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ACHIEVING MORE PRECISE BOUNDS BASED ON DOUBLE AND TRIPLE INTEGRAL AS PROPOSED BY GENERALIZED PROPORTIONAL FRACTIONAL OPERATORS IN THE HILFER SENSE

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Abstract

A user-friendly approach depending on nonlocal kernel has been constituted in this study to model nonlocal behaviors of fractional differential and difference equations, which is known as a generalized proportional fractional operator in the Hilfer sense. It is deemed, for differentiable functions, by a fractional integral operator applied to the derivative of a function having an exponential function in the kernel. This operator generalizes a novel version of Čebyšev-type inequality in two and three variables sense and furthers the result of existing literature as a particular case of the Čebyšev inequality is discussed. Some novel special cases are also apprehended and compared with existing results. The outcome obtained by this study is very broad in nature and fits in terms of yielding an enormous number of relating results simply by practicing the proportionality indices included therein. Furthermore, the outcome of our study demonstrates that the proposed plans are of significant importance and computationally appealing to deal with comparable sorts of differential equations. Taken together, the results can serve as efficient and robust means for the purpose of investigating specific classes of integrodifferential equations.

Keywords: Integral Inequality; Generalized Proportional Fractional Operator in the Hilfer Sense; Čebyšev Inequality; Generalized Riemann–Liouville Fractional Integral.

1. INTRODUCTION

In the recent years, some researchers have demonstrated a lot of interest in the field of fractional calculus which reports the derivatives and integrals with any order. In fact, this deal has sprung out by the impression of the significant outcomes derived when the researchers utilized the tools in this calculus so as to concentrate a few models from this present reality.[1,2] Moreover, the role of the fractional derivative operator can be displayed the nonlinear oscillation of earthquake and the fluid-dynamic traffic can wipe out the lack emerging from the suspicion of the continuum traffic stream. A few equations emerge from mathematical models of most real-life situations. Hence, electromagnetic theory on Maxwell's equations, quantum mechanics depend on Schrödinger's equations and fluid mechanics in different forms of Navier–Stokes' equations. Analytical and numerical solution strategies for fractional differential equations have been quite concentrated in the literature.[3,4] There is a wide range of methods for characterizing fractional derivatives and fractional integrals such as Riemann–Liouville, Caputo, Saigo, Hilfer, Marchaud, tempered, Katugamploa, Atangana–Baleanu, to give some examples.[5,6] Various definitions might be sorted into general classes as indicated by their structure and properties.[7,8] Almeida[9] and Kilbas *et al.*[10] introduced the Caputo and generalized fractional derivative of a function in the sense of another function reported fertile applications in Laplace, Fourier and Mellin transform. Recently, another methodology was initially proposed by Jarad *et al.*,[11] to determine nondifferentiable issues in a fractional Schrödinger equation, and its significant properties were created. Later on, Rashid *et al.*[12] proposed a more general version of generalized proportional fractional integral operator, which has become progressively famous and attained significant progression owing to its remarkable properties in demonstrating complex nonlinear dynamical frameworks in various parts of scientific material science, such as integrodifferential equations, heat transforms, probability density functions and others.[13–23]

Fractional integral inequalities[24–31] have a particularly worthwhile place in the development of integrodifferential and difference equations. Their significant role in understanding the universe by employing the classical fractional integral, derivative operators and their speculative ideas in this matter is available to find the uniqueness and existence of the linear and nonlinear differential equations.[32,33] In this account, fundamental integral inequality by Chebyshev[34] contributes a significant part in statistical theory, numerical analysis and approximation theory in order to find the probability density functions and solutions of integral equations. Recently, Nisar *et al.*[35] obtained certain Čebyšev-type inequalities involving fractional conformable integral operators. In Ref. 36, the authors contemplated the extended version of Čebyšev-type inequalities via generalized fractional integral operator. Some new inequalities involving generalized $\mathcal{K}$-fractional integral operators are found in the work of Rashid *et al.*.[37] Also, Deniz *et al.*[38] introduced several new extensions of Čebyšev-type inequalities using conformable fractional integral operator via Pólya–Szegö inequality.

The well-known celebrated functional for two integrable functions is stated as

$$\mathfrak{T}(\mathcal{U},\mathcal{V}) = \frac{1}{q_2-q_1}\int_{q_1}^{q_2}\mathcal{U}(\varsigma)\mathcal{V}(\varsigma)d\varsigma - \left(\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\mathcal{U}(\varsigma)d\varsigma\right) \times\left(\frac{1}{q_2-q_1}\int_{q_1}^{q_2}\mathcal{V}(\varsigma)d\varsigma\right), \quad (1)$$

where $\mathcal{U}$ and $\mathcal{V}$ are two integrable functions on $[q_1,q_2]$. If $\mathcal{U}$ and $\mathcal{V}$ are synchronous, i.e.

$$(\mathcal{U}(\varsigma)-\mathcal{U}(\omega))(\mathcal{V}(\varsigma)-\mathcal{V}(\omega)) \geq 0$$

for any $\varsigma,\omega \in [q_1,q_2]$, then $\mathfrak{T}(\mathcal{U},\mathcal{V}) \geq 0$. Recently, the functional (1) has been tremendously improved by the insistence of their significance and intrinsic incentive in various parts of the applied sciences.

Our point in this paper is to consolidate the concepts of the generalized proportional fractional integral operator in the Hilfer sense and the proportional derivative in a new way, to make a novel rendition of Čebyšev-type inequalities related for two and three variables. In fact, we propose certain novel generalizations with power for Remark 7, which can be utilized as a prevailing tool to demonstrate the classifications of integral inequalities and integral equations. For our definitions, in this paper, we will discover a relationship via an elementary fractional differential equation to a bivariate exponential kernel which is gaining prominence nowadays in various applications.

In this work, a new concept of generalized proportional fractional operators in form of two and three variables in the Hilfer sense are proposed and its properties are elucidated in Sec. 2. In Sec. 3, we present the Čebyšev-type inequalities of a new fractional scheme and investigate the several new cases. Section 4 also contains the results of Čebyšev-type inequalities of the generalized proportional fractional integral operator in the Hilfer sense in three variable form. Finally, conclusions of this work describe the adaptive technique for the suggested fractional operators and related applications in other stimulating areas of research.

2. PRELIMINARIES

In this section, we introduce the basic definitions and properties of fractional calculus.[6,10]

Now, we present a new fractional operator which is known as the the generalized proportional fractional integral in the Hilfer sense which is proposed by Rashid *et al.*[12]

Definition 1 (Ref. 12). Let $(q_1,q_2)(-\infty \leq q_1 < q_2 \leq \infty)$ be a finite or infinite real interval with $\rho > 0$. Let $\Phi(x)$ is an increasing, positive monotone function with continuous derivative $\Psi'(x)$ defined on $(q_1,q_2]$. Then, the left-sided and right-sided Φ-generalized proportional fractional integral operators of $\mathcal{U}$ are defined as

$$^{\Phi}\mathcal{J}_{q_1}^{\rho,\delta}\mathcal{U}(\varsigma) = \frac{1}{\delta^{\rho}\Gamma(\rho)}\int_{q_1}^{\varsigma}\frac{\exp[\frac{\delta-1}{\delta}(\Phi(\varsigma)-\Phi(x))]\Phi'(x)}{(\Phi(\varsigma)-\Phi(x))^{1-\rho}} \times\mathcal{U}(x)dx, \quad q_1<\varsigma \quad (2)$$

and

$$
{}^{\Phi}\mathcal{J}^{\rho,\delta}_{q_2}\mathcal{U}(\varsigma) = \frac{1}{\delta^{\rho}\Gamma(\rho)}\int_{\varsigma}^{q_2} \frac{\exp[\frac{\delta-1}{\delta}(\Phi(x)-\Phi(\varsigma))]\Phi'(x)}{(\Phi(x)-\Phi(\varsigma))^{1-\rho}} \times \mathcal{U}(x)dx, \quad \varsigma < q_2, \tag{3}
$$

where the proportionality index $\delta \in (0,1], \rho \in \mathbb{C}, \Re(\rho) > 0$ and $\Gamma(\varsigma) = \int_0^\infty x^{\varsigma-1}e^{-x}dx$ is the Gamma function.

Jarad *et al.*[39] gave the concept of general left- and right-sided fractional derivative of $\mathcal{U}$ with respect to Φ as follows.

Definition 2 (Ref. 39). Let $(q_1, q_2)(-\infty \le q_1 < q_2 \le \infty)$ be a finite or infinite real interval and $\rho > 0$. Let $\Phi(x)$ be an increasing, positive monotone function with continuous derivative $\Phi'(x) > 0$ defined on $(q_1, q_2]$. Then, the left-sided and right-sided generalized proportional fractional derivative operators of $\mathcal{U}$ are defined as

$$
{}^{\Phi}\mathcal{D}^{n,\rho,\delta}_{q_1}\mathcal{U}(\varsigma) = {}^{\Phi}\mathcal{D}^{n,\rho}\,{}^{\Phi}\mathcal{J}^{n-\delta,\rho}_{q_1}\mathcal{U}(x) = \frac{{}^{\Phi}\mathcal{D}^{n-\delta,\rho}_{x}}{\delta^{n-\rho}\Gamma(n-\rho)} \times \int_{q_1}^{\varsigma} \frac{\exp[\frac{\delta-1}{\delta}(\Phi(\varsigma)-\Phi(x))]\Phi'(x)}{(\Phi(\varsigma)-\Phi(x))^{1+\rho-n}}\mathcal{U}(x)dx \tag{4}
$$

and

$$
{}^{\Phi}\mathcal{D}^{\rho,\delta}_{q_2}\mathcal{U}(\varsigma) = {}^{\Phi}_{\ominus}\mathcal{D}^{n,\rho}\,{}^{\Phi}\mathcal{J}^{n-\delta,\rho}_{q_1}\mathcal{U}(x) = \frac{{}^{\Phi}_{\ominus}\mathcal{D}^{n-\delta,\rho}_{x}}{\delta^{n-\rho}\Gamma(n-\rho)} \times \int_{\varsigma}^{q_2} \frac{\exp[\frac{\delta-1}{\delta}(\Phi(x)-\Phi(\varsigma))]\Phi'(x)}{(\Phi(x)-\Phi(\varsigma))^{1+\rho-n}}\mathcal{U}(x)dx, \tag{5}
$$

where $n = \Re(\delta) + 1$.

Now, we are in a position to present the generalized proportional fractional partial integral and derivative operator of a function in the Hilfer sense as follows.

Definition 3. Let $\vartheta = (p_1, p_2)$ and $\rho = (\rho_1, \rho_2)$ with $\rho_1, \rho_2 \in (0,1]$. Consider an interval $\mathcal{I} = [p_1, q_1] \times [p_2, q_2]$ where $p_i, q_i > 0$ for $i = 1,2$ and let $\Phi(x)$ is an increasing, positive monotone function with continuous derivative $\Phi'(x)$ defined on $(p_1, q_1] \times (p_2, q_2]$. Then, the left-sided generalized proportional fractional integral operator of $\mathcal{U}$ of order $\rho > 0$ in the Hilfer sense is defined as

$$
{}^{\Phi}\mathcal{J}^{\rho,\delta}_{\vartheta}\mathcal{U}(\varsigma_1,\varsigma_2) = \frac{1}{\delta^{\rho_1}\delta^{\rho_2}\Gamma(\rho_1)\Gamma(\rho_2)} \times \int_{p_1}^{\varsigma_1}\int_{p_2}^{\varsigma_2} \frac{\exp[\frac{\delta-1}{\delta}(\Phi(\varsigma_1)-\Phi(x))]\Phi'(x)}{(\Phi(\varsigma_1)-\Phi(x))^{1-\rho_1}} \times \frac{\exp[\frac{\delta-1}{\delta}(\Phi(\varsigma_2)-\Phi(y))]\Phi'(y)}{(\Phi(\varsigma_2)-\Phi(y))^{1-\rho_2}}\mathcal{U}(x,y)dxdy. \tag{6}
$$

The generalized proportional partial fractional derivative is defined as follows.

