl(\rho(z, w)\bigr)$$

for each $z, w \in \mathcal{W}$, where $\mathcal{P}_{\mathrm{cl,bd}}(\mathcal{W})$ is a nonempty collection of all closed and bounded subsets of $\mathcal{W}$, and $H_{\rho}(\cdot,\cdot)$ denotes for the Hausdorff metric on $\mathcal{P}_{\mathrm{cl,bd}}$ given as

$$H_{\rho}(Q, D) := \max\Bigl\{\sup_{q\in Q} \rho(q, D), \sup_{d\in D} \rho(d, Q)\Bigr\},$$

where $\rho(Q, d) = \inf_{q\in Q} \rho(q, d)$ and $\rho(q, D) = \inf_{d\in D} \rho(q, d)$. We call an element $z \in \mathcal{W}$ a fixed point of S, if $z \in Sz$, and an endpoint or stationary point if $Sz = \{z\}$. The set of all fixed points of S is denoted by $\mathrm{Fix}(S)$, and $\mathrm{End}(S)$ stands for the set of all endpoints of S. We say that S fulfills the approximate endpoint property if $\inf_{z\in\mathcal{W}} \sup_{w\in Sz} \rho(z, w) = 0$.

Lemma 3.2 ([31]) *Let $\mathcal{W}$ be a complete space endowed with a metric ρ, and $S : \mathcal{W} \to \mathcal{P}_{\mathrm{cl,bd}}$ be a multivalued φ-weak contractive. If S verifies the approximate endpoint property, then S has an endpoint. Moreover, we have* $\mathrm{End}(S) = \mathrm{Fix}(S)$.

Definition 3.3 A function $u \in \mathrm{AC}_{\psi}^{2}([1, T], \mathbb{R})$ is called a solution of the inclusion problem (1.1) if there exists a function $l \in L^1([1, T], \mathbb{R})$ with $l(y) \in Z(y, u(y))$, a.e. $y \in [1, T]$, such that

u satisfies conditions (1.2) and

$$u(y) = I_\psi^\sigma l(y) + \sum_{i=0}^{m} \lambda_i I_\psi^\varrho h\big(\eta_i, u(\eta_i)\big), \quad y \in [1,T], \tag{3.1}$$

where $\psi \in C^2(J,\mathbb{R})$ such that $\psi' > 0$ on $[1,T]$.

We set an operator $\mathcal{L} : \mathcal{W} \to \mathcal{P}(\mathcal{W})$ associated with the problem (1.1)–(1.2) as

$$\mathcal{L}(u) : \left\{ f \in \mathcal{W} : f(y) = I_\psi^\sigma l(y) + \sum_{i=0}^{m} \lambda_i I_\psi^\varrho h\big(\eta_i, u(\eta_i)\big), l \in S_{Z,u} \right\}, \tag{3.2}$$

where $S_{Z,u}$ is the set of selections for Z by

$$S_{Z,u} = \big\{ l \in L^1\big([1,T],\mathbb{R}\big), l(y) \in Z\big(y,u(y)\big), \text{a.e. } y \in [1,T] \big\}.$$

Theorem 3.4 *Let $\varphi \in \Psi$. Assume that the following hypotheses hold*:

(H 1) $Z : [1,T] \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ *is a Carathéodory bounded multivalued map, where* $\mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ *is the collection of all nonempty compact subsets of* $\mathbb{R}$.

(H 2) *For $u, \bar{u} \in \mathbb{R}$, we have*

$$H_d\big(Z(y,u), Z(y,\bar{u})\big) \le \frac{\Gamma(\sigma+1)}{(\psi(T))^\sigma}\big(\big|u(y) - \bar{u}(y)\big| - \varphi\big(\big|u(y) - \bar{u}(y)\big|\big)\big).$$

(H 3) *There exists $0 < \theta < 1$, such that*

$$\big|h(y,u) - h(y,\bar{u})\big| \le \theta |u - \bar{u}|.$$

If Z verifies the approximate endpoint property, then the inclusion problem (1.1)–(1.2) *has a solution on $[1,T]$, provided that*

$$-1 < \left(\theta \sum_{i=0}^{m} \lambda_i \frac{(\psi(\eta_i))^\varrho}{\Gamma(\varrho+1)} \right) \le 0.$$

Proof The proof will be given in two steps, where we show that $\mathcal{L} : \mathcal{W} \to \mathcal{P}(\mathcal{W})$ given in (3.2) has an endpoint.

