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Alpha fractional frequency Laplace transform through multiseries

Meganathan Murugesan[1], Thabet Abdeljawad[2,3,4*], Britto Antony Xavier Gnanaprakasam[1] and Fahd Jarad[5]

*Correspondence:
tabdeljawad@psu.edu.sa
[2]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia
[3]Department of Medical Research, China Medical University, Taichung, Taiwan
Full list of author information is available at the end of the article

Abstract

Our main goal in this work is to derive the frequency Laplace transforms of the products of two and three functions with tuning factors. We propose the Laplace transform for certain types of multiseries of circular functions as well. For use in numerical results, we derive a finite summation formula and *m*-series formulas. Moreover, we discuss various explanatory examples.

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Keywords: Circular function; Multiseries; Laplace transform; Fractional frequency; Tuning factor

1 Introduction

Digital signal processing (DSP) has revolutionized many areas in science and engineering such as space, medicine, commerce, military, technology, and communication. The Laplace transform (LT) and discrete Laplace transform (DLT) effectively change a signal (function) from time domain to frequency domain with the factor e^{-st}. Several applications of LT and DLT were discussed by many authors [6, 8–10]. The applications of the n-dimensional Laplace transform appear in heat equations, wave equations, and modeling in fluid dynamics [13, 14, 21]. In [22, 23], the authors considered some mathematical logistic models of fractional operators. Recently, the authors found the solutions of fractional difference equations [24–26]. Some more findings on fractional order models with the numerical simulation are discussed in [27–33]. For recent development in the theory of fractional difference operators, we refer to [15–20].

The LT and DLT are respectively defined as $\mathcal{L}[u(t)] = \int_0^\infty u(t)e^{-st}\,dt$ and $\mathcal{L}[u(n)] = \sum_{n=0}^\infty u(n)e^{-sn}, s > 0$. From the basic difference identity $\Delta^{-1} x_n|_0^\infty = \sum_{n=0}^\infty x_n$ [4] the DLT can be expressed as $\mathcal{L}[u(n)] = \Delta^{-1} u(n)e^{-sn}$. In the literature the Laplace transform in discrete calculus comes from the time scale definition of the Laplace transform and has a strong relation with the Z-transform. Here we define a different kind of Laplace transform because it has two different kinds of solutions and is more suitable with the literature. Let $u(t)$ be an input signal (function), and let h be a shift value. Then we define the alpha fractional frequency Laplace transform (LFT) with tuning factor α and frequency $s^{1/\nu}$ as

$$\mathcal{L}_{\alpha(h)}u(t) = \mathop{\Delta}_{\alpha(h)}^{-1} u(t)e^{-s^{1/\nu}t}\Big|_0^\infty = h\sum_{r=0}^{\infty}\alpha^{-(r+1)}u(rh)e^{-s^{1/\nu}rh}. \tag{1}$$

When $\alpha = 1$ and $h = 1$, transform (1) becomes the discrete Laplace transform. When $\alpha = 1$ and $h \to 0$, (1) becomes the Laplace transform [5, 6]. To develop LFT, we can study the operators Δ_h and $\Delta_{\alpha(h)}$ and their inverses [7, 11].

In 2011, the authors in [1] defined the alpha difference operator as

$$\mathop{\Delta}_{\alpha(h)} u(t) = \frac{u(t+h) - \alpha u(t)}{h} = u(t), \quad t \in [0,\infty), h \in (0,\infty). \tag{2}$$

In this research work, we extend the results on multiseries by using α-difference operators and then analyze LFT for signals of algebraic and geometric type functions.

2 Preliminaries

In [3] the authors introduced $t_h^{(m)} = \prod_{r=0}^{m-1}(t - rh)$ and obtained the following expressions:

$$\text{(i)}\quad \mathop{\Delta}_{h} t_h^{(m)} = (mh)t_h^{(m-1)}, \qquad \text{(ii)}\quad t_h^{(m)} = \sum_{r=1}^{m} s_r^m h^{m-r} t^r,$$
$$\text{(iii)}\quad t^m = \sum_{r=1}^{m} S_r^m h^{m-r} t_h^{(r)}, \tag{3}$$

where s_r^m and S_r^m denote the Stirling numbers.

Lemma 2.1 ([2]) *If* $\lim_{r_i\to\infty}\frac{1}{\alpha_i^{r_i}}\Delta_{\alpha_i(h_i)}^{-1}u(t + r_ih_i) = 0$, *then the* α_i*-difference equation*

$$\mathop{\Delta}_{\alpha_i(h_i)} v(t) = u(t), \quad t \in [0,\infty), h_i > 0, \tag{4}$$

has a solution in the following infinite series form:

$$\mathop{\Delta}_{\alpha_i(h_i)}^{-1} u(t) = \frac{-h_i}{\alpha_i}\sum_{r_i=0}^{\infty}\alpha_i^{-r_i}u(t + r_ih_i). \tag{5}$$

Lemma 2.2 ([12]) *Let* $p > 0$ *and* $1 + \alpha^2 - \cos ph \neq 0$. *Then*

$$\mathop{\Delta}_{\alpha(h)}^{-1} \sin pt = h\frac{\sin p(t-h) - \alpha\sin pt}{1 + \alpha^2 - \cos ph} + c_{h(t)} \tag{6}$$

and

$$\mathop{\Delta}_{\alpha(h)}^{-1} \cos pt = h\frac{\cos p(t-h) - \alpha\cos pt}{1 + \alpha^2 - \cos ph} + c_{h(t)}.r. \tag{7}$$

3 Summation operator

Remark 3.1 Hereafter we take $P = p(n_1 - 2r_1) + q(n_2 - 2r_2)$ and $\overline{P} = p(n_1 - 2r_1) - q(n_2 - 2r_2)$, so that P and $\overline{P}$ depend on n_1, n_2, r_1, r_2, p, and q. We use the notation $\{(, o\}$ for odd numbers and $\{[, e\}$ for even numbers.

(i) $\mathbb{T}_{oo} = \{(1,0),(0,1)\}$,

(ii) $\mathbb{T}_{oe} = \{(1,0),(0,1),(1,1)\}$,

(iii) $\mathbb{T}_{eo} = \{(1,0),(0,1),(1,-1)\}$,

(iv) $\mathbb{T}_{ee} = \{(1,0),(0,1),(1,1),(1,-1)\}$,

(v) $((n_2)) = \binom{n_2}{\frac{n_2}{2}}^{-uv(\frac{u-v}{u^2+v^2})}$,

(vi) $((n_3)) = \binom{n_3}{\frac{n_3}{2}}^{uv(\frac{u+v}{u^2+v^2})}$.

