Research article

Analytical solutions for free convection flow of Casson nanofluid over an infinite vertical plate

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Abstract: This research article is design to elaborate the rule and significance of fractional derivative for heat transport in drilling of nanofluid. The respective nanofluid formed by the suspension of clay nanoparticles in the base fluids namely Casson fluid. The physical flow phenomenon is demonstrated with the help of partial differential equations by utilizing the respective thermophysical properties of nanoparticles. Also the geometric and thermal conditions are imposed in flow domain. In the governing equations, the partial derivative with respect to time replaced by new hybrid fractional derivative and then solved analytically for temperature and velocity field with the help of Laplace transformed. The obtained solutions for temperature and velocity are presented geometrically by Mathcad software to see the effectiveness of potent parameters. The temperature and velocity present a significant increasing trend for increasing volume fraction parameter. The obtained results for temperature as well as velocity are also compared with the existing literature and it is concluded that field variables with new hybrid fractional derivative, show more decaying trend as compare to the results with Caputo and Caputo-Fabrizio fractional derivatives.

Keywords: Casson fluid; Clay nanoparticles; new; vertical plate; heat transfer; hybrid fractional derivative
Mathematics Subject Classification: 26A33, 74F10

Nomenclature

$u-$Velocity $[ms^{-1}]$

U_0-Constant velocity

$s-$Non-dimensional velocity $[ms^{-1}]$

$t-$Time $[s]$

$\tau-$Non-dimensional time

$y-$Special variable $[m]$

$\xi-$Non-dimensional special variable

$\rho_{nf}-$Density of nanofluid, $[kgm^{-3}]$

$\mu_{nf}-$Dynamic viscosity of nanofluid $[kgm^{-1}s^{-1}]$

$T-$Temperature of Nanofluid, $[K]$

k_f-Thermal conductivity of base fluid

k_s-Thermal conductivity of solid particle

$k_{nf}-$Thermal conductivity of the nanofluid, $[Wm^{-2}K^{-1}]$

ρ_f-Density base fluid, $[kgm^{-3}]$

ρ_s-Density solid nanoparticle, $[kgm^{-3}]$

$\upsilon-$Concentration of clay nanoparticle, $[kgm^{-3}]$

$g-$Acceleration due to gravity, $[ms^{-2}]$

$\gamma-$Casson fluid parameter

β_c-non-dimensional Casson parameter

Gr$-$Grashof number for heat transfer

$\beta-$ Volumetric thermal expansion of $[K^{-1}]$

Pr$-$Prandtl number

$\phi-$Non-dimensionless temperature

$T_\infty-$Temperature at infinity, $[K]$

T_W-Temperature at plate, $[K]$

$\alpha-$fractional parameter

$q-$Laplace variable

1. Introduction

Clay nanoparticles have an important rule for drilling of oil and gases because of its significant thermal conductivity. The addition of clay nanoparticles in base fluid, boosted the thermal conductivity and viscosity of nanofluid that make it capable to resist the higher temperatures. The nanofluids with clay nanoparticles fulfil the challenges to improve the thermal conductivity and improved thermal conductivity boosted the thermal execution in respective devices. Now a days this nanofluids become a potent research area of nanotechnology [1–3] and attract the researchers of physical sciences. An interesting work on free convection flow of nanofluid has presented by Asma et al. [4] and explained the effect of volume fraction of nanoparticles over the nanofluid flow with fractional derivative.Hussanan et al. [5] investigated the free convection flow of sodium Carboxymethyl Cellulose based nanofluid with Tiatania nanoparticles, with non-homogeneous thermal and mechanical conditions. Bhattad et al. [6] carried an a compressive review on the formation methods and performance of nanofluids in the

view of applicable processes in engineering. Farhana et al. [7] applied the properties of nanofluids in to enhance the performance of solar collector. Some other investigations about nanofluids is have been listed in [8–15].

Fractional calculus is the generalization classical calculus. In fractional calculus the derivative of real order are consider instead of integer order derivative and have been applied efficiently to explain the several real life problem. Several investigation are reported regarding nanofluid and ordinary fluid flows with different definitions of fractional derivatives. Imran et al. [16] presented a comparative study of viscous fluid flow with two definitions of fractional derivatives. Hristov [17] used generalized Fourier law for thermal balance for steady state conduction. Aman [18] discussed the nanofluid flow and obtained the numerical and analytical solutions by applying non-integer derivative. Sheikh et al. [19] carried out the results for free convection flow of Casson fluid with AtanganaBaleanu and Caputo-Fabrizio fractional derivatives. Azhar et al. [20] bring out the results for the flow of nanofluid with homogenous thermal flux by utilizing the fractional derivatives. Some more interesting studies with application of fractional derivatives are investigated in [21–30]. Atangana et al. [31] have presented new fractional derivatives. Thabet et al. [32] introduced the new class of fractional operators by taking the Discrete Mittag-Leffler Kernels. Jarad et al. [33] have studied On a new class of fractional operators. Atangana et al. [34], Akgül et al. [35], Mirbahar et al. [36], and Abro et al. [37] have successfully apply the fractional derivative approaches to different real life physical phenomenons. Khan et al. [38] investigated the heat transfer model for the flow of nanofluid formed by clay nanoparticles. Recently Baleanu [39] proposed the new approach of fractional operator by combining the proportional and classical diferintegrals. For more details see [40–43].

