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Advances on the fixed point results via simulation function involving rational terms

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Abstract

In this paper, we propose two new contractions via simulation function that involves rational expression in the setting of partial b-metric space. The obtained results not only extend, but also generalize and unify the existing results in two senses: in the sense of contraction terms and in the sense of the abstract setting. We present an example to indicate the validity of the main theorem.

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1 Introduction and preliminaries

The origin of the fixed point theory goes back a century, to the pioneer work of Banach. Since the first study of Banach, researchers have been extended, improved, and generalized this very simple stated but at the same time very powerful theorem. For this purpose, the terms of the contraction inequality and the abstract structure of Banach's theorem have been investigated. In this paper, we shall combine these two trends and introduce two new type contraction via simulation functions involving rational terms in the more general setting, partial-b-metric space.

For the sake of the completeness of the manuscript, we shall recall some basic results and concepts here.

Theorem 1 ([1]) *Let* $(\mathcal{A},\delta)$ *be a complete metric space and* $\mho:\mathcal{A}\to\mathcal{A}$ *be a mapping. If there exist* $\Bbbk_1,\Bbbk_2\in[0,1)$, *with* $\kappa_1+\kappa_2<1$ *such that*

$$\delta(\mho v,\mho\omega)\leq\kappa_1\cdot\delta(\omega,\mho\omega)\frac{1+\delta(v,\mho v)}{1+\delta(v,\omega)}+\kappa_2\cdot\delta(v,\omega),\tag{1.1}$$

for all $v,\omega\in\mathcal{A}$, *then* $\mho$ *has a unique fixed point* $\mathsf{u}\in\mathcal{A}$ *and the sequence* $\{\mho^n x\}$ *converges to the fixed point* u *for all* $x\in\mathcal{A}$.

Theorem 2 ([2]) *Let $(\mathcal{A},\delta)$ be a complete metric space and $\mathcal{O}:\mathcal{A}\to\mathcal{A}$ be a continuous mapping. If there exist $\kappa_1,\kappa_2\in[0,1)$, with $\kappa_1+\kappa_2<1$ such that*

$$\delta(\mathcal{O}v,\mathcal{O}\omega)\le\kappa_1\cdot\frac{\delta(v,\mathcal{O}v)\delta(\omega,\mathcal{O}\omega)}{\delta(v,\omega)}+\kappa_2\cdot\delta(v,\omega), \tag{1.2}$$

for all distinct $v,\omega\in\mathcal{A}$, then $\mathcal{O}$ possesses a unique fixed point in $\mathcal{A}$.

We mention that over the last few years many interesting and different generalizations for rational contractions have been provided; see, for example [3–8].

Let Γ be the set of all non-decreasing and continuous functions $\psi:[0,+\infty)\to[0,+\infty)$. such that $\psi(0)=0$.

Definition 1 ([9]) A function $\eta:\mathbb{R}_0^+\times\mathbb{R}_0^+\to\mathbb{R}$ is a *ψ-simulation function* if there exists $\psi\in\Gamma$ such that the following conditions hold:

(η_1) $\eta(\mathsf{r},\mathsf{t})<\psi(\mathsf{t})-\psi(\mathsf{r})$ for all $\mathsf{r},\mathsf{t}\in\mathbb{R}^+$;

(η_2) if $\{\mathsf{r}_n\},\{\mathsf{t}_n\}$ are two sequences in $[0,+\infty)$ such that $\lim_{n\to+\infty}\mathsf{r}_n=\lim_{n\to+\infty}\mathsf{t}_n>0$, then

$$\limsup_{n\to+\infty}\eta(\mathsf{r}_n,\mathsf{t}_n)<0. \tag{1.3}$$

We will denote by $\mathcal{Z}_\psi$ the family of all ψ-simulation functions; see e.g. [10–22]. It is clear, due to the axiom (η_1), that

$$\sigma(\mathsf{r},\mathsf{r})<0\quad\text{for all }\mathsf{r}>0. \tag{1.4}$$

Definition 2 ([23]) On a non-empty set $\mathcal{A}$, a function $\rho:\mathcal{A}\times\mathcal{A}\to\mathbb{R}_0^+$ is a *partial metric* if the following conditions:

(ρ_1) $v=\omega$ iff $\rho(v,v)=\rho(v,\omega)=\rho(\omega,\omega)$;

(ρ_2) $\rho(v,v)\le\rho(v,\omega)$;

(ρ_3) $\rho(v,\omega)=\rho(\omega,v)$;

(ρ_4) $\rho(v,\omega)\le\rho(v,\mathsf{z})+\rho(\mathsf{z},\omega)-\rho(\mathsf{z},\mathsf{z})$;

hold for all $v,\omega,\mathsf{z}\in\mathcal{A}$.

The pair $(\mathcal{A},\rho)$ is called a partial-metric space.

Every partial metric ρ on $\mathcal{A}$ generates a T_0 topology on $\mathcal{A}$, that has a base of the set of all open balls $\mathsf{B}_\rho(v)$, where an open ball for a partial metric ρ on $\mathcal{A}$ is defined [23] as

$$\mathsf{B}_\rho^{\mathsf{e}}(v)=\bigl\{\omega\in\mathcal{A}:\rho(v,\omega)<\rho(v,v)+\mathsf{e}\bigr\},$$

for each $v\in\mathcal{A}$ and $\mathsf{e}>0$.

If $(\mathcal{A},\rho)$ is a partial-metric space and $\{v_m\}$ a sequence in $\mathcal{A}$, then:

- $\{v_m\}$ is convergent to a limit $\mathsf{u}\in\mathcal{A}$, if $\lim_{m\to+\infty}\rho(v_m,\mathsf{u})=\rho(\mathsf{u},\mathsf{u})$;
- $\{v_m\}$ is a Cauchy sequence if $\lim_{m,q\to+\infty}\rho(v_m,v_q)$ exists and is finite.

Moreover, we say that the partial-metric space $(\mathcal{A},\rho)$ is complete if every Cauchy sequence $\{v_m\}$ in $\mathcal{A}$ converges to a point $\mathsf{u}\in\mathcal{A}$, that is,

$$\lim_{m\to+\infty}\rho(v_m,\mathsf{u})=\rho(\mathsf{u},\mathsf{u})=\lim_{m,q\to+\infty}\rho(v_m,v_q).$$

Remark 1 The limit in a partial metric space may not be unique. For a sequence $\{v_m\}$ on $(\mathcal{A},\rho)$, we denote by $\mathcal{L}(\{v_m\})$ the set of limit points (if there exist any),

$$\mathcal{L}(\{v_m\}) = \Bigl\{\mathsf{u}\in\mathcal{A} : \lim_{m\to+\infty}\rho(v_m,\mathsf{u}) = \rho(\mathsf{u},\mathsf{u})\Bigr\}.$$

We recall some results in the context of partial-metric spaces, necessary in our following considerations.

Lemma 1 *Let $(\mathcal{A},\rho)$ be a partial-metric space and $\{v_m\}$ be a sequence in $\mathcal{A}$ such that $\lim_{m\to+\infty}\rho(v_m,v_{m+1}) = 0$. If $\lim_{m,q\to+\infty}\rho(v_m,v_q)\neq 0$, then there exist $\mathsf{e}>0$ and subsequences $\{v_{m_l}\}$, $\{v_{q_l}\}$ of $\{v_m\}$ such that*

$$\begin{aligned}\lim_{l\to+\infty}\rho(v_{m_l},v_{q_l}) = \lim_{l\to+\infty}\rho(v_{m_l},v_{q_l+1}) = \lim_{l\to+\infty}\rho(v_{m_l+1},v_{q_l})\\ = \lim_{l\to+\infty}\rho(v_{m_l+1},v_{q_l+1}) = \mathsf{e}.\end{aligned}\tag{1.5}$$

Lemma 2 ([24]) *Let $\{v_m\}$ be a Cauchy sequence on a complete partial-metric space $(\mathcal{A},\rho)$. If there exists $\chi\in\mathcal{L}(\{v_m\})$ with $\rho(\chi,\chi)=0$, then $\chi\in\mathcal{L}(\{v_{m_l}\})$, for every subsequence $\{v_{m_l}\}$ of $\{v_m\}$.*

Lemma 3 ([25]) *If $\{v_m\}$, $\{\omega_m\}$ are two sequences in a partial-metric space $(\mathcal{A},\rho)$ such that*

$$\lim_{m\to+\infty}\rho(v_m,\chi) = \lim_{m\to+\infty}\rho(v_m,v_m) = \rho(\chi,\chi),$$
$$\lim_{m\to+\infty}\rho(\omega_m,y) = \lim_{m\to+\infty}\rho(\omega_m,\omega_m) = \rho(y,y),$$

then $\lim_{m\to+\infty}\rho(v_m,\omega_m) = \rho(\chi,y)$. Moreover, $\lim_{m\to+\infty}\rho(v_m,\mathsf{u}) = \rho(\chi,\mathsf{u})$, for each $\mathsf{u}\in\mathcal{A}$.

On a partial-metric space $(\mathcal{A},\rho)$, a mapping $\mathcal{O}:\mathcal{A}\to\mathcal{A}$ is *continuous* at v_0 if and only if for every $\mathsf{e}>0$, there exists $\delta>0$ such that

$$\mathcal{O}\bigl(\mathsf{B}_\rho^\delta(v_0)\bigr)\subseteq\mathsf{B}_\rho^{\mathsf{e}}\bigl(\mathcal{O}(v_0)\bigr).$$

($\mathcal{O}$ is continuous if it is continuous at every point $v\in\mathcal{A}$.)

Lemma 4 ([24]) *On a complete partial-metric space $(\mathcal{A},\rho)$, let $\mathcal{O}$ be a continuous mapping and $\{v_m\}$ be a Cauchy sequence in $\mathcal{A}$. If there exists $\chi\in\mathcal{L}(\{v_m\})$ with $\rho(\chi,\chi)=0$, then $\mathcal{O}\chi\in\mathcal{L}(\{\mathcal{O}v_m\})$.*

Definition 3 ([26]) Let $\mathcal{A}$ be a non-empty set and $\mathsf{s}\geq 1$. A function $\rho_{\mathsf{b}}:\mathcal{A}\times\mathcal{A}\to\mathbb{R}_0^+$ is a *partial b-metric* with a coefficient s if the following conditions hold for all $v,\omega,\mathsf{z}\in\mathcal{A}$

($\rho_b 1$) $v=\omega$ iff $\rho_{\mathsf{b}}(v,v)=\rho_{\mathsf{b}}(v,\omega)=\rho_{\mathsf{b}}(\omega,\omega)$;
($\rho_b 2$) $\rho_{\mathsf{b}}(v,v)\leq\rho_{\mathsf{b}}(v,\omega)$;
($\rho_b 3$) $\rho_{\mathsf{b}}(v,\omega)=\rho_{\mathsf{b}}(\omega,v)$;
($\rho_b 4$) $\rho_{\mathsf{b}}(v,\omega)\leq\mathsf{s}[\rho_{\mathsf{b}}(v,\mathsf{z})+\rho_{\mathsf{b}}(\mathsf{z},\omega)]-\rho_{\mathsf{b}}(\mathsf{z},\mathsf{z})$.

