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Analysis and applications of the proportional Caputo derivative

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Abstract
In this paper, we investigate the analysis of the proportional Caputo derivative that recently has been constructed. We create some useful relations between this new derivative and beta function. We discretize the new derivative. We investigate the stability and obtain a stability condition for the new derivative.

1 Introduction

Fractional calculus is an emerging field of mathematics [1] having important contributions in modeling the dynamics of complex systems [2, 3] from various fields of science and engineering [4, 5]. Nowadays a huge debate was opened by asking the simple" question: can we classify the fractional operators?" Curiously the answer of this question is not simple and, so far, several answers seemed to be possible [6–11]. A new non-singular fractional operator was proposed by Caputo and Fabrizio [12] and their result was generalized by Atangana and Baleanu [13] and applied successfully to a lot of complex phenomena including biological ones.

Khalid et al. [14] have studied the computational research of the Caputo time fractional Allen–Cahn equation. Owolabi [15] has studied by analysis and numerical simulation a multicomponent system with the Atangana–Baleanu fractional derivative. Akgül [16] has presented a novel method for a fractional derivative with non-local and non-singular kernel. Akgül [17] has investigated the solutions of differential equations with the generalized fractional derivatives. Atangana et al. [18] have investigated the analysis of the fractal fractional derivatives in detail. Fernandez et al. [19] investigated the series representations for fractional-calculus operators involving generalized Mittag-Leffler functions. Wu et al. [20] have investigated the fractional impulsive differential equations including the exact solutions, integral equations and short memory case. Some inequalities were investigated within the proportional fractional operators [21, 22] and in [23] was investigated the proportional derivatives of a function with respect to another function. Very recently, a new fractional operator has been constructed in [24]:

$$ {}_0^{PC}D_t^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \left(k_1(\alpha,\tau)f(\tau) + k_0(\alpha,\tau)\frac{df(\tau)}{d\tau}\right)(t-\tau)^{-\alpha}\,d\tau. \tag{1.1}$$

In this paper, we aim to analyze the above derivative in detail for $k_0(\alpha,t) = (\alpha t^{1-\alpha})c^{2\alpha}$ and $k_1(\alpha,t) = (1-\alpha)t^{\alpha}$. Here c is as a constant of the time dimension t for the two terms involved in the new derivative (1.1).

The new fractional operator in the Caputo sense is a generalization of the classical proportional derivative introduced by [24] which has deep applications in control theory. The new fractional operator will provide better applications in control theory. Due to the physical meaning of the initial conditions we concentrate here on the Caputo fractional generalization. For more details see [25–28].

We construct the paper as follows. We give some scientific theorems for the new derivative in Sect. 2. We present the discretization and the applications of the proportional Caputo derivative in Sect. 3. We show the stability analysis in Sect. 4. We demonstrate the numerical results in Sect. 5. We discuss the conclusion in the last section.

2 Analysis of the proportional Caputo derivative

We present the following scientific results for the new derivative.

Lemma 2.1 *We have the following relation for the new derivative given by* (1.1):

$$\begin{aligned}\left|{}_0^{PC}D_t^{\alpha}f(t)\right| &< \frac{t(1-\alpha)}{\Gamma(1-\alpha)}\left\|f(\tau)\right\|_{\infty}B(\alpha+1,1-\alpha)\\ &\quad+\frac{\alpha c^{2\alpha}}{\Gamma(1-\alpha)}t^{2-2\alpha}\left\|\frac{df(\tau)}{d\tau}\right\|_{\infty}B(2-\alpha,1-\alpha).\end{aligned}$$

Proof We have

$$\begin{aligned}\left|{}_0^{PC}D_t^{\alpha}f(t)\right| &= \frac{1}{\Gamma(1-\alpha)}\left|\int_0^t\left((1-\alpha)\tau^{\alpha}f(\tau)+c^{2\alpha}\alpha\tau^{1-\alpha}\frac{df(\tau)}{d\tau}\right)(t-\tau)^{-\alpha}\,d\tau\right|\\ &\le \frac{1}{\Gamma(1-\alpha)}\left|\int_0^t\left((1-\alpha)\tau^{\alpha}f(\tau)\right)(t-\tau)^{-\alpha}\,d\tau\right|\\ &\quad+\frac{c^{2\alpha}}{\Gamma(1-\alpha)}\left|\int_0^t\left(\alpha\tau^{1-\alpha}\frac{df(\tau)}{d\tau}\right)(t-\tau)^{-\alpha}\,d\tau\right|\\ &< \frac{1}{\Gamma(1-\alpha)}\int_0^t\left|f(\tau)\right|(1-\alpha)\tau^{\alpha}(t-\tau)^{-\alpha}\,d\tau\\ &\quad+\frac{c^{2\alpha}}{\Gamma(1-\alpha)}\int_0^t\left|\frac{df(\tau)}{d\tau}\right|\alpha\tau^{1-\alpha}(t-\tau)^{-\alpha}\,d\tau\\ &< \frac{1}{\Gamma(1-\alpha)}\int_0^t\sup_{\tau\in[0,t]}\left|f(\tau)\right|(1-\alpha)\tau^{\alpha}(t-\tau)^{-\alpha}\,d\tau\\ &\quad+\frac{c^{2\alpha}}{\Gamma(1-\alpha)}\int_0^t\sup_{\tau\in[0,t]}\left|\frac{df(\tau)}{d\tau}\right|\alpha\tau^{1-\alpha}(t-\tau)^{-\alpha}\,d\tau.\end{aligned}$$