The focal attention of this research is to extend the work [38] for Casson fluid and apply the new hybrid fractional derivative namely Constant Caputo fractional derivative [39]. According to our best knowledge there is no such result regarding Casson nanofluid formed by clay nanoparticles with constant proportional Caputo fractional derivatives established. The Casson clay nanofluid which is composed by clay nanoparticles and Casson fluid (base fluid) have supposed to flow over a vertical plate moving with constant velocity at constant surface temperature. The PDEs that governing our flow model will solved by Laplace transform technique. The established results can be applied to different engineering processes of relevant field.

2. Mathematical formulation

It is considered that clay nanoparticles are spread out to the incompressible Casson fluid (base fluid) and resulting clay nanofluid suppose to be flow in vicinity of a flat surface, which is situated in the xz-plane in such a way that y-axis is normal to the surface. for the time $t \leq 0$ the the whole system is in rest with thermal and mechanical homogeneity. At the moment $t > 0$ suddenly the temperature of plate raises up to T_w and the same time plate begins to move with a constant velocity U_0. Due to this non-homogenous mechanical and thermal situation the fluid adjoining to the moving plate also start to move with the same velocity of plate. The respective convectional flow is governs by the following set of partial differential equations [4]:
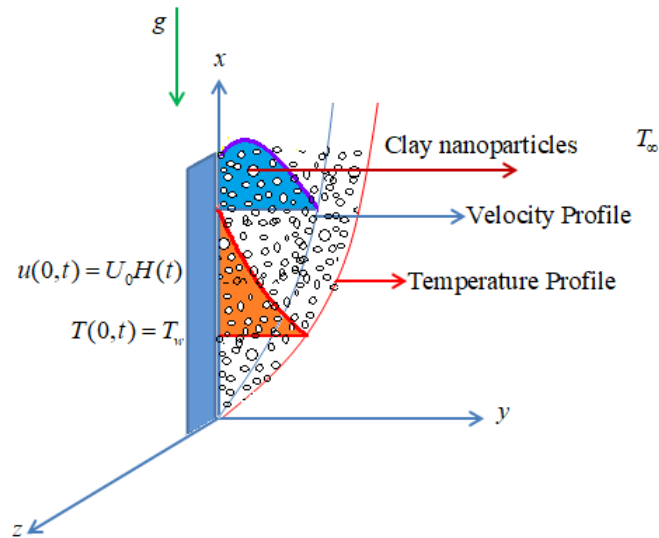
Figure 1. Flow geometry and coordinate system.

$$\rho_{nf}\frac{\partial u(y,t)}{\partial t} = \mu_{nf}(1 + \frac{1}{\gamma})\frac{\partial^2 u(y,t)}{\partial y^2} + g(\rho\beta)_{nf}[T(y,t) - T_\infty], \tag{1}$$

$$(\rho C_p)_{nf}\frac{\partial T(y,t)}{\partial t} = k_{nf}\frac{\partial^2 T(y,t)}{\partial y^2}, y, t > 0, \tag{2}$$

where the effective density of nanofluid is given by [44, 45]

$$\rho_{nf} = (1 - \upsilon)\rho_f + \upsilon\rho_s, \tag{3}$$

effective dynamic viscosity define as [46]

$$\mu_{nf} = \frac{\mu_f}{(1 - \upsilon)^{2.5}}, \tag{4}$$

where as

$$(\rho C_p)_{nf} = (1 - \upsilon)(\rho C_p)_f + \upsilon(\rho C_p)_s, \tag{5}$$

and

$$\frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\upsilon(k_f - k_s)}{k_s + 2k_f + \upsilon(k_f - k_s)}, \tag{6}$$

are the heat capacitance and effective thermal conductivity of nanofluid and υ is the volume fraction of clay nanoparticles. The specification of field variables at boundary are as

$$u(y, 0) = 0, \ T(y, 0) = T\infty, \ y \geq 0, \tag{7}$$

$$u(0, t) = U_0 H(t), \ T(0, t) = T_w \ t > 0, \tag{8}$$

$$u(y, t) \to 0, \ T(y, t) \to 0, \ \text{as } y \to \infty. \tag{9}$$

In order to construct a shape free flow model, the following dimensionless relations are inserted in Eqs (1)–(5)

$$s = \frac{u}{U_o}, \ \xi = \frac{yU_0}{\nu_f}, \ \tau = \frac{tU_0^2}{\nu_f}, \ \phi = \frac{T - T_\infty}{T_w - T_\infty}, \tag{10}$$

and obtain the following dimensionless flow model

$$\left((1 - \upsilon) + \upsilon \frac{\rho_s}{\rho_f}\right) \frac{\partial s(\xi, \tau)}{\partial \tau} = \frac{1}{(1 - \upsilon)^{2.5}} (1 + \frac{1}{\gamma}) \frac{\partial^2 s(\xi, \tau)}{\partial \xi^2} + \left((1 - \upsilon) + \upsilon \frac{(\rho\beta)_s}{(\rho\beta)_f}\right) \text{Gr}\phi(\xi, \tau), \ \xi, \tau > 0, \tag{11}$$

$$\left((1 - \upsilon) + \upsilon \frac{(\rho C_p)_s}{(\rho C_p)_f}\right) \text{Pr} \frac{\partial \phi(\xi, \tau)}{\partial \tau} = \frac{k_{nf}}{k_f} \frac{\partial^2 \phi(\xi, \tau)}{\partial \xi^2}, \ \xi, \tau > 0, \tag{12}$$

and associated dimensionless conditions are

$$s(\xi, 0) = 0, \ \phi(\xi, 0) = 0, \ \xi \geq 0, \tag{13}$$

$$s(0, \tau) = 1, \ \phi(0, \tau) = 1, \ \tau > 0, \tag{14}$$

$$s(\xi, \tau) \to 0, \ \phi(\xi, \tau) \to 0, \ \text{as } \xi \to \infty, \tag{15}$$

where $\text{Gr} = \frac{g(\beta)_f(T_w - T_\infty)}{U_0^3}$, $\text{Pr} = \frac{\mu_f(C_p)_f}{k_f}$.

