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Coupled transform method for time-space fractional Black-Scholes option pricing model

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Abstract This paper presents analytical solutions of a time-space-fractional Black-Scholes model (TSFBSM) using a coupled technique referred to as Fractional Complex Transform (FCT) with the aid of a modified differential transform method. The nature of the derivatives is in the sense of Jumarie. The considered cases and applications show more consistency of the TSFBSM with an actual integer and fractional data when compared with the classical Black-Scholes model. The method is noted to be very effective, even with little knowledge of fractional calculus. Extension of this to multi-factor models formulated in terms of stochastic dynamics is highly recommended.

1. Introduction

From an industrial and academic point of view, the classical option pricing model referred to the Black-Scholes Model (BSM) is a partial differential equation (PDE) for the calculation of theoretical values of financial options based on some specific parameters such as current stock prices, expected volatility, time to expiration, strike prices, and expected dividends. The BSM remains a crucial financial tool for option valuation and pricing [1-5]. The relaxation of assumptions governing the BSM most times leads to a non-linear version of the BSM or jump-diffusion models involving spikes [6-8]. In this work, the generalized form of the classical BSM for option pricing will be considered regarding time–space fractional order. The BSM for option pricing is of the form:

$$\frac{\partial \Lambda}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 \Lambda}{\partial S^2} + rS \frac{\partial \Lambda}{\partial S} - r\Lambda = 0 \tag{1}$$

where$\Lambda = \Lambda(S,t)$ represents the option value, σ the volatility parameter, S, the price of underlying, r the interest rate (risk-free), and t as time. The associated payoff function, $p_\Lambda(S,t)$, and the price of expiration, E is denoted as:

$$p_\Lambda(S,t) = \pm c(S-E)^+ \tag{2}$$

for $c = \pm 1$ indicating European call and put option respectively, where $(S_*, 0)^+$ signifies the maximum between $S_*\&0$, σ as a parameter stands for the underlying asset $(S = S(t))$

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volatility, r is the interest rate (in risk-free term), and T is the time at maturity. Suppose σ, the volatility parameter, and r the interest rate (risk-free) are constants, such that:

$$\left.\begin{array}{r} S = \varsigma e^{x}, \\ \Lambda = \Lambda(S,t) = \varsigma\omega(x,\tau), \\ 2\tau = (T-t)\sigma^2. \end{array}\right\} \quad (3)$$

Then, little algebra gives:

$$\left.\begin{array}{r} \frac{\partial\Lambda}{\partial t} = -\varsigma\frac{\sigma^2}{2}\frac{\partial\omega}{\partial\tau}, \\ \frac{\partial\Lambda}{\partial S} = \frac{\varsigma}{S}\frac{\partial\omega}{\partial x}, \\ \frac{\partial^2\Lambda}{\partial S^2} = -\frac{\varsigma}{S^2}\frac{\partial\omega}{\partial x} + \frac{\varsigma}{S^2}\frac{\partial^2\omega}{\partial x^2}. \end{array}\right\} \quad (4)$$

Thus, (4) in (1) gives:

$$-\varsigma\frac{\sigma^2}{2}\frac{\partial\omega}{\partial\tau} + \frac{1}{2}S^2\sigma^2\left(-\frac{\varsigma}{S^2}\frac{\partial\omega}{\partial x} + \frac{\varsigma}{S^2}\frac{\partial^2\omega}{\partial x^2}\right) + rS\left(\frac{\varsigma}{S}\frac{\partial\omega}{\partial x}\right) - r(\varsigma\omega(x,\tau)) = 0.$$

$$\Rightarrow \begin{cases} \frac{\partial\omega}{\partial\tau} = \frac{\partial^2\omega}{\partial x^2} + (\kappa-1)\frac{\partial\omega}{\partial x} - \kappa\omega, \ \kappa = \frac{2r}{\sigma^2}, \\ \omega(x,0) = h(x). \end{cases} \quad (5)$$

Equation (5) has received the attention of so many researchers in terms of effective semi-analytical numerical and analytical solution methods, which are of vital aid when handling differential models. Recently, in [9] the bond price model was extended to time-fractional order using conformable decomposition for analytical solutions. Edeki and Adinya [10] applied Fractional Complex Transform to a time-fractional one-factor Markovian model for bond pricing. Several methods of solution like variable separable method, direct integration, Laplace, Picard iteration method, differential transform method (DTM), Adomian decomposition method (ADM), Elzaki transform method (ETM), Sumudu decomposition method are applicable to PDEs [11-16].

