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Research article

Efficient computations for weighted generalized proportional fractional operators with respect to a monotone function

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Abstract: In this paper, we propose a new framework of weighted generalized proportional fractional integral operator with respect to a monotone function Ψ, we develop novel modifications of the aforesaid operator. Moreover, contemplating the so-called operator, we determine several notable weighted Chebyshev and Grüss type inequalities with respect to increasing, positive and monotone functions Ψ by employing traditional and forthright inequalities. Furthermore, we demonstrate the applications of the new operator with numerous integral inequalities by inducing assumptions on ω and Ψ verified the superiority of the suggested scheme in terms of efficiency. Additionally, our consequences have a potential association with the previous results. The computations of the proposed scheme show that the approach is straightforward to apply and computationally very user-friendly and accurate.

Keywords: weighted generalized proportional fractional integrals; weighted Chebyshev inequality; Grüss type inequality; Cauchy Schwartz inequality
Mathematics Subject Classification: 26A33, 26A51, 26D07, 26D10, 26D15

1. Introduction

In recent years, a useful extension has been proposed from the classical calculus by permitting derivatives and integrals of arbitrary orders is known as fractional calculus. It emerged from a celebrated logical conversation between Leibniz and L'Hopital in 1695 and was enhanced by different scientists like Laplace, Abel, Euler, Riemann, and Liouville [1]. Fractional calculus has gained popularity on the account of diverse applications in various areas of science and technology [2–4]. The concept of this new calculus was applied in several distinguished areas previously with excellent developments in the frame of novel approaches and posted scholarly papers, see [5–18]. Various notable generalized fractional integral operators such as the Riemann-Liouville, Hadamard, Caputo, Marichev-Saigo-Maeda, Riez, the Gaussian hypergeometric operators and so on, their attempts helpful for researchers to recognize the real world phenomena. Therefore, the Caputo and Riemann-Liouville was the most used fractional operators having singular kernels. It is remarkable that all the above mentioned operators are the particular cases of the operators investigated by Jarad et al. [19]. The utilities to weighted generalized fractional operators are undertaking now.

Adopting the excellency of the above work, we introduce a new weighted framework of generalized proportional fractional integral operator with respect to monotone function Ψ. Also, some new characteristics of the aforesaid operator are apprehended to explore new ideas to amplify the fractional operators and acquire fractional integral inequalities via generalized fractional operators (see Remark 2 and 3 below).

Recently, by employing the fractional integral operators, several researchers have established a bulk of fractional integral inequalities and their variant forms with fertile applications. These sorts of speculations have noteworthy applications in fractional differential/difference equations and fractional Schrödinger equations [20, 21]. By the use of Riemann-Liouville fractional integral operator, Belarbi and Dahmani [22] contemplated the subsequent integral inequalities as follows:

If f_1 and g_1 are two synchronous functions on $[0, \infty)$, then

$$\Omega^{\alpha}(f_1g_1)(\varkappa) \leq \frac{\Gamma(\alpha+1)}{\varkappa^{\alpha}}\Omega^{\alpha}(f_1)(\varkappa)\Omega^{\alpha}(g_1)(\varkappa) \tag{1.1}$$

and

$$\frac{\varkappa^{\alpha}}{\Gamma(\alpha+1)}\Omega^{\beta}(f_1g_1)(\varkappa) + \frac{\varkappa^{\beta}}{\Gamma(\beta+1)}\Omega^{\alpha}(f_1g_1)(\varkappa) \leq \Omega^{\alpha}(f_1)(\varkappa)\Omega^{\beta}(g_1)(\varkappa) + \Omega^{\beta}(f_1)(\varkappa)\Omega^{\alpha}(g_1)(\varkappa), \tag{1.2}$$

for all $\varkappa > 0, \alpha, \beta > 0$. Butt et al. [23], Rashid et al. [24] and Set et al. [25] established the fractional integral inequalities via generalized fractional integral operator having Raina's function, generalized K-fractional integral and Katugampola fractional integral inequalities similar to the variants (1.1) and (1.2), respectively. Here we should emphasize that, inequalities (1.1) and (1.2) are a remarkable instrument for reconnoitering plentiful scientific regions of investigation encompassing probability theory, statistical analysis, physics, meteorology, chaos and henceforth.