Here s, p and q are real numbers.

(1) For odd positive integers n_2 and n_3, we denote

$$\sum_{(n_2,n_3)}^{s,c} = \frac{(-1)^{\frac{n_2-1}{2}}}{2^{n_2+n_3-1}} \sum_{s_2=0}^{\frac{n_2-1}{2}} \sum_{s_3=0}^{\frac{n_3-1}{2}} (-1)^{s_2} \frac{n_2^{(s_2)}}{s_2!} \frac{n_3^{(s_3)}}{s_3!}.$$

(2) For even positive integers n_2 and n_3, we denote

$$\sum_{[n_2,n_3]}^{s,c} = \frac{(-1)^{\frac{n_2}{2}}}{2^{n_2+n_3-1}} \sum_{s_2=0}^{\frac{n_2-2}{2}} \sum_{s_3=0}^{\frac{n_3-2}{2}} (-1)^{s_2} \frac{n_2^{(s_2)}}{s_2!} \frac{n_3^{(s_3)}}{s_3!}.$$

The operator used for products of circular functions with exponential functions alone is defined as

$$\sum_{(n_2,n_3)}^{s,c,m} = \frac{(-1)^{\frac{n_2-1}{2}}}{2^{n_2+n_3-1}} \sum_{s_1=0}^{m} \sum_{s_2=0}^{\frac{n_2-1}{2}} \sum_{s_3=0}^{\frac{n_3-1}{2}} (-1)^{s_1+s_2} \frac{m^{(s_1)}}{s_1!} \frac{n_2^{(s_2)}}{s_2!} \frac{n_3^{(s_3)}}{s_3!}.$$

The operator used for products of circular functions with t-factorial and also for products of circular, t-factorial, and exponential functions is defined as

$$\sum_{n_1(n_2,n_3)}^{s,c,m+s_1} = \frac{(-1)^{\frac{n_2-1}{2}}}{2^{n_2+n_3-1}} \sum_{s_1=0}^{n_1} \sum_{s_2=0}^{\frac{n_2-1}{2}} \sum_{s_3=0}^{\frac{n_3-1}{2}} \sum_{s_4=0}^{m+s_1} \frac{n_1^{(s_1)} n_2^{(s_2)} n_3^{(s_3)} (m+s_1)^{(s_4)}}{(-1)^{s_1+s_2+s_4} s_1! s_2! s_3! s_4!}.$$

In the $m(\alpha)$-series formula, we use

$$\sum_{(r)_1\to i}^{[t]} = \sum_{r_1=0}^{[\frac{t}{h}]} \sum_{r_2=0}^{[\frac{t-r_1h}{h}]} \cdots \sum_{r_i=0}^{[\frac{t-(r_1+r_2\cdots r_i)h}{h}]}, \qquad (UV) = \left(\frac{uP + v\overline{P}}{u^2+v^2}\right).$$

4 Multiseries inverse of product of two and three functions

In this section, we derive a finite summation formula and m-series formula. Also, we present the m-series inverse of the product of two and three functions.

Theorem 4.1 (Finite summation formula) *Let $\alpha \neq 1$ and $m > 1$. Then we have*

$$\sum_{r=1}^{m} \alpha^{r-1} h^r u(t-rh) = \underset{\alpha(h)}{\overset{-1}{\Delta}} u(t) - \alpha^{m+1} h^m \underset{\alpha(h)}{\overset{-1}{\Delta}} u(t-mh). \tag{8}$$

Note: (8) can be represented as $\Delta_{\alpha(h)}^{-1} u(t)|_{t-mh}^{t}$.

Proof From the definitions of $\Delta_{\alpha(h)}$ and $\Delta_{\alpha(h)}^{-1}$ we have

$$\underset{\alpha(h)}{\Delta} v(t) = u(t) \text{ implies } v(t) = \underset{\alpha(h)}{\overset{-1}{\Delta}} u(t), \tag{9}$$

$$\text{(i.e.)} \quad \frac{v(t+h) - \alpha v(t)}{h} = u(t) \Rightarrow v(t+h) = h\big[u(t) + \alpha v(t)\big] \tag{10}$$

Replacing t by $t-h, t-2h, \dots, t-mh$ in (10), we get expressions for $v(t), v(t-h), \dots, v(t-mh)$. Successively substituting all these expressions into (10), we arrive at

$$hu(t-h) + \alpha h^2 u(t-2h) + \cdots + \alpha^{m-1} h^m u(t-mh) = v(t) - \alpha^{m+1} h^r v(t-mh). \tag{11}$$

Now (8) follows from the equality $v(t) = \Delta_{\alpha(h)}^{-1} u(t-mh)$. □

Remark 4.2 We denote $\Delta_{\alpha(h)}^{-1} u(t) - \alpha^{m+1} h^r \Delta_{\alpha(h)}^{-1} u(t-mh) = \Delta_{\alpha(h)}^{-1} u(t)|_{t-mh}^{t}$.

Lemma 4.3 *Let $t \in [h, \infty)$, $a^h \neq \alpha$, and $h(t) = (t-mh)$. Then we have*

$$\underset{\alpha(h)}{\overset{-1}{\Delta}} a^t \Big|_{h(t)}^{t} = \frac{h a^{t+h}}{(a^h - \alpha)} - \alpha^{m+1} h^{r+1} \frac{a^h}{(a^{h(t)} - \alpha)}. \tag{12}$$

Proof Using (2) and Remark 4.2, we get (12). □

Theorem 4.4 *For the functions $u(t)$ and $v(t)$, we have*

$$\underset{\alpha(h)}{\overset{-1}{\Delta}} \big[u(t)v(t)\big] = u(t) \underset{\alpha(h)}{\overset{-1}{\Delta}} v(t) - \underset{\alpha(h)}{\overset{-1}{\Delta}} \Big[\underset{\alpha(h)}{\overset{-1}{\Delta}} v(t+h) \underset{h}{\Delta} u(t)\Big]. \tag{13}$$

Proof From the definition of $\Delta_{\alpha(h)}$ we have

$$\underset{\alpha(h)}{\Delta} \big[u(t)w(t)\big] = u(t) \underset{\alpha(h)}{\Delta} w(t) + w(t+h) \underset{h}{\Delta} u(t). \tag{14}$$