Definition 1. The constant proportional Caputo fractional derivative of order $\alpha \in (0, 1)$ is define as [39]

$$^{CPC}D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \left(k_1(\alpha)f(\tau) + k_0(\alpha)f'(\tau)\right)(t - \tau)^{-\alpha} d\tau, \tag{16}$$

Definition 2. The Laplace transform of the constant proportional Caputo fractional derivative of the function $f(t)$ is define as [39]

$$£\{^{CPC}D_t^\alpha f(t)\} = \left[\frac{k_1(\alpha)}{q} + k_0(\alpha)\right]q^\alpha \bar{f}(q) - k_0(\alpha)q^{\alpha-1}f(0). \tag{17}$$

3. Model with constant proportional Caputo derivative

After replacing ordinary time derivative with constant proportional Caputo fractional derivative $^{CPC}D_\tau^\alpha$ in Eqs (11) and (12), we obtain the following fractionalized governing equations as follows

$$r_1 \, ^{CPC}D_\tau^\alpha s(\xi, \tau) = r_2 \beta_c \frac{\partial^2 s(\xi, \tau)}{\partial \xi^2} + r_3 \text{Gr}\phi(\xi, \tau), \ \xi, \tau > 0, \tag{18}$$

$$r_4 \text{Pr}^{CPC}D_\tau^\alpha \phi(\xi, \tau) = r_5 \frac{\partial^2 \phi(\xi, \tau)}{\partial \xi^2}, \ \xi, \tau > 0, \tag{19}$$

where $\beta_c = (1 + \frac{1}{\gamma})$, $r_1 = \left((1 - \upsilon) + \upsilon \frac{\rho_s}{\rho_f}\right)$, $r_2 = \frac{1}{(1-\upsilon)^{2.5}}$, $r_3 = \left((1 - \upsilon) + \upsilon \frac{(\rho\beta)_s}{(\rho\beta)_f}\right)$, $r_4 = \left((1 - \upsilon) + \upsilon \frac{(\rho C_p)_s}{(\rho C_p)_f}\right)$, $r_5 = \frac{k_{nf}}{k_f}$.

4. Solution of problem

4.1. Temperature distribution

Equation (19) is transformed to following ordinary differential equation by imposing Laplace transform

$$r_4 \Pr\left(\frac{k_1(\alpha)}{q} + k_0(\alpha)\right)q^\alpha \bar{\phi}(\xi, q) = r_5 \frac{\partial^2 \bar{\phi}(\xi, q)}{\partial \xi^2}, \quad \xi > 0, \tag{20}$$

where $\bar{\phi}(\xi, q)$ is the transformed image of $\phi(\xi, \tau)$ with respect to Laplace transform also satisfies the following boundary conditions

$$\bar{\phi}(\xi, q) = \frac{1}{q}, \quad \text{and } \bar{\phi}(\xi, q) \to 0, \text{ as } \xi \to \infty. \tag{21}$$

The temperature field as the solution of Eq (20) subject to conditions (21) can be written as

$$\bar{\phi}(\xi, q) = \frac{1}{s} \exp\left(-\xi \sqrt{\frac{r_4}{r_5} \Pr\left(\frac{k_1(\alpha)}{q} + k_0(\alpha)\right)q^\alpha}\right). \tag{22}$$

The exponential form of Eq (22) is so complicated that Laplace transformed cannot inverted by ordinary inversion formulae therefore it is express in summation notation as follows

$$\bar{\phi}(\xi, q) = \frac{1}{q} + \sum_{a_1=1}^{\infty} \sum_{a_2=0}^{\infty} \frac{\left(-\xi \sqrt{\frac{r_4}{r_5}\Pr}\right)^{a_1} (k_1(\alpha))^{a_2}}{a_1! a_2! (k_0(\alpha))^{a_2 - \frac{a_1}{2}} q^{1 - \frac{\alpha a_1}{2} + a_2}} \frac{\Gamma(\frac{a_1}{2} + 1)}{\Gamma(\frac{a_1}{2} + 1 - a_2)}. \tag{23}$$

by applying the inverse Laplace transformation we have

$$\phi(\xi, \tau) = 1 + \sum_{a_1=1}^{\infty} \sum_{a_2=0}^{\infty} \frac{\left(-\xi \sqrt{\frac{r_4}{r_5}\Pr}\right)^{a_1} (k_1(\alpha))^{a_2} \tau^{-\frac{\alpha a_1}{2} + a_2}}{a_1! a_2! (k_0(\alpha))^{a_2 - \frac{a_1}{2}}} \frac{\Gamma(\frac{a_1}{2} + 1)}{\Gamma(\frac{a_1}{2} + 1 - a_2)\Gamma(1 - \frac{\alpha a_1}{2} + a_2)} \tag{24}$$

