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Boundary value problem for nonlinear fractional differential equations of variable order via Kuratowski MNC technique

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Abstract

In the present research study, for a given multiterm boundary value problem (BVP) involving the Riemann-Liouville fractional differential equation of variable order, the existence properties are analyzed. To achieve this aim, we firstly investigate some specifications of this kind of variable-order operators, and then we derive the required criteria to confirm the existence of solution and study the stability of the obtained solution in the sense of Ulam-Hyers-Rassias (UHR). All results in this study are established with the help of the Darbo's fixed point theorem (DFPT) combined with Kuratowski measure of noncompactness (KMNC). We construct an example to illustrate the validity of our observed results.

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1 Introduction

The idea of fractional calculus is replacing the natural numbers in the derivative order with rational ones. Although it seems an elementary consideration, it has an interesting correspondence in explaining some physical phenomena. In the last two decades, significant research studies appeared on this topic, and some papers dealt with the existence of solutions to the problems of variable order; see, for example, [1–7].

Whereas many researchers investigated the existence of solutions for fractional constant-order problems, the existence of solutions of variable-order problems is rarely mentioned in the literature (we refer to [8–13]).

As a result of our investigation in this interesting research field, our findings are unique and noteworthy.

Furthermore, all of the findings in this paper have a great potential to be applied in a variety of transdisciplinary science applications. With the support of our original findings in this research study, we are able to do further research on this open research topic. In other words, the proposed BVP can be extended to more sophisticated real mathematical fractional models in the future.

In particular, Bai et al. [14] studied the following problem:

$$\begin{cases} {}^cD^u_{0^+}x(t) = f(t, x(t), I^u_{0^+}x(t)), \quad t \in J := [a, b], u \in]0, 1], \\ x(a) = x_a, \end{cases}$$

where ${}^cD^u_{0^+}$ and $I^u_{0^+}$ stand for the Caputo–Hadamard derivative and Hadamard integral operators of order u, respectively, f is a given function, $x_a \in \mathbb{R}$, and $0 < a < b < \infty$.

Inspired by [14] and [1–5], we deal with the boundary value problem (BVP)

$$\begin{cases} D^{u(t)}_{0^+}x(t) = f_1(t, x(t), I^{u(t)}_{0^+}x(t)), \quad t \in J := [0, T], \\ x(0) = 0, \qquad x(T) = 0, \end{cases} \tag{1}$$

where $1 < u(t) \leq 2$, $f_1 : J \times X \times X \to X$ is a continuous function, and $D^{u(t)}_{0^+}$ and $I^{u(t)}_{0^+}$ are the Riemann–Liouville fractional derivative and integral of variable order $u(t)$.

In this paper, we investigate the solution of (1). Further, we study the stability of the obtained solution of (1) in the Ulam–Hyers–Rassias (UHR) sense.

2 Preliminaries

In this section, we introduce some important fundamental definitions that will be needed for obtaining our results in the next sections.

By $C(J, X)$ we denote the Banach space of continuous functions $\varkappa : J \to X$ with the norm

$$\|\varkappa\| = \sup\{\|\varkappa(t)\| : t \in J\},$$

where X is a real (or complex) Banach space.

For $-\infty < a_1 < a_2 < +\infty$, we consider the mappings $u(t) : [a_1, a_2] \to (0, +\infty)$ and $v(t) : [a_1, a_2] \to (n - 1, n)$. Then the left Riemann–Liouville fractional integral (RLFI) of variable order $u(t)$ for function $h_1(t)$ is [15–17]

$$I^{u(t)}_{a_1^+}h_1(t) = \int_{a_1}^t \frac{(t - s)^{u(t)-1}}{\Gamma(u(t))} h_1(s)\, ds, \quad t > a_1, \tag{2}$$

and the left Riemann–Liouville fractional derivative (RLFD) of variable-order $v(t)$ for function $h_1(t)$ is [15–17]

$$D^{v(t)}_{a_1^+}h_1(t) = \left(\frac{d}{dt}\right)^n I^{n-v(t)}_{a_1^+}h_1(t) = \left(\frac{d}{dt}\right)^n \int_{a_1}^t \frac{(t - s)^{n-v(t)-1}}{\Gamma(n - v(t))} h_1(s)\, ds, \quad t > a_1. \tag{3}$$

In case of constant $u(t)$ and $v(t)$, RLFI and RLFD coincide with the standard Riemann–Liouville fractional derivative and integral, respectively; see, for example, [15, 16, 18].

Let us recall the following pivotal observation.

Lemma 2.1 ([18]) *Let $\alpha_1, \alpha_2 > 0$, $a_1 > 0$, $h_1 \in L(a_1, a_2)$, and $D^{\alpha_1}_{a_1^+}h_1 \in L(a_1, a_2)$. Then the differential equation*

$$D^{\alpha_1}_{a_1^+}h_1 = 0$$

has the unique solution

$$h_1(t) = \omega_1(t-a_1)^{\alpha_1-1} + \omega_2(t-a_1)^{\alpha_1-2} + \cdots + \omega_n(t-a_1)^{\alpha_1-n},$$

and

$$I_{a_1^+}^{\alpha_1} D_{a_1^+}^{\alpha_1} h_1(t) = h_1(t) + \omega_1(t-a_1)^{\alpha_1-1} + \omega_2(t-a_1)^{\alpha_1-2} + \cdots + \omega_n(t-a_1)^{\alpha_1-n}$$

with $n-1 < \alpha_1 \leq n$, $\omega_\ell \in \mathbb{R}$, $\ell = 1, 2, \ldots, n$.