More general version of inequalities (1.1) and (1.2) proposed by Dahmani [26] by employing Riemann-Liouville fractional integral operator.

Let f_1 and g_1 be two synchronous functions on $[0, \infty)$ and let $r, s : [0, \infty) \to [0, \infty)$. Then

$$\Omega^{\alpha}\mathcal{P}(\varkappa)\Omega^{\alpha}(\mathcal{Q}f_1g_1)(\varkappa) + \Omega^{\alpha}\mathcal{Q}(\varkappa)\Omega^{\alpha}(\mathcal{P}f_1g_1)(\varkappa)$$

$$\geq \Omega^{\alpha}(\mathcal{Q}f_1)(\varkappa)\Omega^{\alpha}(\mathcal{P}g_1)(\varkappa)+\Omega^{\alpha}(\mathcal{P}f_1)(\varkappa)\Omega^{\alpha}(\mathcal{Q}g_1)(\varkappa) \tag{1.3}$$

and

$$\begin{aligned}&\Omega^{\alpha}\mathcal{P}(\varkappa)\Omega^{\beta}(\mathcal{Q}f_1g_1)(\varkappa)+\Omega^{\beta}\mathcal{Q}(\varkappa)\Omega^{\alpha}(\mathcal{P}f_1g_1)(\varkappa)\\ &\geq \Omega^{\alpha}(\mathcal{Q}f_1)(\varkappa)\Omega^{\beta}(\mathcal{P}g_1)(\varkappa)+\Omega^{\beta}(\mathcal{P}f_1)(\varkappa)\Omega^{\alpha}(\mathcal{Q}g_1)(\varkappa)\end{aligned} \tag{1.4}$$

for all $\varkappa > 0, \alpha, \beta > 0$. Chinchane and Pachpatte [27], Brahim and Taf [28] and Shen et al. [29] explored the Hadamard fractional integral inequalities, the fractional version of integral inequalities in two variable quantum deformation and the Riemann-Liouville fractional integral operator on time scale analysis coincide to variants (1.3) and (1.4), respectively.

Let us define the most distinguished Chebyshev functional [30]:

$$\mathfrak{T}(f_1,g_1)=\frac{1}{b_1-a_1}\int_{a_1}^{b_1}f_1(\varkappa)g_1(\varkappa)d\varkappa-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}f_1(\varkappa)d\varkappa\frac{1}{b_1-a_1}\int_{a_1}^{b_1}g_1(\varkappa)d\varkappa, \tag{1.5}$$

where f_1 and g_1 are two integrable functions on $[a_1, b_1]$. In [31], Grüss proposed the well-known generalization:

$$\left|\mathfrak{T}(f_1,g_1)\right|\leq\frac{1}{4}(\Phi-\phi)(\Upsilon-\gamma), \tag{1.6}$$

where f_1 and g_1 are two integrable functions on $[a_1, b_1]$ satisfying the assumptions

$$\phi\leq f_1(\varkappa)\leq\Phi,\qquad \gamma\leq g_1(\varkappa)\leq\Upsilon,\qquad \phi,\Phi,\gamma,\Upsilon\in\mathbb{R},\varkappa\in[a_1,b_1]. \tag{1.7}$$

The inequality (1.6) is known to be Grüss inequality. In recent years, the Grüss type integral inequality has been the subject of very active research. Mathematicians and scientists can see them in research papers, monographs, and textbooks devoted to the theory of inequalities [32–35] such as, Dragomir [36] demonstrated certain variants with the supposition of vectors and continuous mappings of selfadjoint operators in Hilbert space similar to (1.6). In this context, f_1 and g_1 are holding the assumptions (1.7), Dragomir [37] derived several functionals in two and three variable sense as follows:

$$\left|\mathfrak{S}(f_1,g_1,\mathcal{P})\right|\leq\frac{1}{4}(\Phi-\phi)(\Upsilon-\gamma)\Big(\int_{a_1}^{b_1}\mathcal{P}_1(\varkappa)d\varkappa\Big)^2, \tag{1.8}$$