Now taking $\Delta_{\alpha(h)} w(t) = v(t)$ and $w(t) = \Delta_{\alpha(h)}^{-1} v(t)$ in equation (14), we obtain (13). □

Theorem 4.5 (Product of two functions) *For odd positive integers n_2 and n_3,*

$$\begin{aligned}
&\underset{\alpha(h)}{\overset{-m}{\Delta}} \big(e^{-s^{1/v} t} \sin^{n_2} pt \cos^{n_3} qt\big) \\
&\quad = \sum_{(n_2,n_3)}^{s,c,m} \sum_{(u,v)\in \mathbb{T}_{oo}} \frac{e^{s_1 s^{1/v} h}}{e^{s^{1/v} t}} \frac{\alpha^{s_1} \sin(UV)(t-(m-s_1)h)}{(e^{-s^{1/v} h} + \alpha^2 e^{s^{1/v} h} - 2\cos(UV)h)^m}.
\end{aligned} \tag{15}$$

Proof After changing the powers of sin and cos into linear, we obtain

$$\underset{\alpha(h)}{\overset{-1}{\Delta}}\big(e^{-s^{1/\nu}t}\sin^{n_2}pt\cos^{n_3}qt\big) = \sum_{(n_2,n_3)}^{s,c}\underset{\alpha(h)}{\overset{-1}{\Delta}}\big(e^{-s^{1/\nu}t}(\sin Pt+\sin\overline{P}t)\big)$$
$$= \text{Im part of}\sum_{(n_2,n_3)}^{s,c}\times\underset{\alpha(h)}{\overset{-1}{\Delta}}\big(e^{-s^{1/\nu}t}\big(e^{iPt}+e^{i\overline{P}t}\big)\big)$$
$$= \text{Im part of}\sum_{(n_2,n_3)}^{s,c}\bigg(\frac{e^{(iP-s^{1/\nu})t}}{e^{(iP-s^{1/\nu})h}-\alpha}+\frac{e^{(i\overline{P}-s^{1/\nu})t}}{e^{(i\overline{P}-s^{1/\nu})h}-\alpha}\bigg).$$

After simplification, we get

$$\underset{h}{\overset{-1}{\Delta}}\big(e^{-s^{1/\nu}t}\sin^{n_2}pt\cos^{n_3}qt\big)$$
$$= \sum_{(n_2,n_3)}^{s,c,1}\sum_{(u,v)\in\mathbb{T}_{oo}} e^{-s^{1/\nu}t}\alpha^{s_1}e^{s_1 s^{1/\nu}h}\frac{\sin(UV)(t-(1-s_1)h)}{(e^{-s^{1/\nu}h}+\alpha^2 e^{s^{1/\nu}h}-2\cos(UV)h)^m}. \tag{16}$$

Applying $\Delta_{\alpha(h)}^{-1}$ to both sides of equation (16), we get

$$\underset{\alpha(h)}{\overset{-2}{\Delta}}\big(e^{-s^{1/\nu}t}\sin^{n_2}pt\cos^{n_3}qt\big)$$
$$= \sum_{(n_2,n_3)}^{s,c,2}\sum_{(u,v)\in\mathbb{T}_{oo}} e^{-s^{1/\nu}t}\alpha^{s_1}e^{s_1 s^{1/\nu}h}\frac{\sin(UV)(t-(2-s_1)h)}{(e^{-s^{1/\nu}h}+\alpha^2 e^{s^{1/\nu}h}-2\cos(UV)h)^m}.$$

Continuing this process up to m-inverse, we get (15). □

Theorem 4.6 (Product of three functions) *For odd positive integers n_2 and n_3,*

$$\underset{\alpha(h)}{\overset{-m}{\Delta}}\big(t_h^{(n_1)}e^{-s^{1/\nu}t}\sin^{n_2}pt\cos^{n_3}qt\big)$$
$$= \sum_{n_1(n_2,n_3)}^{s,c,m+s_1}\sum_{(u,v)\in\mathbb{T}_{oo}}\frac{t_h^{(n_1-s_1)}m_{(s_1)}\alpha^{s_4}}{h^{-s_1}e^{s^{1/\nu}(t+s_1h)}}\frac{e^{s_4 s^{1/\nu}h}\sin(UV)(t-(m-s_4)h)}{(e^{-s^{1/\nu}h}+\alpha^2 e^{s^{1/\nu}h}-2\alpha\cos(UV)h)^{m+s_1}}. \tag{17}$$

Proof Let $f_1(t) = t_h^{(1)}e^{-s^{1/\nu}t}\sin^{n_2}pt\cos^{n_3}qt$

$$\underset{\alpha(h)}{\overset{-1}{\Delta}}f_1(t) = \sum_{(n_2,n_3)}^{s,c}\underset{\alpha(h)}{\overset{-1}{\Delta}}\big(t_h^{(1)}e^{-s^{1/\nu}t}(\sin Pt+\sin\overline{P}t)\big)$$
$$= \sum_{1(n_2,n_3)}^{s,c,1+s_1}\sum_{(u,v)\in\mathbb{T}_{oo}}\frac{t_h^{(1-s_1)}1_{(s_1)}\alpha^{s_4}}{h^{-s_1}e^{s^{1/\nu}(t+s_1h)}}\frac{e^{s_4 s^{1/\nu}h}\sin(UV)(t-(1-s_4)h)}{(e^{-s^{1/\nu}h}+\alpha^2 e^{s^{1/\nu}h}-2\alpha\cos(UV)h)^{1+s_1}}.$$

Applying Δ_h^{-1} to both sides, we get

$$\underset{h}{\overset{-2}{\Delta}}f_1(t) = \sum_{1(n_2,n_3)}^{s,c,2+s_1}\sum_{(u,v)\in\mathbb{T}_{oo}}\frac{t_h^{(1-s_1)}2_{(s_1)}\alpha^{s_4}}{h^{-s_1}e^{s^{1/\nu}(t+s_1h)}}\frac{e^{s_4 s^{1/\nu}h}\sin(UV)(t-(2-s_4)h)}{(e^{-s^{1/\nu}h}+\alpha^2 e^{s^{1/\nu}h}-2\alpha\cos(UV)h)^{2+s_1}}.$$

