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Criteria for existence of solutions for a Liouville–Caputo boundary value problem via generalized Gronwall's inequality

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Abstract

In this research, we first investigate the existence of solutions for a new fractional boundary value problem in the Liouville–Caputo setting with mixed integro-derivative boundary conditions. To do this, Kuratowski's measure of noncompactness and Sadovskii's fixed point theorem are our tools to reach this aim. In the sequel, we discuss the continuous dependence of solutions on parameters by means of the generalized Gronwall inequality. Moreover, we consider an inclusion version of the given boundary problem in which we study its existence results by means of the endpoint theory. Finally, we prepare two simulative numerical examples to confirm the validity of the analytical findings.

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Keywords: Boundary value problem; Measure of noncompactness; The approximate endpoint property; The generalized Gronwall inequality

1 Introduction

Fractional models have recently attracted the attention of many scientists and engineers of the modeling fields. This amount of attention can be a motivation for researchers and mathematicians to study the various new fractional models for the complex natural phenomena in the world around. By taking into account the importance and effectiveness of the mathematical modelings, researchers design various novel categories of fractional structures of initial and boundary value problems and study the existence, uniqueness, and stability of solutions by using some analytical and numerical techniques. Furthermore, many researchers endeavor to approximate the obtained solutions by utilizing some numerical and iterative algorithms. The most important to researchers today is the understanding of some qualitative properties of the solutions for such fractional dynamical systems. In this direction, numerous research articles on the existence and stability of solutions have been published, including [1–21].

As we know, the inequalities play an important role in all branches of the pure and applied mathematics, and we can use some of their properties to infer different desired identities and estimates. By following this path, one can observe different published research works in the recent years in which the researchers discuss the existence of solutions for the given boundary problems by combining some analytical methods and some well-known inequalities. For instance, we point out some papers, including [22–25]. Applications of the generalized Gronwall inequality can be found in some papers such as [26–29].

In 2016, Ahmad et al. [30] studied the existence of solutions for a mixed fractional initial value problem in the inclusion version involving Hadamard fractional derivative and Riemann–Liouville fractional integrals given by

$$\begin{cases} {}^{\mathcal{H}}\mathfrak{D}^{\alpha}(\nu(z) - \sum_{i=1}^{m} {}^{\mathcal{RL}}\mathcal{I}^{\beta_i} h_i(z, \nu(z))) \in F(z, \nu(z)) \quad (z \in J := [1, T]), \\ \nu(1) = 0, \end{cases}$$

where ${}^{\mathcal{H}}\mathfrak{D}^{\alpha}$ is the Hadamard fractional derivative of order $\alpha \in (0,1]$ and ${}^{\mathcal{RL}}\mathcal{I}^{\phi}$ denotes the Riemann–Liouville fractional integral of order $0 < \phi \in \{\beta_1, \beta_2, \dots, \beta_m\}$, $h_i \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ with $h_i(1,0) = 0$ ($i = 1, 2, \dots, m$), and finally $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multifunction. The authors used the notions defined in the endpoint theory and proved the main existence results for the above initial inclusion problem with the aid of an approximate endpoint property [30].

Some existing structures in the works mentioned above motivate us to design a new fractional structure of the boundary value problem in the Liouville–Caputo framework (Lio–CapBvp) furnished with mixed integro-derivative conditions as

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_0^{\xi}\nu(z) = \Phi_*(z, \nu(z)) \\ \nu(0) = s_1^* {}^{\mathcal{C}}\mathfrak{D}_0^1\nu(0) + s_2^* {}^{\mathcal{C}}\mathfrak{D}_0^1\nu(1), \qquad {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1}\nu(1) = \int_0^1 {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1}\nu(\varsigma)\,\mathrm{d}\varsigma \end{cases} \tag{1}$$

so that $z \in [0,1]$, $s_1^*, s_2^* \in \mathbb{R}^+$, ${}^{\mathcal{C}}\mathfrak{D}_0^{\xi}$ stands for the Liouville–Caputo derivative of order $\xi \in (1,2)$, ${}^{\mathcal{C}}\mathfrak{D}_0^1 = \frac{\mathrm{d}}{\mathrm{d}z}$ and $\Phi_* \in \mathcal{C}([0,1] \times \mathbb{R}, \mathbb{R})$. To derive the existence criteria of solutions for the proposed fractional Lio–CapBvp (1), we invoke the notion of the well-known measure of noncompactness attributed to Kuratowski. In this direction, we construct a condensing operator and establish our existence result by utilizing a well-known fixed point result due to Sadovskii.

In the sequel, we investigate some estimates by utilizing the generalized Gronwall inequality. Moreover, we discuss an inclusion version of the given boundary problem in which the proof process is based on the approximate endpoint property and some properties of inequalities in relation to the Pompeiu–Hausdorff metric defined for multifunctions. It is notable that our proposed fractional Lio–CapBvp (1) involves new mixed integro-derivative boundary conditions simultaneously, and this formulated framework is novel at this moment.

An outline of the current research manuscript is prepared as follows: In Sect. 2, we assemble some auxiliary and key notions and theorems which are essential for the rest of the paper. In Sect. 3, an analytical criterion about the existence of solutions for the fractional Lio–CapBvp (1) is derived based on the fixed point result attributed to Sadovskii. In the following, we check the dependence of solutions by utilizing the generalized Gronwall inequality. In Sect. 4, we regard an inclusion version of the given structure (1) and find endpoints of this system by applying some inequalities in relation to multifunctions. In Sect. 5, we prepare two simulative numerical examples to confirm the validity of the obtained analytical findings.

2 Preliminaries

In this part of the article, we intend to recall some auxiliary and key concepts and theorems which are necessary in the rest of this manuscript. Take $\xi > 0$. The Riemann–Liouville integral (RL-integral) for a real-valued function ν on $[a,b]$ in the fractional setting is given by

$$ {}^{\mathcal{RL}}\mathcal{I}_0^{\xi}\nu(z) = \int_0^z \frac{(z-\varsigma)^{\xi-1}}{\Gamma(\xi)}\nu(\varsigma)\,\mathrm{d}\varsigma, $$

provided that the integral above is finite-valued (see [31, 32]). At this stage, let $n-1 < \xi < n$ or $n = [\xi] + 1$. The Liouville–Caputo derivative (Lio–Cap derivative) of a given function $\nu \in \mathcal{C}_{\mathbb{R}}^{(n)}([a,b])$ in the fractional setting is defined by

$$ {}^{\mathcal{C}}\mathfrak{D}_0^{\xi}\nu(z) = \int_0^z \frac{(z-\varsigma)^{n-\xi-1}}{\Gamma(n-\xi)}\nu^{(n)}(\varsigma)\,\mathrm{d}\varsigma $$

so that the right-hand side integral is finite-valued (see [31, 32]). In [33], one can see that the structure of the general series solution for the homogeneous differential equation ${}^{\mathcal{C}}\mathfrak{D}_0^{\xi}\nu(z) = 0$ is $\nu(z) = \alpha_*^{(0)} + \alpha_*^{(1)}z + \alpha_*^{(2)}z^2 + \cdots + \alpha_*^{(n-1)}z^{n-1}$, and the following holds:

$$ {}^{\mathcal{RL}}\mathcal{I}_0^{\xi}\big({}^{\mathcal{C}}\mathfrak{D}_0^{\xi}\nu(z)\big) = \nu(z) + \sum_{k=0}^{n-1}\alpha_*^{(k)}z^k = \nu(z) + \alpha_*^{(0)} + \alpha_*^{(1)}z + \alpha_*^{(2)}z^2 + \cdots + \alpha_*^{(n-1)}z^{n-1}, $$

where $\alpha_*^{(0)}, \ldots, \alpha_*^{(n-1)} \in \mathbb{R}$ and $n = [\xi] + 1$. In the sequel, some important inequalities and analytical notions are reviewed. The measure of noncompactness Ω attributed to Kuratowski is introduced by

$$ \Omega(\mathfrak{B}) := \inf\left\{\epsilon > 0 : \mathfrak{B} = \bigcup_{k=1}^{n}\mathfrak{B}_k \text{ and } \operatorname{diam}(\mathfrak{B}_k) \le \epsilon \text{ for } k = 1, \ldots, n\right\}, $$

so that $\operatorname{diam}(\mathfrak{B}_k) = \sup\{|\nu - \nu'| : \nu, \nu' \in \mathfrak{B}_k\}$ and $\mathfrak{B}$ is assumed to be a bounded subset of the Banach space $\mathbb{V}$. Additionally, it is known that $0 \le \Omega(\mathfrak{B}) \le \operatorname{diam}(\mathfrak{B}) < +\infty$ (see [34]).

Lemma 1 ([34]) *Let $\mathbb{V}$ be an arbitrary real Banach space and $\mathfrak{B}, \mathfrak{B}_1, \mathfrak{B}_2 \subseteq \mathbb{V}$ be bounded subsets of $\mathbb{V}$. Then the following are valid*:

(Con1) $\mathfrak{B}$ *is precompact iff* $\Omega(\mathfrak{B}) = 0$;

(Con2) $\Omega(\mathfrak{B}) = \Omega(\overline{\mathfrak{B}}) = \Omega(\operatorname{cnvx}(\mathfrak{B}))$, *where* $\operatorname{cnvx}(\mathfrak{B})$ *and* $\overline{\mathfrak{B}}$ *stand for the convex hull and the closure of* $\mathfrak{B}$, *respectively*;

Moreover, the following inequalities hold:

(Con3) *if* $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$, *then* $\Omega(\mathfrak{B}_1) \le \Omega(\mathfrak{B}_2)$;

(Con4) $\Omega(\mu + \mathfrak{B}) \le \Omega(\mathfrak{B})$ *for each* $\mu \in \mathbb{R}$;

(Con5) $\Omega(\mu\mathfrak{B}) = |\mu|\Omega(\mathfrak{B})$ *for all* $\mu \in \mathbb{R}$;

(Con6) $\Omega(\mathfrak{B}_1 + \mathfrak{B}_2) \le \Omega(\mathfrak{B}_1) + \Omega(\mathfrak{B}_2)$ *so that* $\mathfrak{B}_1 + \mathfrak{B}_2 = \{\nu_1 + \nu_2; \nu_1 \in \mathfrak{B}_1, \nu_2 \in \mathfrak{B}_2\}$;

(Con7) $\Omega(\mathfrak{B}_1 \cup \mathfrak{B}_2) \le \max\{\Omega(\mathfrak{B}_1), \Omega(\mathfrak{B}_2)\}$.