Fractional calculus (FC) is used in different aspects of mathematics and applied sciences such as finance, biomedical, physics, and so on. FC helps in capturing the memory effect on the system.

With regard to FC and its applications, Morales-Delgado *et. al.,* [17] studied a non-integer order dynamics in relation to oxygen diffusion via the application of Laplace homotopy method. Zhang *et al.,* in [18] applied Laplace fractional homotopy perturbation method for the solution of heat conduction equations of non-homogeneous type involving fractional domains both in time- and space- derivatives. In [19], pathway fractional integer operator was considered with the aid of Mittag-Leffler functions. The authors in [20] presented some composition formulae for Marichev-Saigo-Maeda fractional operator, and obtained Laplace transforms of the corresponding composition formulae. In [21], some useful generalizations of fractional integro-differential operators on the basis of omega-parameter were introduced. Fractional kinetic equations based on hypergeometric functions were obtained in [22]. Damping characteristics of a fractional oscillator were considered in [23], while in [24], fractional oscillator model was used to describe damped vibrations. In [25], generalized fractional integral was utilized for obtaining Hermite-Hadamard-type inequalities for coordinated convex function. Several new properties of Mittag-Leffler function were recently obtained based on Konhauser and Laguerre polynomials [26], and [27], respectively. Jain *et. al.,* [28], considered certain Hermite-Hadamard inequalities on the basis of convex logarithmic functions and derived new inequalities in terms of poly gamma functions.

For related researches on fractional models and solution methods with regard to detail and recent views, we make reference to [29-43] and the references therein.

The generalized version of (5) to time–space fractional-order to be considered takes the following form:

$$\begin{cases} \frac{\partial^{\alpha}\omega}{\partial\tau^{\alpha}} = \frac{\partial^{2\gamma}\omega}{\partial x^{2\gamma}} + (\kappa-1)\frac{\partial\omega}{\partial x} - \kappa\omega, \ \alpha\in(0,1], \ 2\gamma\in(0,2], \\ \omega(x,0) = h(x). \end{cases} \quad (6)$$

This paper, therefore, aims at providing approximate-analytical-solutions to the TSFBSM as contained in (6) for option pricing by applying the proposed semi-approximate technique known as Coupled Method. Similar PDEs can be considered by semi-analytical, numerical, and approximate methods in terms of solutions [44-49]. The fractional derivative in (6) is defined in the sense of Jumarie.

In financial mathematics, the fractional Black-Scholes model (FBSM) serves as a unique tool with respect to the modern theory of option pricing. The FBSM is a generalization of the classical BSM due to the modification(s) of the initial constancy assumptions. The exact/analytical solutions of which are difficult to obtain.

A lot of semi-analytical methods have been developed by numerical analysts for integer related mathematical and physical models. Meanwhile, modifications of some of these methods solve time-fractional models. It is, therefore, imperative to consider effective and reliable methods for space-fractional and time–space fractional models. Hence, the motivation for this research. The main feature of the proposed method of solution is hinged on the notion that it transforms the fractional model to its integer counterparts for ease of approach. This helps in employing some known classical semi-analytical methods such as the ADM, Differential Transform Method (DTM), Natural Decomposition Method (NDM), and so on.

2. Remarks on Jumarie's fractional derivative

Reference is made to Fractional Derivative (JFD) in terms of Jumarie as a modified version Riemann-Liouville fractional derivatives [50,51]. The following is considered:

Consider $\Psi(z)$ as a real-valued function of z (continuous), and $D_z^{\alpha}\Psi = \frac{\partial^{\alpha}\Psi}{\partial z^{\alpha}}$ as JFD of $\Psi(z)$, whose order is α with respect to z. So,

$$D_z^{\alpha}\Psi = \begin{cases} \frac{1}{\Gamma(-\alpha)}\frac{d}{dz}\int_0^z (z-\zeta)^{-\alpha-1}(\Psi(\zeta)-\Psi(0))d\zeta, \ \alpha\in(-\infty,0) \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z (z-\zeta)^{-\alpha}(\Psi(\zeta)-\Psi(0))d\zeta, \ \alpha\in(0,1) \\ \left(\Psi^{(\alpha-\eta)}(z)\right)^{(\eta)}, \ \alpha\in[\eta,\eta+1), \ \eta\geqslant 1 \end{cases} \quad (7)$$

where $\Gamma(\cdot)$ represents a gamma function. The main features of JFD [51] are as follows:

(i). $D_z^{\alpha}c = 0, \ \alpha > 0,$ for a constant c
(ii). $D_z^{\alpha}(c\Psi(z)) = cD_z^{\alpha}\Psi(z), \ \alpha > 0,$
(iii). $D_z^{\alpha}z^{\beta} = \left\{\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}z^{\beta-\alpha}, \text{ for } \beta\neq 1-\alpha.\right.$
(iv). $D_z^{\alpha}(\Psi_1(z)\Psi_2(z)) = D_z^{\alpha}\Psi_1(z)(\Psi_2(z)) + \Psi_1(z)D_z^{\alpha}\Psi_2(z),$
(v). $D_z^{\alpha}(\Psi(z(g))) = D_z^1\Psi\cdot D_g^{\alpha}z.$

The properties in (i) through (v) are basic in applications.

