Research Article

Erdal Karapınar* and V. M. L. Hima Bindu

Discussions on the almost $\mathcal{Z}$-contraction

https://doi.org/10.1515/math-2020-0174
received October 23, 2019; accepted April 28, 2020

Abstract: In this paper, we introduce a new contraction, namely, almost $\mathcal{Z}$ contraction with respect to $\zeta \in \mathcal{Z}$, in the setting of complete metric spaces. We proved that such contraction possesses a fixed point and the given theorem covers several existing results in the literature. We consider an example to illustrate our result.

Keywords: almost $\mathcal{Z}$ contraction, simulation function, almost contraction, α admissible, E contraction

MSC 2010: 55M20, 54H25, 47H10

1 Introduction and preliminaries

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ represents the set of positive integers. As usual $\mathbb{R}$ indicates the set of all real numbers. Furthermore, we set $\mathbb{R}_0^+ := [0, \infty)$.

Definition 1.1. (See [1]). A simulation function is a mapping $\zeta: [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_2) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0. \tag{1}$$

Observe first that (ζ_1) implies

$$\zeta(t, t) < 0 \quad \text{for all} \quad t > 0. \tag{2}$$

Indeed, this simulation function is obtained by the abstraction of the Banach contraction mapping principle. We underline that in [1], there was an additional axiom $\zeta(0, 0) = 0$. Since it is a superfluous condition, we omit it. Throughout the paper, the letter $\mathcal{Z}$ denotes the family of all functions $\zeta: [0, \infty) \times [0, \infty) \to \mathbb{R}$. A function $\zeta(t, s) := ks - t$, where $k \in [0, 1)$ for all $s, t \in [0, \infty)$, is an instantaneous example of a simulation function. For further and more interesting examples, we refer e.g. [1–6] and related references therein. In particular, in [7,8] the simulation functions work for establishing also common fixed points and coincidence points, both in a metric space and in a partial metric space.

We say that a self-mapping f, defined on a metric space (X, d), is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ [1], if

$$\zeta(d(fx, fy), d(x, y)) \geq 0 \quad \text{for all} \quad x, y \in X. \tag{3}$$

*** Corresponding author: Erdal Karapınar,** Department of Medical Research, China Medical University Hospital, China Medical University, 40402, Taichung, Taiwan, e-mail: karapinar@mail.cmuh.org.tw; Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey, e-mail: erdalkarapinar@yahoo.com

V. M. L. Hima Bindu: Department of Mathematics, Koneru Lakshmaiah Educational Foundation, Vaddeswaram, Guntur 522 502, Andhra Pradesh, India, e-mail: v.m.l.himabindu@gmail.com

The following is the main result of [1]:

Theorem 1.1. *Every $\mathcal{Z}$-contraction on a complete metric space has a unique fixed point.*

It is clear that the immediate example $\zeta(t, s) := ks - t$ is obtained by abstraction of the Banach contraction mapping principle. In other words, with this function $\zeta(t, s) := ks - t$, Theorem 1.1 yields the Banach contraction mapping principle.

Lemma 1.1. [9] *Let (X, d) be a metric space and let $\{p_n\}$ be a sequence in X such that $d(p_{n+1}, p_n)$ is non-increasing and that*

$$\lim_{n\to\infty} d(p_{n+1}, p_n) = 0.$$

If $\{p_{2n}\}$ is not a Cauchy sequence then there exist a $\delta > 0$ and two strictly increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to δ when $k \to \infty$:

$$d\left(p_{2m_k}, p_{2n_k}\right),\ \ d\left(p_{2m_k}, p_{2n_{k+1}}\right),\ \ d\left(p_{2m_{k-1}}, p_{2n_k}\right),\ \ d\left(p_{2m_{k-1}}, p_{2n_{k+1}}\right),\ \ d\left(p_{2m_{k+1}}, p_{2n_{k+1}}\right).$$

One of the interesting notions, α-admissibility was introduced by Samet-Vetro-Vetro [10], see also [11]. This study, which attracts the attention of many researchers, has been developed and generalized in many respects. In particular, [12,13], the author depicts applications of fixed point methodologies to the solution of a first-order periodic differential problem, converting such a problem into an integral equation. Moreover, in [14], the authors prove an existence theorem producing a periodic solution of some non-linear integral equations, using the Krasnoselskii-Schaefer-type method and technical assumptions.