Furthermore,

$$D_{a_1^+}^{\alpha_1} I_{a_1^+}^{\alpha_1} h_1(t) = h_1(t)$$

and

$$I_{a_1^+}^{\alpha_1} I_{a_1^+}^{\alpha_2} h_1(t) = I_{a_1^+}^{\alpha_2} I_{a_1^+}^{\alpha_1} h_1(t) = I_{a_1^+}^{\alpha_1+\alpha_2} h_1(t).$$

Remark 2.1 ([19–21]) Note that the semigroup property is not fulfilled for general functions $u(t)$, $v(t)$, that is,

$$I_{a_1^+}^{u(t)} I_{a_1^+}^{v(t)} h_1(t) \neq I_{a_1^+}^{u(t)+v(t)} h_1(t).$$

Example 2.1 Let

$$u(t) = t, \quad t \in [0,4], \qquad v(t) = \begin{cases} 2, & t \in [0,1], \\ 3, & t \in]1,4], \end{cases} \qquad h_1(t) = 2, \quad t \in [0,4],$$

$$\begin{aligned} I_{0^+}^{u(t)} I_{0^+}^{v(t)} h_1(t) &= \int_0^t \frac{(t-s)^{u(t)-1}}{\Gamma(u(t))} \int_0^s \frac{(s-\tau)^{v(s)-1}}{\Gamma(v(s))} h_1(\tau)\, d\tau\, ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[\int_0^1 \frac{(s-\tau)}{\Gamma(2)} 2\, d\tau + \int_1^s \frac{(s-\tau)^2}{\Gamma(3)} 2\, d\tau \right] ds \\ &= \int_0^t \frac{(t-s)^{t-1}}{\Gamma(t)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds, \end{aligned}$$

and

$$I_{0^+}^{u(t)+v(t)} h_1(t)| = \int_0^t \frac{(t-s)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))} h_1(s)\, ds.$$

So we get

$$I_{0^+}^{u(t)} I_{0^+}^{v(t)} h_1(t)|_{t=3} = \int_0^3 \frac{(3-s)^2}{\Gamma(3)} \left[2s - 1 + \frac{(s-1)^3}{3} \right] ds = \frac{21}{10},$$

$$\begin{aligned} I_{0^+}^{u(t)+v(t)} h_1(t)|_{t=3} &= \int_0^3 \frac{(3-s)^{u(t)+v(t)-1}}{\Gamma(u(t)+v(t))} h_1(s)\,ds \\ &= \int_0^1 \frac{(3-s)^4}{\Gamma(5)} 2\,ds + \int_1^3 \frac{(3-s)^5}{\Gamma(6)} 2\,ds \\ &= \frac{1}{12} \int_0^1 \left(s^4 - 12s^3 + 54s^2 - 108s + 81\right) ds \\ &\quad + \frac{1}{60} \int_1^3 \left(-s^5 + 15s^4 - 90s^3 + 270s^2 - 405s + 243\right) ds \\ &= \frac{665}{180}. \end{aligned}$$

Therefore we obtain

$$I_{0^+}^{u(t)} I_{0^+}^{v(t)} h_1(t)|_{t=3} \neq I_{0^+}^{u(t)+v(t)} h_1(t)|_{t=3}.$$

Lemma 2.2 ([22]) *Let $u : J \to (1,2]$ be a continuous function. Then for*

$$h_1 \in C_\delta(J,X) = \left\{h_1(t) \in C(J,X), t^\delta h_1(t) \in C(J,X)\right\} \quad \left(0 \leq \delta \leq \min_{t\in J}\left|u(t)\right|\right),$$

the variable-order fractional integral $I_{0^+}^{u(t)} h_1(t)$ exists for any points on J.

Lemma 2.3 ([22]) *Let $u : J \to (1,2]$ be a continuous function. Then*

$$I_{0^+}^{u(t)} h_1(t) \in C(J,X) \quad \textit{for } h_1 \in C(J,X).$$

Definition 2.1 ([23–25]) A set $I \subset \mathbb{R}$ is called a generalized interval if it is either an interval, or $\{a_1\}$, or $\{\,\}$.

A finite set $\mathcal{P}$ of generalized intervals is called a partition of I if each $x \in I$ lies in exactly one generalized interval E in $\mathcal{P}$.

A function $g : I \to X$ is called piecewise constant with respect to partition $\mathcal{P}$ of I if for any $E \in \mathcal{P}$, g is constant on E.

2.1 Measure of noncompactness

In this subsection, we discuss some necessary background information about KMNCs.

Definition 2.2 ([26]) Let X be a Banach space, and let Ω_X be the bounded subsets of X. A KMNC is a mapping $\zeta : \Omega_X \to [0,\infty]$ constructed as follows:

$$\zeta(D) = \inf\left\{\epsilon > 0 : D(\in \Omega_X) \subseteq \cup_{\ell=1}^n D_\ell, \operatorname{diam}(D_\ell) \leq \epsilon\right\},$$

where

$$\operatorname{diam}(D_\ell) = \sup\left\{\|x-y\| : x, y \in D_\ell\right\}.$$

The following properties are valid for KMNCs.

Proposition 2.1 ([26, 27]) *Let X be a Banach space, and let D, D_1, and D_2 be bounded subsets of X. Then*:

1. $\zeta(D) = 0 \Longleftrightarrow D$ *is relatively compact.*
2. $\zeta(\phi) = 0$.
3. $\zeta(D) = \zeta(\overline{D}) = \zeta(convD)$.
4. $D_1 \subset D_2 \Longrightarrow \zeta(D_1) \leq \zeta(D_2)$.
5. $\zeta(D_1 + D_2) \leq \zeta(D_1) + \zeta(D_2)$.
6. $\zeta(\lambda D) = |\lambda|\zeta(D)$, $\lambda \in \mathbb{R}$.
7. $\zeta(D_1 \cup D_2) = \mathrm{Max}\{\zeta(D_1), \zeta(D_2)\}$.
8. $\zeta(D_1 \cap D_2) = \mathrm{Min}\{\zeta(D_1), \zeta(D_2)\}$.
9. $\zeta(D + x_0) = \zeta(D)$ *for all* $x_0 \in X$.

Lemma 2.4 ([28]) *If $U \subset C(J, X)$ is an equicontinuous and bounded set, then*:

(i) *the function $\zeta(U(t))$ is continuous for $t \in J$, and*

$$\widehat{\zeta}(U) = \sup_{t\in J} \zeta\bigl(U(t)\bigr);$$

(ii) $\zeta(\int_0^T x(\theta)\, d\theta : x \in U) \leq \int_0^T \zeta(U(\theta))\, d\theta$,

where

$$U(s) = \bigl\{x(s) : x \in U\bigr\}, \quad s \in J.$$

Theorem 2.1 (DFPT [26]) *Let Λ be nonempty, closed, bounded, and convex subset of a Banach space X, and let $\digamma : \Lambda \longrightarrow \Lambda$ be a continuous operator satisfying*

$$\zeta\bigl(\digamma(S)\bigr) \leq k\zeta(S) \quad \textit{for any } (S \neq \emptyset) \subset \Lambda, k \in [0, 1),$$

that is, $\digamma$ is a k-set contraction.

Then $\digamma$ has at least one fixed point in Λ.