Step 1: $\mathcal{L}$ is closed multivalued of $\mathcal{P}(\mathcal{W})$.

Let $u_n \in \mathcal{W}$ such that $u_n \to u$, and $(f_n)_{n\ge1} \in \mathcal{L}(u)$ be a sequence such that $f_n \to f^*$ whenever $n \to +\infty$. Then there exists a $l_n \in S_{Z,u_n}$ such that, for each $y \in [1,T]$, we get

$$f_n(y) = I_\psi^\sigma l_n(y) + \sum_{i=0}^{m} \lambda_i I_\psi^\varrho h\big(\eta_i, u_n(\eta_i)\big).$$

Since Z has compact values, the sequence $(l_n)_{n\ge1}$ has a sub-sequence, still denoted by $(l_n)_{n\ge1}$, which converges strongly to some $l \in L^1([1,T],\mathbb{R})$, and hence $l \in S_{Z,u}$. For every $\nu \in Z(y,u(y))$, we have

$$\big|l_n(y) - l(y)\big| \le \big|l_n(y) - \nu\big| + \big|\nu - l(y)\big|,$$

which implies

$$\left|l_n(y)-l(y)\right| \le H_d\big(Z(y,u_n),Z(y,u)\big) \le \frac{\Gamma(\sigma+1)}{(\psi(T))^{\sigma}}\big(\|u_n-u\|-\varphi\big(\|u_n-u\|\big)\big).$$

Since $\|u_n-u\| \to 0$ then $\varphi(\|u_n-u\|) \to 0$ and h is a continuous function then, for each $y \in [1,T]$,

$$f_n(y) \to f^*(y) = I_{\psi}^{\sigma} l(y) + \sum_{i=0}^{m} \lambda_i I_{\psi}^{\varrho} h\big(\eta_i, u(\eta_i)\big).$$

So $f^* \in \mathcal{L}$ and $\mathcal{L}$ is closed multivalued.

Step 2: $\mathcal{L}$ is φ-weak contraction multivalued, i.e. for $u,\bar{u} \in \mathcal{W}$, we show

$$H_{\rho}\big(\mathcal{L}(u),\mathcal{L}(\bar{u})\big) \le \|u-\bar{u}\| - \varphi\big(\|u-\bar{u}\|\big).$$

Let $u,\bar{u} \in C(J,\mathbb{R})$ and $f_1 \in \mathcal{L}(u)$. Then, there exists $l_1(y) \in \mathcal{S}_{Z,u}$ such that, for each $y \in [1,T]$,

$$f_1(y) = I_{\psi}^{\sigma} l_1(y) + \sum_{i=0}^{m} \lambda_i I_{\psi}^{\varrho} h\big(\eta_i, u(\eta_i)\big).$$

From (H 2) it follows that

$$H_d\big(Z(y,u),Z(y,\bar{u})\big) \le \frac{\Gamma(\sigma+1)}{(\psi(T))^{\sigma}}\big(\left|u(y)-\bar{u}(y)\right| - \varphi\big(\left|u(y)-\bar{u}(y)\right|\big)\big).$$

Thus, there exists $w \in Z(y,\bar{u}(y))$ provided that

$$\left|l_1(y)-w\right| \le \frac{\Gamma(\sigma+1)}{(\psi(T))^{\sigma}}\big(\left|u(y)-\bar{u}(y)\right| - \varphi\big(\left|u(y)-\bar{u}(y)\right|\big)\big), \quad y \in J.$$

Define $U:[1,T] \to \mathcal{P}(\mathbb{R})$ given by

$$U(y) = \left\{ w \in \mathbb{R} : \left|l_1(y)-w\right| \le \frac{\Gamma(\sigma+1)}{(\psi(T))^{\sigma}}\big(\left|u(y)-\bar{u}(y)\right| - \varphi\big(\left|u(y)-\bar{u}(y)\right|\big)\big)\right\}.$$

Since $U(y) \cap Z(y,\bar{u})$ is measurable, then we can find a measurable selection $l_2(y)$ for $U(y) \cap Z(y,\bar{u})$. Thus $l_2(y) \in Z(y,\bar{u}(y))$, and, for each $y \in [1,T]$, we have

$$\left|l_1(y)-l_2(y)\right| \le \frac{\Gamma(\sigma+1)}{(\psi(T))^{\sigma}}\big(\left|u(y)-\bar{u}(y)\right| - \varphi\big(\left|u(y)-\bar{u}(y)\right|\big)\big).$$