In this case, we say that $(\mathcal{A},\rho_{\mathsf{b}},\mathsf{s})$ is a *partial b-metric space.*

Example 1 ([26]) Let $\mathcal{A}$ be a non-empty set and $v,\omega\in\mathcal{A}$.

- if ρ is a partial metric on $\mathcal{A}$, then the function ρ_{b} defined as

  $$\rho_{\mathsf{b}}(v,\omega) = \bigl[\rho(v,\omega)\bigr]^{\lambda} \tag{1.6}$$

  is a partial b-metric on $\mathcal{A}$, with $\mathsf{s} = 2^{\lambda-1}$, for $\lambda > 1$.
- if b is a b-metric and ρ is a partial metric on $\mathcal{A}$, then the function

  $$\rho_{\mathsf{b}}(v,\omega) = \rho(v,\omega) + \mathsf{b}(v,\omega) \tag{1.7}$$

  is a partial b-metric on $\mathcal{A}$.

A sequence $\{v_m\}$ in a partial b-metric space $(\mathcal{A},\rho_{\mathsf{b}},\mathsf{s})$ is said to be ρ_{b}-*convergent* to a point $\mathsf{u} \in \mathcal{A}$ if

$$\lim_{m\to+\infty} \rho_{\mathsf{b}}(v_m,\mathsf{u}) = \rho_{\mathsf{b}}(\mathsf{u},\mathsf{u}). \tag{1.8}$$

If the limit $\lim_{m,q\to+\infty} \rho_{\mathsf{b}}(v_m, v_q)$ exists and it is finite, the sequence $\{v_m\}$ is said to be ρ_{b}-*Cauchy*. Moreover, if every ρ_{b}-Cauchy sequence in $\mathcal{A}$ is ρ_{b}-convergent to $\mathsf{u} \in \mathcal{A}$, that is

$$\lim_{m,q\to+\infty} \rho_{\mathsf{b}}(v_m, v_q) = \lim_{m\to+\infty} \rho_{\mathsf{b}}(v_m,\mathsf{u}) = \rho_{\mathsf{b}}(\mathsf{u},\mathsf{u}), \tag{1.9}$$

we say that the partial b-metric space $(\mathcal{A},\rho_{\mathsf{b}},\mathsf{s})$ is ρ_{b}-complete.

Remark 2 In [27] it is proved that a partial b-metric induces a b-metric, say δ_{b}, with

$$\delta_{\mathsf{b}}(v,\omega) = 2\rho_{\mathsf{b}}(v,\omega) - \rho_{\mathsf{b}}(v,v) - \rho_{\mathsf{b}}(\omega,\omega), \tag{1.10}$$

for all $v,\omega \in \mathcal{A}$.

On the other hand, in [28], the notion of 0-ρ_{b}-completeness was introduced and the relation between 0-ρ_{b}-completeness and ρ_{b}-completeness of a partial b-metric was established.

Definition 4 ([28]) A sequence $\{v_m\}$ on a partial b-metric space $(\mathcal{A},\rho_{\mathsf{b}},\mathsf{s})$ is 0-ρ_{b}-*Cauchy* if $\lim_{m,q\to+\infty} \rho_{\mathsf{b}}(v_m, v_q) = 0$. Moreover, the space $(\mathcal{A},\rho_{\mathsf{b}},\mathsf{s})$ is said to be 0-ρ_{b}-*complete* if for each 0-ρ_{b}-Cauchy sequence in $\mathcal{A}$, there is $\mathsf{u} \in \mathcal{A}$, such that

$$\lim_{m,q\to+\infty} \rho_{\mathsf{b}}(v_m, v_q) = \lim_{m\to+\infty} \rho_{\mathsf{b}}(v_m,\mathsf{u}) = \rho_{\mathsf{b}}(\mathsf{u},\mathsf{u}) = 0. \tag{1.11}$$

Lemma 5 ([28]) *If the partial b-metric space* $(\mathcal{A},\rho_{\mathsf{b}},\mathsf{s})$ *is* ρ_{b}-*complete, then it is* 0-ρ_{b}-*complete.*

Lemma 6 ([29]) *Let* $(\mathcal{A},\rho_{\mathsf{b}},\mathsf{s})$ *be a partial b-metric space. If* $\rho_{\mathsf{b}}(v,\omega) = 0$ *then* $v = \omega$ *and* $\rho_{\mathsf{b}}(v,\omega) > 0$ *for all* $v \neq \omega$.

The next result is important in our future considerations.

Lemma 7 ([30]) *Let $(\mathcal{A}, \rho_{\mathsf{b}}, \mathsf{s} \geq 1)$ be a partial b-metric space, $O : \mathcal{A} \to \mathcal{A}$ a mapping and a number $\kappa \in [0,1)$. If $\{v_m\}$ is a sequence in $\mathcal{A}$, where $v_m = Ov_{m-1}$ and*

$$\rho_{\mathsf{b}}(v_m, v_{m+1}) \leq \kappa \rho_{\mathsf{b}}(v_{m-1}, v_m), \tag{1.12}$$

for each $m \in \mathbb{N}$, then the sequence $\{v_m\}$ is 0-ρ_{b}-Cauchy.

2 Main results

We start with the definition of simulation function for partial b-metric spaces.

Definition 5 Let $(\mathcal{A}, \rho_{\mathsf{b}}, \mathsf{s} \geq 1)$ be a partial b-metric space. A *b-ψ-simulation function* is a function $\eta_{\mathsf{b}} : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ satisfying:

(η_{b1}) $\eta_{\mathsf{b}}(\mathsf{r},\mathsf{t}) < \psi(\mathsf{t}) - \psi(\mathsf{r})$ for all $\mathsf{r},\mathsf{t} \in \mathbb{R}^+$;

(η_{b2}) if $\{\mathsf{r}_n\}, \{\mathsf{t}_n\}$ are two sequences in $[0,+\infty)$, such that for $p > 0$

$$\limsup_{n\to+\infty} \mathsf{t}_n = \mathsf{s}^p \lim_{n\to+\infty} \mathsf{r}_n > 0, \tag{2.1}$$

then

$$\limsup_{n\to+\infty} \eta_{\mathsf{b}}\big(\mathsf{s}^p \mathsf{r}_n, \mathsf{t}_n\big) < 0. \tag{2.2}$$

We shall denote by $\mathcal{Z}_{\psi_b}$ the family of all b-ψ-simulation functions.

Example 2 Let $\psi \in \Gamma$ and $\gamma : [0,+\infty) \to [0,+\infty)$ be a function such that $\limsup_{\mathsf{t}\to\mathsf{t}_0} \gamma(\mathsf{t}) < 1$ for every $\mathsf{t}_0 > 0$ and $\phi(\mathsf{t}) = 0$ if and only if $\mathsf{t} = 0$. Then $\eta_{\mathsf{b}}(\mathsf{r},\mathsf{t}) = \gamma(\mathsf{t})\psi(\mathsf{t}) - \psi(\mathsf{r})$, for $\mathsf{r},\mathsf{t} \geq 0$ is a b-ψ-simulation function.

Example 3 Let $\psi \in \Gamma$ and $\phi : [0,+\infty) \to [0,+\infty)$ be a function such that $\lim_{\mathsf{t}\to\mathsf{t}_0} \phi(\mathsf{t}) > 0$ for every $\mathsf{t}_0 > 0$ and $\phi(\mathsf{t}) = 0$ if and only if $\mathsf{t} = 0$. Then $\eta_{\mathsf{b}}(\mathsf{r},\mathsf{t}) = \psi(\mathsf{t}) - \phi(\mathsf{t}) - \psi(\mathsf{r})$, for $\mathsf{r},\mathsf{t} \geq 0$ is a b-ψ-simulation function.

Obviously, (η_{b1}) holds. Now, considering two sequences $\{\mathsf{r}_n\}$ and $\{\mathsf{t}_n\}$ in $(0,+\infty)$ such that (2.1) holds, we have

$$\lim_{n\to+\infty} \eta_{\mathsf{b}}\big(\mathsf{s}^p \mathsf{r}_n, \mathsf{t}_n\big) = \lim_{n\to+\infty} \psi(\mathsf{t}_n) - \phi(\mathsf{t}_n) - \psi\big(\mathsf{s}^p \mathsf{r}_n\big) \leq -\phi(\mathsf{t}_n) < 0.$$

Thus, also (η_{b2}) holds, that is $\eta_{\mathsf{b}} \in \mathcal{Z}_{\psi_b}$.