3. The Reduced differential Transform [30]

Let $\omega(x,t)$ be a continuously differentiable and analytic function on a given domain, D, so the differential transform of $\omega(x,t)$ is expressed as:

$$\Omega_k(x) = \frac{1}{k!}\left[\frac{\partial^k \omega(x,t)}{\partial t^k}\right]_{t=0} \tag{8}$$

where $\omega(x,t)$ and $\Omega_k(x)$ are referred to as the original and the transformed functions in that order. Hence, the differential inverse transform (DIT) of $\Omega_k(x)$ is given as:

$$\omega(x,t) = \sum_{k=0}^{\infty} \Omega_k(x) t^k \tag{9}$$

3.1. DTM and its basic properties

D1: If $\omega(x,t) = \alpha q(x,t) \pm \beta p(x,t)$, then $\Omega_k(x) = \alpha Q_k(x) \pm \beta P_k(x)$.

D2: If $\omega(x,t) = \frac{c\partial^m q(x,t)}{\partial t^m}$, $m \in N$, then $\Omega_k(x) = \frac{c(k+m)!}{k!} Q_{k+m}(x)$.

D3: If $\omega(x_*,t) = \frac{m(x_*)\partial^\eta q(x_*,t)}{\partial x_*^\eta}$, $\eta \in N$, then $\Omega_k(x_*) = \frac{M(x_*)\partial^\eta Q_k(x_*)}{\partial x_*^\eta}$, $\eta \in N$.

D4: If $\omega(x,t) = p(x,t)q(x,t)$, then $\Omega_k(x) = \sum_{\eta=0}^{k} P_\eta(x) Q_{k-\eta}(x)$.

D5: If $\omega(x,t) = x^{n_1} t^{n_2}$, then $\Omega_c = x^{n_1}\delta(c - n_2), \delta(c) = \begin{cases} 0, & c \neq 0, \\ 1, & c = 0 \end{cases}$.

3.2. The fractional Complex Transform

The general fractional differential equation of the following form is considered:

$$h\left(\upsilon, D_t^\alpha \upsilon, D_x^\beta \upsilon, D_y^\lambda \upsilon, D_z^\gamma \upsilon\right) = 0, \ \upsilon = \upsilon(t,x,y,z) \tag{10}$$

while the FCT as considered in [30,31], and the references therein is defined as follows:

$$T = \frac{at^\alpha}{\Gamma(1+\alpha)}, \ \alpha \in (0,1]. \tag{11}$$

Thus, using (iii) in section 2, we have:

$$D_z^\alpha z^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} z^{\beta-\alpha}, \ 0 < \alpha \leqslant \beta,$$

$$\therefore D_t^\alpha T == \frac{a}{\Gamma(1+\alpha)}\left[\frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\alpha+1)}\right] t^{\alpha-\alpha} = a. \tag{12}$$

Hence,

$$\begin{cases} D_\tau^\alpha \omega = D_\tau^\alpha \omega(T(\tau)) = D_T^1 \omega \cdot D_\tau^\alpha T = a\frac{\partial \omega}{\partial T}, \\ D_x^\gamma \omega = D_x^\gamma \omega(X(t)) = D_X^1 \omega \cdot D_X^\gamma X = b\frac{\partial \omega}{\partial X}. \end{cases} \tag{13}$$

4. Applications

In this section, the proposed FCT is applied to time–space fractional Black–Scholes models (TSFBSMs) as follows:

Case I- Consider the following form of TSFBSM:

$$\begin{cases} \frac{\partial^\alpha \omega}{\partial \tau^\alpha} = \frac{\partial^{2\gamma}\omega}{\partial x^{2\gamma}} + (\kappa - 1)\frac{\partial \omega}{\partial x} - \kappa\omega, \ \alpha \in (0,1], \ 2\gamma \in (0,2], \\ \omega(x,0) = (\exp(x) - 1)^+. \end{cases} \tag{14}$$