Definition 4. Let $\vartheta = (p_1, p_2)$ and $\rho = (\rho_1, \rho_2)$ with $\rho_1, \rho_2 \in (0,1]$. Consider an interval $\mathcal{I} = [p_1, q_1] \times [p_2, q_2]$ where $p_i, q_i > 0$ for $i = 1,2$ and let $\Phi(x)$ is an increasing, positive monotone function with continuous derivative $\Phi'(x) > 0$ defined on $(p_1, q_1] \times (p_2, q_2]$. Then, the left-sided and right-sided generalized proportional fractional integral operators of $\mathcal{U}$ of order $\rho > 0$ in the Hilfer sense are defined as

$$
{}^{\Phi}\mathcal{D}^{n,\rho,\delta}_{\vartheta}\mathcal{U}(\varsigma_1,\varsigma_2) = {}^{\Phi}\mathcal{J}^{2-\delta,\rho}_{\vartheta}\left(\frac{1}{\Phi'(\varsigma_1)\Phi'(\varsigma_2)}\frac{\partial^2\rho}{\partial\varsigma_2\partial\varsigma_1}\mathcal{U}\right)(\varsigma_1,\varsigma_2).
$$

We use the following notation:

$$
{}^{\Phi}\mathcal{D}^{\rho,\delta}_{\vartheta}\mathcal{U}(\varsigma_1,\varsigma_2) = \frac{\partial^{2\rho}_{\Phi}\mathcal{U}}{\partial_{\Phi}\varsigma_2^{\delta}\partial_{\Phi}\varsigma_1^{\delta}}(\varsigma_1,\varsigma_2).
$$

We define the norm for a function of two variables as follows:

$$
\|{}^{\Phi}\mathcal{D}^{\rho,\delta}_{\vartheta}\mathcal{U}\|_{\infty} = \sup|{}^{\Phi}\mathcal{D}^{\rho,\delta}_{\vartheta}\mathcal{U}(\varsigma_1,\varsigma_2)|.
$$

In a similar way, we introduced the definition of three variable versions in generalized proportional fractional integral and derivative of a function in the Hilfer sense as follows.

Definition 5. Let $\vartheta = (p_1, p_2, p_3)$ and $\rho = (\rho_1, \rho_2, \rho_3)$ with $\rho_1, \rho_2, \rho_3 \in (0,1]$. Consider an interval $\mathcal{I} = [p_1, q_1] \times [p_2, q_2] \times [p_3, q_3]$ where $p_i, q_i > 0$ for $i = 1,2,3$ and let $\Phi(x)$ is an increasing, positive monotone function with continuous derivative $\Phi'(x)$ defined on $(p_1, q_1] \times (p_2, q_2] \times (p_3, q_3]$. Then, the left-sided generalized proportional fractional integral operator of $\mathcal{U}$ of order $\rho > 0$ in the Hilfer sense is defined as

$$\begin{aligned}&{}^{\Phi}\mathcal{J}_{\vartheta}^{\rho,\delta}\mathcal{U}(\varsigma_1,\varsigma_2,\varsigma_3)\\&\quad=\frac{1}{\delta^{\rho_1}\delta^{\rho_2}\delta^{\rho_3}\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)}\\&\qquad\times\int_{p_1}^{\varsigma_1}\int_{p_2}^{\varsigma_2}\int_{p_3}^{\varsigma_3}\frac{\exp[\frac{\delta-1}{\delta}(\Phi(\varsigma_1)-\Phi(x))]\Phi'(x)}{(\Phi(\varsigma_1)-\Phi(x))^{1-\rho_1}}\\&\qquad\times\frac{\exp[\frac{\delta-1}{\delta}(\Phi(\varsigma_2)-\Phi(y))]\Phi'(y)}{(\Phi(\varsigma_2)-\Phi(y))^{1-\rho_2}}\\&\qquad\times\frac{\exp[\frac{\delta-1}{\delta}(\Phi(\varsigma_3)-\Phi(z))]\Phi'(z)}{(\Phi(\varsigma_3)-\Phi(z))^{1-\rho_3}}\\&\qquad\times\mathcal{U}(x,y,z)dxdydz.\end{aligned}\tag{7}$$

The generalized proportional partial fractional derivative is defined as follows.

Definition 6. Let $\vartheta = (p_1, p_2, p_2)$ and $\rho = (\rho_1, \rho_2, \rho_3)$ with $\rho_1, \rho_2, \rho_3 \in (0, 1]$. Consider an interval $\mathcal{I} = [p_1, q_1] \times [p_2, q_2] \times [p_3, q_3]$ where $p_i, q_i > 0$ for $i = 1, 2, 3$ and let $\Phi(x)$ is an increasing, positive monotone function with continuous derivative $\Phi'(x) > 0$ defined on $(p_1, q_1] \times (p_2, q_2] \times (p_3, q_3]$. Then, the left-sided generalized proportional fractional integral operator of $\mathcal{U}$ of order $\rho > 0$ in the Hilfer sense is defined as

$$\begin{aligned}{}^{\Phi}\mathcal{D}_{\vartheta}^{\rho,\delta}\mathcal{U}(\varsigma_1,\varsigma_2,\varsigma_3)={}^{\Phi}\mathcal{J}_{\vartheta}^{3-\delta,\rho}\Bigg(&\frac{1}{\Phi'(\varsigma_1)\Phi'(\varsigma_2)\Phi'(\varsigma_3)}\\&\times\frac{\partial^3\rho}{\partial\varsigma_3\partial\varsigma_2\partial\varsigma_1}\mathcal{U}\Bigg)(\varsigma_1,\varsigma_2,\varsigma_3).\end{aligned}$$

We use the following notation:

$${}^{\Phi}\mathcal{D}_{\vartheta}^{\rho,\delta}\mathcal{U}(\varsigma_1,\varsigma_2,\varsigma_3) = \frac{\partial_{\Phi}^{3\rho}\mathcal{U}}{\partial_{\Phi}\varsigma_3^{\delta}\partial_{\Phi}\varsigma_2^{\delta}\partial_{\Phi}\varsigma_1^{\delta}}(\varsigma_1,\varsigma_2,\varsigma_3).$$

We define the norm for a function of two variables as follows:

$$\|{}^{\Phi}\mathcal{D}_{\vartheta}^{\rho,\delta}\mathcal{U}\|_{\infty} = \sup|{}^{\Phi}\mathcal{D}_{\theta}^{\rho,\delta}\mathcal{U}(\varsigma_1,\varsigma_2,\varsigma_3)|.$$

Remark 7. From Definitions 1–6 we clearly observe the following:

(1) They turn into the both-sided generalized proportional fractional operators[11] if $\Phi(x) = x$.
(2) They turn into the both-sided generalized Riemann–Liouville fractional operators[6,10] if $\Phi(x) = x$ and $\delta = 1$.
(3) They lead to both-sided Riemann–Liouville fractional operators[6,10] if $\delta = 1$ and $\Phi(x) = x$.
(4) They lead to the both-sided generalized proportional Hadamard fractional operators[40] if $\delta = 1$ and $\Phi(x) = x$.
(5) They reduce to the both-sided Hadamard fractional operators[6,10] if $\delta = 1$ and $\Phi(x) = x$.
(6) They reduce to the both-sided generalized fractional operators in the Katugampola sense[41] if $\delta = 1$ and $\Phi(x) = \frac{x^{\tau}}{\tau}$ $(\tau > 0)$.

3. ČEBYŠEV FUNCTIONALS WITHIN GENERALIZED PROPORTIONAL FRACTIONAL OPERATORS

Presently, we exhibit the two-variable Čebyšev-type inequality via generalized proportional fractional derivative operator in the Hilfer sense as follows.

Theorem 8. *Consider two continuous functions $\mathcal{U}, \mathcal{V} : [p_1, q_1] \times [p_2, q_2] \to \mathbb{R}$ defined on $[p_1, q_1] \times [p_2, q_2]$ and $\frac{\partial^{2\eta}\mathcal{U}}{\partial_{\Phi}v^{\eta}\partial_{\Phi}u^{\eta}}$, $\frac{\partial^{2\eta}\mathcal{V}}{\partial_{\Phi}v^{\eta}\partial_{\Phi}u^{\eta}}$ existing as continuous and bounded on $[p_1, q_1] \times [p_2, q_2]$ and $\eta = (\eta_1, \eta_2)$. Then,*

$$\begin{aligned}&\Bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\mathcal{U}(u,v)\mathcal{V}(u,v)-\frac{1}{2}[\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{V}(u,v)\\&\qquad+\mathcal{A}_1(\mathcal{V}(u,v))\mathcal{U}(u,v)]dvdu\Bigg|\\&\quad\le\frac{1}{8}e^{[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(p_1))]}e^{[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(p_2))]}\\&\qquad\times(\Phi(q_1)-\Phi(p_1))(\Phi(q_2)-\Phi(p_2))\\&\qquad\times\int_{p_1}^{q_1}\int_{p_2}^{q_2}[|\mathcal{V}(u,v)|||{}^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}||_{\infty}\\&\qquad+\mathcal{V}(u,v)||{}^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}||_{\infty}]dvdu,\end{aligned}\tag{8}$$

where

$$\begin{aligned}&\mathcal{A}_1(\mathcal{U}(u,v))\\&\quad=\frac{1}{2}[\mathcal{U}(p_1,v)+\mathcal{U}(u,q_2)+\mathcal{U}(u,p_2)+\mathcal{U}(q_1,v)]\\&\qquad-\frac{1}{4}[\mathcal{U}(p_1,p_2)+\mathcal{U}(p_1,q_2)+\mathcal{U}(q_1,p_2)\\&\qquad+\mathcal{U}(q_1,q_2)]\end{aligned}\tag{9}$$

and

$$\Delta\left(\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi v^\eta \partial_\Phi u^\eta}(u,v)\right)$$
$$= \frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}\left[\int_{p_1}^{u}\int_{p_2}^{v} \Phi'(\tau)\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(u)-\Phi(\tau))\right]}e^{\left[\frac{\delta-1}{\delta}(\Phi(v)-\Phi(\zeta))\right]} \times (\Phi(u)-\Phi(\tau))^{\eta_1-1}(\Phi(v)-\Phi(\zeta))^{\eta_2-1} \times \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau - \int_{p_1}^{u}\int_{v}^{q_2} \Phi'(\tau)\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(u)-\Phi(\tau))\right]}e^{\left[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))\right]} \times (\Phi(u)-\Phi(\tau))^{\eta_1-1}(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1} \times \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau - \int_{u}^{q_1}\int_{p_2}^{v} \Phi'(\tau)\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))\right]}e^{\left[\frac{\delta-1}{\delta}(\Phi(v)-\Phi(\zeta))\right]} \times (\Phi(q_1)-\Phi(\tau))^{\eta_1-1}(\Phi(v)-\Phi(\zeta))^{\eta_2-1} \times \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau + \int_{u}^{q_1}\int_{v}^{q_2} \Phi'(\tau)\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))\right]}e^{\left[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))\right]} \times(\Phi(q_1)-\Phi(\tau))^{\eta_1-1}(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1} \times \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau\right]. \tag{10}$$