Definition 6 Let $(\mathcal{A}, \rho_{\mathsf{b}}, \mathsf{s} \geq 1)$ be a partial b-metric space. A mapping $O : \mathcal{A} \to \mathcal{A}$ is called (η_{b})-*rational contraction of type A* if there exists a function $\eta_{\mathsf{b}} \in \mathcal{Z}_{\psi_b}$ such that

$$\begin{aligned} &\frac{1}{2\mathsf{s}} \min\big\{\rho_{\mathsf{b}}(v, Ov), \rho_{\mathsf{b}}(\omega, O\omega)\big\} \leq \rho_{\mathsf{b}}(v,\omega), \quad \text{which implies} \\ &\eta\big(\mathsf{s}^p \rho_{\mathsf{b}}(Ov, O\omega), \mathcal{D}_A(v,\omega)\big) \geq 0, \end{aligned} \tag{2.3}$$

for every $v, \omega \in \mathcal{A}$, where $\mathcal{D}_A$ is defined as

$$\mathcal{D}_A(v,\omega) = \max\left\{\delta(v,\omega), \delta(v, Ov), \delta(\omega, O\omega), \frac{\delta(\omega, O\omega)[1 + \delta(v, Ov)]}{1 + \delta(v,\omega)}\right\}. \tag{2.4}$$

With the purpose to simplify the demonstrations, we prefer in the sequel, to discuss separately, the cases

Theorem 3 *Let* $(\mathcal{A}, \rho_{\mathsf{b}}, \mathsf{s} > 1)$ *be a* ρ_{b}*-complete partial b-metric space and* $\mathcal{O} : \mathcal{A} \to \mathcal{A}$ *be a* (η_{b})*-rational contraction of type* A. *Then* $\mathcal{O}$ *admits exactly one fixed point.*

Proof Let $\upsilon_0 \in \mathcal{A}$ be an arbitrary but fixed point and $\{\upsilon_m\}$ be the sequence in $\mathcal{A}$ defined as follows:

$$\upsilon_m = \mathcal{O}\upsilon_{m-1}, \quad \forall \in \mathbb{N}. \tag{2.5}$$

Thus, we can assume that $\upsilon_{m-1} \neq \upsilon_m$ for every $m \in \mathbb{N}$. Indeed, if we suppose that there exists $m_0 \in \mathbb{N}$ such that $\upsilon_{m_0-1} = \upsilon_{m_0}$. Taking into account (2.5) we get $\upsilon_{m_0-1} = \mathcal{O}\upsilon_{m_0-1}$, that is, υ_{m_0-1} is a fixed point of $\mathcal{O}$. Therefore, substituting $\upsilon = \upsilon_{m-1}$ and $\omega = \upsilon_m$ in (2.4), we have

$$\begin{aligned}
\mathcal{D}_A(\upsilon_{m-1}, \upsilon_m) &= \max\left\{\begin{array}{c}\rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m), \rho_{\mathsf{b}}(\upsilon_{m-1}, \mathcal{O}\upsilon_{m-1}), \rho_{\mathsf{b}}(\upsilon_m, \mathcal{O}\upsilon_m), \\ \frac{\rho_{\mathsf{b}}(\upsilon_m, \mathcal{O}\upsilon_m)[1+\rho_{\mathsf{b}}(\upsilon_{m-1}, \mathcal{O}\upsilon_{m-1})]}{1+\rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m)}\end{array}\right\} \\
&= \max\left\{\begin{array}{c}\rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m), \rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m), \rho_{\mathsf{b}}(\upsilon_m, \upsilon_{m+1}), \\ \frac{\rho_{\mathsf{b}}(\upsilon_m, \upsilon_{m+1})[1+\rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m)]}{1+\rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m)}\end{array}\right\} \\
&= \max\bigl\{\rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m), \rho_{\mathsf{b}}(\upsilon_m, \upsilon_{m+1})\bigr\}.
\end{aligned}$$

Moreover, by (2.3) we get

$$\begin{aligned}
&\frac{1}{2\mathsf{s}}\min\bigl\{\rho_{\mathsf{b}}(\upsilon_{m-1}, \mathcal{O}\upsilon_{m-1}), \rho_{\mathsf{b}}(\upsilon_m, \mathcal{O}\upsilon_m)\bigr\} \\
&\quad = \frac{1}{2\mathsf{s}}\min\bigl\{\rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m), \rho_{\mathsf{b}}(\upsilon_m, \upsilon_{m+1})\bigr\} \\
&\quad \leq \rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m), \quad \text{for all } m \in \mathbb{N},
\end{aligned}$$

which implies

$$\eta_{\mathsf{b}}\bigl(\mathsf{s}^p\rho_{\mathsf{b}}(\mathcal{O}\upsilon_{m-1}, \mathcal{O}\upsilon_m), \mathcal{D}_A(\upsilon_{m-1}, \upsilon_m)\bigr) \geq 0.$$

Now, taking into account (η_{b1}), the above inequality yields

$$0 < \psi\bigl(\mathcal{D}_A(\upsilon_{m-1}, \upsilon_m)\bigr) - \psi\bigl(\mathsf{s}^p\rho_{\mathsf{b}}(\mathcal{O}\upsilon_{m-1}, \mathcal{O}\upsilon_m)\bigr),$$

or, equivalently,

$$\psi\bigl(\mathsf{s}^p\rho_{\mathsf{b}}(\upsilon_m, \upsilon_{m+1})\bigr) < \psi\bigl(\mathcal{D}_A(\upsilon_{m-1}, \upsilon_m)\bigr) = \psi\bigl(\max\bigl\{\rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m), \rho_{\mathsf{b}}(\upsilon_m, \upsilon_{m+1})\bigr\}\bigr).$$

Consequently, due to the monotony of the function ψ, we obtain

$$\mathsf{s}^p\rho_{\mathsf{b}}(\upsilon_m, \upsilon_{m+1}) < \max\bigl\{\rho_{\mathsf{b}}(\upsilon_{m-1}, \upsilon_m), \rho_{\mathsf{b}}(\upsilon_m, \upsilon_{m+1})\bigr\}. \tag{2.6}$$

If there exists $m_1 \in \mathbb{N}$ such that $\max\{\rho_{\mathsf{b}}(v_{m_1-1}, v_{m_1}), \rho_{\mathsf{b}}(v_{m_1}, v_{m_1+1})\} = \rho_{\mathsf{b}}(v_{m_1}, v_{m_1+1})$, (2.6) becomes $\mathsf{s}^p \rho_{\mathsf{b}}(v_{m_1}, v_{m_1+1}) < \rho_{\mathsf{b}}(v_{m_1}, v_{m_1+1})$, which is a contradiction (because $\mathsf{s} > 1$). Therefore, for any $m \in \mathbb{N}$ we have

$$\mathsf{s}^p \rho_{\mathsf{b}}(v_m, v_{m+1}) < \rho_{\mathsf{b}}(v_{m-1}, v_m),$$

or

$$\rho_{\mathsf{b}}(v_m, v_{m+1}) < \frac{1}{\mathsf{s}^p} \rho_{\mathsf{b}}(v_{m-1}, v_m). \tag{2.7}$$

Denoting $\frac{1}{\mathsf{s}^p}$ by κ, we have $\rho_{\mathsf{b}}(v_m, v_{m+1}) < \kappa \rho_{\mathsf{b}}(v_{m-1}, v_m)$, with $0 \le \kappa < 1$. Thus, by Lemma 7 we see that the sequence$\{v_n\}$ is a 0-ρ_{b}-Cauchy sequence on the ρ_{b}-complete partial b-metric space. Since by Lemma 5, the space is also 0-ρ_{b}-complete, it follows that there exists $\mathsf{u} \in \mathcal{A}$ such that

$$\lim_{m,q\to+\infty} \rho_{\mathsf{b}}(v_m, v_q) = \lim_{m\to+\infty} \rho_{\mathsf{b}}(v_m, \mathsf{u}) = \rho_{\mathsf{b}}(\mathsf{u}, \mathsf{u}) = 0. \tag{2.8}$$

Now, we claim that

$$\frac{1}{2\mathsf{s}} \rho_{\mathsf{b}}(v_m, v_{m+1}) \le \rho_{\mathsf{b}}(v_m, \mathsf{u}) \quad \text{or} \quad \frac{1}{2\mathsf{s}} \rho_{\mathsf{b}}(v_{m+1}, v_{m+2}) \le \rho_{\mathsf{b}}(v_{m+1}, \mathsf{u}).$$

Assuming the contrary, we can find $m_0 \in \mathbb{N}$ such that

$$\begin{aligned}
\rho_{\mathsf{b}}(v_{m_0}, v_{m_0+1}) &\le \mathsf{s}\big[\rho_{\mathsf{b}}(v_{m_0}, \mathsf{u}) + \rho_{\mathsf{b}}(\mathsf{u}, v_{m_0+1})\big] - \rho_{\mathsf{b}}(\mathsf{u}, \mathsf{u}) \\
&< \mathsf{s}\left[\frac{1}{2\mathsf{s}} \rho_{\mathsf{b}}(v_{m_0}, v_{m_0+1}) + \frac{1}{2\mathsf{s}} \rho_{\mathsf{b}}(v_{m_0+1}, v_{m_0+2})\right] \\
&= \frac{1}{2}\big[\rho_{\mathsf{b}}(v_{m_0}, v_{m_0+1}) + \rho_{\mathsf{b}}(v_{m_0+1}, v_{m_0+2})\big] \quad \big(\text{taking (2.7) into account}\big) \\
&< \rho_{\mathsf{b}}(v_{m_0}, v_{m_0+1}),
\end{aligned}$$

which is a contradiction. Thus, there exists a subsequence $\{v_{m(l)}\}$ of $\{v_m\}$ such that

$$\frac{1}{2\mathsf{s}} \min\big\{\rho_{\mathsf{b}}(v_{m(l)}, \mathcal{O}v_{m(l)}), \rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u})\big\} = \frac{1}{2\mathsf{s}} \rho_{\mathsf{b}}(v_{m(l)}, v_{m(l)+1}) \le \rho_{\mathsf{b}}(v_{m(l)}, \mathsf{u}),$$

which implies

$$\eta_{\mathsf{b}}\big(\mathsf{s}^p \rho_{\mathsf{b}}(\mathcal{O}v_{m(l)}, \mathcal{O}\mathsf{u}), \mathcal{D}_A(v_{m(l)}, \mathsf{u})\big) \ge 0,$$

where

$$\begin{aligned}
\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u}) \le \mathcal{D}_A(v_{m(l)}, \mathsf{u})) &= \max\left\{\begin{matrix}\rho_{\mathsf{b}}(v_{m(l)}, \mathsf{u}), \rho_{\mathsf{b}}(v_{m(l)}, \mathcal{O}v_{m(l)}), \rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u}), \\ \frac{\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u})[1+\rho_{\mathsf{b}}(v_{m(l)}, \mathcal{O}v_{m(l)})}{1+\rho_{\mathsf{b}}(v_{m(l)}, \mathsf{u})}\end{matrix}\right\} \\
&= \max\left\{\begin{matrix}\rho_{\mathsf{b}}(v_{m(l)}, \mathsf{u}), \rho_{\mathsf{b}}(v_{m(l)}, v_{m(l)+1}), \rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u}), \\ \frac{\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u})[1+\rho_{\mathsf{b}}(v_{m(l)}, v_{m(l)+1})]}{1+\rho_{\mathsf{b}}(v_{m(l)}, \mathsf{u})}.\end{matrix}\right\}
\end{aligned}$$