It is remarked from FCT, that $T = \frac{a\tau^\alpha}{\Gamma(1+\alpha)}$, $X = \frac{ax^\gamma}{\Gamma(1+\gamma)}$, which according to section 3 gives

$$\frac{\partial^\alpha \omega}{\partial \tau^\alpha} = \frac{\partial \omega}{\partial T}, \ \frac{\partial^\gamma \omega}{\partial x^\gamma} = \frac{\partial \omega}{\partial X}, \ a = b = 1.$$

Hence, (12) becomes:

$$\begin{cases} \frac{\partial \omega}{\partial T} = \frac{\partial^2 \omega}{\partial X^2} + (\kappa - 1)\frac{\partial \omega}{\partial X} - \kappa\omega, \\ \omega(X,0) = \max(e^X - 1, 0). \end{cases} \tag{15}$$

Applying the RDTM to (15) gives:

$$\begin{cases} W_{k+1} = \frac{1}{(1+k)}\left(W''_{X,k} + (\kappa - 1)W'_{X,k} - \kappa W_k\right), \ k \geqslant 0, \\ W(X,0) = \max(e^X - 1, 0), \end{cases} \tag{16}$$

where the prime notations denote derivatives w.r.t. variable X. As a result, the recursive relation in (16) yields:

$$\begin{cases} W_0 = \max\{\exp(X) - 1, 0\}, \\ W_1 = \kappa\{-\max(\exp(X) - 1, 0) + \max(\exp(X), 0)\}, \\ W_2 = \frac{-\kappa^2}{2}\{-\max(\exp(X) - 1, 0) + \max(\exp(X), 0)\}, \\ W_3 = \frac{\kappa^3}{6}\{-\max(\exp(X) - 1, 0) + \max(\exp(X), 0)\}, \\ \vdots \\ W_p = \frac{(-1)^{p+1}(\kappa)^p}{p!}\left\{\max \begin{matrix} (0, \exp(X)) \\ -\max(0, -1\exp(X)) \end{matrix}\right\}, p = 1, 2, \cdots. \end{cases} \tag{17}$$

Hence,

$$\omega(X,T) = \sum_{m=0}^{\infty} W_m T^m = W_0 + \sum_{m=1}^{\infty} W_m T^m$$
$$= \max(\exp(X) - 1, 0) + \sum_{m=1}^{\infty} (-1)^{m+1}\frac{(\kappa T)^m}{m!}\left\{\begin{matrix} \max(\exp(X), 0) \\ -\max(\exp(X) - 1, 0) \end{matrix}\right\} \tag{18}$$

Therefore, the solution of (14) is as follows:

$$\omega(x,\tau) = \begin{cases} \max\left(\exp\left(\frac{x^\gamma}{\Gamma(1+\gamma)}\right) - 1, 0\right) \\ + \sum_{m=1}^{\infty} (-1)^{m+1}\frac{1}{m!}\left\{\frac{\kappa\tau^\alpha}{\Gamma(1+\alpha)}\right\}^m \left\{\begin{matrix} \max\left(\exp\left(\frac{x^\gamma}{\Gamma(1+\gamma)}\right), 0\right) \\ -\max\left(\exp\left(\frac{x^\gamma}{\Gamma(1+\gamma)}\right) - 1, 0\right) \end{matrix}\right\}. \end{cases} \tag{19}$$

In (19), the solution to the TSFBSM for option pricing (14) is provided. At $\alpha = \gamma = 1$, the solutions obtained in [52,53] serve as particular cases to this present work.

Case II- Consider the following TSFBSM of the form:

$$\begin{cases} \frac{\partial^\alpha \omega}{\partial \tau^\alpha} = \omega - \left(\frac{x}{2}\frac{\partial \omega}{\partial x} + x^2\frac{\partial^{2\gamma}\omega}{\partial x^{2\gamma}}\right), \ \alpha \in (0,1], \ 2\gamma \in (0,2], \\ \omega(x,0) = \max(x^3, 0) = \begin{cases} x^3, & x > 0 \\ 0, & x \leqslant 0. \end{cases} \end{cases} \tag{20}$$

This coincides with *problem 1* of [54] for $\gamma = 1$ and [55] for $\alpha = \gamma = 1$ as particular cases of this present work.