where

$$\begin{aligned}\mathfrak{S}(f_1,g_1,\mathcal{P}) &= \frac{1}{2}\mathfrak{T}(f_1,g_1,\mathcal{P})\\ &=\int_{a_1}^{b_1}\mathcal{P}(\varkappa)d\varkappa\int_{a_1}^{b_1}\mathcal{P}(\varkappa)f_1(\varkappa)g_1(\varkappa)d\varkappa-\int_{a_1}^{b_1}\mathcal{P}(\varkappa)f_1(\varkappa)d\varkappa\int_{a_1}^{b_1}\mathcal{P}(\varkappa)g_1(\varkappa)d\varkappa\end{aligned} \tag{1.9}$$

and

$$\mathfrak{T}(f_1, g_1, \mathcal{P}, Q) = \int_{a_1}^{b_1} Q(\varkappa)d\varkappa \int_{a_1}^{b_1} \mathcal{P}(\varkappa)f_1(\varkappa)g_1(\varkappa)d\varkappa + \int_{a_1}^{b_1} \mathcal{P}(\varkappa)d\varkappa \int_{a_1}^{b_1} Q(\varkappa)f_1(\varkappa)g_1(\varkappa)d\varkappa$$
$$- \int_{a_1}^{b_1} Q(\varkappa)f_1(\varkappa)d\varkappa \int_{a_1}^{b_1} \mathcal{P}(\varkappa)g_1(\varkappa)d\varkappa - \int_{a_1}^{b_1} \mathcal{P}(\varkappa)f_1(\varkappa)d\varkappa \int_{a_1}^{b_1} Q(\varkappa)g_1(\varkappa)d\varkappa. \tag{1.10}$$

In [37], Dragomir established the inequality:
If $f_1', g_1' \in L_\infty(a_1, b_1)$, then

$$\left|\mathfrak{S}(f_1, g_1, \mathcal{P})\right| \leq \|f_1'\|_\infty \|g_1'\|_\infty \Big(\int_{a_1}^{b_1} \mathcal{P}(\varkappa)d\varkappa \int_{a_1}^{b_1} \varkappa^2 \mathcal{P}(\varkappa)d\varkappa - \Big(\int_{a_1}^{b_1} \varkappa \mathcal{P}(\varkappa)d\varkappa \Big)^2 \Big). \tag{1.11}$$

Moreover, author [37] proved numerous variants for Lipschitzian functions as follows:

If f_1 is L-g_1-Lipschitzian on $[a_1, b_1]$, that is

$$\left|f_1(\mu) - f_\nu\right| \leq L\left|g_1(\mu) - g_1(\nu)\right|, \quad L > 0,\ \mu, \nu \in [a_1, b_1]. \tag{1.12}$$

and

$$\left|\mathfrak{S}(f_1, g_1, \mathcal{P})\right| \leq L\Big(\int_{a_1}^{b_1} \mathcal{P}(\varkappa)d\varkappa \int_{a_1}^{b_1} g_1^2(\varkappa)\mathcal{P}(\varkappa)d\varkappa - \Big(\int_{a_1}^{b_1} g_1(\varkappa)\mathcal{P}(\varkappa)d\varkappa \Big)^2 \Big). \tag{1.13}$$

Furthermore, if f_1 and g_1 are L_1 and L_2-Lipschitzian functions on $[a_1, b_1]$, then

$$\left|\mathfrak{S}(f_1, g_1, \mathcal{P})\right| \leq L_1 L_2 \Big(\int_{a_1}^{b_1} \mathcal{P}(\varkappa)d\varkappa \int_{a_1}^{b_1} \varkappa^2 \mathcal{P}(\varkappa)d\varkappa - \Big(\int_{a_1}^{b_1} \varkappa \mathcal{P}(\varkappa)d\varkappa \Big)^2 \Big). \tag{1.14}$$

Owing to the above tendency, Dhamani et al. [38] proposed the fractional integral inequalities in the Riemann-Liouville parallel to variant (1.6) with the suppositions (1.7). Additionally, Dahamani and Benzidane [39] introduced weighted Grüss type inequality via (α,β)-fractional q-integral inequality resemble to (1.8) under the hypothesis of (1.5). Author [40, 41] derived the extended functional of (1.10) by employing Riemann-Liouville integral corresponds to variants (1.11), (1.13) and (1.14), respectively. In this flow, Set et al. [42] contemplated the Grüss type inequalities considering the generalized K-fractional integral. Chen et al. [43] obtained the novel refinements of Hermite-Hadamard type inequalities for n-polynomial p-convex functions within the generalized fractional integral operators. Abdeljawad et al. [44] derived the Simpson's type inequalities for generalized p-convex functions involving fractal set. Jarad et al. [45] investigated the properties of the more general form of generalized proportional fractional operators in Laplace transforms.