Continuing this process, we get

$$\mathop{\Delta}_{\alpha(h)}^{-m} f_1(t) = \sum_{1(n_2,n_3)}^{s,c,m+s_1} \sum_{(u,v)\in\mathbb{T}_{oo}} \frac{t_h^{(1-s_1)} m_{(s_1)} \alpha^{s_4}}{h^{-s_1} e^{s^{1/v}(t+s_1h)}} \frac{e^{s_4 s^{1/v}h} \sin(UV)(t-(m-s_4)h)}{(e^{-s^{1/v}h} + \alpha^2 e^{s^{1/v}h} - 2\alpha\cos(UV)h)^{m+s_1}}.$$

Similarly, we can obtain

$$\begin{aligned}
&\mathop{\Delta}_{\alpha(h)}^{-m} \bigl(t_h^{(2)} e^{-s^{1/v}t} \sin^{n_2} pt \cos^{n_3} qt\bigr) \\
&\quad = \sum_{2(n_2,n_3)}^{s,c,m+s_1} \sum_{(u,v)\in\mathbb{T}_{oo}} \frac{t_h^{(2-s_1)} m_{(s_1)} \alpha^{s_4}}{h^{-s_1} e^{s^{1/v}(t+s_1h)}} \frac{e^{s_4 s^{1/v}h} \sin(UV)(t-(m-s_4)h)}{(e^{-s^{1/v}h} + \alpha^2 e^{s^{1/v}h} - 2\alpha\cos(UV)h)^{m+s_1}}.
\end{aligned}$$

Continuing this process up to m-inverse for n_1, we get equation (17). □

Corollary 4.7 *For odd positive integers n_2 and n_3, we have*

$$\begin{aligned}
&\mathop{\Delta}_{\alpha(h)}^{-m} \bigl(t^{n_1} e^{-s^{1/v}t} \sin^{n_2} pt \cos^{n_3} qt\bigr) \\
&\quad = \sum_{r_1=1}^{n_1} \sum_{n_1(n_2,n_3)}^{s,c,m+s_1} \sum_{(u,v)\in\mathbb{T}_{oo}} \frac{S_{r_1}^{n_1} t_h^{(r_1-s_1)} m_{(s_1)} \alpha^{s_4}}{h^{r_1-n_1} h^{-s_1} e^{s^{1/v}(t+s_1h)}} \\
&\qquad \times \frac{e^{s_4 s^{1/v}h} \sin(UV)(t-(m-s_4)h)}{(e^{-s^{1/v}h} + \alpha^2 e^{s^{1/v}h} - 2\alpha\cos(UV)h)^{m+s_1}}. \qquad (18)
\end{aligned}$$

Proof The proof follows by applying (ii) of (3) in Theorem 4.6. □

Corollary 4.8 *For odd positive integers n_2 and n_3, we have*

$$\begin{aligned}
&\mathop{\Delta}_{\alpha(h)}^{-m} \bigl(t_h^{(n_1)} \sin^{n_2} pt \cos^{n_3} qt\bigr) \\
&\quad = \sum_{n_1(n_2,n_3)}^{s,c,m+s_1} \sum_{(u,v)\in\mathbb{T}_{oo}} t_h^{(n_1-s_1)} m_{(s_1)} h^{s_1} \alpha^{s_4} \frac{\sin(UV)(t-(m-s_4)h)}{(1+\alpha^2 - 2\alpha\cos(UV)h)^{m+s_1}}. \qquad (19)
\end{aligned}$$

Theorem 4.9 (m-series formula) *For $m \in \mathbb{N}(2)$ and $t \in [mh, \infty)$, we have*

$$\begin{aligned}
&\sum_{r=0}^{[\frac{t}{h}]} \frac{(r+1)_{m-1}}{(m-1)!} \alpha^r u(t-rh) + \sum_{i=1}^{m-1} \sum_{(r)_{1\to i}}^{[t]} \alpha^{[\frac{t-\sum_{j=1}^{i} r_j h}{h}] + \sum_{j=1}^{i} r_j + 1} \\
&\qquad \times \mathop{\Delta}_{\alpha(h)}^{-(m-i)} u\bigl(h(t) + (m-i-1)h\bigr) \\
&\quad = \mathop{\Delta}_{\alpha(h)}^{-m} u(t+mh) - \alpha^{[\frac{t}{h}]+1} \mathop{\Delta}_{\alpha(h)}^{-m} u\bigl(h(t) + (m-1)h\bigr). \qquad (20)
\end{aligned}$$

Proof Replacing t by $t-h$ in (8) and multiplying both sides by α, we get

$$\begin{aligned}
&\alpha\bigl\{u(t-h) + \alpha u(t-2h) + \alpha^2 u(t-3h) + \cdots + \alpha^{[\frac{t-h}{h}]} u\bigl(h(t)\bigr)\bigr\} \\
&\quad = \alpha\Bigl\{\mathop{\Delta}_{\alpha(h)}^{-1} u(t) - \alpha^{[\frac{t-h}{h}]+1} \mathop{\Delta}_{\alpha(h)}^{-1} u\bigl(h(t)\bigr)\Bigr\}.
\end{aligned}$$

Replacing t by $t-2h$ in (8) and multiplying both sides by α^2, we get

$$\alpha^2\{u(t-2h)+\alpha u(t-2h)+\alpha^2 u(t-3h)+\cdots+\alpha^{[\frac{t-2h}{h}]}u(h(t))\}$$
$$=\alpha^2\{\underset{\alpha(h)}{\overset{-1}{\Delta}}u(t-h)-\alpha^{[\frac{t-2h}{h}]+1}\underset{\alpha(h)}{\overset{-1}{\Delta}}u(h(t))\}.$$

Proceeding similarly, replacing t by $h(t)$ in (8), and multiplying both sides by $\alpha^{[\frac{t}{h}]}$, we get $\alpha^{[\frac{t}{h}]}u(h(t))=\alpha^{[\frac{t}{h}]}\{\Delta^{-1}_{\alpha(h)}u(h(t)+h)-\alpha^{[\frac{h(t)}{h}]+1}\Delta^{-1}_{\alpha(h)}u(h(t))\}$.