Definition 1.2. Let f be a self-mapping on a non-empty set X, and $\alpha\colon X \times X \to [0, \infty)$ be mapping. We say that f is extended-α-admissible if, for all $x, y \in X$, we have

$$\alpha(x, fx) \geq 1 \text{ implies } \alpha(fx, f^{1+p}x) \geq 1, \quad \text{for all } p \in \mathbb{N}. \tag{4}$$

In some sources, f is called α-admissible if we let $p = 1$ in (4). On the other hand, if f is extended-α-admissible, then we can conclude that

$$\alpha(f^n x, f^m x) \geq 1 \quad \text{for all } m, n \in \mathbb{N} \text{ with } m > n. \tag{5}$$

Indeed, it is a straightforward conclusion. First, we, recursively, get that $\alpha(f^{n-1}x, f^n x) \geq 1$ and then we observe $\alpha(f^n x, f^{n+p}x) \geq 1$ by applying the extended-α-admissibility of f, where we let $m = n + p > n$.

2 Main results

Definition 2.1. Let f be a self-mapping, defined on a metric space (X, d), and $\alpha\colon X \times X \to [0, \infty)$ be a function. Here, $f\colon X \to X$ is called an almost-$\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$ if there exists $\zeta \in \mathcal{Z}$, and $\beta \in G$, and $L \geq 0$ such that for all $x, y \in X$

$$\alpha(x, y) \geq 1 \Rightarrow \zeta(d(fx, fy), K(x, y)) \geq 0, \tag{6}$$

where

$$K(x, y) := \beta(E(x, y))E(x, y) + LN(x, y),$$

with

$$E(x, y) = d(x, y) + |d(x, fx) - d(y, fy)|$$

and

$$N(x, y) = \min\{d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}.$$

Now we prove our main result.

Theorem 2.1. *Suppose that a self-mapping f, defined on a complete metric space (X, d), forms an almost-$\mathcal{Z}$-contraction. Furthermore, we suppose, for all $x, y \in X$, that*
(i) *f is an extended-α-admissible pair*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$*
(iii) *either*
(iiia) *f is continuous,*
or
(iiib) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and as $n \to \infty$, then there exists a subsequence $\{x_n(k)\}$ of x_n such that $\alpha(x_n(k), x) \geq 1$ for all k.*
Then, f has a fixed point.

Proof. On account of (ii), there is a point $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. By starting this initial point, we shall build a sequence $\{x_n\}_{n \in \mathbb{N}_0}$ in X by $x_{n+1} = fx_n$ for all $n \geq 0$. Throughout the proof, without loss of generality, we shall assume that

$$x_{n+1} \neq x_n \quad \text{for all } n \in N.$$

Indeed, in the opposite case, where $x_{n_0} = x_{n_0+1}$ for some $n_0 \in N$, we conclude that x_0 is the desired fixed point, i.e., $x_{n_0+1} = fx_{n_0} = x_{n_0}$. This implies a trivial solution that is not interesting and that is why we exclude this case.

On the other hand, by taking both (i) and (ii) into account, we observe that

$$\alpha(x_0, fx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq 1,$$

continuing in this way we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in N_0.$$

Furthermore, by regarding (5), we derive that

$$\alpha(x_n, x_m) \geq 1 \quad \text{for all } n, m \in N_0 \text{ with } m > n.$$

As a first step, we want to conclude that $\{d(x_n, x_{n+1})\}$ is non-increasing. Suppose, in contrast, that $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$. Since $\alpha(x_n, x_{n+1}) \geq 1$, we find that

$$0 \leq \zeta(d(fx_n, fx_{n+1}), K(x_n, x_{n+1})) = \zeta(d(x_{n+1}, x_{n+2}), K(x_n, x_{n+1})) < K(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2}), \tag{7}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leq K(x_n, x_{n+1}) = \beta(E(x_n, x_{n+1}))E(x_n, x_{n+1}) + LN(x_n, x_{n+1}),$$

where

$$\begin{aligned} N(x_n, x_{n+1}) &= \min\{d(x_n, fx_n), d(x_{n+1}, fx_{n+1}), d(x_n, fx_{n+1}), d(x_{n+1}, fx_n)\} \\ &= \min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} E(x_n, x_{n+1}) &= d(x_n, x_{n+1}) + |d(x_n, fx_n) - d(x_{n+1}, fx_{n+1})| \\ &= d(x_n, x_{n+1}) + |d(x_n, x_{n+1}) - d(x_{n+1}, x_{n+2})| \\ &= d(x_{n+1}, x_{n+2}). \end{aligned}$$

Hence, inequality (7) turns into

$$d(x_{n+1}, x_{n+2}) \leq \beta(E(x_n, x_{n+1}))E(x_n, x_{n+1}) = \beta(d(x_{n+1}, x_{n+2}))d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_{n+2}), \tag{8}$$

a contradiction. Consequently, we deduce that $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$, for each n. Since the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing.