The motivation of this paper is twofold. First, we propose a novel framework named weighted generalized proportional fractional integral operator based on characteristics, as well as considering the boundedness and semi-group property and able to be widely applied to many scientific results. Second, the current operator employed to the extended weighted Chebyshev and Grüss type inequalities for exploring the analogous versions of (1.5) and (1.6). Some special cases are pictured with new fractional operators which are not computed yet. Interestingly, particular cases are designed for Riemann-Liouville fractional integral, generalized Riemann-Liouville fractional integral and generalized proportional fractional integral inequalities. It is worth mentioning that these operators have the ability to recapture several generalizations in the literature by considering suitable assumptions of Ψ, ω and ρ.

2. Prelude

In this section, we demonstrate the space where the weighted fractional integrals are bounded and also, provide certain specific features of these operators.

Definition 2.1. ([19])Let $\omega \neq 0$ be a mapping defined on $[a_1, b_1]$, g_1 is a differentiable strictly increasing function on $[a_1, b_1]$. The space $\chi^p_\omega(a_1, b_1)$, $1 \leq p < \infty$ is the space of all Lebesgue measurable functions f_1 defined on $[a_1, b_1]$ for which $\|f_1\|_{\chi^p_\omega}$, where

$$\|f_1\|_{\chi^p_\omega} = \Big(\int_{a_1}^{b_1} \Big|\omega(\varkappa) f_1(\varkappa)\Big|^p g_1'(\varkappa) d\varkappa\Big)^{\frac{1}{p}}, \quad 1 < p < \infty \tag{2.1}$$

and

$$\|f_1\|_{\chi^p_\omega} = ess \sup_{a_1 \leq \varkappa \leq b_1} \Big|\omega(\varkappa) f_1(\varkappa)\Big| < \infty. \tag{2.2}$$

Remark 1. Clearly we see that $f_1 \in \chi^p_\omega(a_1, b_1) \implies \omega(\varkappa) f_1(\varkappa)(g_1^{-1}(\varkappa))^{1/p} \in L_p(a_1, b_1)$ for $1 \leq p < \infty$ and $f_1 \in \chi^p_\omega(a_1, b_1) \implies \omega(\varkappa) f_1(\varkappa) \in L_\infty(a_1, b_1)$.

Now, we show a novel fractional integral operator which is known as the weighted generalized proportional fractional integral operator with respect to monotone function Ψ.

Definition 2.2. Let $f_1 \in \chi^p_\omega(a_1, b_1)$ and $\omega \neq 0$ be a function on $[a_1, b_1]$. Also, assume that Ψ is a continuously differentiable function on $[a_1, b_1]$ with $\psi' > 0$ on $[a_1, b_1]$. Then the left and right-sided weighted generalized proportional fractional integral operator with respect to another function Ψ of order $\alpha > 0$ are described as:

$$ {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\rho^\alpha \Gamma(\alpha)} \int_{a_1}^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa) - \Psi(\mu))]}{(\Psi(\varkappa) - \Psi(\mu))^{1-\alpha}} f_1(\mu)\omega(\mu)\Psi'(\mu) d\mu, \ a_1 < \varkappa \tag{2.3}$$

and

$$ {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{b_1} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\rho^\alpha \Gamma(\alpha)} \int_{\varkappa}^{b_1} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\mu) - \Psi(\varkappa))]}{(\Psi(\mu) - \Psi(\varkappa))^{1-\alpha}} f_1(\mu)\omega(\mu)\Psi'(\mu) d\mu, \ \varkappa < b_1, \tag{2.4}$$

where $\rho \in (0,1]$ is the proportionality index, $\alpha \in C, \Re(\alpha) > 0$ and $\Gamma(\varkappa) = \int_0^\infty \mu^{\varkappa-1} e^{-\mu} d\mu$ is the Gamma function.