Adding all left- and right-hand sides with (8), we get

$$\sum_{r=0}^{[\frac{t}{h}]}(r+1)\alpha^r u(t-rh)+\sum_{r_1=0}^{[\frac{t}{h}]}\alpha^{[\frac{t-r_1h}{h}]+r_1+1}\underset{\alpha(h)}{\overset{-1}{\Delta}}u(h(t))$$
$$=\underset{\alpha(h)}{\overset{-2}{\Delta}}u(t+2h)-\alpha^{[\frac{t}{h}]+1}\underset{\alpha(h)}{\overset{-2}{\Delta}}u(h(t)+h). \tag{21}$$

Similarly, by replacing t by $t-h,t-2h,\dots,h(t)$ in (21), multiplying by $\alpha,\alpha^2,\dots\alpha^{[\frac{t}{h}]}$, respectively, and adding all with (21), we arrive at

$$\sum_{r=0}^{[\frac{t}{h}]}\frac{(r+1)(r+2)}{2!}\alpha^r u(t-rh)+\sum_{r_1=0}^{[\frac{t}{h}]}\alpha^{[\frac{t-r_1h}{h}]+r_1+1}\underset{\alpha(h)}{\overset{-2}{\Delta}}u(h(t)+h)+\sum_{r_1=0}^{[\frac{t}{h}]}\sum_{r_2=0}^{[\frac{t-r_1h}{h}]}$$
$$\times\alpha^{[\frac{t-(r_1+r_2)h}{h}]+r_1+r_2+1}\underset{\alpha(h)}{\overset{-1}{\Delta}}u(h(t))$$
$$=\underset{\alpha(h)}{\overset{-3}{\Delta}}u(t+3h)-\alpha^{[\frac{t}{h}]+1}\underset{\alpha(h)}{\overset{-3}{\Delta}}u(h(t)+2h). \tag{22}$$

Proceeding similarly, we finally obtain (20). □

Theorem 4.10 *For odd positive integers n_1 and n_2, the m-series corresponding to* (15) *is*

$$\sum_{r=0}^{[\frac{t}{h}]}\frac{(r+1)_{(m-1)}}{(m-1)!}\alpha^r(t-rh)_h^{(n_1)}e^{-s^{1/\nu}(t-rh)}\sin^{n_2}p(t-rh)\cos^{n_3}q(t-rh)$$
$$+\sum_{i=0}^{m-1}\sum_{(r)_{1\to i}}\sum_{n_1(n_2,n_3)}^{s,c,m-i+s_1}\sum_{(u,v)\in\mathbb{T}_{oo}}\alpha^{[\frac{t-\sum_{j=1}^{i}r_jh}{h}]+\sum_{j=1}^{i}r_jh+1}\big(h(t_1)\big)_h^{(n_1-s_1)}\alpha^{s_4}$$
$$\times\frac{(m-i)_{(s_1)}h^{s_1}}{e^{s^{1/\nu}(h(t_1)+s_1h)}}\frac{e^{s_4s^{1/\nu}h}\sin(UV)(h(t)+(s_4-i-1)h)}{(e^{-s^{1/\nu}h}+\alpha^2e^{s^{1/\nu}h}-2\alpha\cos(UV)h)^{m-i+s_1}}$$
$$=\sum_{n_1(n_2,n_3)}^{s,c,m+s_1}\sum_{(u,v)\in\mathbb{T}_{oo}}\left\{\frac{(t+mh)_h^{(n_1-s_1)}m_{(s_1)}}{h^{-s_1}e^{s^{1/\nu}((t+mh)+s_1h)}}\frac{\alpha^{s_4}e^{s_4s^{1/\nu}h}\sin(UV)(t+s_4h)}{(e^{-s^{1/\nu}h}+\alpha^2e^{s^{1/\nu}h}-2\alpha\cos(UV)h)^{m+s_1}}\right.$$
$$\left.-\alpha^{[\frac{t}{h}]+1}\frac{(h(t_2))_h^{(n_1-s_1)}m_{(s_1)}}{h^{-s_1}e^{s^{1/\nu}(h(t_2)+s_1h)}}\frac{\alpha^{s_4}e^{s_4s^{1/\nu}h}\sin(UV)(h(t)+(s_4-1)h)}{(e^{-s^{1/\nu}h}+\alpha^2e^{sh}-2\alpha\cos(UV)h)^{m+s_1}}\right\}, \tag{23}$$

where $h(t_1)=h(t)+(m-(i+1)h)$, $h(t_2)=h(t)+(m-1)h$.

Proof The proof is obtained by replacing $u(t)$ by $(t_h^{(n_1)} e^{-s^{1/\nu} t} \sin^{n_2} pt \cos^{n_3} qt)$ in Theorem 4.9 and using (20). □

Example 4.11 Consider (20), where $m = 3$, $p = 5$, $q = 3$, $\alpha = 2$, $s = 2$, $n_1 = 4$, $n_2 = 4$, $n_3 = 4$, $P = (5(4 - 2r_1) + 3(4 - 2r_2))$, and $\overline{P} = (5(4 - 2r_1) - 3(4 - 2r_2))$,

$$\begin{aligned}\text{LHS} &= \sum_{r=0}^{[\frac{t}{h}]} \frac{(r+1)_{(3-1)}}{(3-1)!} \alpha^r (t - rh)_h^{(4)} e^{-s^{1/\nu}(t-rh)} \sin^4 5(t - rh) \cos^4 3(t - rh) \\ &\quad + \sum_{i=1}^{m-1} \sum_{(r)_{1\to i}}^{[t]} \alpha^{[\frac{t - \sum_{j=1}^{i} r_j h}{h}] + \sum_{j=1}^{i} r_j + 1} \underset{\alpha(h)}{\overset{-(m-i)}{\Delta}} u\big(h(t) + (m - i - 1)h\big) \\ &= \underset{\alpha(h)}{\overset{-m}{\Delta}} u(t + mh) - \alpha^{[\frac{t}{h}]+1} \underset{\alpha(h)}{\overset{-m}{\Delta}} u\big(h(t) + (m-1)h\big). \end{aligned} \tag{24}$$

Here

$$\begin{aligned} u(t) &= \sum_{4[n_2,n_3)}^{s,c,3+s_1} \sum_{(u,v)\in\mathbb{T}_{ee}} \frac{((n_2))((n_3)) t_h^{(4-s_1)}}{h^{-s_1} e^{s^{1/\nu}(t+s_1 h)}} \frac{\alpha^{s_4} e^{s_4 s^{1/\nu} h} h \cos(UV)(t - (3 - s_4)h)}{(e^{-s^{1/\nu} h} + \alpha^2 e^{s^{1/\nu} h} - 2\alpha \cos(UV)h)^{3+s_1}} \\ &\quad + \frac{n_2^{(\frac{n_2}{2})}}{\frac{n_2}{2}!} \frac{n_3^{(\frac{n_3}{2})}}{\frac{n_3}{2}!} \frac{1}{2(e^{-s^{1/\nu} h} - \alpha)^{3+s_1}}. \end{aligned}$$

5 Laplace transforms and its applications

Here,] we derive the fractional frequency Laplace transform for the input functions (signals) in multiseries ($m = 1$) and analyze the results by MATLAB.