Definition 2.3 ([29]) Let $\vartheta \in C(J, X)$. Equation of (1) is UHR stable with respect to ϑ if there exists $c_f > 0$ such that for any $\epsilon > 0$ and every solution $z \in C(J, X)$ of the inequality

$$\bigl\| D_{0^+}^{u(t)} z(t) - f\bigl(t, z(t), I_{0^+}^{u(t)} z(t)\bigr) \bigr\| \leq \epsilon\vartheta(t), \quad t \in J,$$

there exists a solution $x \in C(J, X)$ of equation (1) with

$$\bigl\| z(t) - x(t) \bigr\| \leq c_f \epsilon\vartheta(t), \quad t \in J.$$

3 Existence of solutions

Let us introduce the following assumptions:

(H1) Let $n \in \mathbb{N}$ be an integer, let $\mathcal{P} = \{J_1 := [0, T_1], J_2 := (T_1, T_2], J_3 := (T_2, T_3], \ldots, J_n := (T_{n-1}, T]\}$ be a partition of the interval J, and let $u(t) : J \to (1, 2]$ be a piecewise constant function with respect to

$\mathcal{P}$, that is,

$$u(t) = \sum_{\ell=1}^{n} u_\ell I_\ell(t) = \begin{cases} u_1 & \text{for } t \in J_1, \\ u_2 & \text{for } t \in J_2, \\ \vdots & \\ u_n & \text{for } t \in J_n, \end{cases}$$

where $1 < u_\ell \leq 2$ are constants, and I_ℓ is the indicator of the interval $J_\ell := (T_{\ell-1}, T_\ell]$, $\ell = 1, 2, \dots, n$ (with $T_0 = 0$, $T_n = T$), such that

$$I_\ell(t) = \begin{cases} 1 & \text{for } t \in J_\ell, \\ 0 & \text{elsewhere.} \end{cases}$$

(H2) Let $t^\delta f_1 : J \times X \times X \to X$ be a continuous function ($0 \leq \delta \leq \min_{t\in J} |(u(t))|$). There exist constants $K, L > 0$ such that

$$t^\delta \left\| f_1(t, y_1, z_1) - f_1(t, y_2, z_2) \right\| \leq K\|y_1 - y_2\| + L\|z_1 - z_2\| \quad \text{for all } y_1, y_2, z_1, z_2 \in X \text{ and } t \in J.$$

Remark 3.1 According to the remark of [30] on page 20, we can easily show that condition (H2) and the inequality

$$\zeta\left(t^\delta \left\| f_1(t, B_1, B_2) \right\|\right) \leq K\zeta(B_1) + L\zeta(B_2)$$

are equivalent for any bounded sets $B_1, B_2 \subset X$ and $t \in J$.

Further, for a given set U of functions $u : J \to X$, let us denote

$$U(t) = \left\{u(t), u \in U\right\}, \quad t \in J,$$

and

$$U(J) = \left\{U(t) : v \in U, t \in J\right\}.$$

Let us now prove the existence of solution for the BVP (1) via the concepts of MNCK and DFPT.

For $\ell \in \{1, 2, \dots, n\}$, by $E_\ell = C(J_\ell, X)$ we denote the Banach space of continuous functions $x : J_\ell \to X$ equipped with the norm

$$\|x\|_{E_\ell} = \sup_{t\in J_\ell} \left\| x(t) \right\|.$$

First, we analyze BVP (1).

By (3) the equation of BVP (1) can be expressed as

$$\frac{d^2}{dt^2} \int_0^t \frac{(t-s)^{1-u(t)}}{\Gamma(2-u(t))} x(s)\, ds = f_1\left(t, x(t), I_{0^+}^{u(t)} x(t)\right), \quad t \in J. \tag{4}$$

Taking $(H1)$ into account, equation(4) in the interval $J_\ell, \ell = 1,2,\dots,n$, can be written as

$$\begin{aligned} &\frac{d^2}{dt^2}\left(\int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)}x(s)\,ds + \cdots + \int_{T_{\ell-1}}^t \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)}x(s)\,ds\right) \\ &\quad = f_1\bigl(t,x(t),I_{0^+}^{u_\ell}x(t)\bigr), \quad t \in J_\ell. \end{aligned} \tag{5}$$

Now we introduce the solution to BVP (1).

Definition 3.1 BVP (1) has a solution if there are functions $x_\ell, \ell = 1,2,\dots,n$, such that $x_\ell \in C([0,T_\ell],X)$ fulfills equation (5) and $x_\ell(0) = 0 = x_\ell(T_\ell)$.

According the above observation, BVP (1) can be expressed for any $t \in J_l$, $l = 1,2,\dots,n$, as (5).

For $0 \le t \le T_{\ell-1}$, taking $x(t) \equiv 0$, we can write (5) as

$$D_{T_{\ell-1}^+}^{u_\ell} x(t) = f_1\bigl(t,x(t),I_{T_{\ell-1}^+}^{u_\ell}x(t)\bigr), \quad t \in J_\ell.$$

We will deal with the following BVP:

$$\begin{cases} D_{T_{\ell-1}^+}^{u_\ell} x(t) = f_1(t,x(t),I_{T_{\ell-1}^+}^{u_\ell}x(t)), & t \in J_\ell, \\ x(T_{\ell-1}) = 0, \qquad x(T_\ell) = 0. \end{cases} \tag{6}$$

For our purpose, the following lemma will be the basis of the solution of (6).

Lemma 3.1 *A function $x \in E_\ell$ forms a solution of* (6) *if and only if x fulfills the integral equation*

$$\begin{aligned} x(t) &= -(T_\ell - T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}I_{T_{\ell-1}^+}^{u_\ell}f_1\bigl(T_\ell,x(T_\ell),I_{T_{\ell-1}^+}^{u_\ell}x(T_\ell)\bigr) \\ &\quad + I_{T_{\ell-1}^+}^{u_\ell}f_1\bigl(t,x(t),I_{T_{\ell-1}^+}^{u_\ell}x(t)\bigr). \end{aligned} \tag{7}$$

Proof Let $x \in E_\ell$ be solution of problem (6). Applying the operator $I_{T_{\ell-1}^+}^{u_\ell}$ to both sides of (6), from Lemma 2.1 we find

$$\begin{aligned} x(t) &= \omega_1(t-T_{\ell-1})^{u_\ell-1} + \omega_2(t-T_{\ell-1})^{u_\ell-2} \\ &\quad + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^t (t-s)^{u_\ell-1}f_1\bigl(s,x(s),I_{T_{\ell-1}^+}^{u_\ell}x(s)\bigr)\,ds, \quad t \in J_\ell. \end{aligned}$$

Due to the assumption on the function f_1 along with $x(T_{\ell-1}) = 0$, we conclude that $\omega_2 = 0$.