Remark 2. Some particular fractional operators are the special cases of (2.3) and (2.4).
(1) Setting $\Psi(\varkappa) = \varkappa$, in Definition (2.2), then we get the weighted generalized proportional fractional operators stated as follows:

$$ {}_{\omega}\Omega_{a_1}^{\rho;\alpha} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)} \int_{a_1}^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\varkappa-\mu)]}{(\varkappa-\mu)^{1-\alpha}} f_1(\mu)\omega(\mu)d\mu, \ a_1 < \varkappa \tag{2.5} $$

and

$$ {}_{\omega}\Omega_{b_1}^{\rho;\alpha} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)} \int_{\varkappa}^{b_1} \frac{\exp[\frac{\rho-1}{\rho}(\mu-\varkappa)]}{(\mu-\varkappa)^{1-\alpha}} f_1(\mu)\omega(\mu)d\mu, \ \varkappa < b_1. \tag{2.6} $$

(2) Setting $\Psi(\varkappa) = \varkappa$ and $\rho = 1$ in Definition (2.2), then we get the weighted Riemann-Liouville fractional operators stated as follows:

$$ {}_{\omega}\Omega_{a_1}^{\alpha} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\Gamma(\alpha)} \int_{a_1}^{\varkappa} \frac{f_1(\mu)\omega(\mu)d\mu}{(\varkappa-\mu)^{1-\alpha}}, \ a_1 < \varkappa \tag{2.7} $$

and

$$ {}_{\omega}\Omega_{b_1}^{\alpha} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\Gamma(\alpha)} \int_{\varkappa}^{b_1} \frac{f_1(\mu)\omega(\mu)d\mu}{(\mu-\varkappa)^{1-\alpha}}, \ \varkappa < b_1. \tag{2.8} $$

(3) Setting $\Psi(\varkappa) = \ln \varkappa$ and $a_1 > 0$ in Definition (2.2), we get the weighted generalized proportional Hadamard fractional operators stated as follows:

$$ {}_{\omega}\Omega_{a_1}^{\rho;\alpha} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)} \int_{a_1}^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\ln\frac{\varkappa}{\mu})]}{(\ln\frac{\varkappa}{\mu})^{1-\alpha}} \frac{f_1(\mu)\omega(\mu)}{\mu} d\mu, \ a_1 < \varkappa \tag{2.9} $$

and

$$ {}_{\omega}\Omega_{b_1}^{\rho;\alpha} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)} \int_{\varkappa}^{b_1} \frac{\exp[\frac{\rho-1}{\rho}(\ln\frac{\mu}{\varkappa})]}{(\ln\frac{\mu}{\varkappa})^{1-\alpha}} \frac{f_1(\mu)\omega(\mu)}{\mu} d\mu, \ \varkappa < b_1. \tag{2.10} $$

(4) Setting $\Psi(\varkappa) = \ln \varkappa$ and $a_1 > 0$ along with $\rho = 1$ in Definition (2.2), then we get the weighted Hadamard fractional operators stated as follows:

$$ {}_{\omega}\Omega_{a_1}^{\alpha} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\Gamma(\alpha)} \int_{a_1}^{\varkappa} \frac{f_1(\mu)\omega(\mu)d\mu}{\mu(\ln\frac{\varkappa}{\mu})^{1-\alpha}}, \ a_1 < \varkappa \tag{2.11} $$

and

$$ {}_{\omega}\Omega^{\alpha}_{b_1} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\Gamma(\alpha)} \int\limits_{\varkappa}^{b_1} \frac{f_1(\mu)\omega(\mu)d\mu}{\mu\big(\ln\frac{\mu}{\varkappa}\big)^{1-\alpha}},\ \varkappa < b_1. \tag{2.12} $$

(5) Setting $\Psi(\varkappa) = \frac{\varkappa^{\tau}}{\tau}$ $(\tau > 0)$ in Definition (2.2), then we get the weighted generalized fractional operators in terms of Katugampola stated as follows:

$$ {}_{\omega}\Omega^{\alpha}_{a_1} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\Gamma(\alpha)} \int\limits_{a_1}^{\varkappa} \Big(\frac{\varkappa^{\tau}-\mu^{\tau}}{\tau}\Big)^{\alpha-1} \frac{f_1(\mu)\omega(\mu)d\mu}{\mu^{1-\tau}},\ a_1 < \varkappa \tag{2.13} $$

and

$$ {}_{\omega}\Omega^{\alpha}_{b_1} f_1(\varkappa) = \frac{\omega^{-1}(\varkappa)}{\Gamma(\alpha)} \int\limits_{\varkappa}^{b_1} \Big(\frac{\mu^{\tau}-\varkappa^{\tau}}{\tau}\Big)^{\alpha-1} \frac{f_1(\mu)\omega(\mu)d\mu}{\mu^{1-\tau}},\ \varkappa < b_1. \tag{2.14} $$