We define $f_2(y)$ for each $y \in [1,T]$, as follows:

$$f_2(y) = I_{\psi}^{\sigma} l_2(y) + \sum_{i=0}^{m} \lambda_i I_{\psi}^{\varrho} h\big(\eta_i, \bar{u}(\eta_i)\big).$$

Then for $y \in [1, T]$

$$
\begin{aligned}
\left|f_1(y) - f_2(y)\right| &\leq \frac{1}{\Gamma(\sigma)} \int_1^y \left|\left(\psi(y) - \psi(\xi)\right)^{\sigma-1} \psi'(\xi)\right| \left|l_1(\xi) - l_2(\xi)\right| d\xi \\
&\quad + \sum_{i=0}^{m} \lambda_i I_\psi^\varrho \left|h\left(\eta_i, u(\eta_i)\right) - h\left(\eta_i, \bar{u}(\eta_i)\right)\right| \\
&\leq \frac{1}{\Gamma(\sigma)} \int_1^y \left|\left(\psi(y) - \psi(\xi)\right)^{\sigma-1} \psi'(\xi)\right| d\xi \\
&\quad \times \frac{\Gamma(\sigma+1)}{(\psi(T))^\sigma} \left(\left|u(y) - \bar{u}(y)\right| - \varphi\left(\left|u(y) - \bar{u}(y)\right|\right)\right) \\
&\quad + \theta \sum_{i=0}^{m} \lambda_i \frac{(\psi(\eta_i))^\varrho}{\Gamma(\varrho+1)} \left|u(\eta_i) - \bar{u}(\eta_i)\right| \\
&\leq \frac{(\psi(T))^\sigma}{\Gamma(\sigma+1)} \frac{\Gamma(\sigma+1)}{(\psi(T))^\sigma} \left(\|u - \bar{u}\| - \varphi\left(\|u - \bar{u}\|\right)\right) \\
&\quad + \theta \sum_{i=0}^{m} \lambda_i \frac{(\psi(\eta_i))^\varrho}{\Gamma(\varrho+1)} \|u - \bar{u}\| \\
&\leq \|u - \bar{u}\| - \varphi\left(\|u - \bar{u}\|\right).
\end{aligned}
$$

Therefore,

$$
\|f_1 - f_2\| \leq \|u - \bar{u}\| - \varphi\left(\|u - \bar{u}\|\right).
$$

It follows that $H_\rho(\mathcal{L}(u), \mathcal{L}(\bar{u})) \leq \|u(y) - \bar{u}(y)\| - \varphi(\|u(y) - \bar{u}(y)\|)$, for all $u, \bar{u} \in \mathcal{W}$. By hypothesis, since the operator Z has an approximate endpoint property, then by Lemma 3.2 $\mathcal{L}$ has an endpoint $u^* \in \mathcal{W}$, i.e. $\mathcal{L}u^* = \{u^*\}$, which is also a fixed point. Consequently, the problem (1.1)–(1.2) has a solution u^* and the proof is now complete. □

4 An example

Let $\mathcal{W} = C([1, e], \mathbb{R})$ be the space of all continuous functions defined on $[1, e]$ and $u \in \mathcal{W}$. Consider the following fractional BVP of differential inclusion:

$$
\begin{cases}
{}^c\mathcal{D}_\psi^\sigma u(y) \in Z(y, u(y)), & y \in [1, e], 1 < \sigma \leq 2, \\
u(1) = \frac{1}{4} I_\psi^{\frac{1}{3}} h(\frac{3}{4}, u(\frac{3}{4})) - \frac{2}{3} I_\psi^{\frac{1}{3}} h(\frac{1}{4}, u(\frac{1}{4})), & \delta_\psi u(e) = \delta_\psi u(1) = 0,
\end{cases}
\tag{4.1}
$$

where $h(y, u) = \frac{u}{e^y(u+1)}$, and $\psi(y) = y^3$. Obviously ψ is differentiable and an increasing function on $[1, e]$ with $\psi'(y) = 3y^2$, which is a continuous function on $[1, e]$. Here $Z : [1, e] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$
Z\left(y, u(y)\right) = \left[0, \frac{\sin(u)}{(1+y)}\right]. \tag{4.2}
$$