Lemma 2 ([35]) *Let $\mathbb{V}$ be a Banach space. For each bounded subset $\mathfrak{B}$ of $\mathbb{V}$, a countable set $\mathfrak{B}_0$ of $\mathfrak{B}$ exists so that $\Omega(\mathfrak{B}) \le 2\Omega(\mathfrak{B}_0)$.*

Lemma 3 ([34]) *Let $\mathbb{V}$ be a Banach space. If $\mathfrak{B} \subseteq \mathcal{C}_{\mathbb{V}}([a,b])$ is assumed to be a bounded and equicontinuous set, then $\Omega(\mathfrak{B}(z))$ is continuous on $[a,b]$ and $\Omega(\mathfrak{B}) = \sup_{z\in[a,b]} \Omega(\mathfrak{B}(z))$.*

Lemma 4 ([34]) *Let $\mathbb{V}$ be a Banach space. If $\mathfrak{B} = \{\nu_n\}_{n\geq 1} \subseteq \mathcal{C}_{\mathbb{V}}([a,b])$ is assumed to be a bounded and countable set, then $\Omega(\mathfrak{B}(z))$ is Lebesgue integrable on $[a,b]$ and the following inequality is valid*:

$$\Omega\left(\left\{\int_0^z \nu_n(\varsigma)\,\mathrm{d}\varsigma\right\}_{n\geq 1}\right) \leq 2\int_0^z \Omega\left(\left\{\nu_n(\varsigma)\right\}_{n\geq 1}\right)\mathrm{d}\varsigma.$$

Definition 5 ([34]) Let $\mathbb{V}$ be a Banach space and $\Phi_* : \mathbb{D} \subset \mathbb{V} \to \mathbb{V}$ be continuous and bounded. Then the operator Φ_* is called condensing if $\Omega(\Phi_*(\mathfrak{B})) < \Omega(\mathfrak{B})$ for any bounded closed set $\mathfrak{B} \subseteq \mathbb{D}$.

Theorem 6 ([34], Sadovskii's fixed point theorem) *Let $\mathbb{V}$ be a Banach space. Also assume that $\mathfrak{B} \subseteq \mathbb{V}$ is a bounded closed convex set and the continuous map $\Phi_* : \mathfrak{B} \to \mathfrak{B}$ is condensing. Then the map Φ_* has at least one fixed point in $\mathfrak{B}$.*

Theorem 7 ([36], The generalized Gronwall inequality) *Let $\check{r}$ be a nonnegative locally integrable function on the interval $[0,T]$ for some $T \leq \infty$ and h a nonnegative nondecreasing continuous mapping on $[0,T]$ satisfying $h(z) \leq M$ where M is a nonzero constant. In addition, let $\check{\nu}$ be a nonnegative locally integrable function on $[0,T]$ such that*

$$\check{\nu}(z) \leq \check{r}(z) + h(z)\int_0^z (z-\varsigma)^{q-1}\check{\nu}(\varsigma)\,\mathrm{d}\varsigma,$$

where $q > 0$. Then we have

$$\check{\nu}(z) \leq \check{r}(z) + \int_0^z \sum_{k=1}^{\infty}\left[\frac{(h(z)\Gamma(q))^k}{\Gamma(kq)}(z-\varsigma)^{kq-1}\check{r}(\varsigma)\right]\mathrm{d}\varsigma \quad \bigl(z\in[0,T]\bigr).$$

Notation 8 In this manuscript, we denote the normed space by $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ and we introduce the notations $\mathcal{P}(\mathbb{V})$, $\mathcal{P}_{cls}(\mathbb{V})$, $\mathcal{P}_{bnd}(\mathbb{V})$, $\mathcal{P}_{cmp}(\mathbb{V})$, and $\mathcal{P}_{cvx}(\mathbb{V})$ for the collections of all nonempty, closed, bounded, compact, and convex subsets in the space $\mathbb{V}$, respectively.

An element $\nu \in \mathbb{V}$ is said to be a fixed point for an abstract multivalued operator $\mathfrak{Q}_* : \mathbb{V} \to \mathcal{P}(\mathbb{V})$ if the inclusion $\nu \in \mathfrak{Q}_*(\nu)$ is satisfied [37]. At this phase, we denote the family of all existing fixed points of $\mathfrak{Q}_*$ by $\mathfrak{FIX}(\mathfrak{Q}_*)$ [37]. Subsequently, the Pompeiu–Hausdorff metric $\mathbb{H}_{d_{\mathbb{V}}} : \mathcal{P}(\mathbb{V}) \times \mathcal{P}(\mathbb{V}) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\mathbb{H}_{d_{\mathbb{V}}}(\mathcal{M}_1, \mathcal{M}_2) = \max\Bigl\{\sup_{m_1\in\mathcal{M}_1} d_{\mathbb{V}}(m_1, \mathcal{M}_2), \sup_{m_2\in\mathcal{M}_2} d_{\mathbb{V}}(\mathcal{M}_1, m_2)\Bigr\}$$

so that $d_{\mathbb{V}}(\mathcal{M}_1, m_2) = \inf_{m_1\in\mathcal{M}_1} d_{\mathbb{V}}(m_1, m_2)$ and $d_{\mathbb{V}}(m_1, \mathcal{M}_2) = \inf_{m_2\in\mathcal{M}_2} d_{\mathbb{V}}(m_1, m_2)$ (see [37]). A multifunction $\mathfrak{Q}_* : \mathbb{V} \to \mathcal{P}_{cls}(\mathbb{V})$ is called Lipschitz with real constant $\hat{\imath} > 0$ if the inequality

$$\mathbb{H}_{d_{\mathbb{V}}}\bigl(\mathfrak{Q}_*(\nu_1), \mathfrak{Q}_*(\nu_2)\bigr) \leq \hat{\imath} d_{\mathbb{V}}(\nu_1, \nu_2)$$

is valid for each $\nu_1, \nu_2 \in \mathbb{V}$. It is notable that a Lipschitz map $\mathfrak{Q}_*$ is called a contraction if $\hat{\iota} \in (0,1)$ (see [37]). Also, we say that $\mathfrak{Q}_*$ possesses the complete continuity property if $\mathfrak{Q}_*(K)$ has the relative compactness property for any $K \in \mathcal{P}_{bnd}(\mathbb{V})$, and also $\mathfrak{Q}_* : [0,1] \to \mathcal{P}_{cls}(\mathbb{R})$ is measurable if $z \longmapsto d_{\mathbb{V}}(r, \mathfrak{Q}_*(z))$ is measurable for any $r \in \mathbb{R}$ (see [37, 38]). Furthermore, $\mathfrak{Q}_*$ is called upper semicontinuous (u.s.c.) if for every $\nu \in \mathbb{V}$, the set $\mathfrak{Q}_*(\nu)$ belongs to $\mathcal{P}_{cls}(\mathbb{V})$ and for any open set $\mathbb{G}$ which contains $\mathfrak{Q}_*(\nu)$, a neighborhood $\mathcal{U}_0^*$ of ν exists so that $\mathfrak{Q}_*(\mathcal{U}_0^*)$ is contained in $\mathbb{G}$ [37].

It is an evident fact that $\mathfrak{Q}_*$ possesses convex values if $\mathfrak{Q}_*(\nu) \in \mathcal{P}_{cvx}(\mathbb{V})$ for any $\nu \in \mathbb{V}$. We denote the family of all existing selections of $\mathfrak{Q}_*$ at point $\nu \in \mathcal{C}_{\mathbb{R}}([0,1])$ by the following rule:

$$(\mathfrak{S}\mathfrak{E}\mathfrak{L})_{\mathfrak{Q}_*,\nu} := \left\{\hbar \in \mathcal{L}^1_{\mathbb{R}}\big([0,1]\big) : \hbar(z) \in \mathfrak{Q}_*\big(z, \nu(z)\big)\right\}$$

for any $z \in [0,1]$ (a.e.) (see [37, 38]). It is important to pay attention to the issue that by assuming $\mathfrak{Q}_*$ as an arbitrary abstract multivalued operator, we deduce that for each $\nu \in \mathcal{C}_{\mathbb{V}}([0,1])$, the set $(\mathfrak{S}\mathfrak{E}\mathfrak{L})_{\mathfrak{Q}_*,\nu}$ is nonempty if $\dim(\mathbb{V})$ is finite (see [37]).

Definition 9 ([39]) An element $\nu \in \mathbb{V}$ is called an endpoint for the given multivalued operator $\mathfrak{Q}_* : \mathbb{V} \to \mathcal{P}(\mathbb{V})$ if we have $\mathfrak{Q}_*(\nu) = \{\nu\}$.

The multivalued mapping $\mathfrak{Q}_* : \mathbb{V} \to \mathcal{P}(\mathbb{V})$ has an approximate endpoint property (approx-endpoint property) if we have $\inf_{\nu_1 \in \mathbb{V}} \sup_{\nu_2 \in \mathfrak{Q}_*(\nu_1)} d_{\mathbb{V}}(\nu_1, \nu_2) = 0$ [39]. The next result is necessary for us in the rest of the article.