As a next step, $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Indeed, since $\{d(x_n, x_{n+1})\}$ is non-increasing and bounded below, we conclude that it converges to some non-negative real numbers, say r,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

It is evident that

$$\lim_{n\to\infty} E(x_n, x_{n+1}) = r.$$

We assert that $r = 0$. Suppose, in contrast, that $r \neq 0$. Then, from Eq. (8) and (ζ_2), and taking limit as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \beta(E(x_n, x_{n+1})) = 1 \Rightarrow \lim_{n\to\infty} E(x_n, x_{n+1}) = 0.$$

Attendantly, $r = 0$ and also

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{9}$$

In what follows, we claim that sequence $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ and sequences $\{x_{n_k}\}$, $\{x_{m_k}\}$; $n_k > m_k > k$ such that

$$d\left(x_{m_k}, x_{n_k}\right) \geq \varepsilon, \tag{10}$$

$$d\left(x_{m_k}, x_{n_k-1}\right) < \varepsilon. \tag{11}$$

Now take $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (6), we have

$$\alpha\left(x_{m_k-1}, x_{n_k-1}\right) \geq 1 \quad \text{for all } k$$

implies

$$0 \leq \zeta\left(d\left(fx_{m_k-1}, fx_{n_k-1}\right), K\left(x_{m_k-1}, x_{n_k-1}\right)\right) < K\left(x_{m_k-1}, x_{n_k-1}\right) - d\left(fx_{m_k-1}, fx_{n_k-1}\right), \tag{12}$$

where

$$N\left(x_{m_k-1}, x_{n_k-1}\right) = \min\left\{d\left(x_{m_k-1}, x_{m_k}\right), d\left(x_{n_k-1}, x_{n_k}\right), d\left(x_{m_k-1}, x_{n_k}\right), d\left(x_{n_k-1}, x_{m_k}\right)\right\}$$

and

$$E\left(x_{m_k-1}, x_{n_k-1}\right) = d\left(x_{m_k-1}, x_{n_k-1}\right) + \left|d\left(x_{m_k-1}, x_{m_k}\right) - d\left(x_{n_k-1}, x_{n_k}\right)\right|.$$

Due to Lemma 1.1, we have

$$\lim_{k\to\infty} d\left(x_{m_k}, x_{n_k}\right) = \lim_{k\to\infty} d\left(x_{m_k-1}, x_{n_k-1}\right) = \lim_{k\to\infty} d\left(x_{m_k}, x_{n_k-1}\right) = \lim_{k\to\infty} d\left(x_{m_k-1}, x_{n_k}\right) = \varepsilon. \tag{13}$$

Since

$$E\left(x_{m_k-1}, x_{n_k-1}\right) = d\left(x_{m_k-1}, x_{n_k-1}\right) + \left|d\left(x_{m_k-1}, x_{m_k}\right) - d\left(x_{n_k-1}, x_{n_k}\right)\right|$$

using (13) and (9), we have

$$\lim_{k\to\infty} E\left(x_{m_k-1}, x_{n_k-1}\right) = \varepsilon.$$

Let $t_{n_k} = d(x_{m_k}, x_{n_k})$ and $s_{n_k} = K(x_{m_k-1}, x_{n_k-1})$ we have $\lim_{k\to\infty} s_{n_k} = \lim_{k\to\infty} t_{n_k} = \varepsilon$ and letting $k \to \infty$ in (12)

$$0 \leq \limsup_{k\to\infty} \zeta\left(d\left(x_{m_k}, x_{n_k}\right), K\left(x_{m_k-1}, x_{n_k-1}\right)\right) = \limsup_{k\to\infty} \zeta\left(d\left(x_{m_k}, x_{n_k}\right), K\left(x_{m_k-1}, x_{n_k-1}\right)\right) = \limsup_{k\to\infty} \zeta\left(t_{n_k}, s_{n_k}\right). \tag{14}$$

Then, by (13), (14) and keeping (ζ_2) in mind, we have

$$0 \leq \limsup_{k\to\infty} \zeta\left(t_{n_k}, s_{n_k}\right) < \limsup_{k\to\infty} \left[s_{n_k} - t_{n_k}\right] \to [\varepsilon - \varepsilon] = 0,$$

a contradiction. As a result, our claim is correct and $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete metric space, the sequence converges to some point $u \in X$ as $n \to \infty$

$$\lim_{n\to\infty} d(x_n, u) = \lim_{n\to\infty} d(x_{n+1}, u) = 0. \tag{15}$$

Now we shall show that $fu = u$.