Proof. Under the given assumption $((u,v) \in [p_1,q_1]\times[p_2,q_2])$, we have

$$\frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}\int_{p_1}^{u}\int_{p_2}^{v} \Phi'(\tau)\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(u)-\Phi(\tau))\right]}e^{\left[\frac{\delta-1}{\delta}(\Phi(v)-\Phi(\zeta))\right]} \times (\Phi(u)-\Phi(\tau))^{\eta_1-1}(\Phi(v)-\Phi(\zeta))^{\eta_2-1} \times \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau = \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)} \times \int_{p_1}^{u}\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(u)-\Phi(\tau))\right]}(\Phi(u)-\Phi(\tau))^{\eta_1-1} \times \frac{\partial^{\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta}(\tau,\zeta)\Big|_{p_2}^{v}d\zeta = \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)} \times \int_{p_1}^{u}\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(u)-\Phi(\tau))\right]}(\Phi(u)-\Phi(\tau))^{\eta_1-1} \times\left[\frac{\partial^{\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta}(\tau,v) - \frac{\partial^{\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta}(\tau,p_2)\right]d\zeta$$
$$= \mathcal{U}(\tau,v)|_{p_1}^{u} - \mathcal{U}(\tau,v)|_{p_1}^{u} = \mathcal{U}(u,v) - \mathcal{U}(p_1,v) - \mathcal{U}(u,p_2) + \mathcal{U}(p_1,p_2). \tag{11}$$

Similarly, we have

$$\frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}\int_{p_1}^{u}\int_{v}^{q_2} \Phi'(\tau)\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(u)-\Phi(\tau))\right]} \times e^{\left[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))\right]}(\Phi(u)-\Phi(\tau))^{\eta_1-1} \times (\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau$$
$$= -\mathcal{U}(u,v) - \mathcal{U}(p_1,q_2) + \mathcal{U}(u,q_2) + \mathcal{U}(p_1,v), \tag{12}$$

$$\frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}\int_{u}^{q_1}\int_{p_2}^{v} \Phi'(\tau)\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))\right]} \times e^{\left[\frac{\delta-1}{\delta}(\Phi(v)-\Phi(\zeta))\right]}(\Phi(q_1)-\Phi(\tau))^{\eta_1-1} \times (\Phi(v)-\Phi(\zeta))^{\eta_2-1}\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau$$
$$= -\mathcal{U}(u,v) - \mathcal{U}(q_1,p_2) + \mathcal{U}(u,p_2) + \mathcal{U}(q_1,v), \tag{13}$$

$$\frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}\int_{u}^{q_1}\int_{v}^{q_2} \Phi'(\tau)\Phi'(\zeta)e^{\left[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))\right]} \times e^{\left[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))\right]}(\Phi(q_1)-\Phi(\tau))^{\eta_1-1} \times (\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau$$
$$= \mathcal{U}(u,v) + \mathcal{U}(q_1,p_2) - \mathcal{U}(u,p_2) - \mathcal{U}(q_1,v). \tag{14}$$

Summing up the above equalities, we get

$$4\mathcal{U}(u,v) - 2[\mathcal{U}(p_1,v) + \mathcal{U}(u,q_2) + \mathcal{U}(u,p_2) + \mathcal{U}(q_1,v)] + [\mathcal{U}(p_1,p_2) + \mathcal{U}(p_1,q_2) + \mathcal{U}(q_1,p_2) + \mathcal{U}(q_1,q_2)]$$
$$= \frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}\left[\int_{p_1}^{u}\int_{p_2}^{v}\Phi'(\tau)\Phi'(\zeta) \times e^{\left[\frac{\delta-1}{\delta}(\Phi(u)-\Phi(\tau))\right]}e^{\left[\frac{\delta-1}{\delta}(\Phi(v)-\Phi(\zeta))\right]}\right.$$

$$\times (\Phi(u)-\Phi(\tau))^{\eta_1-1}(\Phi(v)-\Phi(\zeta))^{\eta_2-1} \times \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau - \int_{p_1}^{u}\int_{v}^{q_2}\Phi'(\tau)\Phi'(\zeta) \times e^{[\frac{\delta-1}{\delta}(\Phi(u)-\Phi(\tau))]}e^{[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))]} \times (\Phi(u)-\Phi(\tau))^{\eta_1-1}(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1} \times \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau - \int_{u}^{q_1}\int_{p_2}^{v}\Phi'(\tau)\Phi'(\zeta) \times e^{[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))]}e^{[\frac{\delta-1}{\delta}(\Phi(v)-\Phi(\zeta))]} \times (\Phi(q_1)-\Phi(\tau))^{\eta_1-1}(\Phi(v)-\Phi(\zeta))^{\eta_2-1} \times \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau + \int_{u}^{q_1}\int_{v}^{q_2}\Phi'(\tau)\Phi'(\zeta) \times e^{[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))]}e^{[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))]} \times (\Phi(q_1)-\Phi(\tau))^{\eta_1-1}(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1} \times \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)d\zeta d\tau\Bigg]. \tag{15}$$

From (15), we get

$$\mathcal{U}(u,v)-\mathcal{A}_1(\mathcal{U}(u,v)) = \frac{1}{4}\Delta\left(\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right) \tag{16}$$

for $(u,v)\in[p_1,q_1]\times[p_2,q_2]$.

Analogously, we have

$$\mathcal{V}(u,v)-\mathcal{A}_1(\mathcal{V}(u,v)) = \frac{1}{4}\Delta\left(\frac{\partial^{2\eta}\mathcal{V}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right) \tag{17}$$

for $(u,v)\in[p_1,q_1]\times[p_2,q_2]$.

Taking product of (16) by $\mathcal{V}(u,v)$, (17) by $\mathcal{U}(u,v)$ adding them and then integrating over $(u,v)\in[p_1,q_1]\times[p_2,q_2]$, we get

$$\int_{p_1}^{q_1}\int_{p_2}^{q_2}[2\mathcal{U}(u,v)\mathcal{V}(u,v)-\mathcal{V}(u,v)\mathcal{A}_1(\mathcal{U}(u,v)) - \mathcal{U}(u,v)\mathcal{A}_1(\mathcal{V}(u,v))]dvdu = \frac{1}{8}\int_{p_1}^{q_1}\int_{p_2}^{q_2}\left[\Delta\left(\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right)\mathcal{V}(u,v) + \frac{1}{4}\Delta\left(\frac{\partial^{2\eta}\mathcal{V}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right)\mathcal{U}(u,v)\right]dvdu. \tag{18}$$

Taking modulus on both sides, we have

$$\left|\Delta\left(\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right)\right| \leq \frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}\int_{p_1}^{q_1}\int_{p_2}^{q_2}\Phi'(\tau)\Phi'(\zeta) \times e^{[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))]}e^{[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))]} \times (\Phi(q_1)-\Phi(\tau))^{\eta_1-1}(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1} \times \left|\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)\right|d\zeta d\tau \leq (\Phi(q_1)-\Phi(p_1))^{\eta_1}(\Phi(q_2) - \Phi(p_2))^{\eta_2}\|{}^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}\|_\infty, \tag{19}$$

$$\left|\Delta\left(\frac{\partial^{2\eta}\mathcal{V}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right)\right| = \frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}\int_{p_1}^{q_1}\int_{p_2}^{q_2}\Phi'(\tau)\Phi'(\zeta) \times e^{[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))]}e^{[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))]} \times (\Phi(q_1)-\Phi(p_1))^{\eta_1-1}(\Phi(q_2)-\Phi(p_2))^{\eta_2-1} \times \left|\frac{\partial^{2\eta}\mathcal{V}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)\right|d\zeta d\tau \leq (\Phi(q_1)-\Phi(p_1))^{\eta_1}(\Phi(q_2) - \Phi(p_2))^{\eta_2}\|{}^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}\|_\infty. \tag{20}$$

Combining all the above inequalities, we have

$$\left|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\mathcal{U}(u,v)\mathcal{V}(u,v)-\frac{1}{2}[\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{V}(u,v) + \mathcal{A}_1(\mathcal{V}(u,v))\mathcal{U}(u,v)]dvdu\right| \leq \frac{1}{8}\int_{p_1}^{q_1}\int_{p_2}^{q_2}\left[\left|\Delta\left(\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right)\right||\mathcal{V}(u,v)| + \left|\Delta\left(\frac{\partial^{2\eta}\mathcal{V}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right)\right||\mathcal{U}(u,v)|\right]dvdu \leq \frac{1}{8}\int_{p_1}^{q_1}\int_{p_2}^{q_2}\Bigg\{|\mathcal{V}(u,v)|\frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)} \times \Bigg[\int_{p_1}^{q_1}\int_{p_2}^{q_2}\Phi'(\tau)\Phi'(\zeta)e^{[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))]}$$

$$\times e^{[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))]}(\Phi(q_1)-\Phi(\tau))^{\eta_1-1}$$
$$\times(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}\left|\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)\right|d\zeta d\tau\Bigg]$$
$$+|\mathcal{U}(u,v)|\frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}$$
$$\times\Bigg[\int_{p_1}^{q_1}\int_{p_2}^{q_2}\Phi'(\tau)\Phi'(\zeta)e^{[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(\tau))]}$$
$$\times e^{[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(\zeta))]}(\Phi(q_1)-\Phi(\tau))^{\eta_1-1}$$
$$\times(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}$$
$$\times\left|\frac{\partial^{2\eta}\mathcal{V}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(\tau,\zeta)\right|d\zeta d\tau\Bigg]\Bigg\}dvdu$$
$$\leq\frac{1}{8}e^{[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(p_1))]}e^{[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(p_2))]}$$
$$\times(\Phi(q_1)-\Phi(p_1))^{\eta_1}(\Phi(q_2)-\Phi(p_2))^{\eta_2}$$
$$\times\int_{p_1}^{q_1}\int_{p_2}^{q_2}[|\mathcal{V}(u,v)|||{}^\Phi\mathcal{D}_\vartheta^{\eta,\delta}\mathcal{U}||_\infty$$
$$+|\mathcal{U}(u,v)||^{p_3}|{}^\Phi\mathcal{D}_\vartheta^{\eta,\delta}\mathcal{V}||_\infty]dvdu, \tag{21}$$

the desired result. This completes the proof. □

Some remarkable cases of Theorem 8 are discussed as follows:

(I) If we choose $\Phi(u)=u$, then, we get a new result for generalized proportional fractional operators.