Therefore, letting $l \to +\infty$ and keeping (2.8) in mind we get

$$\lim_{l\to+\infty} \mathcal{D}_A(v_{m(l)}, \mathsf{u})) = \rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u}). \tag{2.9}$$

On one hand, without loss of generality, we assume that $v_m \neq \mathsf{u}$, for infinitely many $m \in \mathbb{N}$. Thus,

$$\eta_{\mathsf{b}}\big(\mathsf{s}^p \rho_{\mathsf{b}}(\mathcal{O}v_m, \mathcal{O}\mathsf{u}), \mathcal{D}_A(v_m, \mathsf{u})\big) \geq 0,$$

which by (η_{b1}) leads us to

$$\psi\big(\mathsf{s}^p \rho_{\mathsf{b}}(\mathcal{O}v_m, \mathcal{O}\mathsf{u})\big) < \psi\big(\mathcal{D}_A(v_m, \mathsf{u})\big).$$

Taking into account the non-decreasing property of ψ

$$\mathsf{s}^p \rho_{\mathsf{b}}(\mathcal{O}v_m, \mathcal{O}\mathsf{u}) < \mathcal{D}_A(v_m, \mathsf{u}).$$

On the other hand,

$$\begin{aligned}
\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u}) &\leq \mathsf{s}\big[\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}v_m) + \rho_{\mathsf{b}}(\mathcal{O}v_m, \mathcal{O}\mathsf{u})\big] - \rho_{\mathsf{b}}(\mathcal{O}v_m, \mathcal{O}v_m) \\
&\leq \mathsf{s}\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}v_m) + \mathsf{s}^p \rho_{\mathsf{b}}(\mathcal{O}v_m, \mathcal{O}\mathsf{u}) - \rho_{\mathsf{b}}(v_{m+1}, v_{m+1}) \\
&< \mathsf{s}\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}v_m) + \mathcal{D}_A(v_m, \mathsf{u}).
\end{aligned}$$

Letting $m \to +\infty$ in the above inequality and keeping in mind (2.8) and (2.9) we get

$$\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u}) \leq \mathsf{s}^p \lim_{m\to+\infty} \rho_{\mathsf{b}}(\mathcal{O}v_m, \mathcal{O}\mathsf{u}) < \lim_{m\to+\infty} \mathcal{D}_A(v_m, \mathsf{u}) = \rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u}).$$

Therefore, $\mathsf{s}^p \lim_{m\to+\infty} \rho_{\mathsf{b}}(\mathcal{O}v_m, \mathcal{O}\mathsf{u}) = \rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u})$. Thus, letting $\mathsf{r}_m = \rho_{\mathsf{b}}(\mathcal{O}v_m, \mathcal{O}\mathsf{u})$ and $\mathsf{t}_m = \mathcal{D}_A(v_m, \mathsf{u})$, by (η_{b2}) it follows $\limsup_{m\to+\infty} \eta_{\mathsf{b}}(\mathsf{s}^p \mathsf{r}_m, \mathsf{t}_m) < 0$, which is a contradiction. Then $\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u}) = 0 = \rho_{\mathsf{b}}(\mathsf{u}, \mathsf{u})$, that is, u is a fixed point of $\mathcal{O}$.

As a last step, we establish uniqueness of the fixed point. Indeed, if we can find another point, $\mathsf{z} \in \mathcal{A}$, $\mathsf{z} \neq \mathsf{u}$ such that $\mathsf{z} = \mathcal{O}\mathsf{z}$,

$$0 = \frac{1}{2\mathsf{s}} \min\big\{\rho_{\mathsf{b}}(\mathsf{z}, \mathcal{O}\mathsf{z}), \rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u})\big\} \leq \rho_{\mathsf{b}}(\mathsf{z}, \mathsf{u}),$$

which implies

$$\begin{aligned}
0 &\leq \eta_{\mathsf{b}}\big(\mathsf{s}^p \rho_{\mathsf{b}}(\mathcal{O}\mathsf{z}, \mathcal{O}\mathsf{u}), \mathcal{D}_A(\mathsf{z}, \mathsf{u})\big) < \psi\big(\mathcal{D}_A(\mathsf{z}, \mathsf{u})\big) - \psi\big(\mathsf{s}^p \rho_{\mathsf{b}}(\mathcal{O}\mathsf{z}, \mathcal{O}\mathsf{u})\big) \\
&= \psi\big(\rho_{\mathsf{b}}(\mathsf{z}, \mathsf{u})\big) - \psi\big(\mathsf{s}^p \rho_{\mathsf{b}}(\mathsf{z}, \mathsf{u})\big),
\end{aligned}$$

which is a contradiction. Thus, $\mathsf{u} = \mathsf{z}$. □

Example 4 Let the set $\mathcal{A} = \{10, 11, 12, 13\}$ and ρ_{b} be the partial b-metric on $\mathcal{A}$ $(\mathsf{s} = 2)$, where $\rho_{\mathsf{b}}(v, \omega) = \begin{cases} 0.000002 & \text{for } v = \omega = 13, \\ |v - \omega|^2 & \text{otherwise}. \end{cases}$ We define the mapping $\mathcal{O} : \mathcal{A} \to \mathcal{A}$, $\mathcal{O}v = \begin{cases} 10 & \text{for } v \in \{10, 11, 12\}, \\ 11 & \text{for } v = 13, \end{cases}$ and we choose $\phi \in \Gamma$, $\phi(\mathsf{t}) = \frac{\mathsf{t}}{2}$ and $\eta_{\mathsf{b}}(\mathsf{r}, \mathsf{t}) = \frac{\frac{15}{16}\mathsf{t} - \mathsf{r}}{2}$. It is easy to see that $\eta_{\mathsf{b}} \in \mathcal{Z}_{\psi_b}$ (by taking $\gamma(\mathsf{t}) = \frac{15}{16}$ in Example 2). We have

v	$\mathcal{O}v$	$\rho_b(v,\mathcal{O}v)$
10	10	0
11	10	1
12	10	4
13	11	4

and shall consider the following cases:

1. For $v,\omega \in \{10,11,12\}$, we have $\rho_b(\mathcal{O}v,\mathcal{O}\omega) = 0$, and then

   $$\frac{1}{2s}\min\bigl\{\rho_b(v,\mathcal{O}v),\rho_b(\omega,\mathcal{O}\omega)\bigr\} \le 1 \le \rho_b(v,\omega),$$

   which implies

   $$2\rho_b(\mathcal{O}v,\mathcal{O}\omega) = 0 \le \frac{15}{16}\mathcal{D}_B(v,\omega).$$

2. For $v = 10, \omega = 13$ we have $\rho_b(v,\omega) = 9$, $\rho_b(10,\mathcal{O}10 = 0$, $\rho_b(13,\mathcal{O}13) = \rho_b(13,11) = 4$, $\rho_b(\mathcal{O}10,\mathcal{O}13) = \rho_b(10,11) = 1$ and then

   $$\frac{1}{4}\min\bigl\{\rho_b(10,aT10),\rho_b(13,\mathcal{O}13)\bigr\} = 0 < 9 = \rho_b(v,\omega),$$

   which implies

   $$2\rho_b(\mathcal{O}10,\mathcal{O}13) = 2 \le \frac{135}{16} = \frac{15}{16}\cdot\rho_b(10,13).$$

3. For $v = 11, \omega = 13$ we have $\rho_b(v,\omega) = 4$, $\rho_b(11,\mathcal{O}11) = 1$, $\rho_b(13,\mathcal{O}13) = \rho_b(13,11) = 4$, $\rho_b(\mathcal{O}11,\mathcal{O}13) = \rho_b(10,11) = 1$ and then

   $$\frac{1}{4}\min\bigl\{\rho_b(11,aT11),\rho_b(13,\mathcal{O}13)\bigr\} = \frac{1}{4} < 4 = \rho_b(v,\omega),$$

   which implies

   $$2\rho_b(\mathcal{O}11,\mathcal{O}13) = 2 \le \frac{15}{4} = \frac{15}{16}\cdot\rho_b(11,13).$$

4. For $v = 12, \omega = 13$ we have $\rho_b(v,\omega) = 1$, $\rho_b(12,\mathcal{O}12) = 4$, $\rho_b(13,\mathcal{O}13) = \rho_b(13,11) = 4$, $\rho_b(\mathcal{O}12,\mathcal{O}13) = \rho_b(10,11) = 1$ and then

   $$\frac{1}{4}\min\bigl\{\rho_b(12,aT12),\rho_b(13,\mathcal{O}13)\bigr\} = 1\rho_b(v,\omega),$$

   which implies

   $$2\rho_b(\mathcal{O}12,\mathcal{O}13) = 2 \le \frac{75}{16} = \frac{15}{16}\cdot\frac{\rho_b(12,\mathcal{O}12)(1+\rho_b(13,\mathcal{O}13))}{1+\rho_b(12,13)} \le \frac{15}{16}\mathcal{D}_A(12,13).$$

   Thus, the hypothesis of Theorem 3 are satisfied and $v = 10$ is the fixed point of the mapping $\mathcal{O}$.