Suppose we have (iiia). Since f is continuous, we derive the desired results obviously, that is,

$$fu = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

Suppose we have (iiib). We shall use the method of *reductio ad absurdum*. Suppose, in contrast, that $fu \neq u$, that is, $d(u, fu) > 0$. By (iiib), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, u) \geq 1$ for all k. It implies that

$$0 \leq \zeta\left(d\left(fx_{n_k-1}, fu\right), K\left(x_{n_k-1}, u\right)\right) < K\left(x_{n_k-1}, u\right) - d\left(fx_{n_k-1}, fu\right), \tag{16}$$

where

$$N\left(x_{n_k-1}, u\right) = \min\left\{d\left(x_{n_k-1}, x_{n_k}\right), d(u, fu), d\left(u, x_{n_k}\right), d\left(fu, x_{n_k-1}\right)\right\},$$
$$\lim_{k\to\infty} N\left(x_{n_k-1}, u\right) = \min\{0, d(u, fu), 0, d(u, fu)\} = 0,$$

and

$$E\left(x_{n_k-1}, u\right) = d\left(x_{n_k-1}, u\right) + \left|d\left(x_{n_k-1}, x_{n_k}\right) - d(u, fu)\right|,$$
$$\lim_{k\to\infty} E\left(x_{n_k-1}, u\right) = 0 + |0 - d(u, fu)| = d(u, fu).$$

By letting $k \to \infty$ in (16), together with the observation above, we have

$$\begin{aligned}0 &\leq \limsup_{k\to\infty} \zeta\left(d\left(fx_{n_k-1}, fu\right), K\left(x_{n_k-1}, u\right)\right) < \limsup_{k\to\infty} K\left(x_{n_k-1}, u\right) - d\left(fx_{n_k-1}, fu\right)\\ &< \limsup_{k\to\infty} \beta\left(E\left(x_{n_k-1}, u\right)\right)E\left(x_{n_k-1}, u\right) - d\left(fx_{n_k-1}, fu\right) < d(u, fu) - d(u, fu) = 0\end{aligned}$$

is a contradiction. Hence, u is a fixed point of f. □

Theorem 2.2. *In addition to the axioms of Theorem 2.1, we assume that*
(iv) *for all $p, q \in S_f(X)$ we have $\alpha(p, q) \geq 1$,*
where $S_f(X) \subset X$ is the set of all fixed points of f. Then, f has a unique fixed point.

Proof. We shall use the method of *reductio ad absurdum* to reach our goal. Suppose that there are two distinct fixed points of f, that is, namely, $p, q \in S_f(X)$ with $fp = p \neq q = fq$. On account of the additional assumption (iv), we have $\alpha(p, q) \geq 1$, which implies

$$0 \leq \zeta(d(fp, fq), K(p, q)) < K(p, q) - d(fp, fq), \tag{17}$$

where

$$K(p, q) := \beta(E(p, q))E(p, q) + LN(p, q),$$

with

$$E(p, q) = d(p, q) + |d(p, fp) - d(q, fq)| = d(p, q)$$

and

$$N(p, q) = \min\{d(p, fp), d(q, fq), d(p, fq), d(q, fp)\} = 0.$$

Hence, expression (17) turns into

$$0 \le \zeta(d(fp, fq), K(p, q)) < K(p, q) - d(fp, fq) = 0, \tag{18}$$

a contradiction. This completes the proof. □

Example 2.1. Let $X = [0, 1]$ be endowed with metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $\zeta(t, s) = s - t$ and considering β: $[0, \infty) \to [0, 1)$, $\beta(t) = \frac{1}{1+t}$ for all $t \ge 0$ and $L \ge 0$. Let $f: X \to X$ be defined by $fx = \frac{x}{2}$ for all $x \in [0, 1]$ and $\alpha: X \times X \to [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Since $\alpha(x, y) = 1$, $x, y \in [0, 1]$ implies

$$\begin{aligned}
\zeta(d(fx, fy), K(x, y)) &= K(x, y) - d(fx, fy) = \beta(E(x, y))E(x, y) + LN(x, y) - \frac{1}{2}|x - y| \\
&= \frac{E(x, y)}{1 + E(x, y)} + LN(x, y) - \frac{1}{2}|x - y| \le \frac{\frac{3}{2}d(x, y)}{1 + \frac{3}{2}d(x, y)} + LN(x, y) - \frac{1}{2}|x - y| \\
&= \frac{\frac{3}{2}|x - y|}{1 + \frac{3}{2}|x - y|} + LN(x, y) - \frac{1}{2}|x - y| \ge 0.
\end{aligned}$$

Therefore, f is almost-$\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. Hence, all the assumptions of Theorem 2.2 are satisfied, and hence f has a unique fixed point.