Remark 3. Several existing integral operators can be derived from Definition 2.2 as follows:
(1) Letting $\omega(\varkappa) = 1$, then we get the Definition 4 proposed by Rashid et al. [46] and Definition 3.2 introduced by Jarad et al. [47], independently.
(2) Letting $\omega(\varkappa) = 1$, $\Psi(\varkappa) = \varkappa$, then we get the Definition 3.4 defined by Jarad et al. [48].
(3) Letting $\omega(\varkappa) = 1$ and $\Psi(\varkappa) = \ln\varkappa$ along with $a_1 > 0$, then we get the Definition 2.1 defined by Rahman et al. [49].
(4) Letting $\omega(\varkappa) = \rho = 1$ and $\Psi(\varkappa) = \ln\varkappa$ along with $a_1 > 0$, then we get the operator defined by Kilbas et al. [3] and Smako et al. [5], respectively.
(5) Letting $\omega(\varkappa) = \rho = 1$ and $\Psi(\varkappa) = \varkappa$, then we get the operator defined by Kilbas et al [3].
(6) Letting $\omega(\varkappa) = 1$ and $\Psi(\varkappa) = \frac{\varkappa^{\tau}}{\tau}$, $(\tau > 0)$, then we get the operator defined by Katugampola et al. [7].
(7) Letting $\omega(\varkappa) = \rho = 1$ and $\Psi(\varkappa) = \frac{\varkappa^{\tau+s}}{\tau+s}$, $\tau \in (0, 1]$, $s \in \mathbb{R}$, then we get the Definition 2 defined by Khan and Khan et al [50].
(8) Letting $\omega(\varkappa) = \rho = 1$ and $\Psi(\varkappa) = \frac{(\varkappa-a_1)^{\tau}}{\tau}$, and $\Psi(\varkappa) = \frac{-(b_1-\varkappa)^{\tau}}{\tau}$, $(\tau > 0)$, then we get the operator defined by Jarad et al. [51].

Theorem 2.3. *For $\alpha > 0, \rho \in (0, 1]$, $1 \le p \le \infty$ and $f_1 \in \chi^p_{\omega}(a_1, b_1)$. Then ${}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1}$ is bounded in $\chi^p_{\omega}(a_1, b_1)$ and*

$$ \|{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1} f_1\|_{\chi^p_{\omega}} \le \frac{(\Psi(b_1)-\Psi(a_1))^{\alpha}\|f_1\|_{\chi^p_{\omega}}}{\rho^{\alpha}\Gamma(\alpha+1)}. $$

Proof. For $1 \le p \le \infty$, we have

$$ \|{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1} f_1\|_{\chi^p_{\omega}} = \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\Big(\int\limits_{a_1}^{b_1}\Big|\int\limits_{a_1}^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}\Psi(\varkappa)-\Psi(\mu)]}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\omega(\mu)f_1(\mu)\Psi'(\mu)d\mu\Big|^p \Psi'(\varkappa)d\varkappa\Big)^{1/p} $$

$$= \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\Big(\int_{\Psi(a_1)}^{\Psi(b_1)}\Big|\int_{\Psi(a_1)}^{t_2}\frac{\exp[\frac{\rho-1}{\rho}(t_2-t_1)]}{(t_2-t_1)^{1-\alpha}}\omega(\Psi^{-1}(t_1))f_1(\Psi^{-1}(t_1))\Big|^p dt_2\Big)^{1/p}.$$