Theorem 10 ([39], Endpoint theorem) *Assume that the metric space $(\mathbb{V}, d_{\mathbb{V}})$ is complete and $\psi : [0,\infty) \to [0,\infty)$ has an upper semicontinuity property provided that $\psi(z) < z$ and $\liminf_{z\to\infty}(z - \psi(z)) > 0$ for any $z > 0$. Additionally, let us assume that $\mathfrak{Q}_* : \mathbb{V} \to \mathcal{P}_{cls,bnd}(\mathbb{V})$ is such that the inequality*

$$\mathbb{H}_{d_{\mathbb{V}}}(\mathfrak{Q}_*\nu_1, \mathfrak{Q}_*\nu_2) \le \psi\big(d_{\mathbb{V}}(\nu_1, \nu_2)\big)$$

is satisfied for any $\nu_1, \nu_2 \in \mathbb{V}$. Then a unique endpoint exists for $\mathfrak{Q}_*$ iff $\mathfrak{Q}_*$ possesses approx-endpoint property.

3 Criterion of existence for the fractional Lio–CapBvp (1)

In the present section, we derive a criterion of existence of solutions for the proposed Lio–CapBvp (1). In order to achieve this goal, we construct the space $\mathbb{V} = \{\nu(z) : \nu(z) \in \mathcal{C}_{\mathbb{R}}([0,1])\}$ furnished with the supremum norm $\|\nu\|_{\mathbb{V}} = \sup_{z\in[0,1]} |\nu(z)|$. Then we can simply check that an ordered pair $(\mathbb{V}, \|\cdot\|_{\mathbb{V}})$ is a Banach space. By following the contents of two previous sections, now we establish a new existence criterion and also discuss to the stability of the fractional Lio–CapBvp (1). To achieve this goal, we resort to the next structural lemma in relation to (1) at this moment.

Lemma 11 *Let $\xi \in (1,2)$, $\xi - 1 \in (0,1)$, $s_1^*, s_2^* \in \mathbb{R}^+$, and $g \in \mathcal{C}([0,1],\mathbb{R})$. Then the solution of the given linear fractional Lio–CapBvp*

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_0^{\xi}\nu(z) = g(z), \quad (z \in [0,1]), \\ \nu(0) = s_1^{*\,\mathcal{C}}\mathfrak{D}_0^1\nu(0) + s_2^{*\,\mathcal{C}}\mathfrak{D}_0^1\nu(1), \qquad {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1}\nu(1) = \int_0^1 {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1}\nu(\varsigma)\,d\varsigma \end{cases} \tag{2}$$

is given by

$$\begin{aligned} v(z) &= \frac{1}{\Gamma(\xi)}\int_0^z (z-\varsigma)^{\xi-1} g(\varsigma)\,\mathrm{d}\varsigma + \frac{s_2^*}{\Gamma(\xi-1)}\int_0^1 (1-\varsigma)^{\xi-2} g(\varsigma)\,\mathrm{d}\varsigma \\ &\quad - \frac{(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1 g(\varsigma)\,\mathrm{d}\varsigma \\ &\quad + \frac{(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^\varsigma g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma. \end{aligned} \tag{3}$$

Proof At first, we assume that v_0^* satisfies the fractional Lio–CapBvp (2). Then, in view of some properties of the Caputo and Riemann–Liouville fractional operators, two arbitrary real constants $\alpha_*^{(0)}$ and $\alpha_*^{(1)}$ exist so that the equation $v_0^*(z) = {}^{\mathcal{RL}}\mathcal{I}_0^{\xi} g(z) + \alpha_*^{(0)} + \alpha_*^{(1)} z$ is valid for any $z \in [0,1]$. Indeed, we get

$$v_0^*(z) = \int_0^z \frac{(z-\varsigma)^{\xi-1}}{\Gamma(\xi)} g(\varsigma)\,\mathrm{d}\varsigma + \alpha_*^{(0)} + \alpha_*^{(1)} z. \tag{4}$$

In this case, we obtain ${}^{\mathcal{C}}\mathfrak{D}_0^1 v_0^*(z) = \int_0^z \frac{(z-\varsigma)^{\xi-2}}{\Gamma(\xi-1)} g(\varsigma)\,\mathrm{d}\varsigma + \alpha_*^{(1)}$ and

$$\begin{aligned} {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1} v_0^*(z) &= \int_0^z g(\varsigma)\,\mathrm{d}\varsigma + \alpha_*^{(1)} \frac{z^{2-\xi}}{\Gamma(3-\xi)}, \\ \int_0^1 {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1} v_0^*(\varsigma)\,\mathrm{d}\varsigma &= \int_0^1\int_0^\varsigma g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma + \alpha_*^{(1)} \frac{1}{\Gamma(4-\xi)}. \end{aligned}$$

Now, by utilizing the given integro-derivative boundary conditions in (2), the real constants $\alpha_*^{(0)}$ and $\alpha_*^{(1)}$ are obtained as follows:

$$\begin{aligned} \alpha_*^{(0)} &= \frac{s_2^*}{\Gamma(\xi-1)}\int_0^1 (1-\varsigma)^{\xi-2} g(\varsigma)\,\mathrm{d}\varsigma - \frac{(s_1^*+s_2^*)\Gamma(4-\xi)}{2-\xi}\int_0^1 g(\varsigma)\,\mathrm{d}\varsigma \\ &\quad + \frac{(s_1^*+s_2^*)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^\varsigma g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma \end{aligned}$$

and

$$\alpha_*^{(1)} = -\frac{\Gamma(4-\xi)}{2-\xi}\int_0^1 g(\varsigma)\,\mathrm{d}\varsigma + \frac{\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^\varsigma g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma.$$

Here, by inserting the obtained values of $\alpha_*^{(0)}$ and $\alpha_*^{(1)}$ into (4), one can observe that v_0^* satisfies the integral equation (3). In other words,

$$\begin{aligned} v_0^*(z) &= \frac{1}{\Gamma(\xi)}\int_0^z (z-\varsigma)^{\xi-1} g(\varsigma)\,\mathrm{d}\varsigma + \frac{s_2^*}{\Gamma(\xi-1)}\int_0^1 (1-\varsigma)^{\xi-2} g(\varsigma)\,\mathrm{d}\varsigma \\ &\quad - \frac{(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1 g(\varsigma)\,\mathrm{d}\varsigma \\ &\quad + \frac{(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^\varsigma g(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma, \end{aligned}$$

and the proof is completed. □

In what follows, we invoke the notion of Kuratowski's measure of noncompactness and establish a new criterion of solution existence for the relevant fractional Lio–CapBvp (1) based on some estimates and known existing properties of inequalities.

Theorem 12 *Let Φ_* be a continuous real-valued function defined on $[0,1]\times\mathbb{V}$. Additionally, suppose a continuous function $\varphi:[0,1]\to\mathbb{R}^+$ exists such that the inequality*

$$\left|\Phi_*\big(z,\nu(z)\big)\right|\le\varphi(z) \tag{5}$$

holds for any $z\in[0,1]$ and $\nu\in\mathbb{V}$. Furthermore, assume that a function $n_{\Phi_*}:[0,1]\to\mathbb{R}^+$ exists so that the inequality

$$\Omega\big(\Phi_*(z,\mathfrak{B})\big)\le n_{\Phi_*}(z)\Omega(\mathfrak{B}) \tag{6}$$

is valid for any bounded set $\mathfrak{B}\subset\mathbb{V}$ a.e. $z\in[0,1]$. Then, the given fractional Lio–CapBvp (1) possesses at least one solution on $[0,1]$ if

$$\frac{n^*_{\Phi_*}}{\Gamma(\xi+1)}+\frac{s^*_2 n^*_{\Phi_*}}{\Gamma(\xi)}+\frac{3n^*_{\Phi_*}(s^*_1+s^*_2+1)\Gamma(4-\xi)}{2(2-\xi)}<\frac{1}{4}, \tag{7}$$

where $n^*_{\Phi_*}=\sup_{z\in[0,1]}|n_{\Phi_*}(z)|$.

Proof To begin the current proof, in relation to Lio–CapBvp (1) and using Lemma 11, we define the mapping $\mathfrak{P}:\overline{\mathfrak{E}_\varepsilon}\to\overline{\mathfrak{E}_\varepsilon}$ as follows:

$$\begin{aligned}\mathfrak{P}(\nu)(z)={}&\frac{1}{\Gamma(\xi)}\int_0^z(z-\varsigma)^{\xi-1}\Phi_*\big(\varsigma,\nu(\varsigma)\big)\,\mathrm{d}\varsigma+\frac{s^*_2}{\Gamma(\xi-1)}\int_0^1(1-\varsigma)^{\xi-2}\Phi_*\big(\varsigma,\nu(\varsigma)\big)\,\mathrm{d}\varsigma\\&-\frac{(s^*_1+s^*_2+z)\Gamma(4-\xi)}{2-\xi}\int_0^1\Phi_*\big(\varsigma,\nu(\varsigma)\big)\,\mathrm{d}\varsigma\\&+\frac{(s^*_1+s^*_2+z)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^\varsigma\Phi_*\big(\tau,\nu(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}\varsigma,\end{aligned} \tag{8}$$

where $\overline{\mathfrak{E}_\varepsilon}:=\{\nu\in\mathbb{V}:\|\nu\|_{\mathbb{V}}\le\varepsilon,\varepsilon\in\mathbb{R}^+\}$ is considered to be a convex, closed, and bounded subset of the Banach space $\mathbb{V}$. In this case, the suggested problem (1) is equivalent to the fixed point problem $\nu=\mathfrak{P}\nu$, and we have to prove that the operator $\mathfrak{P}$ has a fixed point, since the existence of a fixed point for the newly-introduced operator $\mathfrak{P}$ will imply the existence of a solution for the given fractional Lio–CapBvp (1).