Let x satisfy $x(T_\ell) = 0$. Observe that

$$\omega_1 = -(T_\ell - T_{\ell-1})^{1-u_\ell}I_{T_{\ell-1}^+}^{u_\ell}f_1\bigl(T_\ell,x(T_\ell),I_{T_{\ell-1}^+}^{u_\ell}x(T_\ell)\bigr).$$

Then we find

$$\begin{aligned} x(t) &= -(T_\ell - T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}I_{T_{\ell-1}^+}^{u_\ell}f_1\bigl(T_\ell,x(T_\ell),I_{T_{\ell-1}^+}^{u_\ell}x(T_\ell)\bigr) \\ &\quad + I_{T_{\ell-1}^+}^{u_\ell}f_1\bigl(t,x(t),I_{T_{\ell-1}^+}^{u_\ell}x(t)\bigr), \quad t \in J_\ell. \end{aligned}$$

Conversely, let $x \in E_\ell$ be a solution of integral equation (7), Regarding the continuity of the unction $t^\delta f_1$ and Lemma 2.1, we deduce that x is a solution of problem (6). $\square$

Our first existence result is based on Theorem 2.1.

Theorem 3.1 *Assume that conditions* (*H*1) *and* (*H*2) *hold and*

$$\frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\left(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right) < 1. \tag{8}$$

Then problem (6) *possesses at least one solution on* J.

Proof We construct the operator

$$W : E_\ell \to E_\ell$$

as follows:

$$\begin{aligned} Wx(t) &= -(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1} I^{u_\ell}_{T^+_{\ell-1}} f_1\big(T_\ell, x(T_\ell), I^{u_\ell}_{T^+_{\ell-1}} x(T_\ell)\big) \\ &\quad + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^t (t-s)^{u_\ell-1} f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big)\,ds, \quad t \in J_\ell. \end{aligned} \tag{9}$$

It follows from the properties of fractional integrals and the continuity of function $t^\delta f_1$ that the operator W is well defined.

Let

$$R_\ell \geq \frac{\frac{2f^\star(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)}}{1 - \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}(T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)})}$$

with

$$f^\star = \sup_{t\in J_\ell}\big\|f_1(t,0,0)\big\|.$$

We consider the set

$$B_{R_\ell} = \big\{x \in E_\ell, \|x\|_{E_\ell} \leq R_\ell\big\}.$$

Clearly, B_{R_ℓ} is nonempty, closed, convex, and bounded.

Now we demonstrate that W satisfies the assumptions of Theorem 2.1. We shall prove it in four phases.

Step 1: $W(B_{R_\ell}) \subseteq (B_{R_\ell})$.

For $x \in B_{R_\ell}$, by (H2) we get:

$$\begin{aligned} \big\|Wx(t)\big\| &\leq \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell - s)^{u_\ell-1}\big\|f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big)\big\|\,ds \\ &\quad + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^t (t-s)^{u_\ell-1}\big\|f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big)\big\|\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} \big\| f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big) \big\| \, ds \\
&\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} \big\| f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big) - f_1(s,0,0) \big\| \, ds \\
&\quad + \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} \big\| f_1(s,0,0) \big\| \, ds \\
&\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} s^{-\delta} \big(K \big\| x(s) \big\| + L \big\| I^{u_\ell}_{T^+_{\ell-1}} x(s) \big\| \big) \, ds + \frac{2f^\star (T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
&\leq \frac{2(T_\ell - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} \left(K + L \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \right) \big\| x(s) \big\| \, ds \\
&\quad + \frac{2f^\star (T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
&\leq \frac{2(T_\ell - T_{\ell-1})^{u_\ell - 1} (T_\ell^{1-\delta} - T_{\ell-1}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \left(K + L \frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell + 1)} \right) R_\ell \\
&\quad + \frac{2f^\star (T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)} \\
&\leq R_\ell,
\end{aligned}
$$

which means that $W(B_{R_\ell}) \subseteq B_{R_\ell}$.

Step 2: W is continuous.

Let a sequence (x_n) converge to x in E_ℓ, and let $t \in J_\ell$. Then

$$
\begin{aligned}
&\big\| (Wx_n)(t) - (Wx)(t) \big\| \\
&\quad\leq \frac{(T_\ell - T_{\ell-1})^{1-u_\ell} (t - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} \big\| f_1\big(s, x_n(s), I^{u_\ell}_{T^+_{\ell-1}} x_n(s)\big) \\
&\qquad - f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big) \big\| \, ds \\
&\qquad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell - 1} \big\| f_1\big(s, x_n(s), I^{u_\ell}_{T^+_{\ell-1}} x_n(s)\big) - f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big) \big\| \, ds \\
&\quad\leq \frac{(T_\ell - T_{\ell-1})^{1-u_\ell} (T_\ell - T_{\ell-1})^{u_\ell - 1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} \big\| f_1\big(s, x_n(s), I^{u_\ell}_{T^+_{\ell-1}} x_n(s)\big) \\
&\qquad - f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big) \big\| \, ds \\
&\qquad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (t - s)^{u_\ell - 1} \big\| f_1\big(s, x_n(s), I^{u_\ell}_{T^+_{\ell-1}} x_n(s)\big) - f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big) \big\| \, ds \\
&\quad\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell - 1} \big\| f_1\big(s, x_n(s), I^{u_\ell}_{T^+_{\ell-1}} x_n(s)\big) - f_1\big(s, x(s), I^{u_\ell}_{T^+_{\ell-1}} x(s)\big) \big\| \, ds \\
&\quad\leq \frac{2}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} (T_\ell - s)^{u_\ell - 1} \big(K \big\| x_n(s) - x(s) \big\| + L I^{u_\ell}_{T^+_{\ell-1}} \| x_n(s) - x(s)) \| \big) \, ds \\
&\quad\leq \frac{2K}{\Gamma(u_\ell)} \| x_n - x \|_{E_\ell} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} (T_\ell - s)^{u_\ell - 1} \, ds \\
&\qquad + \frac{2L}{\Gamma(u_\ell)} \big\| I^{u_\ell}_{T^+_{\ell-1}} (x_n - x) \big\|_{E_\ell} \int_{T_{\ell-1}}^{T_\ell} s^{-\delta} (T_\ell - s)^{u_\ell - 1} \, ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{2K}{\Gamma(u_\ell)}\|x_n - x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell} s^{-\delta}(T_\ell - s)^{u_\ell-1}\,ds \\
&\quad + \frac{2L(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)}\|x_n - x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell} s^{-\delta}(T_\ell - s)^{u_\ell-1}\,ds \\
&\le \left(\frac{2K}{\Gamma(u_\ell)} + \frac{2L(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell)\Gamma(u_\ell+1)}\right)\|x_n - x\|_{E_\ell}\int_{T_{\ell-1}}^{T_\ell} s^{-\delta}(T_\ell - s)^{u_\ell-1}\,ds \\
&\le \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\left(2K + \frac{2L(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\|x_n - x\|_{E_\ell},
\end{aligned}
$$