Using the fact that $|\exp[\frac{\rho-1}{\rho}(t_2-t_1)]| < 1$. Taking into account the generalized Minkowski inequality [5], we can write

$$\begin{aligned}\|{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1}f_1\|_{\chi^p_\omega} &\leq \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{\Psi(a_1)}^{\Psi(b_1)}\Big(\Big|\omega(\Psi^{-1}(t_1))f_1(\Psi^{-1}(t_1))\Big|^p\int_{t_1}^{\Psi(b_1)}(t_2-t_1)^{p(\alpha-1)}dt_2\Big)^{1/p}dt_1\\ &= \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{\Psi(a_1)}^{\Psi(b_1)}\Big(\Big|\omega(\Psi^{-1}(t_1))f_1(\Psi^{-1}(t_1))\Big|\Big(\frac{(\Psi(b_1)-t_1)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\Big)^{1/p}dt_1.\end{aligned}$$

By employing the well-known Hölder inequality satisfying $p^{-1}+q^{-1}=1$, we obtain

$$\begin{aligned}\|{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1}f_1\|_{\chi^p_\omega} &\leq \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\Big(\int_{\Psi(a_1)}^{\Psi(b_1)}\Big|\omega(\Psi^{-1}(t_1))f_1(\Psi^{-1}(t_1))\Big|^p dt_1\Big)^{1/p}\Big(\int_{\Psi(a_1)}^{\Psi(b_1)}\Big(\frac{(\Psi(b_1)-t_1)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\Big)^{q/p}dt_1\Big)^{1/q}\\ &\leq \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\Big(\int_{a_1}^{b_1}\Big|\omega(\varkappa)f_1(\varkappa)\Big|^p\Psi'(\varkappa)d\varkappa\Big)^{1/p}\Big(\int_{\Psi(a_1)}^{\Psi(b_1)}\Big(\frac{(\Psi(b_1)-t_1)^{p(\alpha-1)+1}}{p(\alpha-1)+1}\Big)^{q/p}dt_1\Big)^{1/q}\\ &\leq \frac{(\Psi(b_1)-\Psi(a_1))^{\alpha}\|f_1\|_{\chi^p_\omega}}{\rho^{\alpha}\Gamma(\alpha+1)}.\end{aligned}$$

Now, for $p=\infty$, we have

$$\begin{aligned}\Big|\omega(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1}f_1(\varkappa)\Big| &= \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{a_1}^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}f_1(\mu)\omega(\mu)\Psi'(\mu)d\mu\\ &\leq \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_{a_1}^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\Big|f_1(\mu)\omega(\mu)\Big|\Psi'(\mu)d\mu,\\ &\qquad Since\Big(\Big|\exp\Big[\frac{\rho-1}{\rho}(t_2-t_1)\Big]\Big|<1\Big)\\ &\leq \frac{\|f_1\|_{\chi^\infty_\omega}}{\rho^{\alpha}\Gamma(\alpha)}\int_{a_1}^{\varkappa}(\Psi(\varkappa)-\Psi(\mu))^{\alpha-1}d\mu\\ &\leq \frac{(\Psi(\varkappa)-\Psi(a_1))^{\alpha}\|f_1\|_{\chi^\infty_\omega}}{\rho^{\alpha}\Gamma(\alpha+1)}\\ &= \frac{(\Psi(b_1)-\Psi(a_1))^{\alpha}\|f_1\|_{\chi^\infty_\omega}}{\rho^{\alpha}\Gamma(\alpha+1)}.\end{aligned}$$

This ends the proof. □

Our next result is the semi group property for weighted generalized proportional fractional integral operator with respect to monotone function.

Theorem 2.4. *For $\alpha,\beta > 0, \rho \in (0,1]$ with $1 \le p \le \infty$ and let $f_1 \in \chi^p_\omega(a_1,b_1)$. Then*

$$\left({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1}\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{a_1}\right)f_1 = \left({}^{\Psi}_{\omega}\Omega^{\rho;\alpha+\beta}_{a_1}\right)f_1. \tag{2.15}$$

Proof.

$$\begin{aligned}\left({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1}\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{a_1}f_1\right)(\varkappa) &= \frac{\omega^{-1}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)}\int_{a_1}^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\omega(\mu)({}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{a_1}f_1)(\mu)\Psi'(\mu)d\mu\\ &= \frac{\omega^{-1}(\varkappa)}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\int_{a_1}^{\varkappa}\int_{a_1}^{\mu}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\mu)-\Psi(\nu))]}{(\Psi(\mu)-\Psi(\nu))^{1-\beta}}\\ &\quad\times\omega(\nu)f_1(\nu)\Psi'(\nu)\Psi'(\mu)d\mu d\nu.\end{aligned}$$