To check all the hypotheses of Theorem 6, we first confirm the continuity of $\mathfrak{P}$ on $\overline{\mathfrak{E}_\varepsilon}$. Let $\{\nu_n\}_{n\ge1}$ be a sequence contained in $\overline{\mathfrak{E}_\varepsilon}$ so that $\nu_n\to\nu$ for each $\nu\in\overline{\mathfrak{E}_\varepsilon}$. Due to the continuity of the function Φ_* on $[0,1]\times\mathbb{V}$, we obtain that $\lim_{n\to\infty}\Phi_*(z,\nu_n(z))=\Phi_*(z,\nu(z))$. In this case, by the help of the dominated convergence theorem of Lebesgue, we may write

$$\begin{aligned}&\lim_{n\to\infty}(\mathfrak{P}\nu_n)(z)\\&\quad=\frac{1}{\Gamma(\xi)}\int_0^z(z-\varsigma)^{\xi-1}\lim_{n\to\infty}\Phi_*\big(\varsigma,\nu_n(\varsigma)\big)\,\mathrm{d}\varsigma\\&\qquad+\frac{s^*_2}{\Gamma(\xi-1)}\int_0^1(1-\varsigma)^{\xi-2}\lim_{n\to\infty}\Phi_*\big(\varsigma,\nu_n(\varsigma)\big)\,\mathrm{d}\varsigma\end{aligned}$$

$$
\begin{aligned}
&- \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \lim_{n\to\infty} \Phi_*\bigl(\varsigma, \nu_n(\varsigma)\bigr)\,\mathrm{d}\varsigma \\
&+ \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \int_0^\varsigma \lim_{n\to\infty} \Phi_*\bigl(\tau, \nu_n(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\varsigma \\
= {} & \frac{1}{\Gamma(\xi)} \int_0^z (z-\varsigma)^{\xi-1} \Phi_*\bigl(\varsigma, \nu(\varsigma)\bigr)\,\mathrm{d}\varsigma + \frac{s_2^*}{\Gamma(\xi-1)} \int_0^1 (1-\varsigma)^{\xi-2} \Phi_*\bigl(\varsigma, \nu(\varsigma)\bigr)\,\mathrm{d}\varsigma \\
&- \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \Phi_*\bigl(\varsigma, \nu(\varsigma)\bigr)\,\mathrm{d}\varsigma \\
&+ \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \int_0^\varsigma \Phi_*\bigl(\tau, \nu(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\varsigma \\
= {} & (\mathfrak{P}\nu)(z)
\end{aligned}
$$

for any $z \in [0,1]$. Hence, we realize that $\lim_{n\to\infty}(\mathfrak{P}\nu_n)(z) = (\mathfrak{P}\nu)(z)$, which confirms the continuity of $\mathfrak{P}$ on $\overline{\mathfrak{E}_\varepsilon}$. In the sequel, we want to check the uniform boundedness of $\mathfrak{P}$ on $\overline{\mathfrak{E}_\varepsilon}$. To reach this aim, consider an element ν belonging to $\overline{\mathfrak{E}_\varepsilon}$. In the light of the condition (5), we get

$$
\begin{aligned}
\bigl|(\mathfrak{P}\nu)(z)\bigr| \le {} & \frac{1}{\Gamma(\xi)} \int_0^z (z-\varsigma)^{\xi-1} \bigl|\Phi_*\bigl(\varsigma, \nu(\varsigma)\bigr)\bigr|\,\mathrm{d}\varsigma \\
&+ \frac{s_2^*}{\Gamma(\xi-1)} \int_0^1 (1-\varsigma)^{\xi-2} \bigl|\Phi_*\bigl(\varsigma, \nu(\varsigma)\bigr)\bigr|\,\mathrm{d}\varsigma \\
&+ \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \bigl|\Phi_*\bigl(\varsigma, \nu(\varsigma)\bigr)\bigr|\,\mathrm{d}\varsigma \\
&+ \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \int_0^\varsigma \bigl|\Phi_*\bigl(\tau, \nu(\tau)\bigr)\bigr|\,\mathrm{d}\tau\,\mathrm{d}\varsigma \\
\le {} & \frac{z^\xi}{\Gamma(\xi+1)} \varphi(z) + \frac{s_2^*}{\Gamma(\xi)} \varphi(z) \\
&+ \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \varphi(z) + \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2(2-\xi)} \varphi(z)
\end{aligned}
$$

for any $z \in [0,1]$. As a consequence, the above estimate becomes $\|\mathfrak{P}\nu\|_{\mathbb{V}} \le \hat{\Psi}\varphi^* < \infty$, where

$$
\hat{\Psi} = \frac{1}{\Gamma(\xi+1)} + \frac{s_2^*}{\Gamma(\xi)} + \frac{3(s_1^* + s_2^* + 1)\Gamma(4-\xi)}{2(2-\xi)}. \tag{9}
$$

Thus it is proved that $\mathfrak{P}(\overline{\mathfrak{E}_\varepsilon})$ is a uniformly bounded set contained in $\mathbb{V}$. Now, we intend to verify that $\mathfrak{P}$ is equicontinuous. To confirm this, let $z_1, z_2 \in [0,1]$ with $z_1 < z_2$ and $\nu \in \overline{\mathfrak{E}_\varepsilon}$. Then, by assuming $\sup_{(z,\nu)\in[0,1]\times\overline{\mathfrak{E}_\varepsilon}} |\Phi_*(z,\nu)| = \widetilde{\Phi}_* > 0$, we obtain

$$
\begin{aligned}
&\bigl|(\mathfrak{P}\nu)(z_2) - (\mathfrak{P}\nu)(z_1)\bigr| \\
&\quad\le \int_0^{z_1} \frac{[(z_2-\varsigma)^{\xi-1} - (z_1-\varsigma)^{\xi-1}]}{\Gamma(\xi)} \bigl|\Phi_*\bigl(\varsigma, \nu(\varsigma)\bigr)\bigr|\,\mathrm{d}\varsigma \\
&\qquad+ \int_{z_1}^{z_2} \frac{(z_2-\varsigma)^{\xi-1}}{\Gamma(\xi)} \bigl|\Phi_*\bigl(\varsigma, \nu(\varsigma)\bigr)\bigr|\,\mathrm{d}\varsigma
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(z_2-z_1)\Gamma(4-\xi)}{2-\xi}\int_0^1\left|\Phi_*\left(\varsigma,\nu(\varsigma)\right)\right|\mathrm{d}\varsigma\\
&+\frac{(z_2-z_1)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^{\varsigma}\left|\Phi_*\left(\tau,\nu(\tau)\right)\right|\mathrm{d}\tau\,\mathrm{d}\varsigma\\
\le{}&\frac{\widetilde{\Phi}_*}{\Gamma(\xi+1)}\left(\left|z_2^{\xi}-z_1^{\xi}\right|+2(z_1-z_2)^{\xi}\right)\\
&+\frac{(z_2-z_1)\Gamma(4-\xi)\widetilde{\Phi}_*}{2-\xi}+\frac{(z_2-z_1)\Gamma(4-\xi)\widetilde{\Phi}_*}{2(2-\xi)}.
\end{aligned}
$$

When z_1 goes to z_2, the right-hand side of the above estimate approaches zero (independent of $\nu\in\overline{\mathfrak{E}_\varepsilon}$). Thus, by assuming $z_1\to z_2$, we get $\|(\mathfrak{P}\nu)(z_2)-(\mathfrak{P}\nu)(z_1)\|_{\mathbb{V}}\to 0$ and so $\mathfrak{P}$ is equicontinuous. Now, by the aid of the well-known theorem due to Arzela–Ascoli, $\mathfrak{P}$ is proved to be completely continuous, and so is compact on $\overline{\mathfrak{E}_\varepsilon}$.

At this moment, we are going to verify that the operator $\mathfrak{P}$ is condensing on $\overline{\mathfrak{E}_\varepsilon}$. By virtue of Lemma 2, we are sure that for each bounded subset $\mathfrak{B}\subset\overline{\mathfrak{E}_\varepsilon}$, a countable set $\mathfrak{B}_0=\{\nu_n\}_{n\ge1}\subset\mathfrak{B}$ exists such that the inequality $\Omega(\mathfrak{P}(\mathfrak{B}))\le 2\Omega(\mathfrak{P}(\mathfrak{B}_0))$ is satisfied. Hence, by invoking Lemmas 1, 3, and 4, we have the following inequalities:

$$
\begin{aligned}
&\Omega\left(\mathfrak{P}\left(\mathfrak{B}(z)\right)\right)\\
&\quad\le 2\Omega\left(\mathfrak{P}\left(\{\nu_n\}_{n\ge1}\right)\right)\\
&\quad\le\frac{2}{\Gamma(\xi)}\int_0^z(z-\varsigma)^{\xi-1}\Omega\left(\Phi_*\left(\varsigma,\left\{\nu_n(\varsigma)\right\}_{n\ge1}\right)\right)\mathrm{d}\varsigma\\
&\qquad+\frac{2s_2^*}{\Gamma(\xi-1)}\int_0^1(1-\varsigma)^{\xi-2}\Omega\left(\Phi_*\left(\varsigma,\left\{\nu_n(\varsigma)\right\}_{n\ge1}\right)\right)\mathrm{d}\varsigma\\
&\qquad+\frac{2(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1\Omega\left(\Phi_*\left(\varsigma,\left\{\nu_n(\varsigma)\right\}_{n\ge1}\right)\right)\mathrm{d}\varsigma\\
&\qquad+\frac{2(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^{\varsigma}\Omega\left(\Phi_*\left(\tau,\left\{\nu_n(\tau)\right\}_{n\ge1}\right)\right)\mathrm{d}\tau\,\mathrm{d}\varsigma\\
&\quad\le\frac{4}{\Gamma(\xi)}\int_0^z(z-\varsigma)^{\xi-1}n_{\Phi_*}(\varsigma)\Omega\left(\left\{\nu_n(\varsigma)\right\}_{n\ge1}\right)\mathrm{d}\varsigma\\
&\qquad+\frac{4s_2^*}{\Gamma(\xi-1)}\int_0^1(1-\varsigma)^{\xi-2}n_{\Phi_*}(\varsigma)\Omega\left(\left\{\nu_n(\varsigma)\right\}_{n\ge1}\right)\mathrm{d}\varsigma\\
&\qquad+\frac{4(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1 n_{\Phi_*}(\varsigma)\Omega\left(\left\{\nu_n(\varsigma)\right\}_{n\ge1}\right)\mathrm{d}\varsigma\\
&\qquad+\frac{4(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^{\varsigma}n_{\Phi_*}(\tau)\Omega\left(\left\{\nu_n(\tau)\right\}_{n\ge1}\right)\mathrm{d}\tau\,\mathrm{d}\varsigma\\
&\quad\le\frac{4n_{\Phi_*}^*\Omega(\mathfrak{B})}{\Gamma(\xi)}\int_0^z(z-\varsigma)^{\xi-1}\,\mathrm{d}\varsigma+\frac{4s_2^*n_{\Phi_*}^*\Omega(\mathfrak{B})}{\Gamma(\xi-1)}\int_0^1(1-\varsigma)^{\xi-2}\,\mathrm{d}\varsigma\\
&\qquad+\frac{4n_{\Phi_*}^*\Omega(\mathfrak{B})(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1\mathrm{d}\varsigma\\
&\qquad+\frac{4n_{\Phi_*}^*\Omega(\mathfrak{B})(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^{\varsigma}\mathrm{d}\tau\,\mathrm{d}\varsigma\\
&\quad\le\frac{4n_{\Phi_*}^*\Omega(\mathfrak{B})}{\Gamma(\xi+1)}+\frac{4s_2^*n_{\Phi_*}^*\Omega(\mathfrak{B})}{\Gamma(\xi)}+\frac{12n_{\Phi_*}^*\Omega(\mathfrak{B})(s_1^*+s_2^*+1)\Gamma(4-\xi)}{2(2-\xi)}.
\end{aligned}
$$