Then we obtain

$$\begin{aligned}\left|{}_0^{PC}D_t^{\alpha}f(t)\right| &< \frac{(1-\alpha)}{\Gamma(1-\alpha)}\left\|f(\tau)\right\|_{\infty}\int_0^t\tau^{\alpha}(t-\tau)^{-\alpha}\,d\tau\\ &\quad+\frac{\alpha c^{2\alpha}}{\Gamma(1-\alpha)}\left\|\frac{df(\tau)}{d\tau}\right\|_{\infty}\int_0^t\tau^{1-\alpha}(t-\tau)^{-\alpha}\,d\tau.\end{aligned}$$

Let $\tau = th$. Then we obtain

$$\begin{aligned}\left|{}_{0}^{PC}D_{t}^{\alpha}f(t)\right| &< \frac{(1-\alpha)}{\Gamma(1-\alpha)}\left\|f(\tau)\right\|_{\infty}\int_{0}^{1}(th)^{\alpha}(t-th)^{-\alpha}t\,dh \\ &\quad + \frac{\alpha c^{2\alpha}}{\Gamma(1-\alpha)}\left\|\frac{df(\tau)}{d\tau}\right\|_{\infty}\int_{0}^{1}(th)^{1-\alpha}(t-th)^{-\alpha}t\,dh \\ &< \frac{t(1-\alpha)}{\Gamma(1-\alpha)}\left\|f(\tau)\right\|_{\infty}\int_{0}^{1}h^{\alpha}(1-h)^{-\alpha}\,dh \\ &\quad + \frac{\alpha c^{2\alpha}}{\Gamma(1-\alpha)}t^{2-2\alpha}\left\|\frac{df(\tau)}{d\tau}\right\|_{\infty}\int_{0}^{1}h^{1-\alpha}(1-h)^{-\alpha}\,dh.\end{aligned}$$

Then we get the desired result:

$$\begin{aligned}\left|{}_{0}^{PC}D_{t}^{\alpha}f(t)\right| &< \frac{t(1-\alpha)}{\Gamma(1-\alpha)}\left\|f(\tau)\right\|_{\infty}B(\alpha+1,1-\alpha) \\ &\quad + \frac{\alpha c^{2\alpha}}{\Gamma(1-\alpha)}t^{2-2\alpha}\left\|\frac{df(\tau)}{d\tau}\right\|_{\infty}B(2-\alpha,1-\alpha).\end{aligned}$$

This completes the proof. □

Remark 2.2 We consider

$$\begin{aligned}&{}_{0}^{PC}D_{x}^{\gamma}\big(u(x)v(x)\big) \\ &\quad = \frac{1}{\Gamma(1-\gamma)}\int_{0}^{x}\left((1-\gamma)t^{\gamma}u(t)v(t)+\gamma c^{2\gamma}t^{1-\gamma}\frac{du(t)v(t)}{dt}\right)(x-t)^{-\gamma}\,dt.\end{aligned} \tag{2.1}$$

If u and v are continuous and bounded, then we get

$$\begin{aligned}{}_{0}^{PC}D_{x}^{\alpha}\big(u(x)v(x)\big) &= \frac{1}{\Gamma(1-\gamma)}\int_{0}^{x}\big((1-\gamma)t^{\gamma}u(t)v(t)\big)(x-t)^{-\gamma}\,dt \\ &\quad + \frac{1}{\Gamma(1-\gamma)}\int_{0}^{x}\gamma c^{2\gamma}t^{1-\gamma}\left(\frac{du(t)}{dt}v(t)+\frac{dv(t)}{dt}u(t)\right)(x-t)^{-\gamma}\,dt.\end{aligned}$$