Corollary 9. *Consider two continuous functions $\mathcal{U},\mathcal{V}:[p_1,q_1]\times[p_2,q_2]\to\mathbb{R}$ defined on $[p_1,q_1]\times[p_2,q_2]$ and $\frac{\partial^{2\eta}\mathcal{U}}{\partial v^\eta\partial u^\eta}$, $\frac{\partial^{2\eta}\mathcal{V}}{\partial v^\eta\partial u^\eta}$ existing as continuous and bounded on $[p_1,q_1]\times[p_2,q_2]$ and $\eta=(\eta_1,\eta_2)$. Then*

$$\Bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\mathcal{U}(u,v)\mathcal{V}(u,v)-\frac{1}{2}[\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{V}(u,v)$$
$$+\mathcal{A}_1(\mathcal{V}(u,v))\mathcal{U}(u,v)]dvdu\Bigg|$$
$$\leq\frac{1}{8}e^{[\frac{\delta-1}{\delta}(q_1-p_1)]}e^{[\frac{\delta-1}{\delta}(q_2-p_2)]}(q_1-p_1)$$
$$\times(q_2-p_2)\int_{p_1}^{q_1}\int_{p_2}^{q_2}[|\mathcal{V}(u,v)|||\mathcal{D}_\vartheta^{\eta,\delta}\mathcal{U}||_\infty$$
$$+|\mathcal{V}(u,v)|||\mathcal{D}_\vartheta^{\eta,\delta}\mathcal{V}||_\infty]dvdu,$$

where

$$\mathcal{A}_1(\mathcal{U}(u,v))=\frac{1}{2}[\mathcal{U}(p_1,v)+\mathcal{U}(u,q_2)$$
$$+\mathcal{U}(u,p_2)+\mathcal{U}(q_1,v)]$$
$$-\frac{1}{4}[\mathcal{U}(p_1,p_2)+\mathcal{U}(p_1,q_2)$$
$$+\mathcal{U}(q_1,p_2)+\mathcal{U}(q_1,q_2)]$$

and

$$\Delta\left(\frac{\partial^{2\eta}\mathcal{U}}{\partial v^\eta\partial u^\eta}(u,v)\right)$$
$$=\frac{1}{\delta^{\eta_1}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}$$
$$\times\Bigg[\int_{p_1}^{u}\int_{p_2}^{v}e^{[\frac{\delta-1}{\delta}(u-\tau)]}e^{[\frac{\delta-1}{\delta}(v-\zeta)]}$$
$$\times(u-\tau)^{\eta_1-1}(v-\zeta)^{\eta_2-1}\frac{\partial^{2\eta}\mathcal{U}}{\partial\zeta^\eta\partial\tau^\eta}(\tau,\zeta)d\zeta d\tau$$
$$-\int_{p_1}^{u}\int_{v}^{q_2}e^{[\frac{\delta-1}{\delta}(u-\tau)]}e^{[\frac{\delta-1}{\delta}(q_2-\zeta)]}$$
$$\times(u-\tau)^{\eta_1-1}(q_2-\zeta)^{\eta_2-1}\frac{\partial^{2\eta}\mathcal{U}}{\partial\zeta^\eta\partial\tau^\eta}(\tau,\zeta)d\zeta d\tau$$
$$-\int_{u}^{q_1}\int_{p_2}^{v}e^{[\frac{\delta-1}{\delta}(q_1-\tau)]}e^{[\frac{\delta-1}{\delta}(v-\zeta)]}$$
$$\times(q_1-\tau)^{\eta_1-1}(v-\zeta)^{\eta_2-1}\frac{\partial^{2\eta}\mathcal{U}}{\partial\zeta^\eta\partial\tau^\eta}(\tau,\zeta)d\zeta d\tau$$
$$+\int_{u}^{q_1}\int_{v}^{q_2}e^{[\frac{\delta-1}{\delta}(q_1-\tau)]}e^{[\frac{\delta-1}{\delta}(q_2-\zeta)]}$$
$$\times(q_1-\tau)^{\eta_1-1}(q_2-\zeta)^{\eta_2-1}$$
$$\times\frac{\partial^{2\eta}\mathcal{U}}{\partial\zeta^\eta\partial\tau^\eta}(\tau,\zeta)d\zeta d\tau\Bigg].$$

(II) If we choose $\Phi(u)=u$ and $\delta=1$, then we have a new result for the Riemann–Liouville fractional integral operator.

Corollary 10. *Consider two continuous functions $\mathcal{U},\mathcal{V}:[p_1,q_1]\times[p_2,q_2]\to\mathbb{R}$ defined on $[p_1,q_1]\times[p_2,q_2]$ and $\frac{\partial^{2\eta}\mathcal{U}}{\partial v^\eta\partial u^\eta}$, $\frac{\partial^{2\eta}\mathcal{V}}{\partial v^\eta\partial u^\eta}$ existing as continuous and bounded on $[p_1,q_1]\times[p_2,q_2]$ and $\eta=(\eta_1,\eta_2)$.*

Then,

$$\left|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\mathcal{U}(u,v)\mathcal{V}(u,v)-\frac{1}{2}[\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{V}(u,v)+\mathcal{A}_1(\mathcal{V}(u,v))\mathcal{U}(u,v)]dvdu\right|$$
$$\leq\frac{1}{8}(q_1-p_1)(q_2-p_2)\int_{p_1}^{q_1}\int_{p_2}^{q_2}\times[|\mathcal{V}(u,v)|||\mathcal{D}_{\vartheta}^{\eta}\mathcal{U}||_\infty+|\mathcal{V}(u,v)|||\mathcal{D}_{\vartheta}^{\eta}\mathcal{V}||_\infty]dvdu,$$

where

$$\mathcal{A}_1(\mathcal{U}(u,v))=\frac{1}{2}[\mathcal{U}(p_1,v)+\mathcal{U}(u,q_2)+\mathcal{U}(u,p_2)+\mathcal{U}(q_1,v)]-\frac{1}{4}[\mathcal{U}(p_1,p_2)+\mathcal{U}(p_1,q_2)+\mathcal{U}(q_1,p_2)+\mathcal{U}(q_1,q_2)]$$

and

$$\Delta\left(\frac{\partial^{2\eta}\mathcal{U}}{\partial v^\eta\partial u^\eta}(u,v)\right)=\frac{1}{\Gamma(\eta_1)\Gamma(\eta_2)}\left[\int_{p_1}^{u}\int_{p_2}^{v}(u-\tau)^{\eta_1-1}\times(v-\zeta)^{\eta_2-1}\frac{\partial^{2\eta}\mathcal{U}}{\partial\zeta^\eta\partial\tau^\eta}(\tau,\zeta)d\zeta d\tau-\int_{p_1}^{u}\int_{v}^{q_2}(u-\tau)^{\eta_1-1}(q_2-\zeta)^{\eta_2-1}\times\frac{\partial^{2\eta}\mathcal{U}}{\partial\zeta^\eta\partial\tau^\eta}(\tau,\zeta)d\zeta d\tau-\int_{u}^{q_1}\int_{p_2}^{v}(q_1-\tau)^{\eta_1-1}\times(v-\zeta)^{\eta_2-1}\frac{\partial^{2\eta}\mathcal{U}}{\partial\zeta^\eta\partial\tau^\eta}(\tau,\zeta)d\zeta d\tau+\int_{u}^{q_1}\int_{v}^{q_2}\times(q_1-\tau)^{\eta_1-1}(q_2-\zeta)^{\eta_2-1}\times\frac{\partial^{2\eta}\mathcal{U}}{\partial\zeta^\eta\partial\tau^\eta}(\tau,\zeta)d\zeta d\tau\right].$$

Remark 11. If we choose $\delta=1$, Theorem 8 leads to a result in Ref. 42.

Theorem 12. *Consider two continuous functions $\mathcal{U},\mathcal{V}:[p_1,q_1]\times[p_2,q_2]\to\mathbb{R}$ defined on $[p_1,q_1]\times[p_2,q_2]$ and $\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}$, $\frac{\partial^{2\eta}\mathcal{V}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}$ existing as continuous and bounded on $[p_1,q_1]\times[p_2,q_2]$ and $\eta=(\eta_1,\eta_2)$. Then, under the assumption of $\mathcal{A}_1(\mathcal{U}(u,v))$, and $\mathcal{A}_1(\mathcal{V}(\mathcal{U}(u,v)))$, the following inequality holds:*

$$\left|\int_{p_1}^{q_1}\int_{p_2}^{q_2}[\mathcal{U}(u,v)\mathcal{V}(u,v)-[\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{V}(u,v)+\mathcal{A}_1(\mathcal{V}(u,v))\mathcal{U}(u,v)-\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{A}_1(\mathcal{V}(u,v))]]dvdu\right|$$
$$\leq\frac{1}{16}[e^{[\frac{\delta-1}{\delta}(\Phi(q_1)-\Phi(p_1)]}e^{[\frac{\delta-1}{\delta}(\Phi(q_2)-\Phi(p_2))]}\times(\Phi(p_1)-\Phi(q_1))^{\eta_1}(\Phi(q_2)-\Phi(p_2))^{\eta_2}]^2\times\|{}^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}\|_\infty\|{}^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}\|_\infty,\tag{22}$$

where $(u,v)\in[p_1,q_1]\times[p_2,q_2]$.

Proof. Multiplying side by side (16) and (17), we obtain

$$\mathcal{U}(u,v)\mathcal{V}(u,v)-[\mathcal{A}_1(\mathcal{V}(u,v))\mathcal{U}(u,v)+\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{V}(u,v)]\leq\frac{1}{16}\Delta\left(\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right)\times\Delta\left(\frac{\partial^{2\eta}\mathcal{V}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right).\tag{23}$$

Integrating the above inequality over $[p_1,q_1]\times[p_2,q_2]$ and applying modulus property, we get

$$\left|\int_{p_1}^{q_1}\int_{p_2}^{q_2}[\mathcal{U}(u,v)\mathcal{V}(u,v)-[\mathcal{A}_1(\mathcal{V}(u,v))\mathcal{U}(u,v)+\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{V}(u,v)]-\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{A}_1(\mathcal{V}(u,v))dvdu\right|$$
$$\leq\frac{1}{16}\int_{p_1}^{q_1}\int_{p_2}^{q_2}\left|\Delta\left(\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right)\right|\times\left|\Delta\left(\frac{\partial^{2\eta}\mathcal{V}}{\partial_\Phi v^\eta\partial_\Phi u^\eta}(u,v)\right)\right|dvdu.\tag{24}$$

Now using (10) and (24) in (23) gives the required inequality (22). □

Some special cases of Theorem 12 are stated as follows:

(I) If we choose $\Phi(u)=u$, then, we get a new result for generalized proportional fractional integral operator.

Corollary 13. *Consider two continuous functions $\mathcal{U},\mathcal{V}:[p_1,q_1]\times[p_2,q_2]\to\mathbb{R}$ defined on $[p_1,q_1]\times[p_2,q_2]$ and $\frac{\partial^{2\eta}\mathcal{U}}{\partial v^\eta\partial u^\eta}$, $\frac{\partial^{2\eta}\mathcal{V}}{\partial v^\eta\partial u^\eta}$ existing as continuous and bounded on $[p_1,q_1]\times[p_2,q_2]$ and $\eta=(\eta_1,\eta_2)$.*

Then, under the assumption of $\mathcal{A}_1(\mathcal{U}(u,v))$, *and* $\mathcal{A}_1(\mathcal{V}(\mathcal{U}(u,v)))$, *the following inequality holds*:

$$\Bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}[\mathcal{U}(u,v)\mathcal{V}(u,v) - [\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{V}(u,v)+\mathcal{A}_1(\mathcal{V}(u,v))\mathcal{U}(u,v) - \mathcal{A}_1(\mathcal{U}(u,v))\mathcal{A}_1(\mathcal{V}(u,v))]]dvdu| \leq \frac{1}{16}[e^{[\frac{\delta-1}{\delta}(q_1-p_1)]}e^{[\frac{\delta-1}{\delta}(q_2-p_2)]}(p_1-q_1)^{\eta_1} \times (q_2-p_2)^{\eta_2}]^2||\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}||_\infty||\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}||_\infty,$$

where $(u,v)\in[p_1,q_1]\times[p_2,q_2]$.

(II) If we choose $\Phi(u)=u$ along with $\delta=1$, then we get a new result for Riemann–Liouville fractional integral operator.