Theorem 5.1 *For odd positive integers n_2 and n_3,*

(i)

$$\begin{aligned} L_{\alpha(h)}\big(\sin^{n_2} pt \cos^{n_3} qt\big) &= -h \sum_{r=0}^{\infty} \big(\alpha^{-(r+1)} e^{-s^{1/\nu} rh} \sin^{n_2} prh \cos^{n_3} qrh\big) \\ &= \sum_{(n_2,n_3)}^{s,c,1} \sum_{(u,v)\in\mathbb{T}_{oo}} \frac{-h e^{s_1 s^{1/\nu} h} \alpha^{s_1} \sin(UV)(s_1 - 1)h}{(e^{-s^{1/\nu} h} + \alpha^2 e^{s^{1/\nu} h} - 2\cos(UV)h)}, \end{aligned}$$

(ii)

$$\begin{aligned} &{}_t L_{\alpha(h)}\big(\sin^{n_2} pt \cos^{n_3} qt\big) \\ &\quad = -h \times \sum_{r=0}^{\infty} \big(\alpha^{-(r+1)} e^{-s^{1/\nu}(t+rh)} \sin^{n_2} p(t + rh) \cos^{n_3} q(t + rh)\big) \\ &\quad = h \sum_{(n_2,n_3)}^{s,c,1} \sum_{(u,v)\in\mathbb{T}_{oo}} \frac{e^{s_1 s^{1/\nu} h}}{e^{s^{1/\nu} t}} \frac{\alpha^{s_1} \sin(UV)(t - (1 - s_1)h)}{(e^{-s^{1/\nu} h} + \alpha^2 e^{s^{1/\nu} h} - 2\cos(UV)h)}. \end{aligned}$$

Proof Taking the limit from 0 to ∞ in (15) gives the Laplace transform of $\sin^{n_2} pt \cos^{n_3} qt$. □

Similarly, we can find results for other cases (odd–even, even–odd, even–even).

In the following example, we analyze LTT using MATLAB.

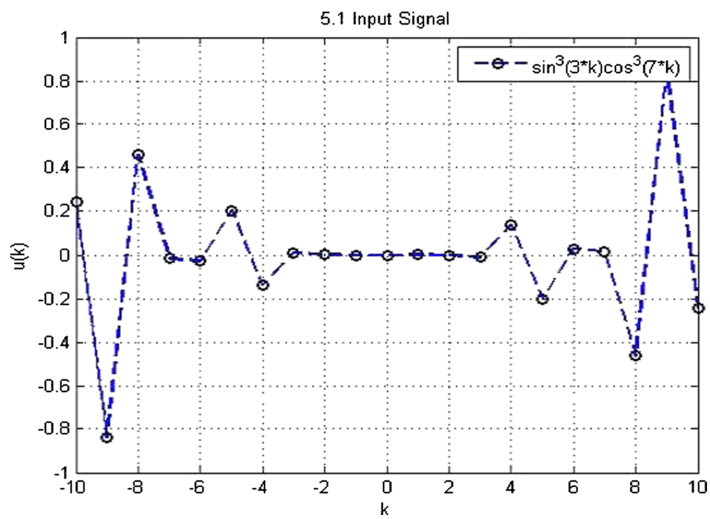

Figure 1 Input Signal

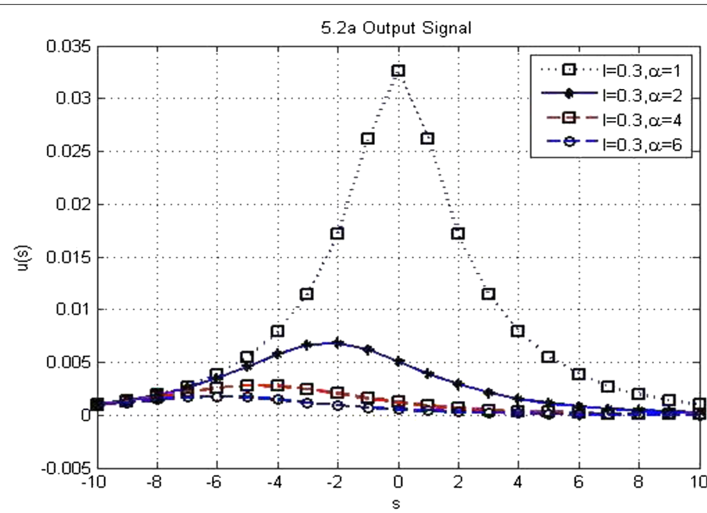

Figure 2 Output signal for $\ell = 0.3$ and α

Example 5.2 Taking $n_2 = 3, n_3 = 3, p = 3$, and $q = 7$ in Theorem 5.1, we obtain

$$L_{\alpha(h)}\big(\sin^3 3t \cos^3 7t\big) = (-h)\sum_{r=0}^{\infty} \alpha^{-(r+1)} e^{-s^{1/\nu} rh} \sin^3 3rh \cos^3 7rh$$

$$= \sum_{(3,3)}^{s,c,1} \sum_{(u,v)\in \mathbb{T}_{oo}} \frac{-he^{s_1 s^{1/nu} h} \alpha^{s_1} \sin(UV)(s_1 - 1)h}{(e^{-s^{1/\nu} h} + \alpha^2 e^{s^{1/\nu} h} - 2\cos(UV)h)},$$

which is verified for $\nu = 0.1, \alpha = 4, h = 0.5$, and $s = 10$ by MATLAB.