Definition 7 Let $(\mathcal{A}, \rho_{\mathsf{b}}, \mathsf{s} > 1)$ be a partial b-metric space. The mapping $O : \mathcal{A} \to \mathcal{A}$ is said to be a (η_{b})-*rational contraction of type B* if there exists $\eta_{\mathsf{b}} \in \mathcal{Z}_{\psi_b}$ such that

$$\begin{aligned} &\frac{1}{2\mathsf{s}} \min\bigl\{\rho_{\mathsf{b}}(v, Ov), \rho_{\mathsf{b}}(\omega, O\omega)\bigr\} \leq \rho_{\mathsf{b}}(v, \omega), \quad \text{which implies} \\ &\eta_{\mathsf{b}}\bigl(\mathsf{s}^p \rho_{\mathsf{b}}(Ov, O\omega), \mathcal{D}_B(v, \omega)\bigr) \geq 0, \end{aligned} \tag{2.10}$$

for all $v, \omega \in \mathcal{A}$, $\rho_{\mathsf{b}}(v, \omega) > 0$, where

$$\mathcal{D}_B(v, \omega) = \max \left\{ \begin{matrix} \rho_{\mathsf{b}}(v, \omega), \rho_{\mathsf{b}}(v, Ov), \rho_{\mathsf{b}}(\omega, O\omega), \frac{\rho_{\mathsf{b}}(v, O\omega) + \rho_{\mathsf{b}}(\omega, Ov)}{2\mathsf{s}}, \\ \frac{\rho_{\mathsf{b}}(\omega, O\omega)\rho_{\mathsf{b}}(v, Ov)}{\rho_{\mathsf{b}}(v, \omega)}. \end{matrix} \right\} \tag{2.11}$$

Theorem 4 *On a* ρ_{b}*-complete partial b-metric space* $(\mathcal{A}, \rho_{\mathsf{b}}, \mathsf{s} > 1)$ *any continuous* (η_{b})*-rational contraction of type B,* $O : \mathcal{A} \to \mathcal{A}$ *admits exactly one fixed point.*

Proof Let the sequence $\{v_m\}$ be defined by (2.5). Since $v_{m-1} \neq v_m$, for each $m \in \mathbb{N}$ (by similar reasoning as in the proof of Theorem 3), we have

$$\begin{aligned} \frac{1}{2\mathsf{s}} \min\bigl\{\rho_{\mathsf{b}}(v_m, Ov_m), \rho_{\mathsf{b}}(v_{m+1}, Ov_{m+1})\bigr\} &= \frac{1}{2\mathsf{s}} \min\bigl\{\rho_{\mathsf{b}}(v_m, v_{m+1}), \rho_{\mathsf{b}}(v_{m+1}, v_{m+2})\bigr\} \\ &\leq \rho_{\mathsf{b}}(v_m, v_{m+1}), \end{aligned}$$

which implies

$$\begin{aligned} 0 &\leq \eta_{\mathsf{b}}\bigl(\mathsf{s}^p \rho_{\mathsf{b}}(Ov_m, Ov_{m+1}), \mathcal{D}_B(v_m, v_{m+1})\bigr) \\ &< \psi\bigl(\mathcal{D}_B(v_m, v_{m+1})\bigr) - \psi\bigl(\mathsf{s}^p \rho_{\mathsf{b}}(Ov_m, Ov_{m+1})\bigr), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \mathcal{D}_B(v_m, v_{m+1}) &= \max \left\{ \begin{matrix} \rho_{\mathsf{b}}(v_m, v_{m+1}), \rho_{\mathsf{b}}(v_{m+1}, v_{m+2}), \frac{\rho_{\mathsf{b}}(v_m, v_{m+2}) + \rho_{\mathsf{b}}(v_{m+1}, v_{m+1})}{2\mathsf{s}}, \\ \frac{\rho_{\mathsf{b}}(v_m, v_{m+1})\rho_{\mathsf{b}}(v_{m+1}, v_{m+2})}{\rho_{\mathsf{b}}(v_m, v_{m+1})} \end{matrix} \right\} \\ &\leq \max \left\{ \begin{matrix} \rho_{\mathsf{b}}(v_m, v_{m+1}), \rho_{\mathsf{b}}(v_{m+1}, v_{m+2}), \\ \frac{\mathsf{s}[\rho_{\mathsf{b}}(v_m, v_{m+1}) + \rho_{\mathsf{b}}(v_{m+1}, v_{m+2})] - \rho_{\mathsf{b}}(v_{m+1}, v_{m+1}) + \rho_{\mathsf{b}}(v_{m+1}, v_{m+1})}{2\mathsf{s}} \end{matrix} \right\} \\ &\leq \max\bigl\{\rho_{\mathsf{b}}(v_m, v_{m+1}), \rho_{\mathsf{b}}(v_{m+1}, v_{m+2})\bigr\}. \end{aligned}$$

Therefore

$$\psi\bigl(\mathsf{s}^p \rho_{\mathsf{b}}(v_{m+1}, v_{m+2})\bigr) < \psi\bigl(\mathcal{D}_B(v_m, v_{m+1})\bigr) \leq \psi\bigl(\max\bigl\{\rho_{\mathsf{b}}(v_m, v_{m+1}), \rho_{\mathsf{b}}(v_{m+1}, v_{m+2})\bigr\}\bigr)$$

and since the function ψ is non-decreasing, we get, for any $m \in \mathbb{N}$,

$$\mathsf{s}^p \rho_{\mathsf{b}}(v_{m+1}, v_{m+2}) < \max\bigl\{\rho_{\mathsf{b}}(v_m, v_{m+1}), \rho_{\mathsf{b}}(v_{m+1}, v_{m+2})\bigr\}.$$

Moreover, if $\max\{\rho_{\mathsf{b}}(v_m, v_{m+1}), \rho_{\mathsf{b}}(v_{m+1}, v_{m+2})\} = \rho_{\mathsf{b}}(v_{m+1}, v_{m+2})$ we get a contradiction, and then it follows that

$$\rho_{\mathsf{b}}(v_{m+1}, v_{m+2}) < \frac{1}{\mathsf{s}^p} \rho_{\mathsf{b}}(v_m, v_{m+1})$$

and by Lemma (7), we conclude that $\{v_m\}$ is a 0-ρ_{b}-Cauchy on a ρ_{b}-complete b-partial-metric space, and there exists $\mathsf{u} \in \mathcal{A}$ such that $\lim_{m\to+\infty} v_m = \mathsf{u}$.

Taking into account the continuity of the mapping $\mathcal{O}$, we have

$$\mathsf{u} = \lim_{m\to+\infty} v_{m+1} = \lim_{m\to+\infty} \mathcal{O}\Big(\lim_{m\to+\infty} v_m\Big) = \mathcal{O}\mathsf{u},$$

that is, u is a fixed point of the mapping $\mathcal{O}$.

We claim that the fixed point of $\mathcal{O}$ is unique. Let $\mathsf{u}, \mathsf{z} \in \mathcal{A}$ be two different fixed point of $\mathcal{O}$. Then

$$0 = \frac{1}{2\mathsf{s}} \min\big\{\rho_{\mathsf{b}}(\mathsf{u}, \mathcal{O}\mathsf{u}), \rho_{\mathsf{b}}(\mathsf{z}, \mathcal{O}\mathsf{z})\big\} < \rho_{\mathsf{b}}(\mathsf{u}, \mathsf{z}),$$

which implies

$$\begin{aligned} 0 &\leq \eta_{\mathsf{b}}\big(\mathsf{s}^p \rho_{\mathsf{b}}(\mathcal{O}\mathsf{u}, \mathcal{O}\mathsf{z}), \mathcal{D}_b(\mathsf{u},\mathsf{z})\big) < \psi\left(\mathcal{D}_b(\mathsf{u},\mathsf{z})\right) - \psi\left(\mathsf{s}^p \rho_{\mathsf{b}}(\mathcal{O}\mathsf{u}, \mathcal{O}\mathsf{z})\right) \\ &= \psi\rho_{\mathsf{b}}(\mathsf{u},\mathsf{z}) - \psi\left(\mathsf{s}^p \rho_{\mathsf{b}}(\mathsf{u},\mathsf{z})\right), \end{aligned}$$

which is a contradiction. Therefore, $\rho_{\mathsf{b}}(\mathsf{u},\mathsf{z}) = 0$, that is (by Lemma 6), $\mathsf{u} = \mathsf{z}$. □

Example 5 Let the set $\mathcal{A} = [0,1]$, and $\rho_{\mathsf{b}} : \mathcal{A} \times \mathcal{A} \to [0,+\infty)$, $\rho_{\mathsf{b}}(v,\omega) = (\max\{v,\omega\})^2$ be a partial b-metric on $\mathcal{A}$. Let the continuous mapping $\mathcal{O} : \mathcal{A} \to \mathcal{A}$ be defined by $\mathcal{O}v = \begin{cases} v^2 & \text{for } v \in [0, \frac{2}{3}], \\ \frac{4}{9} & \text{for } v \in (\frac{2}{3}, 1], \end{cases}$ and the functions $\psi \in \Gamma$, $\eta_{\mathsf{b}} \in \mathcal{Z}_{\psi_b}$, where $\psi(\mathsf{t}) = \frac{\mathsf{t}}{2}$ and $\eta_{\mathsf{b}}(\mathsf{r},\mathsf{t}) = \frac{8}{9}(\frac{\mathsf{t}}{2}) - \frac{\mathsf{r}}{2}$.

We verify that $\mathcal{O}$ is a (η_{b})-ψ-rational contraction of type B.

1. For $v, \omega \in [0, 2/3]$, if $v > \omega$, (the case $v \leq \omega$ is similar), we have

$$\begin{aligned} &\rho_{\mathsf{b}}(v,\omega) = \left(\max\{v,\omega\}\right)^2 = v^2, \qquad \rho_{\mathsf{b}}(v, \mathcal{O}v) = \left(\max\left\{v, v^2\right\}\right)^2 = v^2, \\ &\rho_{\mathsf{b}}(\omega, \mathcal{O}\omega) = \omega^2, \qquad \rho_{\mathsf{b}}(\mathcal{O}v, \mathcal{O}\omega) = \left(\max\left\{v^2, \omega^2\right\}\right)^2 = v^4. \end{aligned}$$

   Therefore,

$$\frac{1}{4} \min\big\{\rho_{\mathsf{b}}(v, \mathcal{O}v), \rho_{\mathsf{b}}(\omega, \mathcal{O}\omega)\big\} = \frac{1}{4} v^2 \leq v^2 = \rho_{\mathsf{b}}(v,\omega),$$

   which implies

$$2\rho_{\mathsf{b}}(\mathcal{O}v, \mathcal{O}\omega) = 2v^4 \leq \frac{8}{9} v^2 \leq \frac{8}{9} \mathcal{D}_B(v,\omega)).$$

2. For $v, \omega \in (2/3, 1]$, if $v > \omega$, (the case $v \leq \omega$ is similar), we have

$$\begin{aligned} &\rho_{\mathsf{b}}(v,\omega) = \left(\max\{v,\omega\}\right)^2 = v^2, \qquad \rho_{\mathsf{b}}(v, \mathcal{O}v) = \left(\max\left\{v, \frac{4}{9}\right\}\right)^2 = v^2, \\ &\rho_{\mathsf{b}}(\omega, \mathcal{O}\omega) = \omega^2, \qquad \rho_{\mathsf{b}}(\mathcal{O}v, \mathcal{O}\omega) = \frac{16}{81}. \end{aligned}$$

   Therefore,

$$\frac{1}{4} \min\big\{\rho_{\mathsf{b}}(v, \mathcal{O}v), \rho_{\mathsf{b}}(\omega, \mathcal{O}\omega)\big\} = \frac{1}{4} v^2 \leq v^2 = \rho_{\mathsf{b}}(v,\omega),$$

which implies

$$2\rho_b(Ov, O\omega) = \frac{32}{81} \le \frac{8}{9}v^2 \le \frac{8}{9}\mathcal{D}_B(v,\omega)).$$

3. For $v \in [0, 2/3], \omega \in (2/3, 1]$, we have

$$\rho_b(v,\omega) = \big(\max\{v,\omega\}\big)^2 = \omega^2, \qquad \rho_b(v, Ov) = v^2,$$

$$\rho_b(\omega, O\omega) = \omega^2, \qquad \rho_b(Ov, O\omega) = \frac{16}{81}.$$

Therefore,

$$\frac{1}{4}\min\big\{\rho_b(v, Ov), \rho_b(\omega, O\omega)\big\} = \frac{1}{4}\omega^2 \le \omega^2 = \rho_b(v,\omega),$$

which implies

$$2\rho_b(Ov, O\omega) = \frac{32}{81} \le \frac{8}{9}\omega^2 = \frac{8}{9}\rho_b(\omega, O\omega) \le \frac{8}{9}\mathcal{D}_B(v,\omega)).$$

Therefore, all the hypotheses of Theorem 2.10 are satisfied and $v = 0$ is the unique fixed point of O.