Lemma 2.3 *Assume that f and g are differentiable and bounded. Then we obtain*

$$\begin{aligned}\left|{}_{0}^{PC}D_{t}^{\alpha}\big(f(t)g(t)\big)\right| &< \frac{t(1-\alpha)}{\Gamma(1-\alpha)}\left\|f(t)\right\|_{\infty}\left\|g(t)\right\|_{\infty}B(\alpha+1,1-\alpha) \\ &\quad + \frac{\alpha t^{2-2\alpha}}{\Gamma(1-\alpha)}\left\|\frac{df(t)}{dt}\right\|_{\infty}\left\|g(t)\right\|_{\infty}B(2-\alpha,1-\alpha) \\ &\quad + \frac{\alpha t^{2-2\alpha}}{\Gamma(1-\alpha)}\left\|\frac{dg(t)}{dt}\right\|_{\infty}\left\|f(t)\right\|_{\infty}B(2-\alpha,1-\alpha).\end{aligned}$$

Proof We have

$$\begin{aligned}\left|{}_0^{PC}D_t^{\alpha}\bigl(f(t)g(t)\bigr)\right| &< \frac{t(1-\alpha)}{\Gamma(1-\alpha)}\bigl\|f(t)\bigr\|_{\infty}\bigl\|g(t)\bigr\|_{\infty}B(\alpha+1,1-\alpha)\\ &\quad+\frac{\alpha t^{2-2\alpha}}{\Gamma(1-\alpha)}\left\|\frac{df(t)}{dt}\right\|_{\infty}\bigl\|g(t)\bigr\|_{\infty}B(2-\alpha,1-\alpha)\\ &\quad+\frac{\alpha t^{2-2\alpha}}{\Gamma(1-\alpha)}\left\|\frac{dg(t)}{dt}\right\|_{\infty}\bigl\|f(t)\bigr\|_{\infty}B(2-\alpha,1-\alpha).\end{aligned}$$

Let $\tau = th$. Then we obtain

$$\begin{aligned}\left|{}_0^{PC}D_t^{\alpha}\bigl(f(t)g(t)\bigr)\right| &< \frac{t(1-\alpha)}{\Gamma(1-\alpha)}\bigl\|f(t)\bigr\|_{\infty}\bigl\|g(t)\bigr\|_{\infty}B(\alpha+1,1-\alpha)\\ &\quad+\frac{\alpha t^{2-2\alpha}}{\Gamma(1-\alpha)}\left\|\frac{df(t)}{dt}\right\|_{\infty}\bigl\|g(t)\bigr\|_{\infty}B(2-\alpha,1-\alpha)\\ &\quad+\frac{\alpha t^{2-2\alpha}}{\Gamma(1-\alpha)}\left\|\frac{dg(t)}{dt}\right\|_{\infty}\bigl\|f(t)\bigr\|_{\infty}B(2-\alpha,1-\alpha).\end{aligned}$$

This completes the proof. □

Lemma 2.4 *If f and g are differentiable and satisfy the following condition*:

$$\left\|\frac{df}{dt}-\frac{dg}{dt}\right\|_{\infty} < K\|f-g\|_{\infty}, \tag{2.2}$$

then we have

$$\bigl\|{}_0^{PC}D_t^{\alpha}f(t)-{}_0^{PC}D_t^{\alpha}g(t)\bigr\|_{\infty} < K\|f-g\|_{\infty}. \tag{2.3}$$

Proof We have

$$\begin{aligned}\bigl\|{}_0^{PC}D_t^{\alpha}f(t)-{}_0^{PC}D_t^{\alpha}g(t)\bigr\|_{\infty} &< \frac{(1-\alpha)}{\Gamma(1-\alpha)}\bigl\|f(t)-g(t)\bigr\|_{\infty}\int_0^t \tau^{\alpha}(t-\tau)^{-\alpha}\,d\tau\\ &\quad+\frac{\alpha c^{2\alpha}}{\Gamma(1-\alpha)}\left\|\frac{df(t)}{dt}-\frac{dg(t)}{dt}\right\|_{\infty}\int_0^t \tau^{1-\alpha}(t-\tau)^{-\alpha}\,d\tau.\end{aligned}$$

Let $\tau = th$. Then we obtain

$$\bigl\|{}_0^{PC}D_t^{\alpha}f(t)-{}_0^{PC}D_t^{\alpha}g(t)\bigr\|_{\infty} < K\bigl\|f(t)-g(t)\bigr\|_{\infty}.$$

This completes the proof. □

Lemma 2.5 *Let f be analytic around* 0, *then we obtain*

$$\begin{aligned}{}_0^{PC}D_t^{\alpha}f(t) &= t(1-\alpha)\sum_{j=0}^{\infty}a_j t^j\frac{\Gamma(j+\alpha+1)}{\Gamma(j+2)} &(2.4)\\ &\quad+t^{1-2\alpha}\alpha c^{2\alpha}\sum_{j=0}^{\infty}ja_j t^j\frac{\Gamma(j-\alpha+1)}{\Gamma(j-2\alpha+2)}. &(2.5)\end{aligned}$$