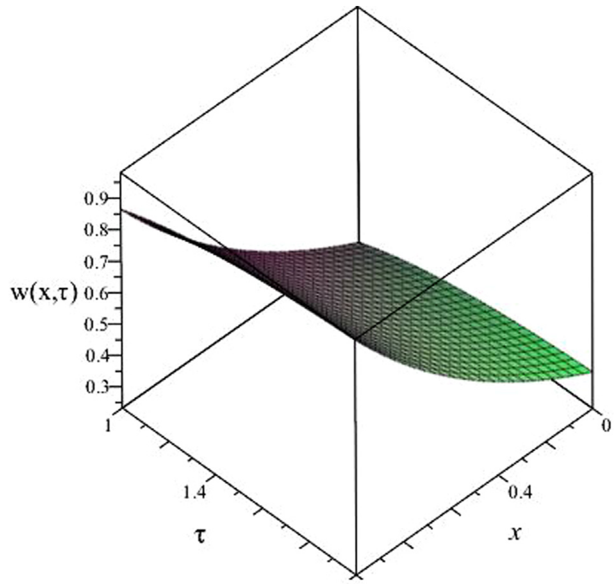

Fig. 1 Case I Solution plot for$\alpha = \gamma = 1$

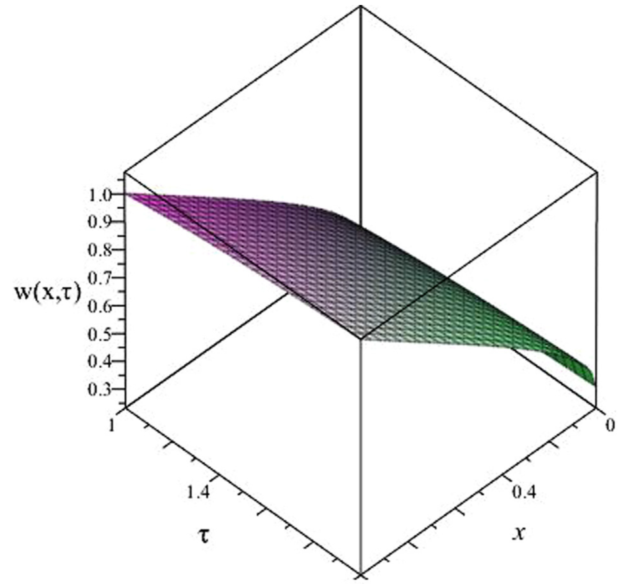

Fig. 3 Case I Solution plot for $\alpha = 1,\ \gamma = 0.5$

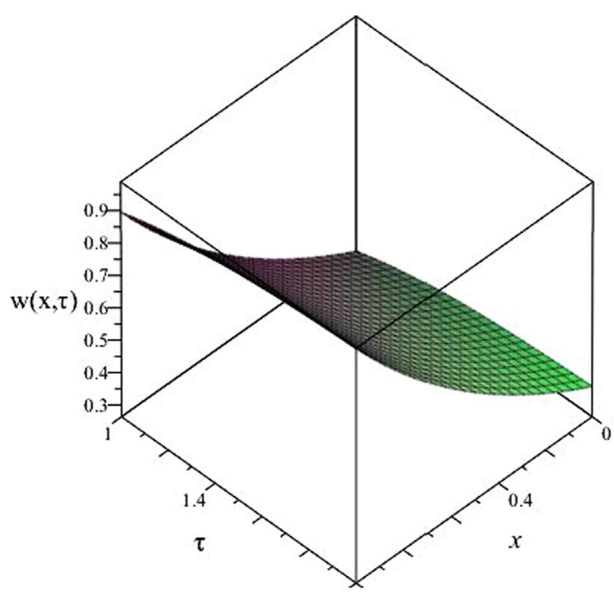

Fig. 2 Case I Solution plot for $\alpha = 0.5,\ \gamma = 1$

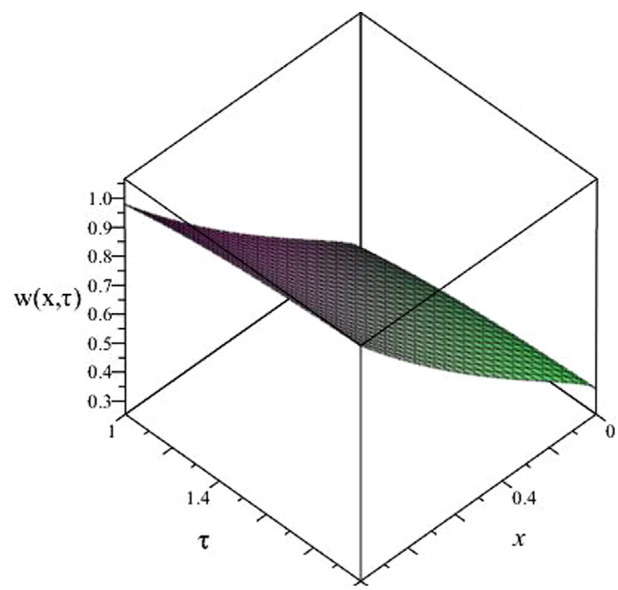

Fig. 4 Case I Solution plot for $\alpha = 0.75,\ \gamma = 0.75$

Thus, following the FCT as in the previous case with:

$$\frac{\partial^\alpha \omega}{\partial \tau^\alpha} = \frac{\partial \omega}{\partial T},\ \frac{\partial^\gamma \omega}{\partial x^\gamma} = \frac{\partial \omega}{\partial X},\ a = b = 1,$$

reforms (20) to:

$$\begin{cases} \frac{\partial \omega}{\partial T} = \omega - \left(\frac{X}{2}\frac{\partial \omega}{\partial X} + X^2 \frac{\partial^2 \omega}{\partial X^2}\right), \\ \omega(X, 0) = \max\left(X^3, 0\right). \end{cases} \tag{21}$$

Applying the RDTM to (21) gives:

$$\begin{cases} W_{k+1} = \frac{1}{(1+k)}\left\{W_k - \left(\frac{X}{2} W'_{X,k} + X^2 W''_{X,k}\right)\right\},\ k \geqslant 0, \\ W(X, 0) = \max\left(X^3, 0\right). \end{cases} \tag{22}$$

As a result, the following are yielded:

$$\begin{cases} W_0 = X^3,\ W_1 = (-6.5)X^3, \\ W_2 = \frac{(-6.5)^2}{2!} X^3, W_3 = \frac{(-6.5)^3}{3!} X^3, \\ W_4 = \frac{(-6.5)^4}{4!} X^3, \\ \vdots \\ W_p = \frac{(-6.5)^p}{p!} X^3, p = 0, 1, 2, \cdots. \end{cases} \tag{23}$$