Hence,

$$\Omega\big(\mathfrak{P}(\mathfrak{B})\big) \leq 4\left[\frac{n^*_{\Phi_*}}{\Gamma(\xi+1)} + \frac{s_2^* n^*_{\Phi_*}}{\Gamma(\xi)} + \frac{3n^*_{\Phi_*}(s_1^*+s_2^*+1)\Gamma(4-\xi)}{2(2-\xi)}\right]\Omega(\mathfrak{B}).$$

As a consequence, by taking into account the given condition (7), we get $\Omega(\mathfrak{P}(\mathfrak{B})) < \Omega(\mathfrak{B})$, and subsequently it is realized that $\mathfrak{P}$ is a condensing operator on $\overline{\mathfrak{E}_\varepsilon}$. Eventually, by resorting to Theorem 6, it is deduced that the operator $\mathfrak{P}$ has at least one fixed point contained in $\overline{\mathfrak{E}_\varepsilon}$ which is regarded as a solution for the given fractional Lio–CapBvp (1), and this ends the proof. □

At this moment, we are going to discuss the dependence of solutions for the given fractional Lio–CapBvp (1). Indeed, in this part we assume that the solution of the fractional Lio–CapBvp (1) depends on some parameters provided that the function Φ_* satisfies the conditions of Theorem 12 which guarantees the existence of solutions and, as we know, the continuous dependence of solutions on these parameters indicates the stability of solutions. So we investigate this property of the solutions of the fractional Lio–CapBvp (1) by making a small change in a parameter of the fractional Lio–CapBvp (1), i.e., its order. To do this, we will apply the generalized Gronwall inequality.

Theorem 13 *Consider $\xi > 0$ so that $1 < \xi - \delta < \xi < 2$. Moreover, assume that $\Phi_* : [0,1] \times \mathbb{V} \to \mathbb{V}$ is continuous and a constant $\rho > 0$ exists such that*

$$\big|\Phi_*\big(z, \nu(z)\big) - \Phi_*\big(z, \nu'(z)\big)\big| \leq \rho\big|\nu(z) - \nu'(z)\big|$$

for any $\nu, \nu' \in \mathbb{V}$ and $z \in [0,1]$. Additionally, let ν be a solution of the fractional Lio–CapBvp (1) and $\bar{\nu}$ be a solution of

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-\delta}\bar{\nu}(z) = \Phi_*(z, \bar{\nu}(z)), \\ \bar{\nu}(0) = s_1^* \, {}^{\mathcal{C}}\mathfrak{D}_0^1\bar{\nu}(0) + s_2^* \, {}^{\mathcal{C}}\mathfrak{D}_0^1\bar{\nu}(1), \qquad {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-\delta-1}\bar{\nu}(1) = \int_0^1 {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-\delta-1}\bar{\nu}(\varsigma)\,\mathrm{d}\varsigma. \end{cases} \tag{10}$$

Then, the following inequality is valid:

$$\|\nu - \bar{\nu}\|_{\mathbb{V}} \leq \frac{\Delta + \Delta\sum_{k=1}^{\infty}\frac{\rho^k}{\Gamma(k(\xi-\delta)+1)}}{1 - \mho - \mho\sum_{k=1}^{\infty}\frac{\rho^k}{\Gamma(k(\xi-\delta)+1)}}$$

provided that $\mho + \mho\sum_{k=1}^{\infty}\frac{\rho^k}{\Gamma(k(\xi-\delta)+1)} < 1$, where

$$\begin{aligned} \Delta = {}& \|\Phi_*\|_{\mathbb{V}} \sup_{z\in[0,1]}\left|\frac{z^\xi}{\Gamma(\xi+1)} - \frac{z^{\xi-\delta}}{\Gamma(\xi-\delta+1)}\right| + \|\Phi_*\|_{\mathbb{V}} \sup_{z\in[0,1]}\left|\frac{s_2^*}{\Gamma(\xi)} - \frac{s_2^*}{\Gamma(\xi-\delta)}\right| \\ & + \|\Phi_*\|_{\mathbb{V}} \sup_{z\in[0,1]}\left|\frac{(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2-\xi} - \frac{(s_1^*+s_2^*+z)\Gamma(4-(\xi-\delta))}{2-(\xi-\delta)}\right| \\ & + \|\Phi_*\|_{\mathbb{V}} \sup_{z\in[0,1]}\left|\frac{(s_1^*+s_2^*+z)\Gamma(4-\xi)}{2(2-\xi)} - \frac{(s_1^*+s_2^*+z)\Gamma(4-(\xi-\delta))}{2(2-(\xi-\delta))}\right| \end{aligned} \tag{11}$$

and

$$\mho = \left(\frac{s_2^*}{\Gamma(\xi - \delta)} + \frac{3(s_1^* + s_2^* + 1)\Gamma(4 - (\xi - \delta))}{2(2 - (\xi - \delta))} \right) \rho, \tag{12}$$

where $\|\Phi_*\|_{\mathbb{V}} = \sup_{z \in [0,1]} |\Phi_*(z, \nu(z))|$.

Proof Prior to deriving the desired inequality, in view of the above facts, it is known that the existence of solutions of both fractional Lio–CapBvps (1) and (10) is guaranteed by a similar argument as above and these solutions are given in the form of (8) and

$$\begin{aligned} \bar{\nu}(z) = {} & \frac{1}{\Gamma(\xi - \delta)} \int_0^z (z - \varsigma)^{\xi - \delta - 1} \Phi_*\big(\varsigma, \bar{\nu}(\varsigma)\big) \, \mathrm{d}\varsigma \\ & + \frac{s_2^*}{\Gamma(\xi - \delta - 1)} \int_0^1 (1 - \varsigma)^{\xi - \delta - 2} \Phi_*\big(\varsigma, \bar{\nu}(\varsigma)\big) \, \mathrm{d}\varsigma \\ & - \frac{(s_1^* + s_2^* + z)\Gamma(4 - (\xi - \delta))}{2 - (\xi - \delta)} \int_0^1 \Phi_*\big(\varsigma, \bar{\nu}(\varsigma)\big) \, \mathrm{d}\varsigma \\ & + \frac{(s_1^* + s_2^* + z)\Gamma(4 - (\xi - \delta))}{2 - (\xi - \delta)} \int_0^1 \int_0^\varsigma \Phi_*\big(\tau, \bar{\nu}(\tau)\big) \, \mathrm{d}\tau \, \mathrm{d}\varsigma, \end{aligned}$$

respectively. Then, an estimate for $\nu - \bar{\nu}$ is calculated as follows:

$$\begin{aligned} & \big|\nu(z) - \bar{\nu}(z)\big| \\ & \quad \leq \left| \int_0^z \left(\frac{1}{\Gamma(\xi)} (z - \varsigma)^{\xi - 1} - \frac{1}{\Gamma(\xi - \delta)} (z - \varsigma)^{\xi - \delta - 1} \right) \Phi_*\big(\varsigma, \nu(\varsigma)\big) \, \mathrm{d}\varsigma \right| \\ & \qquad + \int_0^z \frac{(z - \varsigma)^{\xi - \delta - 1}}{\Gamma(\xi - \delta)} \big| \Phi_*\big(\varsigma, \nu(\varsigma)\big) - \Phi_*\big(\varsigma, \bar{\nu}(\varsigma)\big) \big| \, \mathrm{d}\varsigma \\ & \qquad + \left| \int_0^1 \left(\frac{s_2^*(1 - \varsigma)^{\xi - 2}}{\Gamma(\xi - 1)} - \frac{s_2^*(1 - \varsigma)^{\xi - \delta - 2}}{\Gamma(\xi - \delta - 1)} \right) \Phi_*\big(\varsigma, \nu(\varsigma)\big) \, \mathrm{d}\varsigma \right| \\ & \qquad + \int_0^1 \frac{s_2^*(1 - \varsigma)^{\xi - \delta - 2}}{\Gamma(\xi - \delta - 1)} \big| \Phi_*\big(\varsigma, \nu(\varsigma)\big) - \Phi_*\big(\varsigma, \bar{\nu}(\varsigma)\big) \big| \, \mathrm{d}\varsigma \\ & \qquad + \left| \frac{(s_1^* + s_2^* + z)\Gamma(4 - \xi)}{2 - \xi} \int_0^1 \Phi_*\big(\varsigma, \nu(\varsigma)\big) \, \mathrm{d}\varsigma \right. \\ & \qquad \left. - \frac{(s_1^* + s_2^* + z)\Gamma(4 - (\xi - \delta))}{2 - (\xi - \delta)} \int_0^1 \Phi_*\big(\varsigma, \nu(\varsigma)\big) \, \mathrm{d}\varsigma \right| \\ & \qquad + \frac{(s_1^* + s_2^* + z)\Gamma(4 - (\xi - \delta))}{2 - (\xi - \delta)} \int_0^1 \big| \Phi_*\big(\varsigma, \nu(\varsigma)\big) - \Phi_*\big(\varsigma, \bar{\nu}(\varsigma)\big) \big| \, \mathrm{d}\varsigma \\ & \qquad + \left| \frac{(s_1^* + s_2^* + z)\Gamma(4 - \xi)}{2 - \xi} \int_0^1 \int_0^\varsigma \Phi_*\big(\tau, \nu(\tau)\big) \, \mathrm{d}\tau \, \mathrm{d}\varsigma \right. \\ & \qquad \left. - \frac{(s_1^* + s_2^* + z)\Gamma(4 - (\xi - \delta))}{2 - (\xi - \delta)} \int_0^1 \int_0^\varsigma \Phi_*\big(\tau, \nu(\tau)\big) \, \mathrm{d}\tau \, \mathrm{d}\varsigma \right| \\ & \qquad + \frac{(s_1^* + s_2^* + z)\Gamma(4 - (\xi - \delta))}{2 - (\xi - \delta)} \int_0^1 \int_0^\varsigma \big| \Phi_*\big(\tau, \nu(\tau)\big) - \Phi_*\big(\tau, \bar{\nu}(\tau)\big) \big| \, \mathrm{d}\tau \, \mathrm{d}\varsigma \\ & \quad \leq \Delta + \mho \|\nu - \bar{\nu}\|_{\mathbb{V}} + \int_0^z \frac{(z - \varsigma)^{\xi - \delta - 1}}{\Gamma(\xi - \delta)} \rho \big| \nu(\varsigma) - \bar{\nu}(\varsigma) \big| \, \mathrm{d}\varsigma, \end{aligned}$$

where Δ and $\mho$ are introduced by (11) and (12). Hence with due attention to the generalized Gronwall inequality (Theorem 7) and by letting $\check{\nu}(z) = |\nu(z) - \bar{\nu}(z)|$, $\check{r}(z) = \Delta + \mho\|\nu - \bar{\nu}\|_{\mathbb{V}}$, and $h(z) = \frac{\rho}{\Gamma(\xi-\delta)}$, we reach

$$\left|\nu(z) - \bar{\nu}(z)\right| \leq \Delta + \mho\|\nu - \bar{\nu}\|_{\mathbb{V}} + \int_0^z \sum_{k=1}^{\infty}\left[\frac{\rho^k (z-\varsigma)^{k(\xi-\delta)-1}}{\Gamma(k(\xi-\delta))}\left(\Delta + \mho\|\nu - \bar{\nu}\|_{\mathbb{V}}\right)\right] \mathrm{d}\varsigma.$$

In conclusion, we obtain

$$\|\nu - \bar{\nu}\|_{\mathbb{V}} \leq \frac{\Delta + \Delta\sum_{k=1}^{\infty}\frac{\rho^k}{\Gamma(k(\xi-\delta)+1)}}{1 - \mho - \mho\sum_{k=1}^{\infty}\frac{\rho^k}{\Gamma(k(\xi-\delta)+1)}},$$

and the latter inequality finishes the proof. □

Remark 1 From the above theorem, we deduce that if $\delta = 0$, then $\nu = \bar{\nu}$ and so both considered solutions of the fractional Lio–CapBvp (1) are the same.

4 Inclusion version of the fractional Lio–CapBvp (1)

Here, we continue the above process to establish a criterion of existence of solutions for the inclusion version of the fractional Lio–CapBvp (1) given by

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_0^{\xi}\nu(z) \in \mathfrak{Q}_*(z, \nu(z)), \\ \nu(0) = s_1^{*\,\mathcal{C}}\mathfrak{D}_0^1\nu(0) + s_2^{*\,\mathcal{C}}\mathfrak{D}_0^1\nu(1), \qquad {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1}\nu(1) = \int_0^1 {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1}\nu(\varsigma)\,\mathrm{d}\varsigma \end{cases} \tag{13}$$

so that $z \in [0,1]$, $s_1^*, s_2^* \in \mathbb{R}^+$, ${}^{\mathcal{C}}\mathfrak{D}_0^{\xi}$ stands for the Cap-derivative of order $\xi \in (1,2)$, ${}^{\mathcal{C}}\mathfrak{D}_0^1 = \frac{\mathrm{d}}{\mathrm{d}z}$ and $\mathfrak{Q}_* : [0,1] \times \mathbb{V} \to \mathcal{P}(\mathbb{V})$ is a multivalued mapping. To continue, we utilize the notion of the approx-endpoint property.

Definition 14 An absolutely continuous function $\nu : [0,1] \to \mathbb{R}$ is called a solution for the fractional inclusion Lio–CapBvp (13) whenever there is an $\hbar \in \mathcal{L}^1([0,1], \mathbb{R})$ such that $\hbar(z) \in \mathfrak{Q}_*(z, \nu(z))$ for almost all $z \in [0,1]$ which satisfies integro-derivative boundary conditions

$$\nu(0) = s_1^{*\,\mathcal{C}}\mathfrak{D}_0^1\nu(0) + s_2^{*\,\mathcal{C}}\mathfrak{D}_0^1\nu(1), \qquad {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1}\nu(1) = \int_0^1 {}^{\mathcal{C}}\mathfrak{D}_0^{\xi-1}\nu(\varsigma)\,\mathrm{d}\varsigma$$

and we also have

$$\begin{aligned} \nu(z) = {}& \frac{1}{\Gamma(\xi)}\int_0^z (z-\varsigma)^{\xi-1}\hbar(\varsigma)\,\mathrm{d}\varsigma + \frac{s_2^*}{\Gamma(\xi-1)}\int_0^1 (1-\varsigma)^{\xi-2}\hbar(\varsigma)\,\mathrm{d}\varsigma \\ & - \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi}\int_0^1 \hbar(\varsigma)\,\mathrm{d}\varsigma \\ & + \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^{\varsigma} \hbar(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma \end{aligned}$$

for any $z \in [0,1]$.

For any $\nu \in \mathbb{V}$, the family of all existing selections of $\mathfrak{Q}_*$ is denoted by

$$(\mathfrak{S}\mathfrak{E}\mathfrak{L})_{\mathfrak{Q}_*,\nu} = \{\hbar \in \mathcal{L}^1([0,1]) : \hbar(z) \in \mathfrak{Q}_*(z, \nu(z))\}$$

for almost all $z \in [0,1]$. Now, we define a multifunction $\mathfrak{F} : \mathbb{V} \to \mathcal{P}(\mathbb{V})$ by the following rule:

$$\mathfrak{F}(\nu) = \{\vartheta \in \mathbb{V} : \vartheta(z) = \kappa(z)\}, \tag{14}$$

where

$$\begin{aligned}\kappa(z) = {} & \frac{1}{\Gamma(\xi)} \int_0^z (z-\varsigma)^{\xi-1} \hbar(\varsigma)\,\mathrm{d}\varsigma + \frac{s_2^*}{\Gamma(\xi-1)} \int_0^1 (1-\varsigma)^{\xi-2} \hbar(\varsigma)\,\mathrm{d}\varsigma \\ & - \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \hbar(\varsigma)\,\mathrm{d}\varsigma \\ & + \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \int_0^\varsigma \hbar(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma, \hbar \in (\mathfrak{S}\mathfrak{E}\mathfrak{L})_{\mathfrak{Q}_*,\nu}.\end{aligned}$$

By utilizing the approx-endpoint property for the multivalued map $\mathfrak{F}$, we have the following:

Theorem 15 *Regard $\mathfrak{Q}_* : [0,1] \times \mathbb{V} \to \mathcal{P}_{cmp}(\mathbb{V})$ as a multivalued map. Suppose that*

($\mathfrak{C}_1$) $\psi : [0,\infty) \to [0,\infty)$ *is increasing and u.s.c. with* $\liminf_{z\to\infty}(z - \psi(z)) > 0$ *and* $\psi(z) < z$ *for any* $z > 0$;

($\mathfrak{C}_2$) *the multifunction* $\mathfrak{Q}_* : [0,1] \times \mathbb{V} \to \mathcal{P}_{cmp}(\mathbb{V})$ *is integrable and bounded so that* $\mathfrak{Q}_*(\cdot,\nu) : [0,1] \to \mathcal{P}_{cmp}(\mathbb{V})$ *is measurable for any* $\nu \in \mathbb{V}$;

($\mathfrak{C}_3$) *a function* $\varrho \in \mathcal{C}([0,1],[0,\infty))$ *exists such that*

$$\mathbb{H}_{d_{\mathbb{V}}}\big(\mathfrak{Q}_*(z,\nu_1(z)), \mathfrak{Q}_*(z,\nu_2(z))\big) \le \varrho(z)\psi\big(|\nu_1(z) - \nu_2(z)|\big)\frac{1}{\hat{\mathbb{O}}}$$

for all $z \in [0,1]$ *and* $\nu_1, \nu_2 \in \mathbb{V}$, *where* $\sup_{z\in[0,1]} |\varrho(z)| = \|\varrho\|$ *and*

$$\hat{\mathbb{O}} = \left[\frac{1}{\Gamma(\xi+1)} + \frac{s_2^*}{\Gamma(\xi)} + \frac{3(s_1^* + s_2^* + 1)\Gamma(4-\xi)}{2(2-\xi)}\right]\|\varrho\|; \tag{15}$$

($\mathfrak{C}_4$) *the multifunction* $\mathfrak{F}$ *given by* (14) *has the approx-endpoint property.*

Then a solution exists for the fractional inclusion Lio–CapBvp (13).