The results are analyzed with input and output signals. Figure 1 shows the input signal (function) for the product of sine and cosine functions. Figure 2 shows the output signal

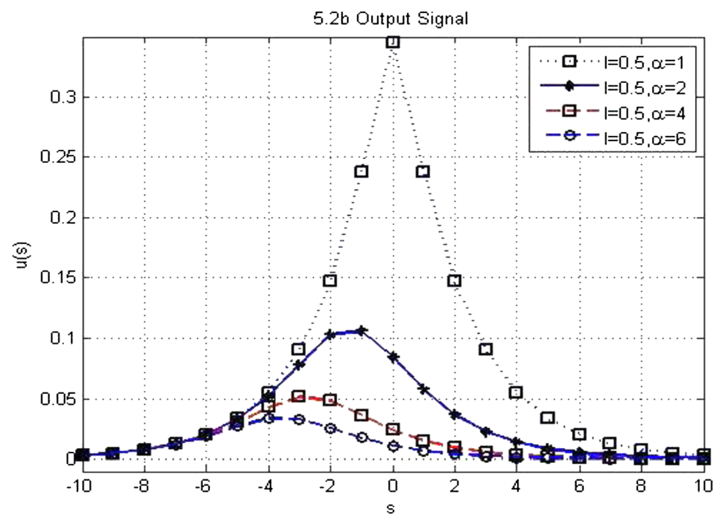

Figure 3 Output signal for $\ell = 0.5$ and α

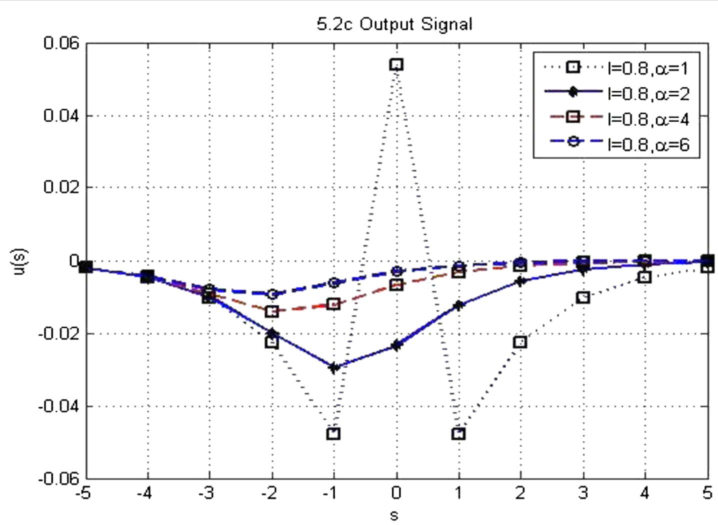

Figure 4 Output signal for $\ell = 0.8$ and α

for $\ell = 0.3$ and varying α. Figure 3 shows the output signal for $\ell = 0.5$ with varying α. Figure 4 is the output signal for $\ell = 0.8$ with varying α.

Theorem 5.3 *For odd positive integers n_2 and n_3, we have*

(i)

$$L_{\alpha(h)}\bigl(t^{n_1}\sin^{n_2}pt\cos^{n_3}qt\bigr)$$

$$= -h\sum_{r=0}^{\infty}\bigl(\alpha^{-(r+1)}(rh)^{n_1}e^{-s^{1/v}rh}\sin^{n_2}prh\cos^{n_3}qrh\bigr)$$

$$= \sum_{r_1=1}^{n_1}\sum_{n_1(n_2,n_3)}^{s,c,1+r_1}\sum_{(u,v)\in\mathbb{T}_{oo}}\frac{S_{r_1}^{n_1}h^{n_1-r_1}1_{(r_1)}\alpha^{s_4}}{h^{-r_1}e^{s^{1/v}r_1h}}$$

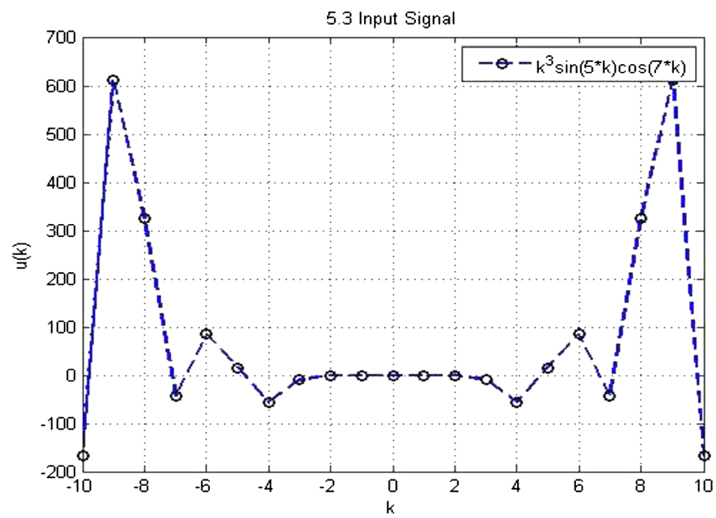

Figure 5 Input signal

$$\times \frac{e^{s_4 s^{1/\nu} h} \sin(UV)(s_4-1)h}{(e^{-s^{1/\nu}h} + \alpha^2 e^{s^{1/\nu}h} - 2\alpha\cos(UV)h)^{1+r_1}}.$$

(ii)

$$\begin{aligned}
&{}_tL_{\alpha(h)}\bigl(t^{n_1}\sin^{n_2} pt\cos^{n_3} qt\bigr)\\
&\quad = -h \times \sum_{r=0}^{\infty}\bigl(\alpha^{-(r+1)}(t+rh)^{n_1} e^{-s^{1/\nu}(t+rh)} \sin^{n_2} p(t+rh)\cos^{n_3} q(t+rh)\bigr)\\
&\quad = \sum_{r_1=1}^{n_1} \sum_{n_1(n_2,n_3)}^{s,c,1+s_1} \sum_{(u,v)\in\mathbb{T}_{oo}}\\
&\qquad \times \frac{S_{r_1}^{n_1} t_h^{(r_1-s_1)} 1_{(s_1)} \alpha^{s_4}}{h^{r_1-n_1} h^{-s_1} e^{s^{1/\nu}(t+s_1h)}} \frac{e^{s_4 s^{1/\nu}h}\sin(UV)(t-(1-s_4)h)}{(e^{-s^{1/\nu}h} + \alpha^2 e^{s^{1/\nu}h} - 2\alpha\cos(UV)h)^{1+s_1}}.
\end{aligned}$$

Proof Taking the limit 0 to ∞ in (18) gives the Laplace transform of $t^{n_1}\sin^{n_2} pt\cos^{n_3} qt$. □

Similarly, we can find results for other cases (odd-even, even–odd, even–even).