Further, Eq (24) can be expressed in terms of more general function namely M-function [40],

$$\phi(y, \tau) = 1 + \sum_{a_1=1}^{\infty} \frac{\left(-y \sqrt{\frac{r_4}{r_5}\Pr}\right)^{a_1}}{a_1! (k_0(\alpha))^{-\frac{a_1}{2}}} M_2^1 \left[(\frac{k_1(\alpha)}{k_0(\alpha)})\tau\Big|_{(\frac{a_1}{2}+1,-1),(1-\frac{\alpha a_1}{2},1)}^{(\frac{a_1}{2}+1,0)}\right] \tag{25}$$

4.2. Velocity field

The Eq (18) is a partial differential equation and can be transform to ordinary differential equation by solving partial derivative with respect to time τ, with the help of integral transform namely Laplace transform as

$$r_1\left(\frac{k_1(\alpha)}{q} + k_0(\alpha)\right)q^\alpha \bar{s}(\xi, q) = r_2 \beta_c \frac{\partial^2 s(\xi, q)}{\partial \xi^2} + r_3 \text{Gr}\bar{\phi}(\xi, q), \quad \xi > 0, \tag{26}$$

Equation (26) satisfies the following transformed boundary conditions

$$\bar{s}(0, q) = 0 \ \text{ and } \ \bar{s}(\xi, q) \to 0 \text{ as } \xi \to \infty. \tag{27}$$

Solution of Eq (26) subject to condition (27) is obtain as

$$\bar{s}(\xi, q) = \frac{1}{q} e^{-\xi \sqrt{\frac{r_1}{r_2\beta_c}\left(\frac{k_1(\alpha)}{q} + k_0(\alpha)\right)q^\alpha}} + r_3 \text{Gr} \frac{e^{-\xi \sqrt{\frac{r_1}{r_2\beta_c}\left(\frac{k_1(\alpha)}{q} + k_0(\alpha)\right)q^\alpha}} - e^{-\xi \sqrt{\frac{r_4 \text{Pr}}{r_5}\left(\frac{k_1(\alpha)}{q} + k_0(\alpha)\right)q^\alpha}}}{\left(\frac{r_2 r_4 \text{Pr}\beta_c}{r_5} - 1\right)\left(\frac{k_1(\alpha)}{q} + k_0(\alpha)\right)q^{\alpha+1}}. \tag{28}$$

Equation (28) is also complicated and cannot be inverted by ordinary inversion formulae of Laplace transform, therefore we express it in summation form

$$
\begin{aligned}
\bar{s}(\xi, q) \ = \ & \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \frac{\left(-y\sqrt{\frac{r_1}{r_2\beta_c}}\right)^{b_1}(k_1(\alpha))^{b_2}}{b_1! b_2! (k_0(\alpha))^{b_2-\frac{b_1}{2}} q^{1-\frac{\alpha b_1}{2}+b_2}} \frac{\Gamma(\frac{b_1}{2}+1)}{\Gamma(\frac{b_1}{2}+1-b_2)} \\
& + \frac{r_3\text{Gr}}{\left(\frac{r_2 r_4 \text{Pr}\beta_c}{r_5}-1\right)} \sum_{b_1=0}^{\infty} \sum_{b_3=0}^{\infty} \left[\frac{\left(-y\sqrt{\frac{r_1}{r_2\beta_c}}\right)^{b_1}(k_1(\alpha))^{b_3}}{b_1! b_3! (k_0(\alpha))^{b_3-\frac{b_1}{2}+1} q^{1-\frac{\alpha b_1}{2}+b_3+\alpha}} \frac{\Gamma(\frac{b_1}{2})}{\Gamma(\frac{b_1}{2}-b_3)}\right] \\
& - \frac{r_3\text{Gr}}{\left(\frac{r_2 r_4 \text{Pr}\beta_c}{r_5}-1\right)} \sum_{a_1=0}^{\infty} \sum_{b_3=0}^{\infty} \left[\frac{\left(-y\sqrt{\frac{r_4}{r_5}\text{Pr}}\right)^{a_1}(k_1(\alpha))^{b_3}}{a_1! b_3! (k_0(\alpha))^{b_3-\frac{a_1}{2}+1} q^{1-\frac{\alpha a_1}{2}+b_3+\alpha}} \frac{\Gamma(\frac{a_1}{2})}{\Gamma(\frac{a_1}{2}-b_3)}\right]. \tag{29}
\end{aligned}
$$

Now taking inverse Laplace, have

$$
\begin{aligned}
s(\xi, \tau) \ = \ & \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \frac{\left(-y\sqrt{\frac{r_1}{r_2\beta_c}}\right)^{b_1}(k_1(\alpha))^{b_2}}{b_1! b_2! (k_0(\alpha))^{b_2-\frac{b_1}{2}}} \frac{t^{-\frac{\alpha b_1}{2}+b_2}\Gamma(\frac{b_1}{2}+1)}{\Gamma(\frac{b_1}{2}+1-b_2)\Gamma(1-\frac{\alpha b_1}{2}+b_2)} \\
& + \frac{r_3\text{Gr}}{\left(\frac{r_2 r_4 \text{Pr}\beta_c}{r_5}-1\right)} \sum_{b_1=0}^{\infty} \sum_{b_3=0}^{\infty} \left[\frac{\left(-y\sqrt{\frac{r_1}{r_2\beta_c}}\right)^{b_1}(k_1(\alpha))^{b_3}}{b_1! b_3! (k_0(\alpha))^{b_3-\frac{b_1}{2}+1}} \frac{t^{-\frac{\alpha b_1}{2}+b_3+\alpha}\Gamma(\frac{b_1}{2})}{\Gamma(\frac{b_1}{2}-b_3)\Gamma(1-\frac{\alpha b_1}{2}+b_3+\alpha)}\right] \\
& - \frac{r_3\text{Gr}}{\left(\frac{r_2 r_4 \text{Pr}\beta_c}{r_5}-1\right)} \sum_{a_1=0}^{\infty} \sum_{b_3=0}^{\infty} \left[\frac{\left(-y\sqrt{\frac{r_4}{r_5}\text{Pr}}\right)^{a_1}(k_1(\alpha))^{b_3}}{a_1! b_3! (k_0(\alpha))^{b_3-\frac{a_1}{2}+1}} \frac{t^{-\frac{\alpha a_1}{2}+b_3+\alpha}\Gamma(\frac{a_1}{2})}{\Gamma(\frac{a_1}{2}-b_3)\Gamma(1-\frac{\alpha a_1}{2}+b_3+\alpha)}\right]. \tag{30}
\end{aligned}
$$