Proof In the current proof, we try to confirm the existence of an endpoint for the multivalued map $\mathfrak{F} : \mathbb{V} \to \mathcal{P}(\mathbb{V})$ defined by (14). To proceed, we first need to check that $\mathfrak{F}(\nu)$ is closed for any $\nu \in \mathbb{V}$. In the light of the hypothesis ($\mathfrak{C}_2$), the mapping $z \mapsto \mathfrak{Q}_*(z,\nu(z))$ is a measurable and closed-valued multivalued map for each $\nu \in \mathbb{V}$. As a consequence, $\mathfrak{Q}_*$ has a measurable selection as $(\mathfrak{S}\mathfrak{E}\mathfrak{L})_{\mathfrak{Q}_*,\nu} \neq \emptyset$. Now, we claim that $\mathfrak{F}(\nu) \subseteq \mathbb{V}$ is closed for all $\nu \in \mathbb{V}$. Consider a sequence $(\nu_n)_{n\ge1}$ contained in $\mathfrak{F}(\nu)$ with $\nu_n \to \nu^*$. For each n, an element $\hbar_n \in (\mathfrak{S}\mathfrak{E}\mathfrak{L})_{\mathfrak{Q}_*,\nu}$ exists so that

$$\nu_n(z) = \frac{1}{\Gamma(\xi)} \int_0^z (z-\varsigma)^{\xi-1} \hbar_n(\varsigma)\,\mathrm{d}\varsigma + \frac{s_2^*}{\Gamma(\xi-1)} \int_0^1 (1-\varsigma)^{\xi-2} \hbar_n(\varsigma)\,\mathrm{d}\varsigma$$

$$- \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \hbar_n(\varsigma)\,\mathrm{d}\varsigma + \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \int_0^\varsigma \hbar_n(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma$$

for almost all $z \in [0,1]$. Since $\mathfrak{Q}_*$ is a compact multivalued mapping, we get a subsequence $\{\hbar_n\}_{n\geq 1}$ tending to $\hbar \in \mathcal{L}^1([0,1])$. Hence, we have $\hbar \in (\mathfrak{SEL})_{\mathfrak{Q}_*,\nu}$ and

$$\begin{aligned} \lim_{n\to\infty} \nu_n(z) &= \frac{1}{\Gamma(\xi)} \int_0^z (z-\varsigma)^{\xi-1}\hbar(\varsigma)\,\mathrm{d}\varsigma + \frac{s_2^*}{\Gamma(\xi-1)} \int_0^1 (1-\varsigma)^{\xi-2}\hbar(\varsigma)\,\mathrm{d}\varsigma \\ &\quad - \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \hbar(\varsigma)\,\mathrm{d}\varsigma \\ &\quad + \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \int_0^\varsigma \hbar(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma \\ &= \nu(z) \end{aligned}$$

for any $z \in [0,1]$. Thus $\nu \in \mathfrak{F}$ which confirms that $\mathfrak{F}$ is closed-valued. Therefore, we know that $\mathfrak{F}(\nu)$ is bounded for each $\nu \in \mathbb{V}$ since $\mathfrak{Q}_*$ is assumed to be compact. Now, we check whether the inequality

$$\mathbb{H}_{d_{\mathbb{V}}}\big(\mathfrak{F}(\nu_1), \mathfrak{F}(\nu_2)\big) \leq \psi\big(\|\nu_1 - \nu_2\|\big)$$

holds. Let $\nu_1, \nu_2 \in \mathbb{V}$ and $x_1 \in \mathfrak{F}(\nu_2)$. Select $\hbar_1 \in (\mathfrak{SEL})_{\mathfrak{Q}_*,\nu_2}$ so that

$$\begin{aligned} x_1(z) &= \frac{1}{\Gamma(\xi)} \int_0^z (z-\varsigma)^{\xi-1}\hbar_1(\varsigma)\,\mathrm{d}\varsigma + \frac{s_2^*}{\Gamma(\xi-1)} \int_0^1 (1-\varsigma)^{\xi-2}\hbar_1(\varsigma)\,\mathrm{d}\varsigma \\ &\quad - \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \hbar_1(\varsigma)\,\mathrm{d}\varsigma \\ &\quad + \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi} \int_0^1 \int_0^\varsigma \hbar_1(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma \end{aligned}$$

for almost all $z \in [0,1]$. Since

$$\mathbb{H}_{d_{\mathbb{V}}}\big(\mathfrak{Q}_*\big(z, \nu_1(z)\big), \mathfrak{Q}_*\big(z, \nu_2(z)\big)\big) \leq \varrho(z)\big(\psi\big(\nu_1(z) - \nu_2(z)\big)\big)\frac{1}{\hat{\mathbb{O}}}$$

for any $z \in [0,1]$, we obtain that $\hbar^* \in \mathfrak{Q}_*(z, \nu_1(z))$ exists such that

$$\big|\hbar_1(z) - \hbar^*\big| \leq \varrho(z)\big(\psi\big(\nu_1(z) - \nu_2(z)\big)\big)\frac{1}{\hat{\mathbb{O}}}$$

for any $z \in [0,1]$. In the following, consider the multivalued map $\mathfrak{A} : [0,1] \to \mathcal{P}(\mathbb{V})$ introduced by

$$\mathfrak{A}(z) = \left\{\hbar^* \in \mathbb{V} : \big|\hbar_1(z) - \hbar^*\big| \leq \varrho(z)\big(\psi\big(\nu_1(z) - \nu_2(z)\big)\big)\frac{1}{\hat{\mathbb{O}}}\right\}.$$

Since $\hbar_1$ and $\sigma = \varrho(\psi(\nu_1 - \nu_2))\frac{1}{\hat{\mathbb{O}}}$ are measurable, we can choose $\hbar_2(z) \in \mathfrak{Q}_*(z, \nu_1(z))$ so that

$$\left|\hbar_1(z) - \hbar_2(z)\right| \le \varrho(z)\big(\psi\big(\nu_1(z) - \nu_2(z)\big)\big)\frac{1}{\hat{\mathbb{O}}}$$

for all $z \in [0,1]$. Select $x_2 \in \mathfrak{F}(\nu_1)$ such that

$$\begin{aligned} x_2(z) = {} & \frac{1}{\Gamma(\xi)}\int_0^z (z-\varsigma)^{\xi-1}\hbar_2(\varsigma)\,\mathrm{d}\varsigma + \frac{s_2^*}{\Gamma(\xi-1)}\int_0^1 (1-\varsigma)^{\xi-2}\hbar_2(\varsigma)\,\mathrm{d}\varsigma \\ & - \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi}\int_0^1 \hbar_2(\varsigma)\,\mathrm{d}\varsigma \\ & + \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^\varsigma \hbar_2(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma \end{aligned}$$

for each $z \in [0,1]$. Hence, we get that

$$\begin{aligned} & \left|x_1(z) - x_2(z)\right| \\ & \quad \le \frac{1}{\Gamma(\xi)}\int_0^z (z-\varsigma)^{\xi-1}\left|\hbar_1(\varsigma) - \hbar(\varsigma)\right|\mathrm{d}\varsigma \\ & \qquad + \frac{s_2^*}{\Gamma(\xi-1)}\int_0^1 (1-\varsigma)^{\xi-2}\left|\hbar_1(\varsigma) - \hbar(\varsigma)\right|\mathrm{d}\varsigma \\ & \qquad + \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi}\int_0^1 \left|\hbar_1(\varsigma) - \hbar(\varsigma)\right|\mathrm{d}\varsigma \\ & \qquad + \frac{(s_1^* + s_2^* + z)\Gamma(4-\xi)}{2-\xi}\int_0^1\int_0^\varsigma \left|\hbar_1(\tau) - \hbar(\tau)\right|\mathrm{d}\tau\,\mathrm{d}\varsigma \\ & \quad \le \frac{1}{\Gamma(\xi+1)}\|\varrho\|\psi\big(\|\nu_1 - \nu_2\|\big)\frac{1}{\hat{\mathbb{O}}} + \frac{s_2^*}{\Gamma(\xi)}\|\varrho\|\psi\big(\|\nu_1 - \nu_2\|\big)\frac{1}{\hat{\mathbb{O}}} \\ & \qquad + \frac{(s_1^* + s_2^* + 1)\Gamma(4-\xi)}{2-\xi}\|\varrho\|\psi\big(\|\nu_1 - \nu_2\|\big)\frac{1}{\hat{\mathbb{O}}} \\ & \qquad + \frac{(s_1^* + s_2^* + 1)\Gamma(4-\xi)}{2(2-\xi)}\|\varrho\|\psi\big(\|\nu_1 - \nu_2\|\big)\frac{1}{\hat{\mathbb{O}}} \\ & \quad = \left[\frac{1}{\Gamma(\xi+1)} + \frac{s_2^*}{\Gamma(\xi)} + \frac{3(s_1^* + s_2^* + 1)\Gamma(4-\xi)}{2(2-\xi)}\right]\|\varrho\|\psi\big(\|\nu_1 - \nu_2\|\big)\frac{1}{\hat{\mathbb{O}}} \\ & \quad = \hat{\mathbb{O}}\psi\big(\|\nu_1 - \nu_2\|\big)\frac{1}{\hat{\mathbb{O}}} = \psi\big(\|\nu_1 - \nu_2\|\big). \end{aligned}$$

This gives $\|x_1 - x_2\| \le \psi(\|\nu_1 - \nu_2\|)$ and yields $\mathbb{H}_{d_{\mathbb{V}}}(\mathfrak{F}(\nu_1), \mathfrak{F}(\nu_2)) \le \psi(\|\nu_1 - \nu_2\|)$ for any $\nu_1, \nu_2 \in \mathbb{V}$. Also from $(\mathfrak{C}_4)$, we understand that $\mathfrak{F}$ possesses the approx-endpoint property. Then Theorem 10 gives that $\mathfrak{F}$ has a unique endpoint. In other words, $\nu^* \in \mathbb{V}$ exists so that $\mathfrak{F}(\nu^*) = \{\nu^*\}$. As a consequence, ν^* is a solution of the fractional inclusion Lio–CapBvp (13) and the proof is finished. □

5 Numerical examples

Here, we prepare two numerical examples to confirm the validity of the analytical findings. The first example illustrates Theorem 12.