Removing the condition $\frac{1}{2s}\min\{\rho_b(v, Ov), \rho_b(\omega, O\omega)\} \le \rho_b(v,\omega)$ in Theorem 3, respectively, Theorem 4, we immediately obtain the next results.

Corollary 1 *Let $(\mathcal{A}, \rho_b, s > 1)$ be a ρ_b-complete partial b-metric space and $O : \mathcal{A} \to \mathcal{A}$ be a mapping such that there exists $\eta_b \in \mathcal{Z}_{\psi_b}$ such that*

$$\eta_b\big(s^p\rho_b(Ov, O\omega), \mathcal{D}_A(v,\omega)\big) \ge 0$$

for all $v, \omega \in \mathcal{A}$, where $\mathcal{D}_{\mathcal{A}}$ is defined by (2.4). Then O has a unique fixed point.

Corollary 2 *Let $(\mathcal{A}, \rho_b, s > 1)$ be a ρ_b-complete partial b-metric space and $O : \mathcal{A} \to \mathcal{A}$ be a continuous mapping such that there exists $\eta_b \in \mathcal{Z}_{\psi_b}$ such that*

$$\eta_b\big(s^p\rho_b(Ov, O\omega), \mathcal{D}_B(v,\omega)\big) \ge 0$$

for all distinct $v, \omega \in \mathcal{A}$, where $\mathcal{D}_B$ is defined by (2.11). Then O has a unique fixed point.

Corollary 3 *Let $O : \mathcal{A} \to \mathcal{A}$ be a mapping on a ρ_b-complete partial b-metric space $(\mathcal{A}, \rho_b, s > 1)$. Suppose that $\psi \in \Gamma$ and $\phi : [0, +\infty) \to [0, +\infty)$ is a function such that $\liminf_{t \to t_0} \phi(t) > 0$, for $t_0 > 0$ and $\phi(t) = 0 \Leftrightarrow t = 0$. If for every $r, t \in \mathcal{A}$*

$$\frac{1}{2s}\min\big\{\rho_b(v, Ov), \rho_b(\omega, O\omega)\big\} \le \rho_b(v,\omega),$$

which implies

$$\psi\big(s^p\rho_b(Ov, O\omega)\big) \le \psi\big(\mathcal{D}_A(v,\omega)\big) - \phi\big(\mathcal{D}_A(v,\omega)\big)$$

then O admits a unique fixed point.

Proof Let $\eta_{\mathsf{b}}(\mathsf{r},\mathsf{t}) = \psi(\mathsf{t}) - \phi(\mathsf{t}) - \psi(\mathsf{r})$ in Theorem 3 and take into account Example 2. □

Corollary 4 *Let* $O : \mathcal{A} \to \mathcal{A}$ *be a continuous mapping on a* ρ_{b}*-complete partial b-metric space* $(\mathcal{A}, \rho_{\mathsf{b}}, \mathsf{s} > 1)$*. Suppose that* $\psi \in \Gamma$ *and* $\phi : [0,+\infty) \to [0,+\infty)$ *is a function such that* $\liminf_{\mathsf{t}\to\mathsf{t}_0} \phi(\mathsf{t}) > 0$*, for* $\mathsf{t}_0 > 0$ *and* $\phi(\mathsf{t}) = 0 \Leftrightarrow \mathsf{t} = 0$*. If for every distinct* $\mathsf{r},\mathsf{t} \in \mathcal{A}$

$$\frac{1}{2\mathsf{s}} \min\bigl\{\rho_{\mathsf{b}}(v, Ov), \rho_{\mathsf{b}}(\omega, O\omega)\bigr\} \le \rho_{\mathsf{b}}(v,\omega),$$

which implies

$$\psi\bigl(\mathsf{s}^p \rho_{\mathsf{b}}(Ov, O\omega)\bigr) \le \psi\bigl(\mathcal{D}_B(v,\omega)\bigr) - \phi\bigl(\mathcal{D}_B(v,\omega)\bigr)$$

then O *admits a unique fixed point.*

Proof Let $\eta_{\mathsf{b}}(\mathsf{r},\mathsf{t}) = \psi(\mathsf{t}) - \phi(\mathsf{t}) - \psi(\mathsf{r})$ in Theorem 4 and take into account Example 3. □

Corollary 5 *Let* $O : \mathcal{A} \to \mathcal{A}$ *be a mapping on a* ρ_{b}*-complete partial b-metric space* $(\mathcal{A}, \rho_{\mathsf{b}}, \mathsf{s} > 1)$*. Suppose that* $\psi \in \Gamma$ *and* $\gamma : [0,+\infty) \to [0,1)$ *is a function such that* $\limsup_{\mathsf{t}\to\mathsf{t}_0} \gamma(\mathsf{t}) < 1$*, for* $\mathsf{t}_0 > 0$ *and* $\gamma(\mathsf{t}) = 0 \Leftrightarrow \mathsf{t} = 0$*. If for every* $\mathsf{r},\mathsf{t} \in \mathcal{A}$

$$\frac{1}{2\mathsf{s}} \min\bigl\{\rho_{\mathsf{b}}(v, Ov), \rho_{\mathsf{b}}(\omega, O\omega)\bigr\} \le \rho_{\mathsf{b}}(v,\omega),$$

which implies

$$\psi\bigl(\mathsf{s}^p \rho_{\mathsf{b}}(Ov, O\omega)\bigr) \le \gamma\bigl(\mathcal{D}_A(v,\omega)\bigr)\psi\bigl(\mathcal{D}_A(v,\omega)\bigr)$$

then O *admits a unique fixed point.*

Proof Let $\eta_{\mathsf{b}}(\mathsf{r},\mathsf{t}) = \gamma(\mathsf{t})\psi(\mathsf{t}) - \psi(\mathsf{r})$ in Theorem 3 and take into account Example 2. □

Corollary 6 *Let* $O : \mathcal{A} \to \mathcal{A}$ *be a continuous mapping on a* ρ_{b}*-complete partial b-metric space* $(\mathcal{A}, \rho_{\mathsf{b}}, \mathsf{s} > 1)$*. Suppose that* $\psi \in \Gamma$ *and* $\gamma : [0,+\infty) \to [0,1)$ *is a function such that* $\limsup_{\mathsf{t}\to\mathsf{t}_0} \gamma(\mathsf{t}) < 1$*, for* $\mathsf{t}_0 > 0$ *and* $\gamma(\mathsf{t}) = 0 \Leftrightarrow \mathsf{t} = 0$*. If for every* $\mathsf{r},\mathsf{t} \in \mathcal{A}$*, with* $\rho_{\mathsf{b}}(v,\omega) > 0$*,*

$$\frac{1}{2\mathsf{s}} \min\bigl\{\rho_{\mathsf{b}}(v, Ov), \rho_{\mathsf{b}}(\omega, O\omega)\bigr\} \le \rho_{\mathsf{b}}(v,\omega),$$

which implies

$$\psi\bigl(\mathsf{s}^p \rho_{\mathsf{b}}(Ov, O\omega)\bigr) \le \psi\bigl(\mathcal{D}_B(v,\omega)\bigr) - \phi\bigl(\mathcal{D}_B(v,\omega)\bigr)$$

then O *admits a unique fixed point.*

Proof Let $\eta_{\mathsf{b}}(\mathsf{r},\mathsf{t}) = \gamma(\mathsf{t})\psi(\mathsf{t}) - \psi(\mathsf{r})$ in Theorem 4 and take into account Example 2. □

We will prove below results similar to those stated in Theorems 3, 4 that can be formulated for the case $\mathsf{s} = 1$.

Theorem 5 *Let* $(\mathcal{A}, \rho)$ *be a* ρ_{b}*-complete partial-metric space and* $O : \mathcal{A} \to \mathcal{A}$ *be a mapping. If there exists a function* $\eta \in \mathcal{Z}_{\psi}$ *such that*

$$\begin{aligned} &\frac{1}{2}\min\bigl\{\rho(v, Ov), \rho(\omega, O\omega)\bigr\} \leq \rho(v, \omega), \quad \textit{which implies} \\ &\quad \eta\bigl(\rho(Ov, O\omega), \mathcal{D}^1_A(v, \omega)\bigr) \geq 0, \end{aligned} \tag{2.13}$$

for every distinct $v, \omega \in \mathcal{A}$*, where* $\mathcal{D}^1_A$ *is defined as*

$$\mathcal{D}^1_A(v, \omega) = \max\left\{\rho(v, \omega), \rho(v, Ov), \rho(\omega, O\omega), \frac{\rho(\omega, O\omega)[1 + \rho(v, Ov)]}{1 + \rho(v, \omega)}\right\}, \tag{2.14}$$

then O *admits exactly one fixed point.*

Proof For $v_0 \in \mathcal{A}$, let $\{v_n\}$ be the sequence defined by (2.5), $\rho(v_m, v_{m+1}) > 0$, for any $m \in \mathbb{N}$.