Proof We have

$$
\begin{aligned}
{}_0^{PC}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \left((1-\alpha)\tau^{\alpha} f(\tau) + \alpha c^{2\alpha} \tau^{1+\alpha} \frac{df(\tau)}{d\tau} \right) (t-\tau)^{-\alpha}\, d\tau \\
&= \frac{(1-\alpha)}{\Gamma(1-\alpha)} \sum_{j=0}^{\infty} a_j \int_0^t \tau^{\alpha+j} (t-\tau)^{-\alpha}\, d\tau \\
&\quad + \frac{\alpha c^{2\alpha}}{\Gamma(1-\alpha)} \sum_{j=0}^{\infty} j a_j \int_0^t \tau^{j+\alpha} (t-\tau)^{-\alpha}\, d\tau.
\end{aligned}
$$

We let $\tau = ht$. Then we obtain

$$
\begin{aligned}
{}_0^{PC}D_t^{\alpha} f(t) &= t(1-\alpha) \sum_{j=0}^{\infty} a_j t^j \frac{\Gamma(j+\alpha+1)}{\Gamma(j+2)} \\
&\quad + t^{1-2\alpha} \alpha c^{2\alpha} \sum_{j=0}^{\infty} j a_j t^j \frac{\Gamma(j-\alpha+1)}{\Gamma(j-2\alpha+2)}.
\end{aligned}
$$

This completes the proof. □

3 Discretization and applications of the proportional Caputo derivative

We consider the new derivative [24]:

$$
{}_0^{PC}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \left((1-\alpha)\tau^{\alpha} f(\tau) + \alpha c^{2\alpha} \tau^{1-\alpha} \frac{df(\tau)}{d\tau} \right) (t-\tau)^{-\alpha}\, d\tau. \tag{3.1}
$$

We put $t_n = n\Delta t$, then at t_{n+1}, we have

$$
\begin{aligned}
{}_0^{PC}D_t^{\alpha} f(t_{n+1}) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{n+1}} \left((1-\alpha)\tau^{\alpha} f(\tau) + \alpha c^{2\alpha} \tau^{1-\alpha} \frac{df(\tau)}{d\tau} \right) (t_{n+1}-\tau)^{-\alpha}\, d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \left((1-\alpha) t_j^{\alpha} f^{j+1} + \alpha c^{2\alpha} t_j^{1-\alpha} \frac{f^{j+1}-f^j}{\Delta t} \right) (t_{n+1}-\tau)^{-\alpha}\, d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \left((1-\alpha) t_j^{\alpha} f^{j+1} + \alpha c^{2\alpha} t_j^{1-\alpha} \frac{f^{j+1}-f^j}{\Delta t} \right) \\
&\quad \times \int_{t_j}^{t_{j+1}} (t_{n+1}-\tau)^{-\alpha}\, d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \left((1-\alpha) t_j^{\alpha} f^{j+1} + \alpha c^{2\alpha} t_j^{1-\alpha} \frac{f^{j+1}-f^j}{\Delta t} \right) \\
&\quad \times \left[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha} \right].
\end{aligned}
$$

We take into consideration [18]

$$
{}_0^{PC}D_t^{\alpha} u(x,t) = f\bigl(x,t,u(x,t)\bigr). \tag{3.2}
$$

Here $u(x,0) = g(x)$, $x_m - x_{m-1} = \Delta x$, $t_{n+1} - t_n = \Delta t$, $t_n = n\Delta t$, $x_m = m\Delta x$. The above equation can be approximated as

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{n}\left((1-\alpha)t_j^{\alpha}u_m^{j+1} + \alpha c^{2\alpha}t_j^{1-\alpha}\frac{u_m^{j+1}-u_m^{j}}{\Delta t}\right)\left[(n-j+1)^{1-\alpha}-(n-j)^{1-\alpha}\right]\\&\quad= f\left(x_m, t_{n+1}, u_m^{n+1}\right).\end{aligned}$$

4 Stability analysis

We discretize the following problem and investigate the stability of it. We consider the heat equation,

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{4.1}$$

We change the left hand side of the above equation with the new derivative and we obtain

$${}_{0}^{PC}D_t^{\alpha}u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{4.2}$$