that is,

$$
\big\|(Wx_n) - (Wx)\big\|_{E_\ell} \to 0 \quad \text{as } n \to \infty.
$$

Thus the operator W is continuous on E_ℓ.

Step 3: W is bounded and equicontinuous.

From Step 2 we have $W(B_{R_\ell}) = \{W(x) : x \in B_{R_\ell}\} \subset B_{R_\ell}$, and hence, for each $x \in B_{R_\ell}$, we have $\|W(x)\|_{E_\ell} \le R_\ell$, which means that $W(B_{R_\ell})$ is bounded. It remains to check that $W(B_{R_\ell})$ is equicontinuous.

For $t_1, t_2 \in J_\ell$, $t_1 < t_2$, and $x \in B_{R_\ell}$, we have:

$$
\begin{aligned}
&\big\|(Wx)(t_2) - (Wx)(t_1)\big\| \\
&\quad = \Bigg\| -\frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t_2 - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \\
&\qquad \times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\,ds \\
&\qquad + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_2} (t_2 - s)^{u_\ell-1} f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\,ds + \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t_1 - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \\
&\qquad \times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\,ds \\
&\qquad - \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1} (t_1 - s)^{u_\ell-1} f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\,ds \Bigg\| \\
&\quad \le \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\big((t_2 - T_{\ell-1})^{u_\ell-1} - (t_1 - T_{\ell-1})^{u_\ell-1}\big) \\
&\qquad \times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1}\big\|f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\big\|\,ds \\
&\qquad + \frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1} \big((t_2 - s)^{u_\ell-1} - (t_1 - s)^{u_\ell-1}\big)\big\|f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\big\|\,ds \\
&\qquad + \frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2} (t_2 - s)^{u_\ell-1}\big\|f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\big\|\,ds \\
&\quad \le \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\big((t_2 - T_{\ell-1})^{u_\ell-1} - (t_1 - T_{\ell-1})^{u_\ell-1}\big) \\
&\qquad \times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1}\big\|f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big) - f_1(s, 0, 0)\big\|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(T_\ell-T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\big\|f_1(s,0,0)\big\|\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\big)\big\|f_1\big(s,x(s),I_{T_{\ell-1}^+}^{u_\ell}x(s)\big)-f_1(s,0,0)\big\|\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\big)\big\|f_1(s,0,0)\big\|\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\big\|f_1\big(s,x(s),I_{T_{\ell-1}^+}^{u_\ell}x(s)\big)-f_1(s,0,0)\big\|\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\big\|f_1(s,0,0)\big\|\,ds.\\
\le{}&\frac{(T_\ell-T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\big)\\
&\times\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}s^{-\delta}\big(K\big\|x(s)\big\|+L\big\|I_{T_{\ell-1}^+}^{u_\ell}x(s)\big\|\big)\,ds\\
&+\frac{f^\star(T_\ell-T_{\ell-1})^{1-u_\ell}}{\Gamma(u_\ell)}\big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\big)\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}s^{-\delta}\big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\big)\big(K\big\|x(s)\big\|+L\big\|I_{T_{\ell-1}^+}^{u_\ell}x(s)\big\|\big)\,ds\\
&+\frac{f^\star}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t_1}\big((t_2-s)^{u_\ell-1}-(t_1-s)^{u_\ell-1}\big)\,ds\\
&+\frac{1}{\Gamma(u_\ell)}\int_{t_1}^{t_2}s^{-\delta}(t_2-s)^{u_\ell-1}\big(K\big\|x(s)\big\|+L\big\|I_{T_{\ell-1}^+}^{u_\ell}x(s)\big\|\big)\,ds\\
&+\frac{f^\star}{\Gamma(u_\ell)}\int_{t_1}^{t_2}(t_2-s)^{u_\ell-1}\,ds\\
\le{}&\frac{1}{\Gamma(u_\ell)}\big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\big)\big(K\|x\|_{E_\ell}+L\big\|I_{T_{\ell-1}^+}^{u_\ell}x\big\|_{E_\ell}\big)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\,ds\\
&+\frac{f^\star(T_\ell-T_{\ell-1})}{\Gamma(u_\ell+1)}\big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\big)\\
&+\frac{1}{\Gamma(u_\ell)}\big(K\|x\|_{E_\ell}+L\big\|I_{T_{\ell-1}^+}^{u_\ell}x\big\|_{E_\ell}\big)\int_{T_{\ell-1}}^{t_1}s^{-\delta}\big((t_2-t_1)^{u_\ell-1}\big)\,ds\\
&+\frac{f^\star}{\Gamma(u_\ell)}\bigg(\frac{(t_2-T_{\ell-1})^{u_\ell}}{u_\ell}-\frac{(t_2-t_1)^{u_\ell}}{u_\ell}-\frac{(t_1-T_{\ell-1})^{u_\ell}}{u_\ell}\bigg)\\
&+\frac{(t_2-t_1)^{u_\ell-1}}{\Gamma(u_\ell)}\big(K\|x\|_{E_\ell}+L\big\|I_{T_{\ell-1}^+}^{u_\ell}x\big\|_{E_\ell}\big)\int_{t_1}^{t_2}s^{-\delta}\,ds+\frac{f^\star}{\Gamma(u_\ell)}\frac{(t_2-t_1)^{u_\ell}}{u_\ell}\\
\le{}&\frac{{T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\big)\\
&\times\bigg(K\|x\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|x\|_{E_\ell}\bigg)\\
&+\frac{f^\star(T_\ell-T_{\ell-1})}{\Gamma(u_\ell+1)}\big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\big)\\
&+\bigg(\frac{({t_1}^{1-\delta}-{T_{\ell-1}}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\bigg)\bigg(K\|x\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|x\|_{E_\ell}\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{f^\star}{\Gamma(u_\ell+1)}\big((t_2-T_{\ell-1})^{u_\ell}-(t_2-t_1)^{u_\ell}-(t_1-T_{\ell-1})^{u_\ell}\big)\\
&+ \frac{({t_2}^{1-\delta}-{t_1}^{1-\delta})(t_2-t_1)^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\left(K\|x\|_{E_\ell}+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|x\|_{E_\ell}\right)+\frac{f^\star(t_2-t_1)^{u_\ell}}{\Gamma(u_\ell+1)}\\
\le{}& \left(\frac{{T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\left(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\|x\|_{E_\ell}+\frac{f^\star(T_\ell-T_{\ell-1})}{\Gamma(u_\ell+1)}\right)\\
&\times\big((t_2-T_{\ell-1})^{u_\ell-1}-(t_1-T_{\ell-1})^{u_\ell-1}\big)\\
&+\left(\frac{{t_2}^{1-\delta}-{T_{\ell-1}}^{1-\delta}}{(1-\delta)\Gamma(u_\ell)}\left(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\|x\|_{E_\ell}\right)(t_2-t_1)^{u_\ell-1}\\
&+\frac{f^\star}{\Gamma(u_\ell+1)}\big((t_2-T_{\ell-1})^{u_\ell}-(t_1-T_{\ell-1})^{u_\ell}\big).
\end{aligned}
$$