3 Immediate consequences

The first conclusion of our main results is the following.

Theorem 3.1. *Let f be a self-mapping, defined on a complete metric space (X, d), and $\alpha: X \times X \to [0, \infty)$ be a function. Suppose that there exists $\zeta \in \mathcal{Z}$, and $\beta \in G$, and $L \ge 0$ such that for all $x, y \in X$*

$$\alpha(x, y) \ge 1 \Rightarrow \zeta(d(fx, fy), \beta(d(x, y))d(x, y) + LN(x, y)) \ge 0, \tag{19}$$

where

$$N(x, y) = \min\{d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}.$$

Furthermore, we suppose, for all $x, y \in X$, that

(i) *f is an extended-α-admissible pair;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \ge 1$;*
(iii) *either*
  (iiia) *f is continuous,*
  or
  (iiib) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and as $n \to \infty$, then there exists a subsequence $\{x_n(k)\}$ of x_n such that $\alpha(x_n(k), x) \ge 1$ for all k;*
(iv) *for all $p, q \in S_f(X)$ we have $\alpha(p, q) \ge 1$,*

where $S_f(X) \subset X$ is the set of all fixed points of f. Then, f has a fixed point.

We skip the proof since it is the analog of the proof of Theorem 2.2 (and hence Theorem 2.1).

In the next theorem, we omit the auxiliary function $\alpha: X \times X \to [0, \infty)$ to get a result in the standard metric spaces.

Theorem 3.2. *Let f be a self-mapping, defined on a complete metric space (X, d). Suppose that there exists $\zeta \in \mathcal{Z}$, and $\beta \in G$, and $L \geq 0$ such that for all $x, y \in X$*

$$\zeta\left(d(fx, fy), K(x, y)\right) \geq 0, \tag{20}$$

where $K(x, y)$, $E(x, y)$ and $N(x, y)$ are defined as in Theorem 2.1. Then, f has a unique fixed point.

Proof. It is sufficient to set $\alpha(x, y) = 1$ for all $x, y \in X$, in Theorem 2.2 (and hence Theorem 2.1). □

The trend of searching a fixed point on the partially ordered set was initiated by Turinici [15] in 1986. We shall collect some basic notions here. Let f be a self-mapping on a partially ordered set $(X, \preccurlyeq)$. A mapping f is called non-decreasing with respect to $\preccurlyeq$ if

$$x, y \in X, x \preccurlyeq y \Rightarrow fx \preccurlyeq fy.$$

Analogously, a sequence $\{x_n\} \subset X$ is called non-decreasing with respect to $\preccurlyeq$ if $x_n \preccurlyeq x_{n+1}$ for all n. In addition, suppose that d is a metric on X. The tripled $(X, \preccurlyeq, d)$ is regular if for every non-decreasing sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ as $n \to \infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preccurlyeq x$ for all k.

Theorem 3.3. *Let f be a self-mapping, defined on a complete metric space (X, d) endowed with a partial order $\preccurlyeq$ on X. Suppose that there exists $\zeta \in \mathcal{Z}$, and $\beta \in G$, and $L \geq 0$ such that for all $x, y \in X$ with $x \preccurlyeq y$*

$$\zeta\left(d(fx, fy), K(x, y)\right) \geq 0,$$

where $K(x, y)$, $E(x, y)$ and $N(x, y)$ are defined as in Theorem 2.1. Suppose also that the following conditions hold:
(i) there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$;
(ii) f is continuous or $(X, \preccurlyeq, d)$ is regular.
Then, f has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preccurlyeq z$ and $y \preccurlyeq z$, we have uniqueness of the fixed point.