We obtain

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)}\sum_{p=0}^{s}\left((1-\alpha)t_p^{\alpha}u_m^{p+1} + \alpha c^{2\alpha}t_p^{1-\alpha}\frac{u_m^{p+1}-u_m^{p}}{\Delta t}\right)\left[(s-p+1)^{1-\alpha}-(s-p)^{1-\alpha}\right]\\&\quad= k\frac{u_{m+1}^{s+1}-2u_m^{s+1}+u_{m-1}^{s+1}}{(\Delta x)^2}\end{aligned}$$

at (t_{s+1}, x_m). We put $u_m^s = \delta_s\exp(ik_m x)$. Plugging this into the above equation, we obtain

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)}\sum_{p=0}^{s}\left((1-\alpha)t_p^{\alpha}\delta_{p+1}\exp(ik_m x) + \alpha c^{2\alpha}t_p^{1-\alpha}\frac{\delta_{p+1}\exp(ik_m x)-\delta_p\exp(ik_m x)}{\Delta t}\right)\\&\qquad\times\left[(s-p+1)^{1-\alpha}-(s-p)^{1-\alpha}\right]\\&\quad= k\frac{\delta_{s+1}\exp(ik_m(x+\Delta x))-2\delta_{s+1}\exp(ik_m x)+\delta_{s+1}\exp(ik_m(x-\Delta x))}{(\Delta x)^2}.\end{aligned}$$

After simplification we get

$$\begin{aligned}&\frac{1}{\Gamma(1-\alpha)}\sum_{p=0}^{s}\left((1-\alpha)t_p^{\alpha}\delta_{p+1} + \alpha c^{2\alpha}t_p^{1-\alpha}\frac{\delta_{p+1}-\delta_p}{\Delta t}\right)\\&\qquad\times\left[(s-p+1)^{1-\alpha}-(s-p)^{1-\alpha}\right]\\&\quad= k\frac{\delta_{s+1}\exp(ik_m(\Delta x))-2\delta_{s+1}+\delta_{s+1}\exp(ik_m(-\Delta x))}{(\Delta x)^2}.\end{aligned}$$

For simplicity, we take

$$A_{p,\alpha} = \frac{(1-\alpha)(p\Delta t)^{\alpha}}{\Gamma(1-\alpha)},\qquad B_{p,\alpha} = \frac{\alpha c^{2\alpha}(p\Delta t)^{1-\alpha}}{\Gamma(1-\alpha)\Delta t},\qquad a = \frac{k}{(\Delta x)^2}.$$

Then we obtain

$$\sum_{p=0}^{s}\bigl(A_{p,\alpha}\delta_{p+1}+B_{p,\alpha}(\delta_{p+1}-\delta_p)\bigr)\bigl[(s-p+1)^{1-\alpha}-(s-p)^{1-\alpha}\bigr]$$
$$=a\delta_{s+1}\exp\bigl(ik_m(\Delta x)\bigr)-2a\delta_{s+1}+a\delta_{s+1}\exp\bigl(ik_m(-\Delta x)\bigr).$$

Thus, we obtain

$$\sum_{p=0}^{s}\bigl((A_{p,\alpha}+B_{p,\alpha})\delta_{p+1}-B_{p,\alpha}\delta_p\bigr)\bigl[(s-p+1)^{1-\alpha}-(s-p)^{1-\alpha}\bigr]$$
$$=a\delta_{s+1}\bigl(\exp\bigl(ik_m(\Delta x)\bigr)-2+\exp\bigl(-ik_m(\Delta x)\bigr)\bigr).$$

Using the relation between the trigonometric functions and exponential functions gives

$$\sum_{p=0}^{s}\bigl((A_{p,\alpha}+B_{p,\alpha})\delta_{p+1}-B_{p,\alpha}\delta_p\bigr)\bigl[(s-p+1)^{1-\alpha}-(s-p)^{1-\alpha}\bigr]=-4a\delta_{s+1}\sin^2\biggl(\frac{k_m\Delta x}{2}\biggr).$$

For $s=0$, we obtain

$$\bigl((A_{0,\alpha}+B_{0,\alpha})\delta_1-B_{0,\alpha}\delta_0\bigr)=-4a\delta_1\sin^2\biggl(\frac{k_m\Delta x}{2}\biggr).$$

Here $|\frac{\delta_1}{\delta_0}|<1$ implies

$$\biggl|\frac{B_{0,\alpha}}{A_{0,\alpha}+B_{0,\alpha}+4a\sin^2(\frac{k_m\Delta x}{2})}\biggr|<1.$$