Further, Eq (30) can be expressed in terms of more general function namely M-function [40],

$$
s(\xi, \tau) \ = \ \sum_{b_1=0}^{\infty} \frac{\left(-y\sqrt{\frac{r_1}{r_2\beta_c}}\right)^{b_1}}{b_1! (k_0(\alpha))^{-\frac{b_1}{2}}} M_2^1 \left[(\frac{k_1(\alpha)}{k_0(\alpha)})t\Big|^{(\frac{b_1}{2}+1,0)}_{(\frac{b_1}{2}+1,-1),(1-\frac{\alpha b_1}{2},1)}\right]
$$

$$+ \frac{r_3 \text{Gr}}{\left(\frac{r_2 r_4 \text{Pr} \beta_c}{r_5} - 1\right)} \sum_{b_1=0}^{\infty} \frac{\left(-y\sqrt{\frac{r_1}{r_2 \beta_c}}\right)^{b_1}}{b_1!(k_0(\alpha))^{-\frac{b_1}{2}+1}} M_2^1 \left[(\frac{k_1(\alpha)}{k_0(\alpha)})t \Big|_{(\frac{b_1}{2},-1),(1-\frac{ab_1}{2}+\alpha,1)}^{(\frac{b_1}{2},0)} \right]$$

$$- \frac{r_3 \text{Gr}}{\left(\frac{r_2 r_4 \text{Pr} \beta_c}{r_5} - 1\right)} \sum_{a_1=0}^{\infty} \frac{\left(-y\sqrt{\frac{r_4}{r_5}\text{Pr}}\right)^{a_1}}{a_1!(k_0(\alpha))^{-\frac{a_1}{2}+1}} M_2^1 \left[(\frac{k_1(\alpha)}{k_0(\alpha)})t \Big|_{(\frac{a_1}{2},-1),(1-\frac{\alpha a_1}{2}+\alpha,1)}^{(\frac{a_1}{2},0)} \right]. \tag{31}$$

5. Results and discussions

The main aim of this investigation is to explore the results for the flow of drilling nanofluid with constant proportional Caputo fractional derivative. Casson fluid flow model is consider with clay particles and explicit expression for temperature and velocity fields are established as solutions of governing equations via the Laplace transformation. The subjectivity of fractional parameter and other parameter of interest have been studied numerical by plotting the some 2-D graphs of temperature and velocity fields in Mathcad software. Temperature profile is plotted in Figure 2(a),(b) to see the effect of fractional parameter α for large and small times. An interesting behavior is observed for large and small times that is temperature show increasing trend for large time while it is falling down for small time. The influence of Pr and volume fraction are highlighted in Figure 3(a),(b) respectively. An increasing value of the Pr lessen the thermal conductivity of the fluid and due to this temperature of flowing fluid falls down with the elevated value of Pr. Thermal conductivity of nanofluid increases with the increasing volume fraction υ of clay nanoparticles that is why temperature of flowing fluid raises with increasing υ. Figure 4(a),(b) presents the comparison between the ordinary and fractional temperature profiles. Figure 4(a) shows that temperature profile with constant proportional Caputo fractional derivative has the more decaying nature as compare to the ordinary temperature while it is tends to ordinary with increasing α and finally it overlaps with ordinary temperature as $\alpha \to 1$ as shown in the Figure 4(b).