First of all, we claim that $\lim_{n\to+\infty} \rho(v_m, v_{m+1}) = 0$. From (2.13), we have

$$\frac{1}{2}\min\bigl\{\rho(v_{m-1}, Ov_{m-1}), \rho(v_m, Ov_m\bigr\} = \frac{1}{2}\min\bigl\{\rho(v_{m-1}, v_m), \rho(v_m, v_{m+1}\bigr\} \leq \rho(v_{m-1}, v_m),$$

which implies

$$0 \leq \eta\bigl(\rho(Ov_{m-1}, Ov_m), \mathcal{D}^1_A(v_{m-1}, v_m)\bigr) < \psi\bigl(\mathcal{D}^1_A(v_{m-1}, v_m)\bigr) - \psi\bigl(\rho(Ov_{m-1}, Ov_m)\bigr).$$

Consequently, we get

$$\psi\bigl(\rho(Ov_{m-1}, Ov_m)\bigr) < \psi\bigl(\mathcal{D}^1_A(v_{m-1}, v_m)\bigr),$$

which, since ψ is non-decreasing, implies

$$\rho(v_m, v_{m+1}) = \rho(Ov_{m-1}, Ov_m) < \mathcal{D}^1_A(v_{m-1}, v_m) = \max\bigl\{\rho(v_{m-1}, v_m), \rho(v_m, v_{m+1})\bigr\}.$$

Therefore, the sequence $\{\rho(v_m, v_{m+1})\}$ is decreasing, so, we can find $\theta \geq 0$ such that $\lim_{m\to+\infty} \rho(v_m, v_{m+1}) = \theta$. On the other hand, it is easy to see that $\lim_{m\to+\infty} \mathcal{D}^1_A(v_{m-1}, v_m) = \theta$, as well. Assuming that $\theta > 0$, from (η_2) and (2.13) it follows that

$$0 \leq \limsup_{m\to+\infty} \eta\bigl(\rho(v_m, v_{m+1}), \mathcal{D}^1_A(v_{m-1}, v_m)\bigr) < 0,$$

which is a contradiction. So, we found that

$$\theta = \lim_{m\to+\infty} \rho(v_m, v_{m+1}) = 0. \tag{2.15}$$

We claim that $\{v_m\}$ is a Cauchy sequence. If we suppose that $\lim_{m,q\to+\infty} \rho(v_m, v_q) \neq 0$, there exist two subsequences $\{v_{m_l}\}$, $\{v_{q_l}\}$ of the sequence $\{v_m\}$ and a number $\mathsf{e} > 0$ such that $\rho(v_{m_l}, v_{q_l}) > \mathsf{e}$.

Moreover, by Lemma 1, we have

$$\lim_{l\to+\infty} \rho(v_{m_l}, v_{q_l-1}) = \mathsf{e} = \lim_{l\to+\infty} \rho(v_{m_l+1}, v_{q_l}). \tag{2.16}$$

Looking on the definition of the function $\mathcal{D}_A^1$, we have

$$\rho(v_{m_l}, v_{q_l-1}) \leq \mathcal{D}_A^1(v_{m_l}, v_{q_l-1}) = \max\left\{\begin{array}{c}\rho(v_{m_l}, v_{q_l-1}), \rho(v_{m_l}, v_{m_l+1}), \rho(v_{q_l-1}, v_{q_l}), \\ \frac{\rho(v_{q_l-1}, v_{q_l})[1+\rho(v_{m_l}, v_{m_l+1})]}{1+\rho(v_{m_l}, v_{q_l-1})}\end{array}\right\} \tag{2.17}$$

and keeping in mind (2.15) and (2.16) we get

$$\lim_{l\to+\infty} \mathcal{D}_A^1(v_{m_l}, v_{q_l-1}) = \mathsf{e}. \tag{2.18}$$

Now, letting $\mathsf{r}_l = \rho(v_{m_l+1}, v_{q_l})$ and $\mathsf{t}_l = \mathcal{D}_A^1(v_{m_l}, v_{q_l-1})$, by (η_2) we get

$$\limsup_{l\to+\infty} \eta\bigl(\rho(\mathcal{O}v_{m_l}, \mathcal{O}v_{q_l-1}), \mathcal{D}_A^1(v_{m_l}, v_{q_l-1})\bigr) < 0. \tag{2.19}$$

On the other hand, by (2.15), we have

$$\rho(v_{m_l}, v_{m_l+1}) < \frac{\mathsf{e}}{2} \quad \text{and} \quad \rho(v_{q_l-1}, v_{q_l}) < \frac{\mathsf{e}}{2}. \tag{2.20}$$

Thus, by the triangle inequality and taking into account (2.20), we get

$$\mathsf{e} < \rho(v_{m_l}, v_{q_l}) \leq \rho(v_{m_l}, v_{q_l-1}) + \rho(v_{q_l-1}, v_{q_l}) - \rho(v_{q_l-1}, v_{q_l-1}) < \rho(v_{m_l}, v_{q_l-1}) + \frac{\mathsf{e}}{2}$$

and then $\frac{\mathsf{e}}{2} < \rho(v_{m_l}, v_{q_l-1})$. Therefore,

$$\begin{aligned}\frac{1}{2}\min\bigl\{\rho(v_{m_l}, \mathcal{O}v_{m_l}), \rho(v_{q_l-1}, \mathcal{O}v_{q_l-1})\bigr\} &= \frac{1}{2}\min\bigl\{\rho(v_{m_l}, v_{m_l+1}), \rho(v_{q_l-1}, v_{q_l})\bigr\} \\ &< \frac{\mathsf{e}}{4} < \frac{\mathsf{e}}{2} < \rho(v_{m_l}, v_{q_l-1}),\end{aligned}$$

which implies

$$0 \leq \eta\bigl(\rho(\mathcal{O}v_{m_l}, \mathcal{O}v_{q_l-1}), \mathcal{D}_A^1(v_{m_l}, v_{q_l-1})\bigr),$$

which contradicts (2.19). Thus,

$$\lim_{m,q\to+\infty} \rho(v_m, v_q) = 0$$

and $\{v_m\}$ is a Cauchy sequence in the complete partial-metric space $(\mathcal{A}, \rho)$. This implies that there exists $\mathsf{u} \in \mathcal{A}$ such that

$$\lim_{m,q\to+\infty} \rho(v_m, v_q) = 0 = \lim_{m\to+\infty} \rho(v_m, \mathsf{u}) = \rho(\mathsf{u}, \mathsf{u}). \tag{2.21}$$

We shall prove that $\mathsf{u} = \mathcal{O}\mathsf{u}$. By (ρ_{b2}), we get

$$\frac{1}{2}\min\bigl\{\rho(v_m, \mathcal{O}v_m), \rho(\mathsf{u}, \mathcal{O}\mathsf{u})\bigr\} \leq \rho(v_m, \mathsf{u}),$$

which implies

$$\begin{aligned}0 &\le \eta\big(\rho(\mathcal{O}v_m, \mathcal{O}\mathsf{u}), \mathcal{D}^1_A(v_m, \mathsf{u})\big)\\ &< \psi\big(\mathcal{D}^1_A(v_m, \mathsf{u})\big) - \psi\big(\rho(\mathcal{O}v_m), \mathcal{O}\mathsf{u})\big)\\ &= \psi\big(\max\big\{\rho(v_m, \mathsf{u}), \rho(v_m, v_{m+1}), \rho(\mathsf{u}, \mathcal{O}\mathsf{u})\big\} - \psi\big(\rho(v_{m+1}), \mathcal{O}\mathsf{u})\big).\end{aligned}$$

Thus, by the non-decreasing property of ψ, we obtain

$$\begin{aligned}\rho(\mathsf{u}, \mathcal{O}\mathsf{u}) &\le \rho(\mathsf{u}, v_{m+1}) + \rho(v_{m+1}, \mathcal{O}\mathsf{u}) - \rho(v_{m+1}, v_{m+1})\\ &< \rho(\mathsf{u}, v_{m+1}) + \mathcal{D}^1_A(v_m, \mathsf{u}) - \rho(v_{m+1}, v_{m+1})\\ &< \rho(\mathsf{u}, v_{m+1}) + \max\left\{\begin{matrix}\rho(v_m, \mathsf{u}), \rho(v_m, v_{m+1}), \rho(\mathsf{u}, \mathcal{O}\mathsf{u}),\\ \frac{\rho(\mathsf{u}, \mathcal{O}\mathsf{u})[1+\rho(v_m, v_{m+1})]}{1+\rho(v_m, \mathsf{u})}\end{matrix}\right\}\\ &\quad - \rho(v_{m+1}, v_{m+1})\end{aligned}$$

and using (2.21) we get $\rho(\mathsf{u}, \mathcal{O}\mathsf{u}) = 0$. Thus, $\mathsf{u} = \mathcal{O}\mathsf{u}$ and u is a fixed point of $\mathcal{O}$.

In order to show the uniqueness of the fixed point, let $\mathsf{u}, \mathsf{z} \in \mathcal{A}$ such that $\mathsf{u} = \mathcal{O}\mathsf{u}$ and $\mathsf{z} = \mathcal{O}\mathsf{z}$. We have

$$0 = \frac{1}{2}\min \rho(\mathsf{u}, \mathcal{O}\mathsf{u}), \qquad \rho(\mathsf{z}, \mathcal{O}\mathsf{z}) \le \rho(\mathsf{u}, \mathsf{z}),$$

which implies

$$\begin{aligned}0 &\le \eta\big(\rho(\mathcal{O}\mathsf{u}, \mathcal{O}\mathsf{z}), \mathcal{D}^1_A(\mathsf{u}, \mathsf{z})\big)\\ &< \psi\left(\max\left\{\rho(\mathsf{u}, \mathsf{z}), \rho(\mathsf{u}, \mathcal{O}\mathsf{u}), \rho(\mathsf{z}, \mathcal{O}\mathsf{z}), \frac{\rho(\mathsf{z}, \mathcal{O}\mathsf{z})[1+\rho(\mathsf{u}, \mathcal{O}\mathsf{u})]}{1+\rho(\mathsf{u}, \mathsf{z})}\right\}\right)\\ &\quad - \psi(\rho(\mathcal{O}\mathsf{u}, \mathcal{O}\mathsf{z})\\ &= \rho(\mathsf{u}, \mathsf{z}) - \rho(\mathsf{u}, \mathsf{z}),\end{aligned}$$

which is a contradiction. Thus, we conclude that u is the unique fixed point of $\mathcal{O}$. ☐

Theorem 6 *Let $(\mathcal{A}, \rho)$ be a ρ_{b}-complete partial-metric space and $\mathcal{O} : \mathcal{A} \to \mathcal{A}$ be a continuous mapping. If there exists a function $\eta \in \mathcal{Z}_\psi$ such that*

$$\begin{aligned}&\frac{1}{2}\min\big\{\rho(v, \mathcal{O}v), \rho(\omega, \mathcal{O}\omega)\big\} \le \rho(v, \omega), \quad \textit{which implies}\\ &\quad\eta\big(\rho(\mathcal{O}v, \mathcal{O}\omega), \mathcal{D}^1_B(v, \omega)\big) \ge 0,\end{aligned} \tag{2.22}$$

holds for every $v, \omega \in \mathcal{A}$, $\rho(v, \omega) > 0$ where $\mathcal{D}^1_A$ is defined as

$$\mathcal{D}^1_B(v, \omega) = \max\left\{\begin{matrix}\rho(v, \omega), \rho(v, \mathcal{O}v), \rho(\omega, \mathcal{O}\omega), \frac{\rho(v, \mathcal{O}\omega)+\rho(\omega, \mathcal{O}v)}{2},\\ \frac{\rho(\omega, \mathcal{O}\omega)\rho(v, \mathcal{O}v)}{\rho(v, \omega)}\end{matrix}\right\}, \tag{2.23}$$

then $\mathcal{O}$ admits exactly one fixed point.