This is true for $\forall m$. Thus, we get

$$\biggl|\frac{B_{0,\alpha}}{A_{0,\alpha}+B_{0,\alpha}+4a}\biggr|<1.$$

We assume that $|\frac{\delta_s}{\delta_0}|<1$. We need to show that $|\frac{\delta_{s+1}}{\delta_0}|<1$. We know that

$$\sum_{p=0}^{s}\bigl((A_{p,\alpha}+B_{p,\alpha})\delta_{p+1}-B_{p,\alpha}\delta_p\bigr)\bigl[(s-p+1)^{1-\alpha}-(s-p)^{1-\alpha}\bigr]=-4a\delta_{s+1}\sin^2\biggl(\frac{k_m\Delta x}{2}\biggr).$$

Then we get

$$\biggl|-4a\delta_{s+1}\sin^2\biggl(\frac{k_m\Delta x}{2}\biggr)\biggr|=\biggl|\sum_{p=0}^{s}\bigl((A_{p,\alpha}+B_{p,\alpha})\delta_{p+1}-B_{p,\alpha}\delta_p\bigr)\bigl[(s-p+1)^{1-\alpha}-(s-p)^{1-\alpha}\bigr]\biggr|.$$

Thus, we reach

$$\delta_{s+1}\biggl|-4a\sin^2\biggl(\frac{k_m\Delta x}{2}\biggr)\biggr|<\delta_0\biggl|\sum_{p=0}^{s}\bigl((A_{p,\alpha}+B_{p,\alpha})-B_{p,\alpha}\bigr)\bigl[(s-p+1)^{1-\alpha}-(s-p)^{1-\alpha}\bigr]\biggr|.$$

Here $|\frac{\delta_{s+1}}{\delta_0}| < 1$ implies

$$\frac{\sum_{p=0}^{s} |((A_{p,\alpha} + B_{p,\alpha}) - B_{p,\alpha})[(s-p+1)^{1-\alpha} - (s-p)^{1-\alpha}]|}{|-4a\sin^2(\frac{k_m\Delta x}{2})|} < 1.$$

This is true for $\forall m$. Thus, we get

$$\frac{\sum_{p=0}^{s} |((A_{p,\alpha} + B_{p,\alpha}) - B_{p,\alpha})[(s-p+1)^{1-\alpha} - (s-p)^{1-\alpha}]|}{|-4a|} < 1.$$

Therefore, the method is stable if

$$\min\left(\left|\frac{B_{0,\alpha}}{A_{0,\alpha} + B_{0,\alpha} + 4a}\right|, \frac{\sum_{p=0}^{s} |((A_{p,\alpha} + B_{p,\alpha}) - B_{p,\alpha})[(s-p+1)^{1-\alpha} - (s-p)^{1-\alpha}]|}{|-4a|}\right) < 1.$$

5 Numerical results

We consider the following problem:

$$ {}_{0}^{CPC}D_x^{\alpha} u(x) = \sin(x) \tag{5.1}$$

where

$$ {}_{0}^{CPC}D_x^{\alpha} u(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \left(k_1(\alpha)u(t) + k_0(\alpha)\frac{du(t)}{dt}\right)(x-t)^{-\alpha}\,dt. \tag{5.2}$$

We apply the Laplace transform to Eq. (5.1):

$$L\big({}_{0}^{CPC}D_x^{\alpha} u(x)\big) = L\big(\sin(x)\big). \tag{5.3}$$

Then we obtain

$$\left[\frac{K_1(\alpha)}{s} + K_0(\alpha)\right]s^{\alpha} L\big(u(t)\big) - K_0(\alpha)s^{\alpha-1}u(0) = \frac{1}{1+s^2}. \tag{5.4}$$

After simplification, we get

$$L\big(u(x)\big) = \frac{(1+s^2)K_0(\alpha)s^{\alpha-1}u(0) + 1}{(1+s^2)(s^{\alpha-1}K_1(\alpha) + s^{\alpha}K_0(\alpha))}. \tag{5.5}$$

If we apply the inverse Laplace transform to the above equation, we will obtain

$$u(x) = u(0)\exp\left(\frac{-K_1(\alpha)}{K_0(\alpha)}x\right) + \frac{x^{\alpha}A(x,\alpha)}{(K_1(\alpha)^2 + K_2(\alpha)^2)\Gamma(\alpha)} \tag{5.6}$$

where

$$\begin{aligned} A(x,\alpha) = {} & K_1(\alpha)\exp\left(\frac{-K_1(\alpha)}{K_0(\alpha)}x\right)\left(-\frac{K_1(\alpha)}{K_0(\alpha)}\right)^{-\alpha}\left(-\Gamma(\alpha) + \Gamma\left(\alpha, \frac{-K_1(\alpha)}{K_1(\alpha)}x\right)\right) \\ & + \frac{1}{\alpha}\mathrm{HypergeometricPFQ}\left[\left\{\frac{\alpha}{2}\right\}, \left\{\frac{1}{2}, 1 + \frac{\alpha}{2}\right\}, -\frac{x^2}{4}\right] \end{aligned}$$