Velocity profiles also present a dual behavior due to α variation for large and small times and it is highlighted in Figure 5(a),(b). The effect of Gr and Pr have been explain in Figure 6(a),(b) respectively and is noted that an increasing Gr (more bouncy effect) speeds up the fluid flow while an increasing Pr (more viscosity) create some hindrance to fluid flow which slows down the fluid flow. The influence of volume fraction υ and time t is illustrated in Figure 7(a),(b) and it is found that enhancing volume fraction of clay nanoparticles boosts the viscous effect for the nanofluid consequently it decreases the fluid velocity as depicted in the Figure 7(a). For a long time, the thermal boundary layer become more thick as compare to the momentum boundary layer that aided the fluid flow as shown in Figure 7(b). In Figures 8(a),(b), and 9(a),(b), the fractional and ordinary velocity of viscous and Casson fluid are compared for the different values of fractional parameter α. Fraction nanofluid shows a decaying nature for both viscous and Casson fluid as shown in Figure 8(a) while both fractionalized fluids, viscous and Casson tend toward ordinary fluid and this aspect is bring into light in Figures 8(b) and 9(a). At last all velocity profiles concede with each other as both Casson fluid parameter $\beta_c \to 1$ and fractional parameter $\alpha \to 1$ as shown in Figure 9(b). Finally the temperature and velocity fields obtained by CPC are also compared with that obtained by Caputo fractional derivative and presented graphically in the Figure 10(a), 10(b). From outlines of the figures it is observed that temperature and velocity with CPC are of more decaying nature with respect to temperature and velocity with Caputo fractional derivative.
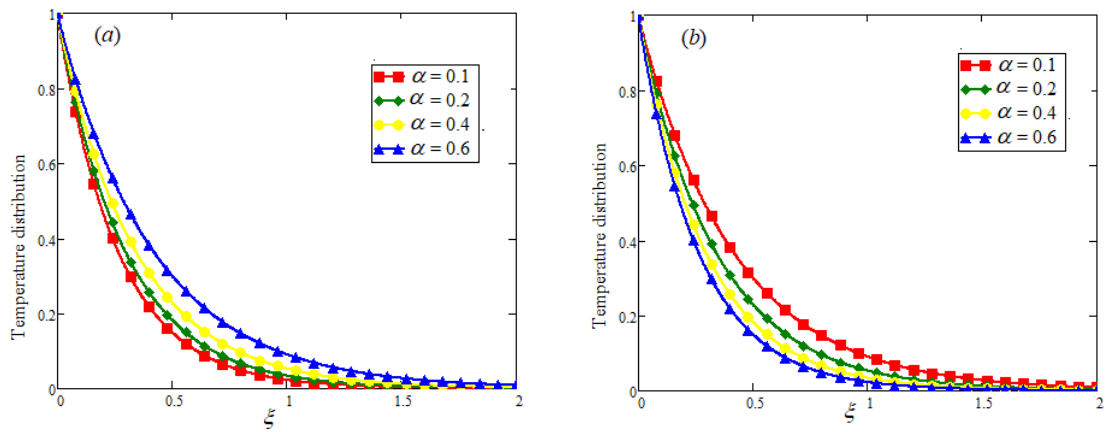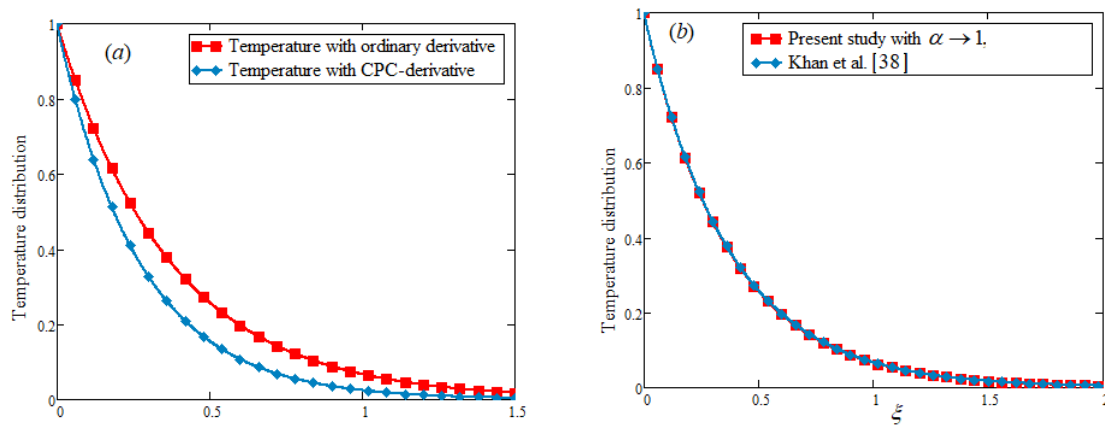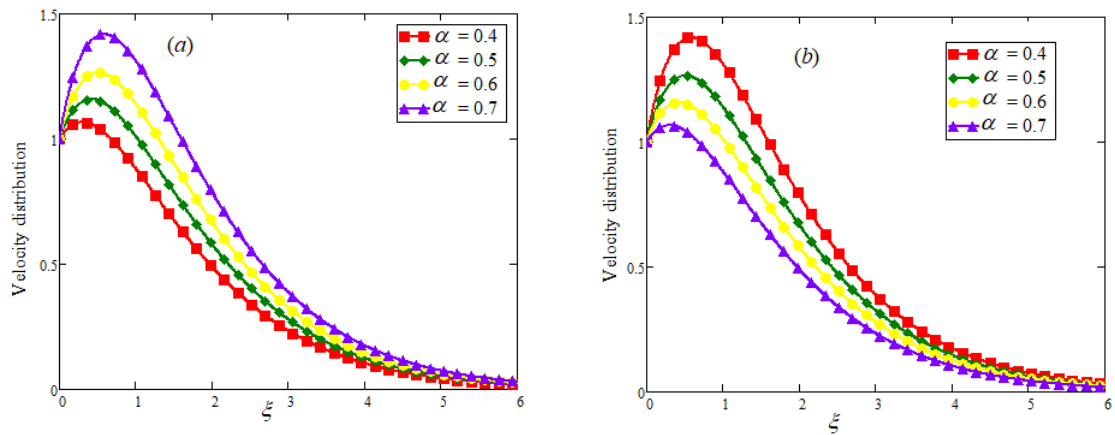
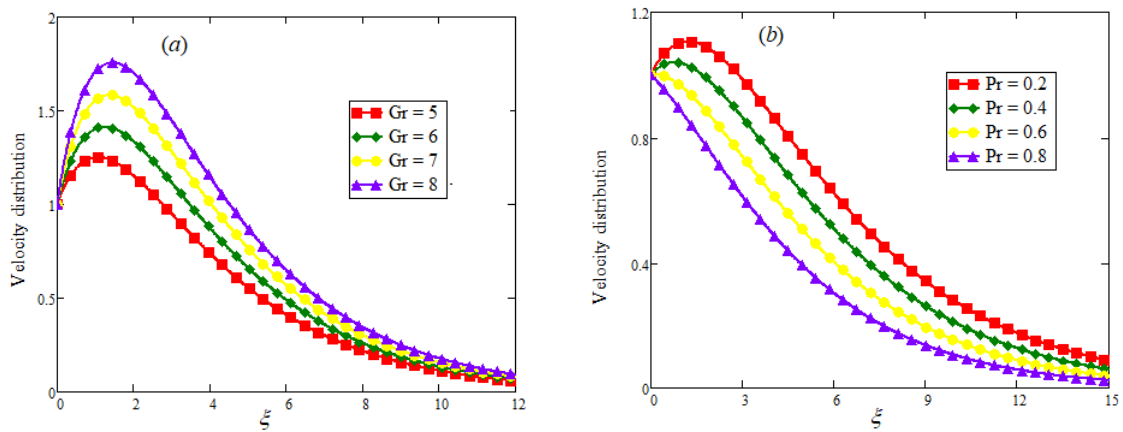**Figure 2.** Temperature profile subject to α variation for large t= 4 and small $t = 0.25$ times.
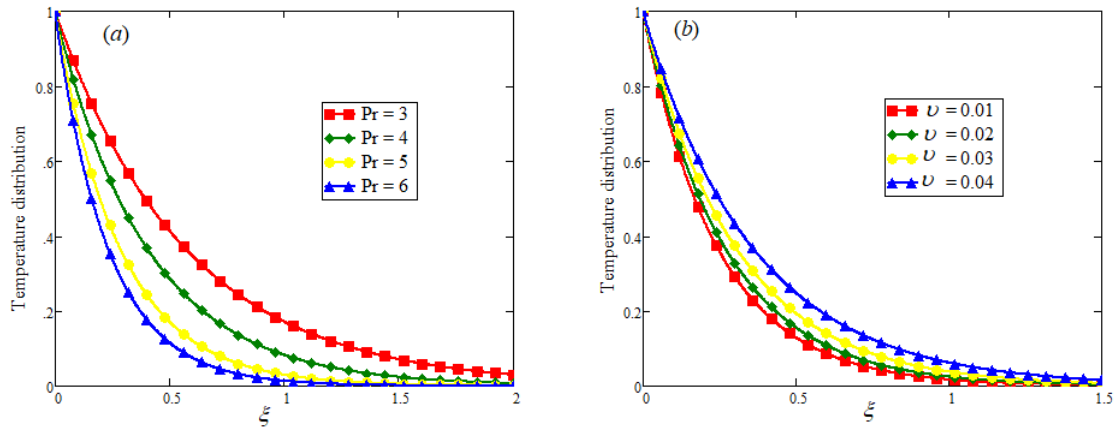
Figure 3. Temperature profile subject to Pr and time and volume fraction υ variation, and $\alpha = 0.5$.