Example 5.4 Taking $n_1 = 3, n_2 = 1, n_3 = 1, p = 5$, and $q = 7$ in Theorem 5.3, we obtain

$$\begin{aligned}
&L_{\alpha(h)}\bigl(t^3\sin 5t\cos 7t\bigr)\\
&\quad = (-h)\sum_{r=0}^{\infty} \alpha^{-(r+1)}(rh)^3 e^{-s^{1/\nu}rh}\sin 5rh\cos 7rh\\
&\quad = \sum_{r_1=1}^{3}\sum_{3(1,1)}^{s,c,1+r_1}\sum_{(u,v)\in\mathbb{T}_{oo}} \frac{S_{r_1}^3 h^{3-r_1} 1_{(r_1)}\alpha^{s_4}}{h^{-r_1} e^{s^{1/\nu}r_1h}} \frac{e^{s_4 s^{1/\nu}h}\sin(UV)(s_4-1)h}{(e^{-s^{1/\nu}h} + \alpha^2 e^{s^{1/\nu}h} - 2\alpha\cos(UV)h)^{1+r_1}},
\end{aligned}$$

which is verified for $\alpha = 5, h = 0.8, \nu = 0.1$, and $s = 15$ by MATLAB.

The results are analyzed with input and output signals. Figure 5 shows the input signal (function) for the product of polynomial, sine, and cosine functions. Figure 6

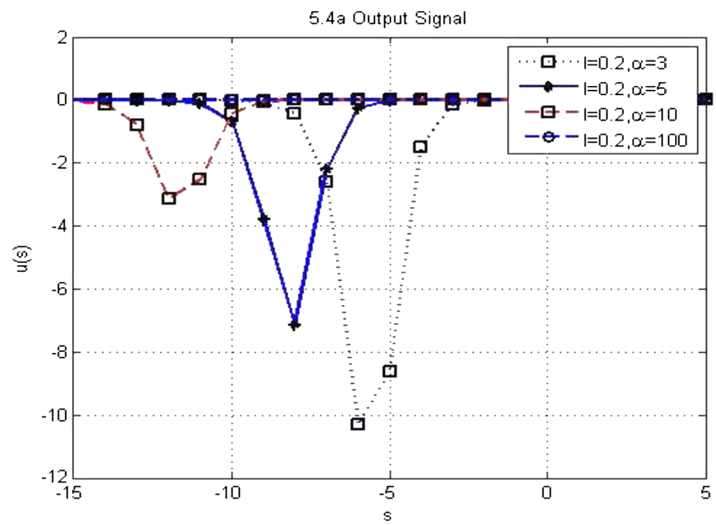

Figure 6 Output signal for $\ell = 0.2$ and α

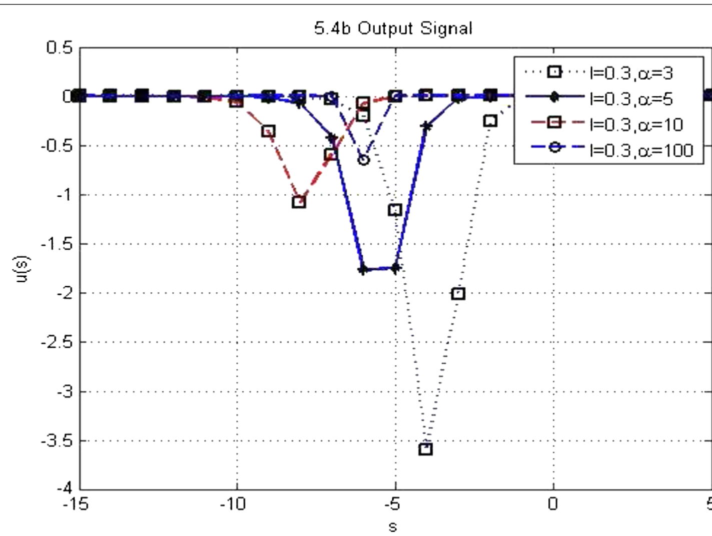

Figure 7 Output signal for $\ell = 0.3$ and α

shows the output signal for $\ell = 0.2$ with varying α. Figure 7 shows the output signal for $\ell = 0.3$ and varying α. Figure 8 shows the output signal for $\ell = 0.8$ with varying α.

6 Conclusions

We proposed formulas for the frequency Laplace transforms of the products of two and three functions and a multiseries formula for circular functions. Further, LFT is employed on circular functions to get appropriate results numerically and also analyzed the findings for different values of tuning factor α and fractional frequency factor $s^{1/\nu}$. We also observed with the help of the diagrams generated by MATLAB that LFT gives innumerable outcomes for the given input signal, and this enables us to make a choice for an optimal

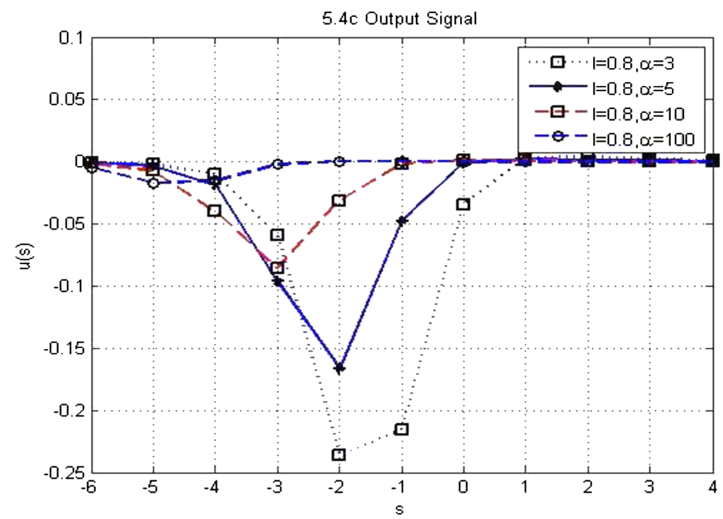

Figure 8 Output signal for $\ell = 0.8$ and α

one. As a very important findingof this research, when $\alpha = \nu = 1$, we get the Laplace transform existing in the literature.

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Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur, India. [2]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, Saudi Arabia. [3]Department of Medical Research, China Medical University, Taichung, Taiwan. [4]Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan. [5]Department of Mathematics, Çankaya University, Ankara, Turkey.

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