Proof. It is sufficient to define the mapping $\alpha\colon X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preccurlyeq y \text{ or } x \succcurlyeq y, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly, f is an almost-$\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$. From condition (i), we have $\alpha(x_0, fx_0) \geq 1$. Moreover, for all $x, y \in X$, from the monotone property of f, we have

$$\alpha(x_0, fx_0) \geq 1 \Leftrightarrow x_0 \preccurlyeq fx_0 \Rightarrow fx_0 \preccurlyeq f^2x_0 \Leftrightarrow \alpha(fx_0, f^2x_0) \geq 1.$$

The rest is satisfied in a straightway. □

Let Ψ be the collection of all auxiliary functions and $\phi\colon [0, \infty) \to [0, \infty)$ be continuous functions with $\phi(t) = 0$ if, and only if, $t = 0$.

Theorem 3.4. *Let f be a self-mapping, defined on a complete metric space (X, d), and $\alpha\colon X \times X \to [0, \infty)$ be a function. Suppose that there exists $\phi_1, \phi_2 \in \Phi$ with $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$, and $\beta \in G$, and $L \geq 0$ such that for all $x, y \in X$*

$$\alpha(x, y) \geq 1 \Rightarrow \phi_2(d(fx, fy)) \leq \phi_1(K(x, y)), \tag{21}$$

where $K(x, y)$, $E(x, y)$ and $N(x, y)$ are defined as in Theorem 2.1. Furthermore, we suppose, for all $x, y \in X$, that
(i) f is an extended-α-admissible pair;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;
(iii) either
(iiia) f is continuous,

or

(iiib) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and as* $n \to \infty$, *then there exists a subsequence* $\{x_n(k)\}$ *of* x_n *such that* $\alpha(x_n(k), x) \geq 1$ *for all* k;

(iv) *for all* $p, q \in S_f(X)$ *we have* $\alpha(p, q) \geq 1$,

where $S_f(X) \subset X$ *is the set of all fixed points of* f. *Then,* f *has a fixed point.*

Proof. Let $\zeta(t, s) = \phi_1(s) - \phi_2(t)$ for all $t, s \geq 0$, where $\phi_1(t) < t \leq \phi_2(t)$ for all $t > 0$. It is clear that $\zeta \in \mathcal{Z}$, see, e.g. [1,2]. Thus, the desired results follow from Theorem 2.2. □

Theorem 3.5. *Let* f *be a self-mapping, defined on a complete metric space* (X, d), *and* $\alpha\colon X \times X \to [0, \infty)$ *be a function. Suppose that there exists* $\phi\Phi$, *and* $\beta \in G$, *and* $L \geq 0$ *such that for all* $x, y \in X$

$$\alpha(x, y) \geq 1 \Rightarrow d(fx, fy) \leq K(x, y) - \phi(K(x, y)), \tag{22}$$

where $K(x, y)$, $E(x, y)$ *and* $N(x, y)$ *are defined as in Theorem 2.1. Furthermore, we suppose, for all* $x, y \in X$, *that*

(i) f *is an extended-*α*-admissible pair;*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$;

(iii) *either*

(iiia) f *is continuous,*

or

(iiib) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and as* $n \to \infty$, *then there exists a subsequence* $\{x_n(k)\}$ *of* x_n *such that* $\alpha(x_n(k), x) \geq 1$ *for all* k;

(iv) *for all* $p, q \in S_f(X)$ *we have* $\alpha(p, q) \geq 1$,

where $S_f(X) \subset X$ *is the set of all fixed points of* f. *Then,* f *has a fixed point.*

Proof. Let $\zeta(t, s) = s - \phi(s) - t$ for all $t, s \geq 0$. It is clear that $\zeta \in \mathcal{Z}$, see, e.g. [1–3]. Thus, the desired results follow from Theorem 2.2. □

Theorem 3.6. *Let* f *be a self-mapping, defined on a complete metric space* (X, d), *and* $\alpha\colon X \times X \to [0, \infty)$ *be a function. Suppose that there exists* $\phi_1, \phi_2 \in \Phi$ *with* $\phi_1(t) < t \leq \phi_2(t)$ *for all* $t > 0$, *and* $\beta \in G$, *and* $L \geq 0$ *such that for all* $x, y \in X$

$$\alpha(x, y) \geq 1 \Rightarrow d(fx, fy) \leq \int_0^{K(x,y)} \mu(u)\mathrm{d}u, \tag{23}$$

where $K(x, y)$, $E(x, y)$ *and* $N(x, y)$ *are defined as in Theorem 2.1. Furthermore, we suppose, for all* $x, y \in X$, *that*

(i) f *is an extended-*α*-admissible pair;*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$;

(iii) *either*

(iiia) f *is continuous,*

or

(iiib) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and as* $n \to \infty$, *then there exists a subsequence* $\{x_n(k)\}$ *of* x_n *such that* $\alpha(x_n(k), x) \geq 1$ *for all* k;

(iv) *for all* $p, q \in S_f(X)$ *we have* $\alpha(p, q) \geq 1$,

where $S_f(X) \subset X$ *is the set of all fixed points of* f. *Then,* f *has a fixed point.*

Proof. Let $\zeta(t, s) = s - \int_0^t \mu(u)\mathrm{d}u$ for all $t, s \geq 0$.