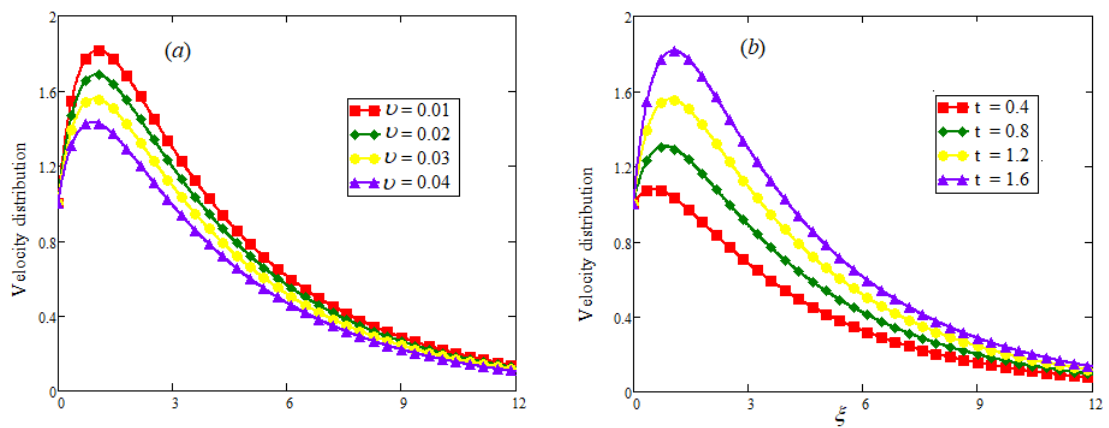
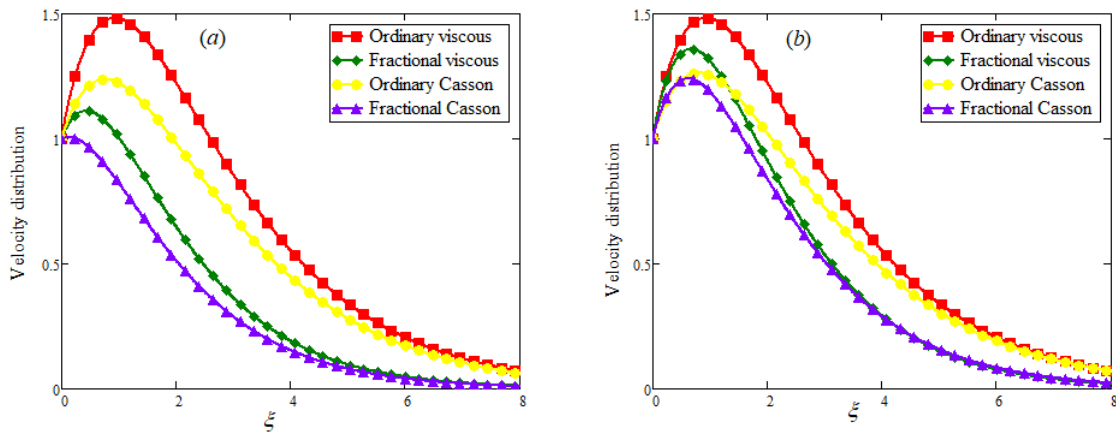**Figure 4.** Comparison Temperature profile for $\alpha = 0.5$ and $\alpha \rightarrow 1$.