Selecting $\varphi(x) = \frac{x}{2}$. It is clear that the function $\varphi \in \Psi$, indeed $\varphi(x) < x$ for all $x \in [1,e]$, $\varphi^{-1}(\{0\}) = 0$, $\varphi(x_n) \to 0$ when $x_n \to 0$

$$\begin{aligned} H_\rho\big(Z(y,\vartheta), Z(y,\bar{\vartheta})\big) &\le \left|\frac{\sin(\vartheta) - \sin(\bar{\vartheta})}{(1+y)}\right| \\ &\le \frac{1}{2}\big|\vartheta(y) - \bar{\vartheta}(y)\big| \\ &< \frac{\Gamma(\sigma+1)}{e^3}\big(\|\vartheta - \bar{\vartheta}\| - \varphi\big(\|\vartheta - \bar{\vartheta}\|\big)\big). \end{aligned}$$

Hence the condition (H 2) holds for $\vartheta, \bar{\vartheta} \in \mathbb{R}$ a.e $\sigma \in (1,2]$. On the other hand, we have

$$\begin{aligned} \big|h(y,\vartheta) - h(y,\bar{\vartheta})\big| &\le \frac{1}{e^y}\left|\frac{\vartheta}{\vartheta+1} - \frac{\bar{\vartheta}}{\bar{\vartheta}+1}\right| \\ &= \frac{1}{e^y}\frac{|\vartheta - \bar{\vartheta}|}{(\vartheta+1)(1+\bar{\vartheta})} \\ &\le \frac{1}{e}|\vartheta - \bar{\vartheta}| \\ &\le \theta|\vartheta - \bar{\vartheta}|. \end{aligned}$$

Therefore condition (H 3) holds. With the given data, it is found that

$$\left(1 + \theta\sum_{i=0}^{m}\lambda_i\frac{(\psi(\eta_i))^\varrho}{\Gamma(\varrho+1)}\right) = 1 + e\left(\frac{\sqrt[3]{3}}{16\Gamma(\frac{4}{3})} - \frac{1}{6\Gamma(\frac{4}{3})}\right),$$

with

$$-1 < e\left(\frac{\sqrt[3]{3}}{16\Gamma(\frac{4}{3})} - \frac{1}{6\Gamma(\frac{4}{3})}\right) \approx -0.2329498 < 0.$$

We define an operator $\mathcal{L} : \mathcal{W} \to \mathcal{P}(\mathcal{W})$

$$\mathcal{L}(u) = \big\{g \in \mathcal{W} : \text{there exists } l \in S_{Z,u}, g(y) = u(y), \text{for all } y \in [1,e]\big\},$$

where

$$u(y) = I_\psi^\sigma l(y) + \sum_{i=0}^{m}\lambda_i I_\psi^\varrho h\big(\eta_i, u(\eta_i)\big).$$

Note that 0 is a unique endpoint of $\mathcal{L}$, i.e. $\mathcal{L}(0) = \{0\}$, which implies that $\sup_{u\in\mathcal{L}(0)}\|u\| = 0$, thus $\inf_{u\in\mathcal{W}}\sup_{g\in\mathcal{L}(u)}\|\vartheta - g\| = 0$. The operator $\mathcal{L}$ as a consequence has the approximate endpoint property. Therefore all conditions of Theorem 3.4 are satisfied, then the inclusion problem (4.1) has at least one solution on $[1,e]$.

5 Conclusion

In the present work, the endpoint theorem for φ-weak contractive maps was used to establish the existence results of solutions for fractional differential inclusion which involves the ψ-Caputo fractional derivative. Systems of fractional differential inclusions with the ψ-Caputo derivative provide more adaptable models, in the sense that by a proper choice of the function ψ, hidden features of real-world phenomena could be extracted. An illustrative example is presented to point out the applicability of our main results. Our results are not only new in the given configuration but also correspond to some new results associated with the specific choice of the function ψ involved in the given problem.

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Author details
[1]Department of Mathematics, University of Batna 2, 05078 Batna, Algeria. [2]Department of Mathematics, Çankaya University, 06790 Etimesgut, Ankara, Turkey. [3]Department of Mathematics and General Sciences, Prince Sultan University, P. O. Box 66833, Riyadh 11586, Saudi Arabia. [4]Department of Medical Research, China Medical University, Taichung 40402, Taiwan. [5]Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan. [6]Department of Elementary Education, Faculty of Education, Adiyaman University, 02040 Adiyaman, Turkey.

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