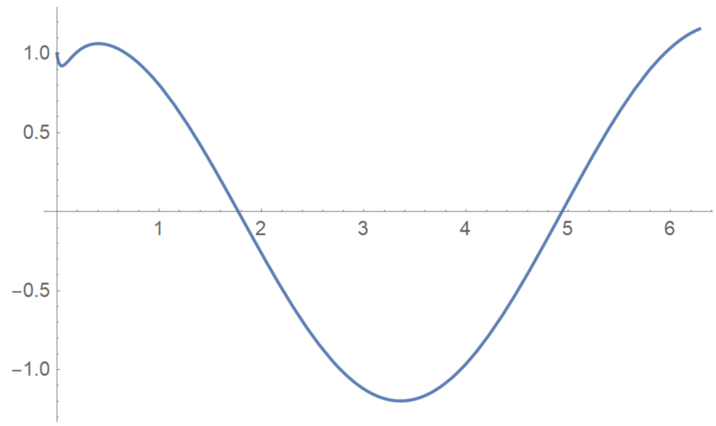

Figure 1 Solution of the problem for $\alpha = 0.1$

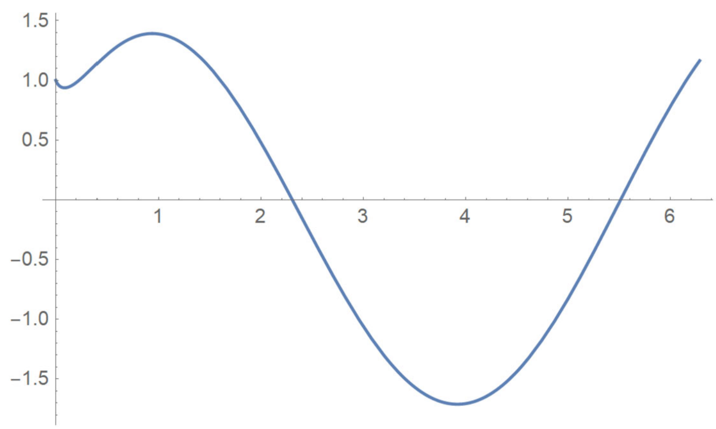

Figure 2 Solution of the problem for $\alpha = 0.3$

$$\times \bigl(K_1(\alpha)\cos(x) + K_0(\alpha)\sin(x)\bigr) + \frac{1}{1+\alpha} x\bigl(-K_0(\alpha)\cos(x) + K_1(\alpha)\sin(x)\bigr)$$
$$\times \text{HypergeometricPFQ}\left[\left\{\frac{1}{2} + \frac{\alpha}{2}\right\}, \left\{\frac{3}{2}, \frac{3}{2}, \frac{\alpha}{2}\right\}, -\frac{x^2}{4}\right].$$

We demonstrate the above solution by the following figures for different values of α. We choose $K_1(\alpha) = (1-\alpha)w^\alpha$, $K_0(\alpha) = \alpha c^{2\alpha} w^{1-\alpha}$, $c = 1$, $w = 0.5$ and $u(0) = 1$ in Figs. 1–6. In Fig. 7, we choose $c = w = \alpha = 0.8$. In these figures, we can see the effect of the fractional order.

6 Conclusion

We presented the analysis of the proportional Caputo derivative in this paper. We presented some scientific theorems for this new derivative. We discretized the new derivative. We presented the stability analysis and experiments. We obtained the stability condition for a problem using the new derivative. We considered a problem with the constant proportional Caputo derivative. We solved the problem by the Laplace transform. We demonstrated the numerical simulations by some figures.