Corollary 14. *Consider two continuous functions* $\mathcal{U},\mathcal{V}:[p_1,q_1]\times[p_2,q_2]\to\mathbb{R}$ *defined on* $[p_1,q_1]\times[p_2,q_2]$ *and* $\frac{\partial^{2\eta}\mathcal{U}}{\partial v^\eta\partial u^\eta}$, $\frac{\partial^{2\eta}\mathcal{V}}{\partial v^\eta\partial u^\eta}$ *existing as continuous and bounded on* $[p_1,q_1]\times[p_2,q_2]$ *and* $\eta=(\eta_1,\eta_2)$. *Then, under the assumption of* $\mathcal{A}_1(\mathcal{U}(u,v))$, *and* $\mathcal{A}_1(\mathcal{V}(\mathcal{U}(u,v)))$, *the following inequality holds*:

$$\Bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}[\mathcal{U}(u,v)\mathcal{V}(u,v) - [\mathcal{A}_1(\mathcal{U}(u,v))\mathcal{V}(u,v)+\mathcal{A}_1(\mathcal{V}(u,v))\mathcal{U}(u,v) - \mathcal{A}_1(\mathcal{U}(u,v))\mathcal{A}_1(\mathcal{V}(u,v))]]dvdu\Bigg| \leq \frac{1}{16}[(p_1-q_1)^{\eta_1}(q_2-p_2)^{\eta_2}]^2 \times||\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}||_\infty||\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}||_\infty,$$

where $(u,v)\in[p_1,q_1]\times[p_2,q_2]$.

Remark 15. If we choose $\delta=1$, Theorem 12 leads to a result in Ref. 42.

4. SOME NOVEL APPROACHES FOR ČEBYŠEV ESTIMATES INVOLVING FUNCTIONS OF THREE VARIABLES

In this section, we present the generalized proportional fractional derivative in the Hilfer sense for obtaining Čebyšev inequality involving functions of three variables. Given notations will be followed throughout this text for the ease of reading:.

$$\begin{aligned}\nabla(\mathcal{Q}(x,y,z)) &= \frac{1}{8}[\mathcal{Q}(p_1,p_2,p_3)+\mathcal{Q}(q_1,q_2,q_3)] - \frac{1}{4}[\mathcal{Q}(x,p_2,p_3)+\mathcal{Q}(x,q_2,q_3) + \mathcal{Q}(x,q_2,p_3)+\mathcal{Q}(x,p_2,q_3)] \\ &\quad - \frac{1}{4}[\mathcal{Q}(p_1,y,p_3)+\mathcal{Q}(q_1,y,q_3) + \mathcal{Q}(p_1,y,q_3)+\mathcal{Q}(q_1,y,p_3)] \\ &\quad - \frac{1}{4}[\mathcal{Q}(p_1,p_2,z)+\mathcal{Q}(q_1,q_2,z) + \mathcal{Q}(q_1,p_2,z)+\mathcal{Q}(p_1,q_2,z)] \\ &\quad + \frac{1}{2}[\mathcal{Q}(p_1,y,z)+\mathcal{Q}(q_1,y,z)] + \frac{1}{2}[\mathcal{Q}(x,p_2,z)+\mathcal{Q}(x,q_2,z)] + \frac{1}{2}[\mathcal{Q}(x,y,p_3)+\mathcal{Q}(x,y,q_3)]\end{aligned} \tag{25}$$

and

$$\begin{aligned}\Upsilon\left(\frac{\partial^{3\eta}\mathcal{Q}}{\partial_\Phi z^\eta\partial_\Phi y^\eta\partial_\Phi x^\eta}(x,y,z)\right) &= \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \times\int_{p_1}^{x}\int_{p_2}^{y}\int_{p_3}^{z}\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta) \\ &\quad \times e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]} \times e^{[\frac{\delta}{\delta-1}(\Phi(z)-\Phi(\tau))]}(\Phi(x)-\Phi(r))^{\eta_1-1} \\ &\quad \times(\Phi(y)-\Phi(\zeta))^{\eta_2-1}(\Phi(z)-\Phi(\tau))^{\eta_3-1} \times\frac{\partial^{3\eta}\mathcal{Q}}{\partial_\Phi r^\eta\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr \\ &\quad - \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \times\int_{p_1}^{x}\int_{p_2}^{y}\int_{p_3}^{q_3}\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta) \\ &\quad \times e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]} \times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]}(\Phi(x)-\Phi(r))^{\eta_1-1} \\ &\quad \times(\Phi(y)-\Phi(\zeta))^{\eta_2-1}(\Phi(q_3)-\Phi(\tau))^{\eta_3-1}\end{aligned}$$

$$
\begin{aligned}
&\times \frac{\partial^{3\eta}\mathcal{Q}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr\\
&- \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)}\\
&\times \int_{p_1}^{x}\int_{y}^{q_2}\int_{p_3}^{z} \Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta)\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))]}\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(z)-\Phi(\tau))]}(\Phi(x)-\Phi(r))^{\eta_1-1}\\
&\times (\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}(\Phi(z)-\Phi(\tau))^{\eta_3-1}\\
&\times \frac{\partial^{3\eta}\mathcal{Q}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr\\
&- \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)}\\
&\times \int_{x}^{q_1}\int_{p_2}^{y}\int_{p_3}^{z} \Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta)\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]}\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(z)-\Phi(\tau))]}(\Phi(q_1)-\Phi(r))^{\eta_1-1}\\
&\times (\Phi(y)-\Phi(\zeta))^{\eta_2-1}(\Phi(z)-\Phi(\tau))^{\eta_3-1}\\
&\times \frac{\partial^{3\eta}\mathcal{Q}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr\\
&+ \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)}\\
&\times \int_{p_1}^{x}\int_{r}^{q_2}\int_{z}^{q_3} \Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta)\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))]}\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]}(\Phi(x)-\Phi(r))^{\eta_1-1}\\
&\times (\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}(\Phi(q_3)-\Phi(\tau))^{\eta_3-1}\\
&\times \frac{\partial^{3\eta}\mathcal{Q}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr\\
&+ \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)}\\
&\times \int_{x}^{q_1}\int_{y}^{q_2}\int_{p_3}^{z} \Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta)\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))]}\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(z)-\Phi(\tau))]}(\Phi(q_1)-\Phi(r))^{\eta_1-1}\\
&\times (\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}(\Phi(z)-\Phi(\tau))^{\eta_3-1}\\
&\times \frac{\partial^{3\eta}\mathcal{Q}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr\\
&+ \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)}\\
&\times \int_{x}^{q_1}\int_{p_2}^{y}\int_{z}^{q_3} \Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta)\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]}\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]}(\Phi(q_1)-\Phi(r))^{\eta_1-1}\\
&\times (\Phi(y)-\Phi(\zeta))^{\eta_2-1}(\Phi(q_3)-\Phi(\tau))^{\eta_3-1}\\
&\times \frac{\partial^{3\eta}\mathcal{Q}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr\\
&+ \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)}\\
&\times \int_{x}^{q_1}\int_{y}^{q_2}\int_{z}^{q_3} \Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta)\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))]}\\
&\times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]}(\Phi(q_1)-\Phi(r))^{\eta_1-1}\\
&\times (\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}(\Phi(q_3)-\Phi(\tau))^{\eta_3-1}\\
&\times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr. \qquad (26)
\end{aligned}
$$

Our next result is the following theorem.

Theorem 16. *Consider two continuous functions* $\mathcal{U},\mathcal{V} : [p_1,q_1]\times[p_2,q_2]\times[p_3,q_3] \to \mathbb{R}$ *defined on* $[p_1,q_1]\times[p_2,q_2]$ *and* $\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi \tau^\eta\partial_\Phi\zeta^\eta\partial_\Phi r^\eta}$, $\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi \tau^\eta\partial_\Phi\zeta^\eta\partial_\Phi r^\eta}$ *existing as continuous and bounded on* $[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ *and* $\eta=(\eta_1,\eta_2,\eta_3)$. *Then,*

$$
\begin{aligned}
&\bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}\mathcal{U}(x,y,z)\mathcal{V}(x,y,z)\\
&\quad -\frac{1}{2}[\nabla(\mathcal{U}(x,y,z))\mathcal{V}(x,y,z)\\
&\quad +\nabla(\mathcal{V}(x,y,z))\mathcal{U}(x,y,z)]dydx\bigg|\\
&\le \frac{1}{16}e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(p_1))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(p_2))]}\\
&\quad \times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(p_3))]}(\Phi(q_1)-\Phi(p_1))^{\eta_1}\\
&\quad \times (\Phi(q_2)-\Phi(p_2))^{\eta_2}(\Phi(q_3)-\Phi(p_3))^{\eta_3}\\
&\quad \times \int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}[|\mathcal{V}(x,y,z)|||^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}||_\infty\\
&\quad +|\mathcal{U}(x,y,z)|||^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}||_\infty]dzdydx, \qquad (27)
\end{aligned}
$$

where ∇,Υ are as given in (16) and (20).

Proof. From the given assumption, we have for $x, y, z \in [p_1, q_1] \times [p_2, q_2] \times [p_3, q_3]$