It is clear that $\zeta \in \mathcal{Z}$, see, e.g. [3,1,2]. Thus, the desired results follow from Theorem 2.2. □

Theorem 3.7. *Let f be a self-mapping, defined on a complete metric space (X, d), and $\alpha\colon X \times X \to [0, \infty)$ be a function, and $\beta \in G$, and $L \geq 0$. Suppose that there exist $\varphi\colon [0, \infty) \to [0, \infty)$ which is upper semi-continuous and such that $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$. Assume, for all $x, y \in X$, that*

$$\alpha(x, y) \geq 1 \Rightarrow d(fx, fy) \leq \varphi(K(x, y)), \tag{24}$$

where $K(x, y)$, $E(x, y)$ and $N(x, y)$ are defined as in Theorem 2.1. Furthermore, we suppose, for all $x, y \in X$, that

(i) *f is an extended-α-admissible pair;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(iii) *either*
 (iiia) *f is continuous*
 or
 (iiib) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and as $n \to \infty$, then there exists a subsequence $\{x_n(k)\}$ of x_n such that $\alpha(x_n(k), x) \geq 1$ for all k;*
(iv) *for all $p, q \in S_f(X)$ we have $\alpha(p, q) \geq 1$,*

where $S_f(X) \subset X$ is the set of all fixed points of f. Then, f has a fixed point.

Proof. Let $\zeta(t, s) = \varphi(s) - t$ for all $t, s \geq 0$. It is clear that $\zeta \in \mathcal{Z}$, see, e.g. [1–3]. Thus, the desired results follow from Theorem 2.2. □

Theorem 3.8. *Let $\{A_i\}_{i=1}^2$ be non-empty closed subsets of a complete metric space (X, d) and $T\colon Y \to Y$ be a given mapping, where $Y = A_1 \cup A_2$ with*

$$T(A_1) \subseteq A_2 \text{ and } T(A_2) \subseteq A_1. \tag{25}$$

Suppose that there exists $\zeta \in \mathcal{Z}$, and $\beta \in G$, and $L \geq 0$ such that for all $(x, y) \in A_1 \times A_2$

$$\zeta(d(fx, fy), K(x, y)) \geq 0, \tag{26}$$

where $K(x, y)$, $E(x, y)$ and $N(x, y)$ are defined as in Theorem 2.1. Then, T has a unique fixed point that belongs to $A_1 \cap A_2$.

Proof. (Y, d) is a complete metric space since both A_1 and A_2 are closed subsets of the complete metric space (X, d). We construct the mapping $\alpha\colon Y \times Y \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in (A_1 \times A_2) \cup (A_2 \times A_1), \\ 0 & \text{otherwise.} \end{cases}$$

From (26) and the definition of α, we can write

$$\zeta(d(fx, fy), K(x, y)) \geq 0,$$

for all $x, y \in Y$. Thus, T is an almost $\mathcal{Z}$-contraction.

Let $(x, y) \in Y \times Y$ such that $\alpha(x, y) \geq 1$. If $(x, y) \in A_1 \times A_2$, from (25), $(Tx, Ty) \in A_2 \times A_1$, which yields $\alpha(Tx, Ty) \geq 1$. If $(x, y) \in A_2 \times A_1$, (25), $(Tx, Ty) \in A_1 \times A_2$, which yields $\alpha(Tx, Ty) \geq 1$. Consequently, in all cases, we find $\alpha(Tx, Ty) \geq 1$. It yields T is α-admissible.

On account of (25), for any $a \in A_1$, we have $(a, Ta) \in A_1 \times A_2$, which implies that $\alpha(a, Ta) \geq 1$.