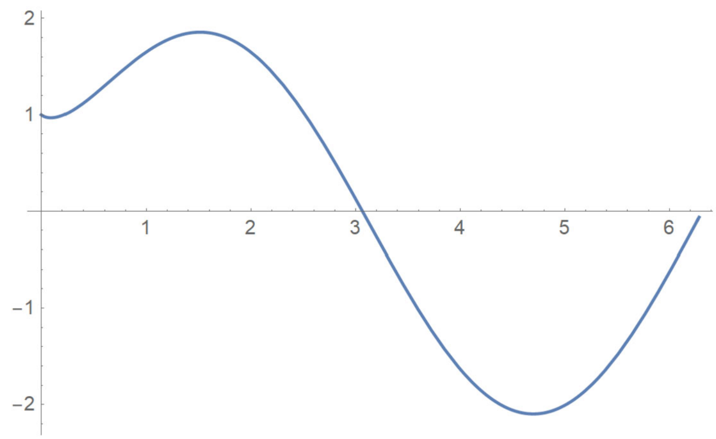

Figure 3 Solution of the problem for $\alpha = 0.5$

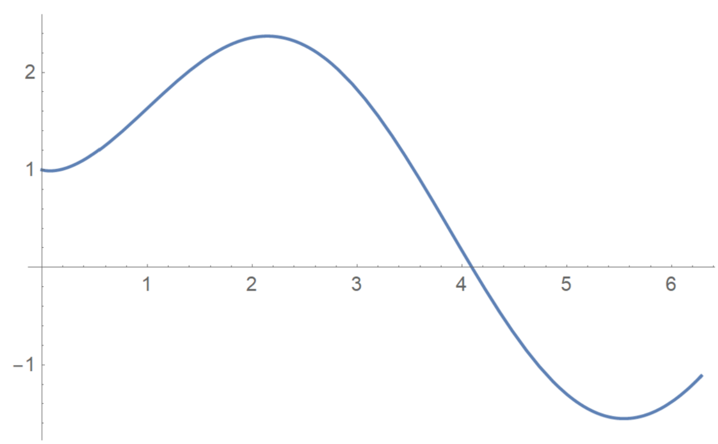

Figure 4 Solution of the problem for $\alpha = 0.7$

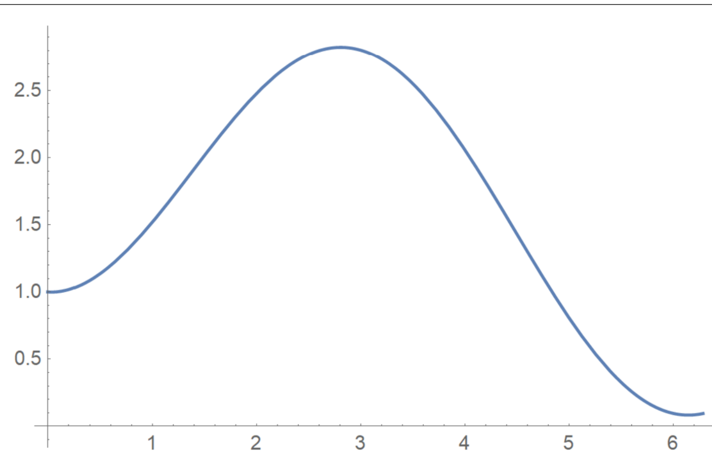

Figure 5 Solution of the problem for $\alpha = 0.9$

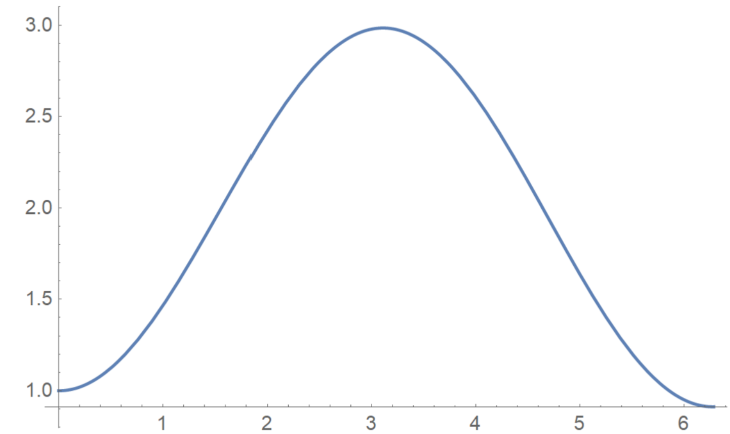

Figure 6 Solution of the problem for $\alpha = 0.99$

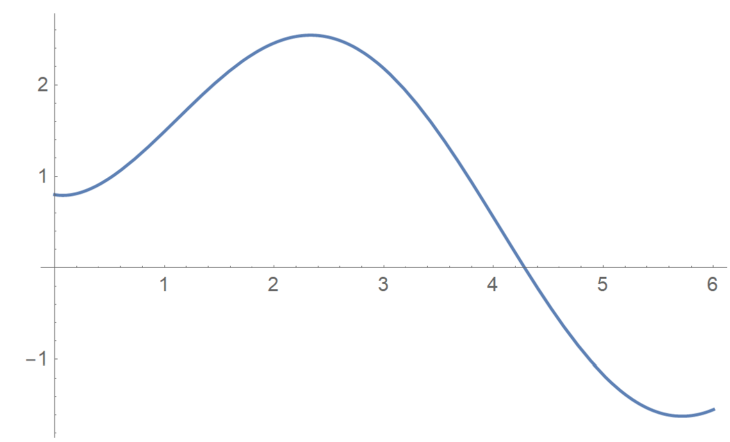

Figure 7 Solution of the problem for $\alpha = w = c = 0.8$

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