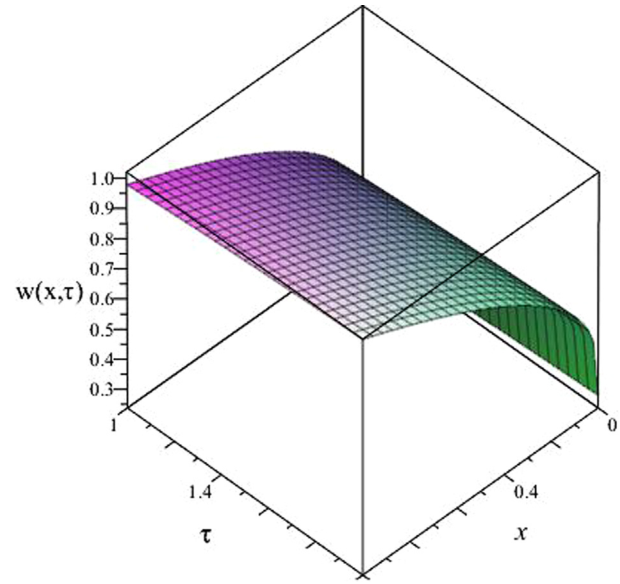

Fig. 5 Case I Solution plot for $\alpha = 0.95,\ \gamma = 0.25$

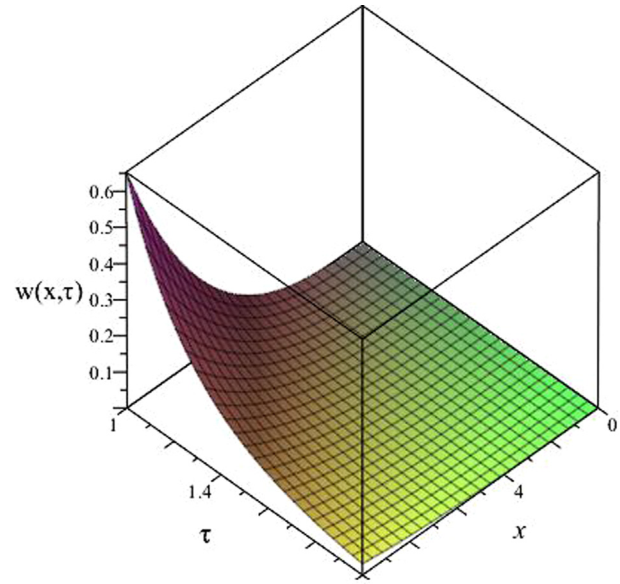

Fig. 7 Case II Solution plot for $\alpha = 0.5,\ \gamma = 1$

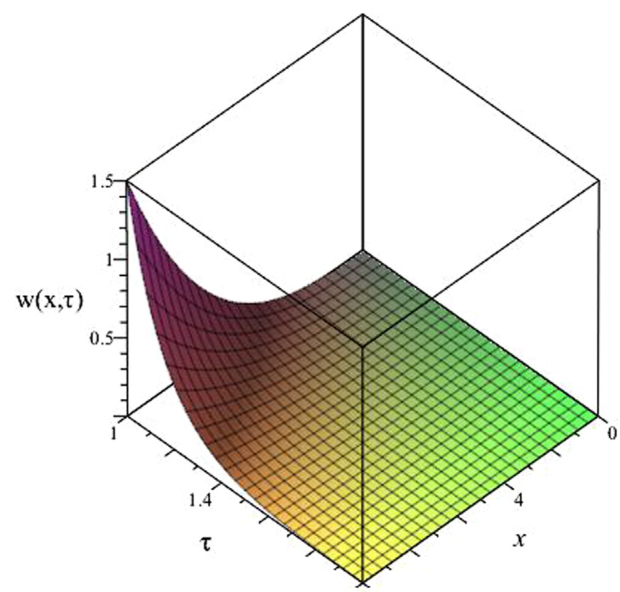

Fig. 6 Case II Solution plot for$\alpha = \gamma = 1$

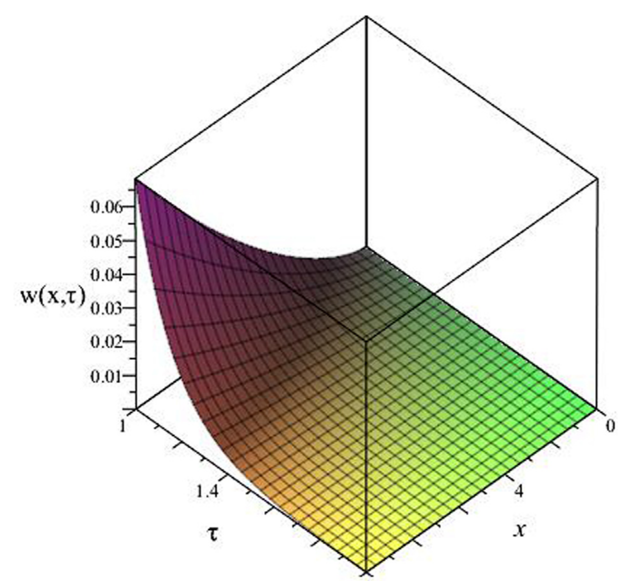

Fig. 8 Case II Solution plot for $\alpha = 1,\ \gamma = 0.5$

Therefore,

$$\begin{aligned}\omega(X,T) &= \sum_{m=0}^{\infty} W_m T^m = W_0 + \sum_{m=1}^{\infty} W_m T^m \\ &= X^3 + \sum_{m=1}^{\infty}\left(\tfrac{(-6.5)^m}{m!}X^3\right)T^m \\ &= X^3\sum_{m=0}^{\infty}\left(\tfrac{(-6.5T)^m}{m!}\right) = X^3\exp(-6.5T).\end{aligned} \tag{24}$$

Hence, the solution of (20) is given as:

$$\omega(x,\tau) = \left(\frac{x^{\gamma}}{\Gamma(1+\gamma)}\right)^3 \exp\left(\frac{-6.5\tau^{\alpha}}{\Gamma(1+\alpha)}\right). \tag{25}$$

Note: At $\alpha = \gamma = 1$, the exact solution [54] is:

$$\omega(x,\tau) = x^3\exp(-6.5\tau). \tag{26}$$

In (26), the solution to the TSFBSM for option pricing (20) is provided. At $\alpha = \gamma = 1$, the solutions obtained in [32,54,56] serve as particular cases to this present work.

4.1. Notes on the numerical solution

It is noteworthy here, that Fig. 1 shows the solution graphic of the model in *case I* for $\alpha = \gamma = 1$, which implies the non-fractional (classical integer) case of the model. Fig. 2 and Fig. 3 denote plots for purely time and purely space fractional cases, respectively. Meanwhile, Fig. 4 and Fig. 5 represent the solution plots of the model for time–space fractional cases.