Now, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. This implies from the definition of α that

$$(x_n, x_{n+1}) \in (A_1 \times A_2) \cup (A_2 \times A_1), \quad \text{for all } n.$$

It is clear that $(A_1 \times A_2) \cup (A_2 \times A_1)$ is a closed set, and hence we find

$$(x, x) \in (A_1 \times A_2) \cup (A_2 \times A_1),$$

which implies that $x \in A_1 \cap A_2$. Attendantly, the definition of α implies that $\alpha(x_n, x) \geq 1$ for all n.

Finally, let $x, y \in \mathrm{Fix}(T)$. Taking (25) into account, we find $x, y \in A_1 \cap A_2$. So, for any $z \in Y$, we have $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Thus, the criteria (iv) is provided.

Since all axioms of Theorem 2.1 are fulfilled, we conclude that T has a unique fixed point in $A_1 \cap A_2$ (25). □

4 Conclusion

It is a well-known fact that the auxiliary function α is a good tool to combine three different theorems, in three distinct constructions: the structure of the standard metric space, the structure of a metric space endowed with a partial metric space and the structure of cyclic mappings via closed subsets of a metric space. Indeed, Theorems 3.2, 3.3 and 3.8 are concrete examples for these constructions derived from Theorem 2.1, respectively. For more details, see e.g. [11]. In particular, we may use these approaches to Theorems 3.1, 3.4, 3.5, 3.6 and 3.7 to get different variants in the aforementioned three structures. We avoid to put all these consequences regarding the length of the paper and the verbatim of the proofs. Moreover, by using the interesting more simulation functions (see, e.g. [1–4,8]), more consequences of Theorems 2.1 and 3.1 can be derived. As a result, our main results combine and cover several existing results in the literature. Since these results are easily predictable from the content and since the main ideas are already mentioned, we avoid to be put all possible consequences.

Acknowledgments: The authors thank anonymous referees for their remarkable comments, suggestion and ideas that help to improve this paper.

References

[1] F. Khojasteh, S. Shukla, and S. Radenović, *A new approach to the study of fixed point theorems via simulation functions*, Filomat **29** (2015), no. 6, 1189–1194.

[2] A.F. Roldán-López-de-Hierro, E. Karapınar, C. Roldán-López-de-Hierro, and J. Martínez-Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. **275** (2015), 345–355.

[3] H. H. Alsulami, E. Karapınar, F. Khojasteh, and A. F. Roldán-López-de-Hierro, *A proposal to the study of contractions in quasi-metric spaces*, Discrete Dyn. Nat. Soc. **2014** (2014), 269286.

[4] A. S. Alharbi, H. Alsulami, and E. Karapınar, *On the power of simulation and admissible functions in metric fixed point theory*, J. Funct. Spaces **2017** (2017), 2068163.

[5] B. Alqahtani, A. Fulga, and E. Karapınar, *Fixed point results on Δ-symmetric quasi-metric space via simulation function with an application to Ulam stability*, Mathematics **6** (2018), no. 10, 208.

[6] H. Aydi, A. Felhi, E. Karapınar, and F. A. Alojail, *Fixed points on quasi-metric spaces via simulation functions and consequences*, J. Math. Anal. **9** (2018), no. 2, 10–24.

[7] F. Vetro, *Points of coincidence that are zeros of a given function*, Results Math. **74** (2019), no. 4, 159.

[8] F. Vetro, *Fixed point results for ordered S-G-contractions in ordered metric spaces*, Nonlinear Anal. Model. Control **23** (2018), no. 2, 269–283.

[9] S. Radenovic, Z. Kadelburg, D. Jandrlic, and A. Jandrlic, *Some results on weakly contractive maps*, Bull. Iranian Math. Soc. **38** (2012), no. 3, 625–645.

[10] B. Samet, C. Vetro, and P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal. TMA **75** (2012), no. 4, 2154–2165.

[11] E. Karapınar and B. Samet, *Generalized α-ψ-contractive type mappings and related fixed point theorems with applications*, Abstract Appl. Anal. **2012** (2012), 793486.

[12] F. Vetro, *Fixed point for α-Θ-Φ-contractions and first-order periodic differential problem*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **113** (2019), no. 3, 1823–1837.

[13] F. Vetro, *μ-Contractions in ordered metric spaces endowed with a ω_0-distance*, Filomat **32** (2018), no. 10, 3725–3730.

[14] C. Vetro and D. Wardowski, *Krasnoselskii-Schaefer type method in the existence problems*, Topol. Methods Nonlinear Anal. **54** (2019), no. 1, 131–139.

[15] M. Turinici, *Abstract comparison principles and multivariable Gronwall-Bellman inequalities*, J. Math. Anal. Appl. **117** (1986), 100–127.