$$\begin{aligned}
&\frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \\
&\quad\times \int_{p_1}^{x}\int_{p_2}^{y}\int_{p_3}^{z} \Phi'(r)\Phi'(\tau)\Phi'(\zeta) \\
&\quad\times e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]} \\
&\quad\times e^{[\frac{\delta}{\delta-1}(\Phi(z)-\Phi(\tau))]} (\Phi(x)-\Phi(r))^{\eta_1-1} \\
&\quad\times (\Phi(y)-\Phi(\zeta))^{\eta_2-1}(\Phi(z)-\Phi(\tau))^{\eta_3-1} \\
&\quad\times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta\partial_\Phi r^\eta}(r,\tau,\zeta)\,d\zeta\, d\tau\, dr \\
&= \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)} \int_{p_1}^{x}\int_{p_2}^{y} \Phi'(r)\Phi'(\zeta) \\
&\quad\times e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]} \\
&\quad\times (\Phi(x)-\Phi(r))^{\eta_1-1} \\
&\quad\times (\Phi(y)-\Phi(\zeta))^{\eta_2-1} \left.\frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi r^\eta}(r,\zeta,\tau)\right|_{p_3}^{z} d\zeta \\
&= \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)} \int_{p_1}^{x}\int_{p_2}^{y} \\
&\quad\times \Phi'(r)\Phi'(\zeta) e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} \\
&\quad\times e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]} (\Phi(x)-\Phi(r))^{\eta_1-1} \\
&\quad\times (\Phi(y)-\Phi(\zeta))^{\eta_2-1} \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi r^\eta}(r,\zeta,z)\,d\zeta\, dr \\
&\quad - \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)} \int_{p_1}^{x}\int_{p_2}^{y} \\
&\quad\times \Phi'(r)\Phi'(\zeta) e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} \\
&\quad\times e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]} (\Phi(x)-\Phi(r))^{\eta_1-1} \\
&\quad\times (\Phi(y)-\Phi(\zeta))^{\eta_2-1} \frac{\partial^{2\eta}\mathcal{U}}{\partial_\Phi\zeta^\eta\partial_\Phi r^\eta}(r,\zeta,p_3)\,d\zeta\, dr \\
&= \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)} \int_{p_1}^{x} \Phi'(r) e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} \\
&\quad\times (\Phi(x)-\Phi(r))^{\eta_1-1} \left.\frac{\partial^{\eta}\mathcal{U}}{\partial_\Phi r^\eta}(r,\zeta,z)\right|_{p_2}^{y} dr \\
&\quad - \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)} \int_{p_1}^{x} \Phi'(r) e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} \\
&\quad\times (\Phi(x)-\Phi(r))^{\eta_1-1} \left.\frac{\partial^{\eta}\mathcal{U}}{\partial_\Phi r^\eta}(r,\zeta,p_3)\right|_{p_2}^{y} dr \\
&= \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)} \int_{p_1}^{x} \Phi'(r) e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} \\
&\quad\times (\Phi(x)-\Phi(r))^{\eta_1-1} \frac{\partial^{\eta}\mathcal{U}}{\partial_\Phi r^\eta}(r,y,z)\,dr \\
&\quad - \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)} \int_{p_1}^{x} \Phi'(r) e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} \\
&\quad\times (\Phi(x)-\Phi(r))^{\eta_1-1} \frac{\partial^{\eta}\mathcal{U}}{\partial_\Phi r^\eta}(r,p_2,z)\,dr \\
&\quad - \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)} \int_{p_1}^{x} \Phi'(r) e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} \\
&\quad\times (\Phi(x)-\Phi(r))^{\eta_1-1} \frac{\partial^{\eta}\mathcal{U}}{\partial_\Phi r^\eta}(r,y,p_3)\,dr \\
&\quad + \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)} \int_{p_1}^{x} \Phi'(r) e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} \\
&\quad\times (\Phi(x)-\Phi(r))^{\eta_1-1} \frac{\partial^{\eta}\mathcal{U}}{\partial_\Phi r^\eta}(r,p_2,p_3)\,dr \\
&= \mathcal{U}(r,y,z)|_{p_1}^{x} - \mathcal{U}(r,p_3,z)|_{p_1}^{x} \\
&\quad - \mathcal{U}(r,y,p_3)|_{p_1}^{x} + \mathcal{U}(r,p_2,p_3)|_{p_1}^{x} \\
&= \mathcal{U}(x,y,z) - \mathcal{U}(p_1,y,z) - \mathcal{U}(x,p_2,z) \\
&\quad + \mathcal{U}(p_1,p_2,z) - \mathcal{U}(x,y,p_3) + \mathcal{U}(p_1,y,p_3) \\
&\quad + \mathcal{U}(x,p_2,p_3) + \mathcal{U}(p_1,p_2,p_3).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathcal{U}(x,y,z) \\
&\quad = \mathcal{U}(p_1,y,z) + \mathcal{U}(x,p_2,z) - \mathcal{U}(p_1,p_2,z) \\
&\qquad + \mathcal{U}(x,y,p_3) - \mathcal{U}(p_1,y,p_3) \\
&\qquad - \mathcal{U}(x,p_2,p_3) - \mathcal{U}(p_1,p_2,p_3) \\
&\frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \\
&\qquad\times \int_{p_1}^{x}\int_{p_2}^{y}\int_{p_3}^{z} \Phi'(r)\Phi'(\tau)\Phi'(\zeta) \\
&\qquad\times e^{[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))]} e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]} \\
&\qquad\times e^{[\frac{\delta}{\delta-1}(\Phi(z)-\Phi(\tau))]} (\Phi(x)-\Phi(r))^{\eta_1-1} \\
&\qquad\times (\Phi(y)-\Phi(\zeta))^{\eta_2-1}(\Phi(z)-\Phi(\tau))^{\eta_3-1} \\
&\qquad\times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(r,\zeta,\tau)\,d\tau\, d\zeta\, dr. \qquad (28)
\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}
\mathcal{U}(x,y,z) &= \mathcal{U}(x,y,q_3)+\mathcal{U}(p_1,y,z)+\mathcal{U}(x,p_2,z) \\
&\quad +\mathcal{U}(p_1,p_2,q_3)-\mathcal{U}(p_1,p_2,z) \\
&\quad -\mathcal{U}(p_1,y,q_3)-\mathcal{U}(y,p_2,q_3) \\
&\quad -\frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \\
&\quad \times \int_{p_1}^{x}\int_{p_2}^{y}\int_{z}^{q_3}\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta) \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))\right]}e^{\left[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))\right]} \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))\right]}(\Phi(x)-\Phi(r))^{\eta_1-1} \\
&\quad \times (\Phi(y)-\Phi(\zeta))^{\eta_2-1}(\Phi(q_3)-\Phi(\tau))^{\eta_3-1} \\
&\quad \times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr,
\end{aligned}\tag{29}$$

$$\begin{aligned}
\mathcal{U}(x,y,z) &= \mathcal{U}(x,q_2,z)+\mathcal{U}(x,y,p_3)+\mathcal{U}(p_1,q_2,p_3) \\
&\quad +\mathcal{U}(p_1,y,z)-\mathcal{U}(x,q_2,p_3) \\
&\quad -\mathcal{U}(p_1,q_2,z)-\mathcal{U}(p_1,y,p_3) \\
&\quad -\frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \\
&\quad \times \int_{p_1}^{x}\int_{y}^{q_2}\int_{p_3}^{z}\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta) \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))\right]}e^{\left[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))\right]} \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(z)-\Phi(\tau))\right]}(\Phi(x)-\Phi(r))^{\eta_1-1} \\
&\quad \times (\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}(\Phi(z)-\Phi(\tau))^{\eta_3-1} \\
&\quad \times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr,
\end{aligned}\tag{30}$$

$$\begin{aligned}
\mathcal{U}(x,y,z) &= \mathcal{U}(q_1,\zeta,\tau)+\mathcal{U}(q_1,p_2,p_3)+\mathcal{U}(x,y,p_3) \\
&\quad +\mathcal{U}(x,p_2,z)-\mathcal{U}(q_1,y,p_3) \\
&\quad -\mathcal{U}(q_1,p_2,z)-\mathcal{U}(x,p_2,p_3) \\
&\quad -\frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \\
&\quad \times \int_{x}^{q_1}\int_{p_2}^{v}\int_{p_3}^{z}\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta) \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))\right]}e^{\left[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))\right]} \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(z)-\Phi(\tau))\right]}(\Phi(q_1)-\Phi(r))^{\eta_1-1} \\
&\quad \times (\Phi(y)-\Phi(\zeta))^{\eta_2-1}(\Phi(z)-\Phi(\tau))^{\eta_3-1} \\
&\quad \times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr,
\end{aligned}\tag{31}$$

$$\begin{aligned}
\mathcal{U}(x,y,z) &= \mathcal{U}(x,q_2,z)+\mathcal{U}(x,y,q_3)+\mathcal{U}(p_1,q_2,q_3) \\
&\quad +\mathcal{U}(p_1,y,z)-\mathcal{U}(x,q_2,q_3) \\
&\quad -\mathcal{U}(p_1,q_2,z)-\mathcal{U}(p_1,y,q_3) \\
&\quad +\frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \\
&\quad \times \int_{p_1}^{x}\int_{y}^{q_2}\int_{z}^{q_3}\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta) \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(x)-\Phi(r))\right]}e^{\left[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))\right]} \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))\right]}(\Phi(x)-\Phi(r))^{\eta_1-1} \\
&\quad \times (\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}(\Phi(q_3)-\Phi(\tau))^{\eta_3-1} \\
&\quad \times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr,
\end{aligned}\tag{32}$$

$$\begin{aligned}
\mathcal{U}(x,y,z) &= \mathcal{U}(r,q_2,\tau)+\mathcal{U}(x,y,p_3)+\mathcal{U}(q_1,\zeta,\tau) \\
&\quad +\mathcal{U}(q_1,q_2,p_3)-\mathcal{U}(q_1,q_2,z) \\
&\quad -\mathcal{U}(q_1,y,p_3)-\mathcal{U}(x,q_2,p_3) \\
&\quad +\frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \\
&\quad \times \int_{x}^{q_1}\int_{y}^{q_2}\int_{p_3}^{z}\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta) \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))\right]}e^{\left[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))\right]} \\
&\quad \times e^{\left[\frac{\delta}{\delta-1}(\Phi(z)-\Phi(\tau))\right]}(\Phi(q_1)-\Phi(r))^{\eta_1-1} \\
&\quad \times (\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}(\Phi(z)-\Phi(\tau))^{\eta_3-1} \\
&\quad \times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta\partial_\Phi\zeta^\eta\partial_\Phi\tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr,
\end{aligned}\tag{33}$$

$$\begin{aligned}
\mathcal{U}(x,y,z) &= \mathcal{U}(q_1,y,z)+\mathcal{U}(q_1,p_2,q_3)+\mathcal{U}(x,y,q_3) \\
&\quad +\mathcal{U}(x,p_2,\tau)-\mathcal{U}(q_1,y,q_3) \\
&\quad -\mathcal{U}(q_1,p_2,z)-\mathcal{U}(x,p_2,q_3) \\
&\quad +\frac{1}{\delta^{\eta_1}\delta^{\eta_2}\delta^{\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)\Gamma(\eta_3)}\int_{x}^{q_1}\int_{p_2}^{y}\int_{z}^{q_3}
\end{aligned}$$

$$\times \Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta)e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))]} \times e^{[\frac{\delta}{\delta-1}(\Phi(y)-\Phi(\zeta))]} \times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]}(\Phi(q_1)-\Phi(r))^{\eta_1-1} \times(\Phi(v)-\Phi(\zeta))^{\eta_2-1}(\Phi(q_3)-\Phi(\tau))^{\eta_3-1} \times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr \tag{34}$$

and

$$\mathcal{U}(x,y,z) = \mathcal{U}(q_1,q_2,q_3)+\mathcal{U}(p_1,y,z)+\mathcal{U}(x,q_2,z) + \mathcal{U}(x,y,q_3)-\mathcal{U}(q_1,q_2,z)-\mathcal{U}(q_1,y,q_3) - \mathcal{U}(x,q_2,q_3)+\frac{1}{\delta^{\eta_1}\delta^{\eta_2}\delta^{\eta_3}\Gamma(\eta_1)\Gamma(\eta_2)\Gamma(\eta_3)} \times \int_x^{q_1}\int_y^{q_2}\int_z^{q_3}\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta) \times e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))]} \times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]}(\Phi(q_1)-\Phi(r))^{\eta_1-1} \times(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}\Phi(q_3)-\Phi(\tau))^{\eta_3-1} \times \frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)d\tau d\zeta dr. \tag{35}$$

Summing up the above identities we have

$$\mathcal{U}(x,y,z)-\nabla(\mathcal{U}(x,y,z)) = \frac{1}{8}\Upsilon\left(\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi z^\eta \partial_\Phi y^\eta \partial_\Phi x^\eta}(x,y,z)\right), \tag{36}$$

for $(x,y,z)\in[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$.

Similarly, we have

$$\mathcal{V}(x,y,z)-\nabla(\mathcal{V}(x,y,z)) = \frac{1}{8}\Upsilon\left(\frac{\partial^{3\eta}\mathcal{V}}{\partial_\Phi z^\eta \partial_\Phi y^\eta \partial_\Phi x^\eta}(x,y,z)\right), \tag{37}$$

for $(x,y,z)\in[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$.