Example 1 With due attention to the proposed fractional Lio–CapBvp (1), we formulate the following structure of the fractional Lio–CapBvp as follows:

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_0^{1.5}\nu(z) = \frac{3+2e^{-z}}{10{,}000}\sin(\nu(z)) \\ \nu(0) = 0.3{}^{\mathcal{C}}\mathfrak{D}_0^{1}\nu(0) + 0.4{}^{\mathcal{C}}\mathfrak{D}_0^{1}\nu(1), \qquad {}^{\mathcal{C}}\mathfrak{D}_0^{0.5}\nu(1) = \int_0^1 {}^{\mathcal{C}}\mathfrak{D}_0^{0.5}\nu(\varsigma)\,\mathrm{d}\varsigma \end{cases} \tag{16}$$

so that $\xi = 1.5$, $\xi - 1 = 0.5$, $s_1^* = 0.3$, $s_2^* = 0.4$, and $z \in [0,1]$. In addition, we introduce a continuous function $\Phi_* : [0,1] \times \mathbb{R} \to \mathbb{R}$ given by

$$\Phi_*\big(z,\nu(z)\big) = \frac{3+2e^{-z}}{10{,}000}\sin\big(\nu(z)\big).$$

Then, for any $\nu \in \mathbb{R}$, one can write

$$\big|\Phi_*\big(z,\nu(z)\big)\big| \le \frac{3+2e^{-z}}{10{,}000}\big|\sin\big(\nu(z)\big)\big| \le \frac{3+2e^{-z}}{10{,}000} = \varphi(z),$$

where a continuous function $\varphi : [0,1] \to \mathbb{R}^+$ is defined by $\varphi(z) = \frac{3+2e^{-z}}{10{,}000}$. On the other hand, for every $\nu_1, \nu_2 \in \mathbb{R}$, we have

$$\begin{aligned} \big|\Phi_*\big(z,\nu_1(z)\big) - \Phi_*\big(z,\nu_2(z)\big)\big| &\le \frac{3+2e^{-z}}{10{,}000}\big|\sin\big(\nu_1(z)\big) - \sin\big(\nu_2(z)\big)\big| \\ &\le \frac{3+2e^{-z}}{10{,}000}\big|\nu_1(z) - \nu_2(z)\big|. \end{aligned}$$

As a consequence, for any bounded set $\mathfrak{B} \subset \mathbb{R}$, we may write

$$\Omega\big(\Phi_*(z,\mathfrak{B})\big) \le \frac{3+2e^{-z}}{10{,}000}\Omega(\mathfrak{B}) := n_{\Phi_*}(z)\Omega(\mathfrak{B})$$

so that $n_{\Phi_*}^* = \sup_{z\in[0,1]} |n_{\Phi_*}| = 0.0005$. Then in view of above values, since

$$\frac{n_{\Phi_*}^*}{\Gamma(\xi+1)} + \frac{s_2^* n_{\Phi_*}^*}{\Gamma(\xi)} + \frac{3n_{\Phi_*}^*(s_1^* + s_2^* + 1)\Gamma(4-\xi)}{2(2-\xi)} \simeq 0.00399152 < \frac{1}{4},$$

the inequality (7) holds. In conclusion, all the hypotheses of Theorem 12 are valid and thus this result implies that there exists at least one solution for the fractional Lio–CapBvp (16).

The next example illustrates Theorem 15.

Example 2 With due attention to (13), we design the fractional inclusion Cap–Bvp in the form

$$\begin{cases} {}^{\mathcal{C}}\mathfrak{D}_0^{1.5}\nu(z) \in \big[0, \frac{e^{-2z}|\arctan(\nu(z))|}{12(50+z)(1+|\arctan(\nu(z))|)}\big], \\ \nu(0) = 0.3{}^{\mathcal{C}}\mathfrak{D}_0^{1}\nu(0) + 0.4{}^{\mathcal{C}}\mathfrak{D}_0^{1}\nu(1), \qquad {}^{\mathcal{C}}\mathfrak{D}_0^{0.5}\nu(1) = \int_0^1 {}^{\mathcal{C}}\mathfrak{D}_0^{0.5}\nu(\varsigma)\,\mathrm{d}\varsigma \end{cases} \tag{17}$$

with the same values $\xi = 1.5$, $\xi - 1 = 0.5$, $s_1^* = 0.3$, $s_2^* = 0.4$ and $z \in [0,1]$. At first, we introduce the Banach space $\mathbb{V} = \{\nu(z) : \nu(z) \in \mathcal{C}_{\mathbb{R}}([0,1])\}$ furnished with $\|\nu\|_{\mathbb{V}} = \sup_{z\in[0,1]} |\nu(z)|$.

In addition, we define a multifunction $\mathfrak{Q}_* : [0,1] \times \mathbb{V} \to \mathcal{P}(\mathbb{V})$ by

$$\mathfrak{Q}_*\big(z, \nu(z)\big) = \left[0, \frac{e^{-2z}|\arctan(\nu(z))|}{12(50+z)(1+|\arctan(\nu(z))|)}\right]$$

for all $z \in [0,1]$. In the next stage, we regard an increasing u.s.c. mapping $\psi : [0,\infty) \to [0,\infty)$ by $\psi(z) = \frac{z}{3}$ for all $z > 0$. It is obvious that $\liminf_{z\to\infty}(z - \psi(z)) > 0$ and $\psi(z) < z$ for all $z > 0$. At this moment, for any $\nu_1, \nu_2 \in \mathbb{V}$, we have

$$\begin{aligned}
&\mathbb{H}_{d_{\mathbb{V}}}\big(\mathfrak{Q}_*\big(z, \nu_1(z)\big), \mathfrak{Q}_*\big(z, \nu_2(z)\big)\big) \\
&\quad\le \frac{e^{-2z}}{12(50+z)}\big(\big|\arctan\big(\nu_1(z)\big) - \arctan\big(\nu_2(z)\big)\big|\big) \\
&\quad\le \frac{e^{-2z}}{12(50+z)}\big(\big|\nu_1(z) - \nu_2(z)\big|\big) \\
&\quad= \frac{e^{-2z}}{4(50+z)}\psi\big(\big|\nu_1(z) - \nu_2(z)\big|\big) \\
&\quad\le \varrho(z)\psi\big(\big|\nu_1(z) - \nu_2(z)\big|\big)\frac{1}{\hat{\mathbb{O}}},
\end{aligned}$$

where

$$\hat{\mathbb{O}} = \left[\frac{1}{\Gamma(\xi+1)} + \frac{s_2^*}{\Gamma(\xi)} + \frac{3(s_1^* + s_2^* + 1)\Gamma(4-\xi)}{2(2-\xi)}\right]\|\varrho\| \simeq 0.0399152785,$$

and we obtain $\varrho \in \mathcal{C}([0,1],[0,\infty))$ given by $\varrho(z) = \frac{e^{-2z}}{4(50+z)}$ for all z. In this phase, we get $\|\varrho\| = \sup_{z\in[0,1]} |\varrho(z)| = 0.005$. Finally, we introduce the multifunction $\mathfrak{F} : \mathbb{V} \to \mathcal{P}(\mathbb{V})$ by

$$\mathfrak{F}(\nu) = \big\{\vartheta \in \mathbb{V} : \text{ there exists } \hbar \in (\mathfrak{S}\mathfrak{C}\mathfrak{L})_{\mathfrak{Q}_*,\nu} \text{ s.t. } \vartheta(z) = \kappa(z), \forall z \in [0,1]\big\},$$

where

$$\begin{aligned}
\kappa(z) = {}& \frac{1}{\Gamma(1.5)}\int_0^z (z-\varsigma)^{1.5-1}\hbar(\varsigma)\,\mathrm{d}\varsigma + \frac{0.4}{\Gamma(0.5)}\int_0^1 (1-\varsigma)^{1.5-2}\hbar(\varsigma)\,\mathrm{d}\varsigma \\
& - \frac{(0.7+z)\Gamma(2.5)}{0.5}\int_0^1 \hbar(\varsigma)\,\mathrm{d}\varsigma + \frac{(0.7+z)\Gamma(2.5)}{0.5}\int_0^1\int_0^\varsigma \hbar(\tau)\,\mathrm{d}\tau\,\mathrm{d}\varsigma.
\end{aligned}$$

Thus we observe that all conditions of Theorem 15 are valid. Hence the fractional inclusion Lio–CapBvp (17) has a solution.

6 Conclusion

In this paper, we designed a new Liouville–Caputo fractional boundary value problem with mixed integro-derivative boundary conditions. To obtain some criteria establishing the existence of solutions for the proposed problem, we used condensing operators and proved the main result with the help of the measure of noncompactness due to Kuratowski. Next, the continuous dependence of solutions was checked by utilizing the generalized Gronwall inequality. In the next step, we considered an inclusion version of the suggested boundary value problem in which we derived existence results based on approximate endpoint property for the defined multifunction. In the final step, we prepared two illustrative examples to show the validity of the results. In spite of many papers published in this field in which the authors use some standard fixed point theorems for deriving the existence results, we here utilized two theoretical techniques contained in measure theory and that of endpoint simultaneously. This defined boundary value problem is an instance of abstract boundary value problem in which one can extend boundary conditions to multipoint multiorder multistrip conditions in the future.

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