Hence $\|(Wx)(t_2)-(Wx)(t_1)\|_{E_\ell}\to 0$ as $|t_2-t_1|\to 0$, which implies that $T(B_{R_\ell})$ is equicontinuous.

Step 4: W is a k-set contraction.

For $U\in B_{R_\ell}$ and $t\in J_\ell$, we have:

$$
\begin{aligned}
\zeta\big(W(U)(t)\big) &= \zeta\big((Wx)(t),x\in U\big)\\
&\le \left\{\frac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\zeta f_1\big(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s)\big)\,ds\right.\\
&\quad\left.+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\zeta f_1\big(s,x(s),I^{u_\ell}_{T^+_{\ell-1}}x(s)\big)\,ds, x\in U\right\}.
\end{aligned}
$$

Then Remark 3.1 implies that, for each $s\in J_i$,

$$
\begin{aligned}
&\zeta\big(W(U)(t)\big)\\
&\le\left\{\frac{(T_\ell-T_{\ell-1})^{1-u_\ell}(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\right.\\
&\quad\times\int_{T_{\ell-1}}^{T_\ell}(T_\ell-s)^{u_\ell-1}\left[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\,ds+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\,ds\right]\\
&\quad+\frac{1}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}(t-s)^{u_\ell-1}\left[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}\,ds\right.\\
&\quad\left.\left.+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}\,ds\right],x\in U\right\}\\
&\le\left\{\frac{(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{T_\ell}\left[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\,ds+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\widehat{\zeta}(U)\int_{T_{\ell-1}}^{T_\ell}s^{-\delta}\,ds\right]\right.\\
&\quad+\frac{(t-T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\int_{T_{\ell-1}}^{t}\left[K\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}\,ds\right.\\
&\quad\left.\left.+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\widehat{\zeta}(U)\int_{T_{\ell-1}}^{t}s^{-\delta}\,ds\right],x\in U\right\}\\
&\le\frac{[({T_\ell}^{1-\delta}-{T_{\ell-1}}^{1-\delta})+(t^{1-\delta}-{T_{\ell-1}}^{1-\delta})](t-T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\left(K+L\frac{(T_\ell-T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\widehat{\zeta}(U)
\end{aligned}
$$

$$\leq \frac{2({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\left(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\widehat{\zeta}(U).$$

Therefore we have:

$$\widehat{\zeta}(WU) \leq \frac{2({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\left(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\widehat{\zeta}(U).$$

Consequently, from (8) we deduce that W forms a set contraction. Hence by Theorem 2.1 problem (6) has at least a solution $\widetilde{x_\ell}$ in B_{R_ℓ}.

Let

$$x_\ell = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell. \end{cases} \tag{10}$$

We know that $x_\ell \in C([0,T_\ell],X)$ defined by (10) satisfies the equation

$$\frac{d^2}{dt^2}\left(\int_0^{T_1} \frac{(t-s)^{1-u_1}}{\Gamma(2-u_1)} x_\ell(s)\,ds + \cdots + \int_{T_{\ell-1}}^{t} \frac{(t-s)^{1-u_\ell}}{\Gamma(2-u_\ell)} x_\ell(s)\,ds\right) = f_1\big(s, x_\ell(s), I_{0^+}^{u_\ell} x_\ell(s)\big)$$

for $t \in J_\ell$, which means that x_ℓ is a solution of (5) with $x_\ell(0) = 0$ and $x_\ell(T_\ell) = \widetilde{x}_\ell(T_\ell) = 0$. Then

$$x(t) = \begin{cases} x_1(t), & t \in J_1, \\ x_2(t) = \begin{cases} 0, & t \in J_1, \\ \widetilde{x}_2, & t \in J_2, \end{cases} \\ \vdots \\ x_n(t) = \begin{cases} 0, & t \in [0, T_{\ell-1}], \\ \widetilde{x}_\ell, & t \in J_\ell, \end{cases} \end{cases}$$

forms a solution of BVP (1). □

4 Ulam–Hyers–Rassias stability

Theorem 4.1 *Assume* $(H1)$, $(H2)$, (8), *and*

(H3) $\vartheta \in C(J_\ell, X)$ *is an increasing function, and there exists* $\lambda_\vartheta > 0$ *such that*

$$I_{T_{\ell-1}^+}^{u_\ell} \vartheta(t) \leq \lambda_{\vartheta(t)} \vartheta(t) \quad \textit{for all } t \in J_\ell.$$

Then equation of (1) *is UHR stable with respect to* ϑ.