Now, taking product of (36) and (37) by $\mathcal{V}(x,y,z)$ and $\mathcal{U}(x,y,z)$, respectively, adding them and integrating over $[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$, we have

$$\left|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}\mathcal{U}(x,y,z)\mathcal{V}(x,y,z) - \frac{1}{2}[\nabla(\mathcal{U}(x,y,z))\mathcal{V}(x,y,z) \times \nabla(\mathcal{V}(x,y,z))\mathcal{U}(x,y,z)]dzdydx\right| \le \frac{1}{16}\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}\left[\mathcal{V}(x,y,z)\Upsilon \times\left(\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi z^\eta \partial_\Phi y^\eta \partial_\Phi x^\eta}(x,y,z)\right)+\mathcal{U}(x,y,z)\Upsilon \times\left(\frac{\partial^{3\eta}\mathcal{V}}{\partial_\Phi z^\eta \partial_\Phi y^\eta \partial_\Phi x^\eta}(x,y,z)\right)\right]dzdydx. \tag{38}$$

Taking modulus on both sides, we have

$$\left|\Upsilon\left(\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi z^\eta \partial_\Phi y^\eta \partial_\Phi x^\eta}(x,y,z)\right)\right| \le \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)} \times\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta) \times e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))]} \times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]}(\Phi(q_1)-\Phi(r))^{\eta_1-1} \times(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1}(\Phi(q_3)-\Phi(\tau))^{\eta_3-1} \times\left|\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,s,t)\right|d\tau d\zeta dr \le e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))]} \times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]}(\Phi(q_1)-\Phi(r))^{\eta_1} \times(\Phi(q_2)-\Phi(\zeta))^{\eta_2}(\Phi(q_3)-\Phi(\tau))^{\eta_3} \times||^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}||_\infty, \tag{39}$$

$$\left|\Upsilon\left(\frac{\partial^{3\eta}\mathcal{V}}{\partial_\Phi z^\eta \partial_\Phi y^\eta \partial_\Phi x^\eta}(x,y,z)\right)\right| \le \frac{1}{\delta^{\eta_1}\Gamma(\eta_1)\delta^{\eta_2}\Gamma(\eta_2)\delta^{\eta_3}\Gamma(\eta_3)}\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3} \times\Phi^{'}(r)\Phi^{'}(\tau)\Phi^{'}(\zeta)e^{[\frac{\delta}{\delta-1}(\Phi(q_q)-\Phi(r))]} \times e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]} \times(\Phi(q_1)-\Phi(r))^{\eta_1-1}(\Phi(q_2)-\Phi(\zeta))^{\eta_2-1} \times(\Phi(q_3)-\Phi(\tau))^{\eta_3-1} \times\left|\frac{\partial^{3\eta}\mathcal{V}}{\partial_\Phi r^\eta \partial_\Phi \zeta^\eta \partial_\Phi \tau^\eta}(r,\zeta,\tau)\right|d\tau d\zeta dr$$

$$\leq e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(r))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(\zeta))]}$$
$$\times e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(\tau))]}$$
$$\times (\Phi(q_1)-\Phi(r))^{\eta_1}(\Phi(q_2)-\Phi(\zeta))^{\eta_2}$$
$$\times(\Phi(q_3)-\Phi(\tau))^{\eta_3}||{}^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}||_\infty. \tag{40}$$

Combining (37)–(39), we get the desired inequality (27). This completes the proof. □

Some remarkable cases of Theorem 16 are discussed as follows:

(I) If we choose $\Phi(u) = u$ then we get a new result for generalized proportional fractional operators.

Corollary 17. *Consider two continuous functions* $\mathcal{U},\mathcal{V} : [p_1,q_1]\times[p_2,q_2]\times[p_3,q_3] \to \mathbb{R}$ *defined on* $[p_1,q_1]\times[p_2,q_2]$ *and* $\frac{\partial^{3\eta}\mathcal{U}}{\partial\tau^\eta\partial\zeta^\eta\partial r^\eta}$, $\frac{\partial^{3\eta}\mathcal{U}}{\partial\tau^\eta\partial\zeta^\eta\partial r^\eta}$ *existing as continuous and bounded on* $[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ *and* $\eta = (\eta_1,\eta_2,\eta_3)$. *Then,*

$$\Bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}\mathcal{U}(x,y,z)\mathcal{V}(x,y,z) - \frac{1}{2}[\nabla(\mathcal{U}(x,y,z))\mathcal{V}(x,y,z) + \nabla(\mathcal{V}(x,y,z))\mathcal{U}(x,y,z)]dydx\Bigg|$$
$$\leq \frac{1}{16}e^{[\frac{\delta}{\delta-1}(q_1-p_1)]}e^{[\frac{\delta}{\delta-1}(q_2-p_2)]}e^{[\frac{\delta}{\delta-1}(q_3-p_3)]}$$
$$\times (q_1-p_1)^{\eta_1}(q_2-p_2)^{\eta_2}(q_3-p_3)^{\eta_3}$$
$$\times\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}[|\mathcal{V}(x,y,z)|||\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}||_\infty + |\mathcal{U}(x,y,z)|||\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}||_\infty]dzdydx,$$

where ∇,Υ *are as given in* (16) *and* (20).

(II) If we choose $\Phi(u) = u$ along with $\delta = 1$, then we get a new result for Riemann–Liouville fractional integral operator.

Corollary 18. *Consider two continuous functions* $\mathcal{U},\mathcal{V} : [p_1,q_1]\times[p_2,q_2]\times[p_3,q_3] \to \mathbb{R}$ *defined on* $[p_1,q_1]\times[p_2,q_2]$ *and* $\frac{\partial^{3\eta}\mathcal{U}}{\partial\tau^\eta\partial\zeta^\eta\partial r^\eta}$, $\frac{\partial^{3\eta}\mathcal{U}}{\partial\tau^\eta\partial\zeta^\eta\partial r^\eta}$ *existing as continuous and bounded on* $[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ *and* $\eta = (\eta_1,\eta_2,\eta_3)$. *Then,*

$$\Bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}\mathcal{U}(x,y,z)\mathcal{V}(x,y,z) - \frac{1}{2}[\nabla(\mathcal{U}(x,y,z))\mathcal{V}(x,y,z) + \nabla(\mathcal{V}(x,y,z))\mathcal{U}(x,y,z)]dydx\Bigg|$$
$$\leq \frac{1}{16}(q_1-p_1)^{\eta_1}(q_2-p_2)^{\eta_2}(q_3-p_3)^{\eta_3}$$
$$\times\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}[|\mathcal{V}(x,y,z)|||\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}||_\infty + |\mathcal{U}(x,y,z)|||\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}||_\infty]dzdydx,$$

where ∇,Υ *are as given in* (16) *and* (20).

Remark 19. If we choose $\delta = 1$, Theorem 16 leads to a result in Ref. 42.

Theorem 20. *Consider two continuous functions* $\mathcal{U},\mathcal{V} : [p_1,q_1]\times[p_2,q_2]\times[p_3,q_3] \to \mathbb{R}$ *defined on* $[p_1,q_1]\times[p_2,q_2]$ *and* $\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi\tau^\eta\partial_\Phi\zeta^\eta\partial_\Phi r^\eta}$, $\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi\tau^\eta\partial_\Phi\zeta^\eta\partial_\Phi r^\eta}$ *existing as continuous and bounded on* $[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ *and* $\eta = (\eta_1,\eta_2,\eta_3)$. *Then,*

$$\Bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}[\mathcal{U}(x,y,z)\mathcal{V}(x,y,z) - [\nabla(\mathcal{U}(x,y,z))\mathcal{V}(x,y,z) \times\nabla(\mathcal{V}(x,y,z))\mathcal{U}(x,y,z) - \nabla(\mathcal{U}(x,y,z))\nabla(\mathcal{V}(x,y,z))]]\Bigg|dzdydx$$
$$\leq \frac{1}{64}[e^{[\frac{\delta}{\delta-1}(\Phi(q_1)-\Phi(p_1))]}e^{[\frac{\delta}{\delta-1}(\Phi(q_3)-\Phi(p_3))]}$$
$$\times e^{[\frac{\delta}{\delta-1}(\Phi(q_2)-\Phi(p_2))]}(\Phi(q_1)-\Phi(p_1))^{\eta_1}$$
$$\times(\Phi(q_2)-\Phi(p_2))^{\eta_2}(\Phi(q_3)-\Phi(p_3))^{\eta_3}]^2$$
$$\times||{}^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{U}||_\infty||{}^{\Phi}\mathcal{D}_{\vartheta}^{\eta,\delta}\mathcal{V}||_\infty \tag{41}$$

for $(r,\zeta,\tau)\in[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ *and* ∇,Υ *are given in* (16) *and* (20).

Proof. Taking product side by side of Eqs. (36) and (37), we have

$$\mathcal{U}(x,y,z)\mathcal{V}(x,y,z) - [\mathcal{U}(x,y,z)\nabla(\mathcal{V}(x,y,z)) + \mathcal{V}(x,y,z)\nabla(\mathcal{U}(x,y,z)) - \nabla(\mathcal{U}(x,y,z))\nabla(\mathcal{V}(x,y,z))]$$

$$\leq \frac{1}{64}\Upsilon\left(\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi z^\eta \partial_\Phi y^\eta \partial_\Phi x^\eta}(x,y,z)\right) \times \Upsilon\left(\frac{\partial^{3\eta}\mathcal{V}}{\partial_\Phi z^\eta \partial_\Phi y^\eta \partial_\Phi x^\eta}(x,y,z)\right). \tag{42}$$

Integrating both sides (42) over $[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ and considering the modulus property, we get

$$\begin{aligned}&\bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}[\mathcal{U}(x,y,z)\mathcal{V}(x,y,z)\\&\quad-[\mathcal{U}(x,y,z)\nabla(\mathcal{V}(x,y,z))\\&\quad+\mathcal{V}(x,y,z)\nabla(\mathcal{U}(x,y,z))\\&\quad-\nabla(\mathcal{U}(x,y,z))\nabla(\mathcal{V}(x,y,z))]]\bigg|dzdydx\\&\leq\frac{1}{64}\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}\left|\Upsilon\left(\frac{\partial^{3\eta}\mathcal{U}}{\partial_\Phi z^\eta\partial_\Phi y^\eta\partial_\Phi x^\eta}(x,y,z)\right)\right|\\&\quad\times\left|\Upsilon\left(\frac{\partial^{3\eta}\mathcal{V}}{\partial_\Phi z^\eta\partial_\Phi y^\eta\partial_\Phi x^\eta}(x,y,z)\right)\right|dzdydx.\end{aligned} \tag{43}$$

Substituting (39) and (40) in (43), we get the required inequality (44). □

Some remarkable cases of Theorem 20 are discussed as follows:

(I) If we choose $\Phi(u)=u$, then, we get a new result for generalized proportional fractional operators.

Corollary 21. *Consider two continuous functions* $\mathcal{U},\mathcal{V}:[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]\to\mathbb{R}$ *defined on* $[p_1,q_1]\times[p_2,q_2]$ *and* $\frac{\partial^{3\eta}\mathcal{U}}{\partial\tau^\eta\partial\zeta^\eta\partial r^\eta}$, $\frac{\partial^{3\eta}\mathcal{U}}{\partial\tau^\eta\partial\zeta^\eta\partial r^\eta}$ *existing as continuous and bounded on* $[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ *and* $\eta=(\eta_1,\eta_2,\eta_3)$. *Then,*

$$\begin{aligned}&\bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}[\mathcal{U}(x,y,z)\mathcal{V}(x,y,z)\\&\quad-[\nabla(\mathcal{U}(x,y,z))\mathcal{V}(x,y,z)\\&\quad\times\nabla(\mathcal{V}(x,y,z))\mathcal{U}(x,y,z)\\&\quad-\nabla(\mathcal{U}(x,y,z))\nabla(\mathcal{V}(x,y,z))]]\bigg|dzdydx\\&\leq\frac{1}{64}[e^{[\frac{\delta}{\delta-1}(q_1-p_1)]}e^{[\frac{\delta}{\delta-1}(q_3-p_3)]}e^{[\frac{\delta}{\delta-1}(q_2-p_2)]}\\&\quad\times(q_1-p_1)^{\eta_1}(q_2-p_2)^{\eta_2}(q_3-p_3)^{\eta_3}]^2\\&\quad\times\|\mathcal{D}_\vartheta^{\eta,\delta}\mathcal{U}\|_\infty\|\mathcal{D}_\vartheta^{\eta,\delta}\mathcal{V}\|_\infty\end{aligned}$$

for $(r,\zeta,\tau)\in[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ *and* ∇,Υ *are given in* (16) *and* (20).

(II) If we choose $\Phi(u)=u$ along with $\delta=1$, then we get a new result for Riemann–Liouville fractional integral operator.