Proof Let $v_0 \in \mathcal{A}$ and consider the sequence $\{v_m\}$, with $v_m = Ov_{m-1}$. We assume that $\rho(v_m, v_{m=1}) > 0$ for each $m \in \mathbb{N}$ because we remark that, on the contrary, if there exits l_0 such that $v_{l_0} = v_{l_0+1} = Ov_{l_0}$, that is v_{l_0} is a fixed point for the mapping O, then by (2.23), for any terms $v = v_m$ and $\omega = v_{m+1}$ we have

$$\begin{aligned}\mathcal{D}_B^1(v_m, v_{m+1}) &= \max\left\{\begin{array}{c}\rho(v_m, v_{m+1}), \rho(v_m, Ov_m), \rho(v_{m+1}, Ov_{m+1}),\\ \frac{\rho(v_m,Ov_{m+1})+\rho(v_{m+1},Ov_m)}{2},\\ \frac{\rho(v_{m+1},Ov_{m+1})\rho(v_m,Ov_m)}{\rho(v_m,v_{m+1})}\end{array}\right\}\\ &= \max\left\{\begin{array}{c}\rho(v_m, v_{m+1}), \rho(v_{m+1}, v_{m+2}), \frac{\rho(v_m,v_{m+2})+\rho(v_{m+1},v_{m+1})}{2},\\ \frac{\rho(v_{m+1},v_{m+2})\rho(v_m,v_{m+1})}{\rho(v_m,v_{m+1})}\end{array}\right\}\\ &\leq \max\left\{\begin{array}{c}\rho(v_m, v_{m+1}), \rho(v_{m+1}, v_{m+2}),\\ \frac{\rho(v_m,v_{m+1})+\rho(v_{m+1},v_{m+2})-\rho(v_{m+1},v_{m+1})+\rho(v_{m+1},v_{m+1})}{2}\end{array}\right\}\\ &= \max\bigl\{\rho(v_m, v_{m+1}), \rho(v_{m+1}, v_{m+2})\bigr\}\end{aligned}$$

On the other hand, by (2.22),

$$\frac{1}{2}\min\bigl\{\rho(v_m, Ov_m), \rho(v_{m+1}, Ov_{m+1})\bigr\} = \frac{1}{2}\min\bigl\{\rho(v_m, v_{m+1}), \rho(v_{m+1}, v_{m+2})\bigr\} \leq \rho(v_m, v_{m+1}),$$

which implies

$$0 \leq \eta\bigl(\rho(Ov_m, Ov_{m+1}), \mathcal{D}_B^1(v_m, v_{m+1})\bigr) < \psi\bigl(\mathcal{D}_B^1(v_m, v_{m+1})\bigr) - \psi\bigl(\rho(v_{m+1}, v_{m+2})\bigr).$$

But $\psi \in \Gamma$ and then

$$\rho(v_{m+1}, v_{m+2}) < \mathcal{D}_B^1(v_m, v_{m+1}) \leq \max\bigl\{\rho(v_m, v_{m+1}), \rho(v_{m+1}, v_{m+2})\bigr\}. \tag{2.24}$$

If for some m, $\max\{\rho(v_{m+1}, v_{m+2}), \rho(v_m, v_{m+1})\} = \rho(v_{m+1}, v_{m+2})$ then (2.24) becomes $\rho(v_{m+1}, v_{m+2}) < \rho(v_{m+1}, v_{m+2})$, which is a contradiction. Then, for each $m \geq 0$, $\max\{\rho(v_{m+1}, v_{m+2}), \rho(v_m, v_{m+1})\} = \rho(v_m, v_{m+1})$, the inequality (2.24) yields

$$\rho(v_{m+1}, v_{m+2}) < \rho(v_m, v_{m+1}).$$

Thus, the sequence $\{\rho(v_m, v_{m+1})\}$ is decreasing, so it is convergent (being bounded from below). In this case, we can find a real number $u \geq 0$ such that $\lim_{m\to+\infty}\rho(v_m, v_{m+1}) = u$. Assume that $u > 0$, let $\mathsf{r}_m = \rho(v_{m+1}, v_{m+2})$ and $\mathsf{t}_m = \mathcal{D}_B^1(v_m, v_{m+1}$. Since

$$\lim_{m\to+\infty}\mathsf{r}_m = \lim_{m\to+\infty}\mathsf{t}_m = u,$$

from (η_2) we have

$$0 \leq \limsup_{m\to+\infty}\eta(\mathsf{r}_m, \mathsf{t}_m) < 0.$$

This is a contradiction, so that

$$\lim_{m\to+\infty}\rho(v_m, v_{m+1}) = 0. \tag{2.25}$$

As a next step, we claim that $\{v_m\}$ is a Cauchy sequence in $(\mathcal{A},\rho)$. Reasoning by contradiction, we suppose that $\lim_{m,q\to+\infty}\rho(v_m,v_q)\neq 0$. Then, by Lemma 1, there exist the subsequences $\{v_{m_l}\}$, $\{v_{q_l}\}$ of the sequence $\{v_m\}$, with $q_l > m_l > l$, and a number $\mathsf{e} > 0$ such that $\rho(v_{m_l},v_{q_l})\geq \mathsf{e}$ and

$$\lim_{l\to+\infty}\rho(v_{m_l},v_{q_l+1}) = \mathsf{e} = \lim_{l\to+\infty}\rho(v_{m_l-1},v_{q_l}).$$

Now, according to (2.25), there exists $n_1\in\mathbb{N}$, such that

$$\rho(v_{m_l-1},v_{m_l}) < \frac{\mathsf{e}}{2}, \quad \text{for any } l > n_1$$

and $n_2\in\mathbb{N}$, such that

$$\rho(v_{q_l},v_{q_l+1}) < \frac{\mathsf{e}}{2}, \quad \text{for any } l > n_2.$$

Therefore, for $l > \max\{n_1,n_2\}$ we have

$$\begin{aligned}\mathsf{e}\leq\rho(v_{m_l},v_{q_l}) &\leq \rho(v_{m_l},v_{m_l-1}) + \rho(v_{m_l-1},v_{q_l}) - \rho(v_{m_l-1},v_{m_l-1})\\ &\leq \rho(v_{m_l-1},v_{q_l}) + \frac{\mathsf{e}}{2} - \rho(v_{m_l-1},v_{m_l-1})\end{aligned}$$

and we can conclude $\frac{\mathsf{e}}{2}\leq\rho(v_{m_l-1},v_{q_l})$. Thus,

$$\frac{1}{2}\min\bigl\{\rho(v_{m_l-1},v_{m_l}),\rho(v_{q_l},v_{q_l+1})\bigr\} < \frac{\mathsf{e}}{4} < \frac{\mathsf{e}}{2}\leq\rho(v_{m_l-1},v_{q_l}),$$

which implies

$$0\leq\limsup_{l\to+\infty}\eta\bigl(\rho(\mathcal{O}v_{m_l-1},\mathcal{O}v_{q_l}),\mathcal{D}_B^1(v_{m_l-1},v_{q_l})\bigr). \tag{2.26}$$

On the other hand,

$$\lim_{l\to+\infty}\mathcal{D}_B^1(v_{m_l-1},v_{q_l}) = \lim_{l\to+\infty}\max\left\{\begin{array}{c}\rho(v_{m_l-1},v_{q_l}),\rho(v_{m_l-1},v_{m_l}),\rho(v_{q_l},v_{q_l+1}),\\ \frac{\rho(v_{m_l-1},v_{q_l+1})+\rho(v_{m_l},v_{q_l})}{2},\\ \frac{\rho(v_{m_l-1},v_{m_l})\rho(v_{q_l},v_{q_l+1})}{\rho(v_{m_l-1},v_{q_l})}\end{array}\right\} = \mathsf{e}$$

and (η_2) implies

$$\limsup_{l\to+\infty}\eta\bigl(\rho(\mathcal{O}v_{m_l-1},\mathcal{O}v_{q_l}),\mathcal{D}_B^1(v_{m_l-1},v_{q_l})\bigr) < 0,$$

which contradicts (2.26). Therefore, $\{v_m\}$ is a Cauchy sequence in a ρ-complete partial-metric space $(\mathcal{A},\rho)$ and there exists $\mathsf{u}\in\mathcal{A}$ such that

$$\rho(\mathsf{u},\mathsf{u}) = \lim_{m\to+\infty}\rho(v_m,\mathsf{u}) = \lim_{m,q\to+\infty}\rho(v_m,v_q) = 0. \tag{2.27}$$

On the other hand, due to the continuity of the mapping $\mathcal{O}$, we get

$$\lim_{m\to+\infty}\rho(v_{m+1},\mathcal{O}\mathsf{u}) = \lim_{m\to+\infty}\rho(\mathcal{O}v_m,\mathcal{O}\mathsf{u}) = 0. \tag{2.28}$$

Consequently, from (2.27), (2.28), on account of Lemma 3, we see that u is a fixed point of $\mathcal{O}$. The uniqueness of the fixed point follows immediately as in the previous theorem. □

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