Proof Let $z \in C(J_\ell, X)$ be a solution of the inequality

$$\left\| D_{T_{\ell-1}^+}^{u_\ell} z(t) - f_1\big(t, z(t), I_{T_{\ell-1}^+}^{u_\ell} z(t)\big) \right\| \leq \epsilon \vartheta(t), \quad t \in J_\ell. \tag{11}$$

Let $x \in C(J_\ell, X)$ be a solution of the problem

$$D_{T_{\ell-1}^+}^{u_\ell} x(t) = f_1\big(t, x(t), I_{T_{\ell-1}^+}^{u_\ell} x(t)\big), \quad t \in J_\ell.$$

$$x(T_{\ell-1}) = 0, \quad x(T_\ell) = 0$$

By Lemma 3.1 we have:

$$\begin{aligned}
x(t) = {}& -\frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\, ds \\
& + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell-1} f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\, ds.
\end{aligned}$$

By integration of (11) from (H3) we obtain:

$$\begin{aligned}
& \bigg\| z(t) + \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\big(s, z(s), I_{T_{\ell-1}^+}^{u_\ell} z(s)\big)\, ds \\
& \quad - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell-1} f_1\big(s, z(s), I_{T_{\ell-1}^+}^{u_\ell} z(s)\big)\, ds \bigg\| \\
& \leq \epsilon \int_{T_{\ell-1}}^{t} \frac{(t-s)^{u(i)-1}}{\Gamma(u(i))} \vartheta(s)\, ds \\
& \leq \epsilon \lambda_{\vartheta(t)} \vartheta(t).
\end{aligned}$$

On the other hand, for each $t \in J_\ell$, we have:

$$\begin{aligned}
& \big\| z(t) - x(t) \big\| \\
& = \bigg\| z(t) + \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\, ds \\
& \quad - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell-1} f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x(s)\big)\, ds \bigg\| \\
& = \bigg\| z(t) + \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} f_1\big(s, z(s), I_{T_{\ell-1}^+}^{u_\ell} z(s)\big)\, ds \\
& \quad - \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell-1} f_1\big(s, z(s), I_{T_{\ell-1}^+}^{u_\ell} z(s)\big)\, ds \bigg\| \\
& \quad + \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} \\
& \quad \big\| f_1\big(s, z(s), I_{T_{\ell-1}^+}^{u_\ell} z\big) - f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x\big) \big\|\, ds \\
& \quad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell-1} \big\| f_1\big(s, z(s), I_{T_{\ell-1}^+}^{u_\ell} z\big) - f_1\big(s, x(s), I_{T_{\ell-1}^+}^{u_\ell} x\big) \big\|\, ds \\
& \leq \lambda_{\vartheta(t)} \epsilon \vartheta(t) + \frac{(T_\ell - T_{\ell-1})^{1-u_\ell}(t - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \\
& \quad \times \int_{T_{\ell-1}}^{T_\ell} (T_\ell - s)^{u_\ell-1} s^{-\delta} \Big(K \big\| z(s) - x(s) \big\| + L I_{T_{\ell-1}^+}^{u_\ell} \big\| z(s) - x(s) \big\| \Big)\, ds \\
& \quad + \frac{1}{\Gamma(u_\ell)} \int_{T_{\ell-1}}^{t} (t - s)^{u_\ell-1} s^{-\delta} \Big(K \big\| z(s) - x(s) \big\| + L I_{T_{\ell-1}^+}^{u_\ell} \big\| z(s) - x(s) \big\| \Big)\, ds \\
& \leq \lambda_{\vartheta(t)} \epsilon \vartheta(t) + \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)} \Big(K \|z - x\|_{E_\ell} + L I_{T_{\ell-1}^+}^{u_\ell} \|z - x\|_{E_\ell}\Big) \int_{T_{\ell-1}}^{T_\ell} s^{-\delta}\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(T_\ell - T_{\ell-1})^{u_\ell-1}}{\Gamma(u_\ell)}\left(K\|z-x\|_{E_\ell} + LI^{u_\ell}_{T^+_{\ell-1}}\|z-x\|_{E_\ell}\right)\int_{T_{\ell-1}}^{t} s^{-\delta}\,ds \\
&\le \lambda_{\vartheta(t)}\epsilon\vartheta(t) + \frac{(T_\ell - T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)} \\
&\quad\times\left(K\|z-x\|_{E_\ell} + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z-x\|_{E_\ell}\right) \\
&\quad+\frac{(T_\ell - T_{\ell-1})^{u_\ell-1}(t^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\left(K\|z-x\|_{E_\ell} + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\|z-x\|_{E_\ell}\right) \\
&\le \lambda_{\vartheta(t)}\epsilon\vartheta(t) + \frac{2(T_\ell - T_{\ell-1})^{u_\ell-1}({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})}{(1-\delta)\Gamma(u_\ell)}\left(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\|z-x\|_{E_\ell}.
\end{aligned}
$$

Then

$$
\|z-y\|_{E_\ell}\left(1-\frac{2({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\left(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\right) \le \lambda_{\vartheta(t)}\epsilon\vartheta(t)
$$

For each $t \in J_\ell$, we obtain:

$$
\begin{aligned}
\|z-y\|_{E_\ell} &\le \frac{\lambda_{\vartheta(t)}\epsilon\vartheta(t)}{(1-\frac{2({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}))} \\
&= \left[1-\frac{2({T_\ell}^{1-\delta} - {T_{\ell-1}}^{1-\delta})(T_\ell - T_{\ell-1})^{u_\ell-1}}{(1-\delta)\Gamma(u_\ell)}\left(K + L\frac{(T_\ell - T_{\ell-1})^{u_\ell}}{\Gamma(u_\ell+1)}\right)\right]^{-1} \\
&\quad\times \lambda_{\vartheta(t)}\epsilon\vartheta(t) := c_{f_1}\epsilon\vartheta(t).
\end{aligned}
$$

Then the equation in (6) is UHR stable with respect to ϑ for each $\ell \in \{1, 2, \dots, n\}$. Consequently, the equation in (1) is UHR stable with respect to ϑ. □

5 Example

In this example, we deal with the fractional boundary value problem

$$
\begin{cases}
D^{u(t)}_{0^+}x(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|x(t)|+|I^{u(t)}_0 x(t)|)}, & t \in J := [0,2], \\
x(0) = 0, \qquad x(2) = 0.
\end{cases}
\tag{12}
$$

Let

$$
\begin{aligned}
&f_1(t,y,z) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+y+z)}, \quad (t,y,z) \in [0,2]\times[0,+\infty)\times[0,+\infty), \\
&u(t) = \begin{cases} \frac{3}{2}, & t \in J_1 := [0,1], \\ \frac{9}{5}, & t \in J_2 :=]1,2]. \end{cases}
\end{aligned}
\tag{13}
$$

Then we have:

$$
t^{\frac{1}{3}}\left|f_1(t,y_1,z_1) - f_1(t,y_2,z_2)\right| = \left|\frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)}\left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right)\right|
$$

$$\begin{aligned}
&\le \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+y_1+z_1)(1+y_2+z_2)} \\
&\le \frac{e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)}\big(|y_1-y_2|+|z_1-z_2|\big) \\
&\le \frac{1}{(e+5)}|y_1-y_2|+\frac{1}{(e+5)}|z_1-z_2|.
\end{aligned}$$

Thus (H2) holds with $\delta=\frac{1}{3}$ and $K=L=\frac{1}{e+5}$.