Corollary 22. *Consider two continuous functions* $\mathcal{U},\mathcal{V}:[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]\to\mathbb{R}$ *defined on* $[p_1,q_1]\times[p_2,q_2]$ *and* $\frac{\partial^{3\eta}\mathcal{U}}{\partial\tau^\eta\partial\zeta^\eta\partial r^\eta}$, $\frac{\partial^{3\eta}\mathcal{U}}{\partial\tau^\eta\partial\zeta^\eta\partial r^\eta}$ *existing as continuous and bounded on* $[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ *and* $\eta=(\eta_1,\eta_2,\eta_3)$. *Then,*

$$\begin{aligned}&\bigg|\int_{p_1}^{q_1}\int_{p_2}^{q_2}\int_{p_3}^{q_3}[\mathcal{U}(x,y,z)\mathcal{V}(x,y,z)\\&\quad-[\nabla(\mathcal{U}(x,y,z))\mathcal{V}(x,y,z)\\&\quad\times\nabla(\mathcal{V}(x,y,z))\mathcal{U}(x,y,z)\\&\quad-\nabla(\mathcal{U}(x,y,z))\nabla(\mathcal{V}(x,y,z))]]\bigg|dzdydx\\&\leq\frac{1}{64}[(q_1-p_1)^{\eta_1}(q_2-p_2)^{\eta_2}(q_3-p_3)^{\eta_3}]^2\\&\quad\times\|\mathcal{D}_\vartheta^{\eta,\delta}\mathcal{U}\|_\infty\|\mathcal{D}_\vartheta^{\eta,\delta}\mathcal{V}\|_\infty\end{aligned}$$

for $(r,\zeta,\tau)\in[p_1,q_1]\times[p_2,q_2]\times[p_3,q_3]$ *and* ∇,Υ *are given in* (16) *and* (20).

Remark 23. If we choose $\delta=1$, then Theorem 20 leads to a result in Ref. 42.

5. CONCLUSION

The main aim in this paper is to determine Čebyšev functionals for two- and three-variable schemes within the generalized proportional fractional integral operator, which is quite useful in establishing the solution of nonlinear-differentiable equations in fractional calculus. Within this framework, we have derived several novel generalizations which are the refinements of the results derived by Ref. 42. Additionally, the proposed operator is the generalization of several existing operators such as generalized Riemann–Liouville, Riemann–Liouville, generalized proportional fractional, Hadamard and conformable fractional integral operators, but they are unified when Remark 7 and the proportionality index $\delta=1$ are taken into consideration. For the sake of obtaining a better understanding of the method, we have discussed the earlier results proposed by Rashid *et al.*[12] and Zhou *et al.*[43] The outcome shows that the proposed plans are extremely important and computationally appealing to deal with comparable sorts of differential equations. As a future research course and direction related to this paper, the new techniques obtained in this paper can be expanded to attain analytical solutions of the fractional Schrödinger equations and in image processing introduced in the works investigated currently connected with high-dimensional fractional equations.[20–23] All in all, the results in our study can be utilized to serve as efficient and robust means to investigate the specific classes of integrodifferential equations.

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REFERENCES

1. T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.* **279** (2017) 57–66.
2. T. Abdeljawad and D. Baleanu, Monotonicity results for fractional difference operators with discrete exponential kernels, *Adv. Differ. Equ.* **2017** (2017) 78.
3. T. Abdeljawad and D. Baleanu, On fractional derivatives with exponential kernel and their discrete versions, *Rep. Math. Phys.* **2017** (2017) 80.
4. D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods* (World Scientific, Singapore, 2012).
5. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley-Interscience, John Wiley and Sons, New York, NY, USA, 1993).
6. S. G. Samko and A. A. Kilbas, Normal densities, *Linear Algebra Appl.* **396** (2005) 317–328.
7. V. Kiryakova, *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics Series, Vol. 301 (Longman Scientific and Technical, Harlow, UK, 1994).
8. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models* (World Scientific, Singapore; Hackensack, NJ, USA, 2010).
9. R. Almeida, A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.* **44** (2017) 460–481.
10. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies (Elsevier, New York, NY, USA, 2006).
11. F. Jarad, T. Abdeljawad and J. Alzabut, Generalized fractional derivatives generated by a class of local proportional derivatives, *Eur. Phys. J. Spec. Top.* **226** (2017) 3457–3471.
12. S. Rashid, F. Jarad, M. A. Noor, H. Kalsoom and Y.-M. Chu, Inequalities by means of generalized proportional fractional integral operators with respect to another function, *Mathematics* **7**(12) (2020) 1225.
13. Z. Khan, S. Rashid, R. Ashraf, D. Baleanu and Y.-M. Chu, Generalized trapezium-type inequalities in the settings of fractal sets for functions having generalized convexity property, *Adv. Differ. Equ.* **2020** (2020) 657.
14. S.-B. Chen, S. Rashid, Z. Hammouch, M. A. Noor, R. Ashraf and Y.-M. Chu, Integral inequalities via Raina's fractional integrals operator with respect to a monotone function, *Adv. Differ. Equ.* **2020** (2020) 647.
15. S. Rashid, R. Ashraf, K. S. Nisar and T. Abdeljawad, Estimation of integral inequalities using the generalized fractional derivative operator in the Hilfer sense, *J. Math.* **2020** (2020) 1626091, doi:10.1155/2020/1626091.
16. S. Rashid, H. Ahmad, A. Khalid and Y.-M. Chu, On discrete fractional integral inequalities for a class of functions, *Complexity* **2020** (2020) 845867, doi:10.1155/2020/8845867.
17. T. Abdeljawad, S. Rashid, Z. Hammouch and Y.-M. Chu, Some new Simpson-type inequalities for generalized p-convex function on fractal sets with applications, *Adv. Differ. Equ.* **2020** (2020) 496.
18. T. Abdeljawad, S. Rashid, A. A. El-Deeb, Z. Hammouch and Y.-M. Chu, Certain new weighted estimates proposing generalized proportional fractional operator in another sense, *Adv. Differ. Equ.* **2020** (2020) 463.
19. S.-B. Chen, S. Rashid, M. A. Noor, R. Ashraf and Y.-M. Chu, A new approach on fractional calculus and probability density function, *AIMS Math.* **5**(6) (2020) 7041–7054.
20. D. Zhang, Y. Zhang, Z. Zhang, N. Ahmed, Y. Zhang, F. Li, M. R. Belic and M. Xiao, Unveiling the link between fractional Schrödinger equation and light propagation in honeycomb lattice, *Ann. Phys.* **529** (2017) 1700149.
21. Y. Zhang, R. Wang, H. Zhong, J. Zhang, M. R. Belic and Y. Zhang, Resonant mode conversions and Rabi oscillations in a fractional Schrödinger equation, *Opt. Express* **25** (2017) 32401.
22. Y. Zhang, H. Zhong, M. R. Belic, Y. Zhu, W. P. Zhong, Y. Zhang, D. N. Christodoulides and M. Xiao, PT symmetry in a fractional Schrödinger equation, *Laser Photonics Rev.* **10** (2017) 526–531.

23. Y. Zhang, X. Xing Liu, M. R. Belic, W. Zhong, Y. P. Zhang and M. Xiao, Propagation dynamics of a light beam in a fractional Schrödinger equation, *Phys. Rev. Lett.* **115** (2015) 180403.
24. S. Rashid, D. Baleanu and Y.-M. Chu, Some new extensions for fractional integral operator having exponential in the kernel and their applications in physical systems, *Open Phys.* **18**(1) (2020) 478–491.
25. S.-S. Zhou, S. Rashid, M. A. Noor, K. I. Noor, F. Safdar and Y.-M. Chu, New Hermite–Hadamard type inequalities for exponentially convex functions and applications, *AIMS Math.* **5**(6) (2020) 6874–6901, doi:10.3934/math.2020441.
26. H. G. Jile, S. Rashid, M. A. Noor, A. Suhail and Y.-M. Chu, Some unified bounds for exponentially *tgs*-convex functions governed by conformable fractional operators, *AIMS Math.* **5**(6) (2020) 6108–6123, doi:10.3934/math.2020392.
27. T. Abdeljawad, S. Rashid, Z. Hammouch and Y.-M. Chu, Some new local fractional inequalities associated with generalized (s, m)-convex functions and applications, *Adv. Differ. Equ.* **2020** (2020) 406.
28. S. Rashid, Z. Hammouch, R. Ashraf, D. Baleanu and K. S. Nisar, New quantum estimates in the setting of fractional calculus theory, *Adv. Differ. Equ.* **2020** (2020) 383.
29. J.-M. Shen, S. Rashid, M. A. Noor, R. Ashraf and Y.-M. Chu, Certain novel estimates within fractional calculus theory on time scales, *AIMS Math.* **5**(6) (2020) 6073–6086, doi:10.3934/math.2020390.
30. L. Xu, Y.-M. Chu, S. Rashid, A. A. El-Deeb and K. S. Nisar, On new unified bounds for a family of functions via fractional q-calculus theory, *J. Funct. Spaces* **2020** (2020) 4984612, doi:10.1155/2020/4984612.
31. S. Rashid, A. Khalid, G. Rahman, K. S. Nisar and Y.-M. Chu, On new modifications governed by quantum Hahn's integral operator pertaining to fractional calculus, *J. Funct. Spaces* **2020** (2020) 8262860, doi:10.1155/2020/8262860.
32. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach, London, UK, 1993).
33. N. Minculete and L. Ciurdariu, A generalized form of Gruss type inequality and other integral inequalities, *J. Inequal. Appl.* **2014** (2015) 119.
34. P. L. Chebyshev, Sur les expressions approximatives des integrales definies par les autres prises entre les memes limites, *Proc. Math. Soc. Charkov* **2** (1882) 93–98.
35. S. K. Nisar, G. Rahman and K. Mehrez, Chebyshev type inequalities via generalized fractional conformable integrals, *J. Inequal. Appl.* **2019** (2019) 245.
36. E. Set, M. E. Özdemir and S. Demirbas, Chebyshev type inequalities involving extended generalized fractional integral operators, *AIMS Math.* **5**(4) (2020) 3573–3583, doi:10.3934/math.2020232.
37. S. Rashid, F. Jarad, H. Kalsoom and Y.-M. Chu, On Pólya–Szegö and Čebyšev type inequalities via generalized k-fractional integrals, *Adv. Differ. Equ.* **2020** (2020) 125, doi:10.1186/s13662-020-02583-3.
38. E. Deniz, A. O. Akdemir and E. Yüksel, New extensions of Čebyšev–Pólya–Szegö type inequalities via conformable integrals, *AIMS Math.* **5**(2) (2020) 956–965, doi:10.3934/math.2020066.
39. F. Jarad, M. A. Alqudah and T. Abdeljawad, On more general forms of proportional fractional operators, *Open Math.* **18**(1) (2020) 167–176, doi:10.1515/math-2020-0014.
40. G. Rahman, T. Abdeljawad, F. Jarad, A. Khan and K. S. Nisar, Certain inequalities via generalized proportional Hadamard fractional integral operators, *Adv. Differ. Equ.* **2019** (2019) 454.
41. U.-N. Katugampola, Approach to a generalized fractional integral, *Appl. Math. Comput.* **218** (2011) 860–866.
42. D. B. Pachpatte, On some Ψ-Caputo fractional Čebyšev like inequalities for functions of two and three variables, *AIMS Math.* **5**(3) (2020) 2244–2260, doi:10.3934/math.2020148.
43. S.-S. Zhou, S. Rashid, F. Jarad, H. Kalsoom and Y.-M. Chu, New estimates considering the generalized proportional Hadamard fractional integral operators, *Adv. Differ. Equ.* **2020** (2020) 275.