Figure 5. Velocity profile subject to α variation for large t=4 and small t=0.25 times, and $\alpha = 0.5$.
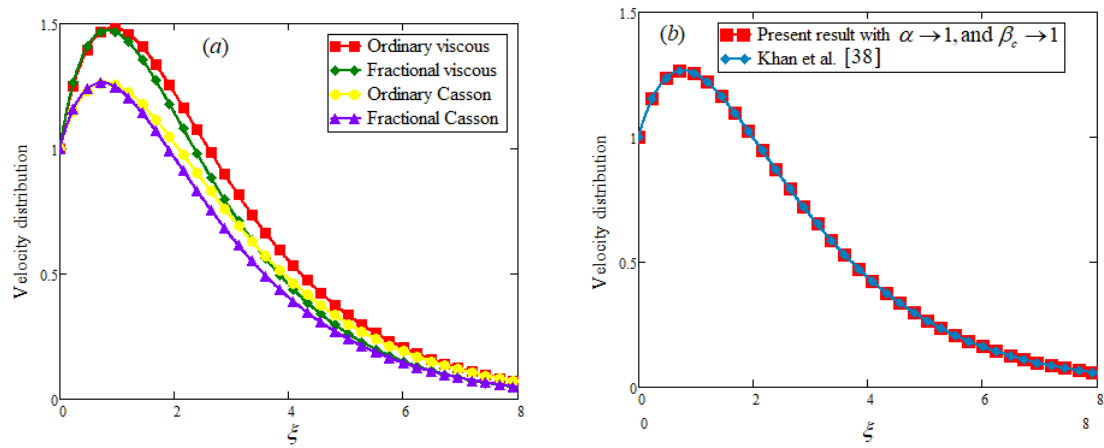

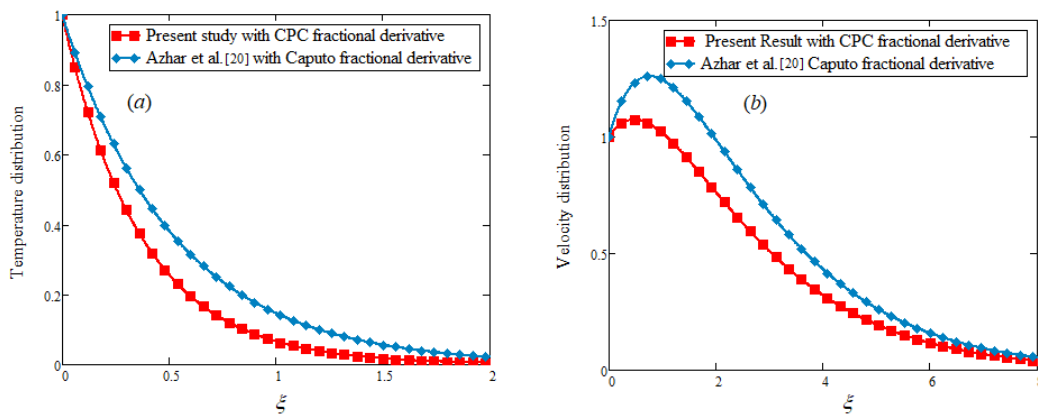**Figure 6.** Velocity profile subject to positive and Pr variation and $\alpha = 0.5$.

Figure 7. Velocity profile subject to volume fraction υ and time t variation, and $\alpha = 0.5$.

Figure 8. Comparison velocity profile for $\alpha = 0.4$ and $\alpha = 0.6$.

Figure 9. Comparison velocity profile for $\alpha = 0.8$ and $\alpha \to 1$.

Figure 10. Comparison of Temperature and velocity profiles for among CPC and Caputo fractional derivatives with $\alpha = 0.5$.

6. Conclusion

The fractional Casson nanofluid flow with clay nanoparticles is modeled by utilizing constant proportional Caputo fractional derivative. Results for temperature and velocity field are established with the help of Laplace transformation. Regarding obtained results for temperature and velocity, the present study can be summarized withe following concluding remarks:

• Temperature and velocity profiles with constant proportional Caputo fractional derivative remain lower than the ordinary profiles.

• Velocity profile is an increasing function for α, Gr, and t.

• Velocity is a decreasing function of Pr and v.

• Temperature is a increasing functions of v while it is decreasing for Pr.

• Velocity of Casson fluid in both cases (fractional and ordinary) show a more restrain as compare to the viscous one.

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Conflict of interest

The authors declare no conflict of interest.

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