By (13) the equation of problem (12) can be divided into two expressions as follows:

$$\begin{aligned}
D_{0^+}^{\frac{3}{2}}x(t) &= \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|x(t)|+|I_0^{\frac{3}{2}}x(t)|)}, \quad t\in J_1, \\
D_{1^+}^{\frac{9}{5}}x(t) &= \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|x(t)|+|I_0^{\frac{9}{5}}x(t)|)}, \quad t\in J_2.
\end{aligned}$$

For $t\in J_1$, problem (12) is equivalent to the problem

$$\begin{cases} D_{0^+}^{\frac{3}{2}}x(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|x(t)|+|I_0^{\frac{3}{2}}x(t)|)}, \quad t\in J_1, \\ x(0)=0, \qquad x(1)=0. \end{cases} \tag{14}$$

Next, we prove that condition (8) is fulfilled.

$$\begin{aligned}
&\frac{2({T_1}^{1-\delta}-{T_0}^{1-\delta})(T_1-T_0)^{u_1-1}}{(1-\delta)\Gamma(u_1)}\left(K+\frac{L(T_1-T_0)^{u_1}}{\Gamma(u_1+1)}\right) \\
&\quad = \frac{2}{\frac{2}{3}(e+5)\Gamma(\frac{3}{2})}\left(1+\frac{1}{\Gamma(\frac{5}{2})}\right)\simeq 0.7685<1.
\end{aligned}$$

Let $\vartheta(t)=t^{\frac{1}{2}}$. Then

$$\begin{aligned}
I_{0^+}^{u_1}\vartheta(t) &= \frac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}}s^{\frac{1}{2}}\,ds \\
&\le \frac{1}{\Gamma(\frac{3}{2})}\int_0^t (t-s)^{\frac{1}{2}}\,ds \\
&\le \frac{2}{3\Gamma(\frac{3}{2})}\vartheta(t) := \lambda_{\vartheta(t)}\vartheta(t).
\end{aligned}$$

Thus (H3) is satisfied with $\vartheta(t)=t^{\frac{1}{2}}$ and $\lambda_{\vartheta(t)}=\frac{2}{3\Gamma(\frac{3}{2})}$.

By Theorem 3.1 problem (14) has a solution $x_1\in\tilde{E}_1$, and by Theorem 4.1 the equation in (14) is UHR stable.

For $t\in J_2$, problem (12) can be written as follows:

$$\begin{cases} D_{1^+}^{\frac{9}{5}}x(t) = \frac{t^{-\frac{1}{3}}e^{-t}}{(e^{e^{\frac{t^2}{1+t}}}+4e^{2t}+1)(1+|x(t)|+|I_0^{\frac{9}{5}}x(t)|)}, \quad t\in J_2, \\ x(1)=0, \qquad x(2)=0. \end{cases} \tag{15}$$

We see that

$$\begin{aligned}&\frac{2({T_2}^{1-\delta}-{T_1}^{1-\delta})(T_2-T_1)^{u_2-1}}{(1-\delta)\Gamma(u_2)}\left(K+\frac{L(T_2-T_1)^{u_2}}{\Gamma(u_2+1)}\right)\\&\quad=\frac{2(2^{\frac{2}{3}}-1)}{\frac{2}{3}\Gamma(\frac{9}{5})}\frac{1}{e+5}\left(1+\frac{1}{\Gamma(\frac{14}{5})}\right)\simeq 0.3913<1.\end{aligned}$$

As a result, condition (8) is satisfied. Moreover,

$$\begin{aligned}I_{1^+}^{u_2}\vartheta(t)&=\frac{1}{\Gamma(\frac{9}{5})}\int_1^t(t-s)^{\frac{4}{5}}s^{\frac{1}{2}}\,ds\\&\le\frac{1}{\Gamma(\frac{9}{5})}\int_1^t(t-s)^{\frac{4}{5}}\,ds\\&\le\frac{5}{9\Gamma(\frac{9}{5})}\vartheta(t):=\lambda_{\vartheta(t)}\vartheta(t).\end{aligned}$$

Thus (H3) is fulfilled with $\vartheta(t)=t^{\frac{1}{2}}$ and $\lambda_{\vartheta(t)}=\frac{5}{9\Gamma(\frac{9}{5})}$.

By Theorem 3.1 problem (15) possesses a solution $\widetilde{x}_2\in E_2$, Further, Theorem 4.1 yields that (15) is UHR stable.

It is known that

$$x_2(t)=\begin{cases}0, & t\in J_1\\ \widetilde{x}_2(t), & t\in J_2.\end{cases}$$

As a result, by Definition 3.1 the boundary value problem (12) has a solution

$$x(t)=\begin{cases}x_1(t), & t\in J_1,\\ x_2(t)=\begin{cases}0, & t\in J_1,\\ \widetilde{x}_2(t), & t\in J_2.\end{cases}\end{cases}$$

In addition, by Theorem 4.1 the equation in (12) is UHR stable.

6 Conclusion

Our proposed multiterm BVP has been successfully investigated in this work via three theorems: The Darbo's fixed point theorem (DFPT), the Kuratowski measure of noncompactness (KMNC), and the Ulam-Hyers-Rassias stability (UHR) to prove the existence and stability of solutions for our proposed BVP. A numerical example is given at the end to support and validate the potentiality of all our obtained results. As a result of our investigation into this particular research subject, our results are new and novel. Furthermore, with the support of our new results in this work, further research works can be investigated on this open research subject. Our proposed BVP can be possibly extended to other fractional models.

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