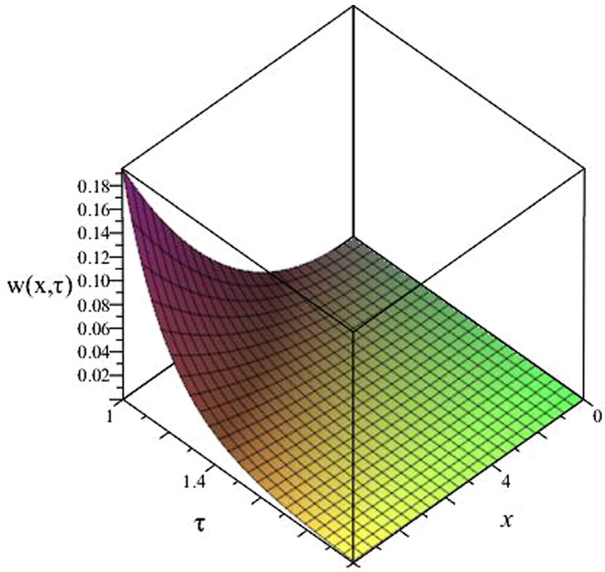

Fig. 9 Case II Solution plot for $\alpha = 0.75,\ \gamma = 0.75$

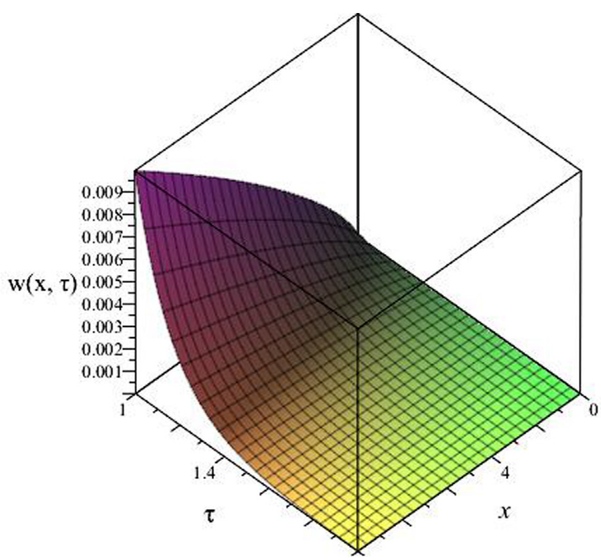

Fig. 10 Case II Solution plot for $\alpha = 0.95,\ \gamma = 0.25$

Similarly, we have in Figs. 6-10 for *case II*. Tables 1 and 2 show the comparison of present results with the exact solution at a particular value of $\gamma = 1$.

5. Concluding remarks

This paper presented a solution method referred to as Coupled Transform Method with combined features of Fractional Complex Transform and Reduced Differential Transform Method. This proposed method is applied for an approximate-analytical-solution of a time–space fractional Black–Scholes model (TSFBSM) for option pricing. It is worthy of remark that this proposed method does not necessarily require a complete knowledge of fractional calculus while computing the solutions of fractional models, yet the level of accuracy is highly maintained.

Table 1 Comparison of the present solution with the exact solution of Case I.

τ	x	$\alpha = 1.0, \kappa = 2$, and $\gamma = 1$	
		Present solution	Exact solution
	0.00	0.3296799	0.3296799
	0.25	0.6137053	0 0.6137053
0.2	0.50	0.9784012	0.9784012
	0.75	1.4466799	1.4466799
	1.00	2.0479617	2.0479617
	0.00	0.5506710	0.5506710
	0.25	0.8346964	0.8346964
0.4	0.50	1.1993923	1.1993923
	0.75	1.6676710	1.6676710
	1.00	2.2689528	2.2689528
	0.00	0.6988057	0.6988057
	0.25	0.9828312	0.9828312
0.6	0.50	1.3475270	1.3475270
	0.75	1.8158058	1.8158058
	1.00	2.4170876	2.4170876

Table 2 Comparison of the present solution with the exact solution of Case II.

τ	x	$\alpha = 1.0$ and $\gamma = 1$	
		Present solution	Exact solution
	0.00	0.0000000000000	0.000000000000
	0.25	0.4258309266E-2	0.4258309266E-2
0.2	0.50	0.3406647413E-1	0.3406647413E-1
	0.75	0.114974350200	0.114974350200
	1.00	0.272531793000	0.272531793000
	0.00	0.0000000000000	0.000000000000
	0.25	0.1160524659E-2	0.1160524659E-2
0.4	0.50	0.9284197273E-2	0.9284197273E-2
	0.75	0.3133416580E-1	0.3133416580E-1
	1.00	0.7427357819E-1	0.7427357819E-1
	0.00	0.000000000000	0.000000000000
	0.25	0.3162798663E-3	0.3162798663E-3
0.6	0.50	0.2530238930E-2	0.2530238930E-2
	0.75	0.8539556390E-2	0.8539556390E-2
	1.00	0.2024191144E-1	0.2024191144E-1

Declaration of Competing Interest

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