By making change of variable technique $\theta = \frac{\Psi(\mu)-\Psi(a_1)}{\Psi(\varkappa)-\Psi(a_1)}$, we can write

$$\begin{aligned}&\left({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1}\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{a_1}f_1\right)(\varkappa)\\ &= \frac{\omega^{-1}(\varkappa)}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\int_0^1\theta^{\beta-1}(1-\theta)^{\alpha-1}d\theta\int_{a_1}^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]}{(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha-\beta}}\omega(\nu)f_1(\nu)\Psi'(\nu)d\nu\\ &= \frac{\omega^{-1}(\varkappa)}{\rho^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_{a_1}^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]}{(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha-\beta}}\omega(\nu)f_1(\nu)\Psi'(\nu)d\nu\\ &= \left({}^{\Psi}_{\omega}\Omega^{\rho;\alpha+\beta}_{a_1}f_1\right)(\varkappa),\end{aligned}$$

where $\mathbb{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1\theta^{\beta-1}(1-\theta)^{\alpha-1}d\theta$ is known to be Euler Beta function. □

3. Main results

This section contains some significant generalizations for weighted integral inequalities by employing weighted generalized proportional fractional integral operator, for the consequences relating to (1.1) and (1.2), it is suppose that all mappings are integrable in the Riemann sense.

Throughout this investigation, we use the following assumptions:

I. Let f_1 and g_1 be two synchronous functions on $[0,\infty)$.

II. Let $\Psi : [0,\infty) \to (0,\infty)$ is an increasing function with continuous derivative Ψ' on the interval $(0,\infty)$.

Lemma 3.1. *If the supposition **I** and **II** are satisfied and let $\mathcal{Q}$ and $\mathcal{P}$ be two non-negative continuous mappings on $[0,\infty)$. Then the inequality*

$$\begin{aligned}&{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P})(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{Q}f_1g_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P}f_1g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{Q})(\varkappa)\\ &\ge {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P}g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{Q}f_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P}f_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{Q}g_1)(\varkappa),\end{aligned} \tag{3.1}$$

holds for all $\rho \in (0,1], \alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$.

Proof. Since f_1 and g_1 are two synchronous functions on $[0, \infty)$, then for all $\mu > 0$ and $\nu > 0$, we have

$$(f_1(\mu) - f_1(\nu))(g_1(\mu) - g_1(\nu)) \geq 0. \tag{3.2}$$

By (3.2), we write

$$f_1(\mu)g_1(\mu) + f_1(\nu)g_1(\nu) \geq g_1(\mu)f_1(\nu) + g_1(\nu)f_1(\mu). \tag{3.3}$$

If we multiply both sides of (3.3) by $\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]Q(\mu)\omega(\mu)\Psi'(\mu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}$ and integrating the resulting inequality with respect to μ from 0 to $\varkappa$, we get

$$\begin{aligned}
&\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]Q(\mu)\omega(\mu)\Psi'(\mu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}f_1(\mu)g_1(\mu)d\mu\\
&+\frac{f_1(\nu)g_1(\nu)}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]Q(\mu)\omega(\mu)\Psi'(\mu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}d\mu\\
&\geq\frac{f_1(\nu)}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]Q(\mu)\omega(\mu)\Psi'(\mu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}g_1(\nu)d\nu\\
&+\frac{g_1(\nu)}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]Q(\mu)\omega(\mu)\Psi'(\mu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}f_1(\mu)d\mu.
\end{aligned} \tag{3.4}$$

Taking product both sides of the above equation by $\omega^{-1}(\varkappa)$ and in view of Definition (2.2), we have

$${}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(Qf_1g_1)(\varkappa) + f_1(\nu)g_1(\nu)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(Q)(\varkappa) \geq g_1(\nu)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(Qf_1)(\varkappa) + f_1(\nu)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(Qg_1)(\varkappa). \tag{3.5}$$

Further, if we multiply both sides of (3.5) by $\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]\mathcal{P}(\nu)\omega(\nu)\Psi'(\nu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha}}$ and integrating the resulting inequality with respect to ν from 0 to $\varkappa$. Then, multiplying by $\omega^{-1}(\varkappa)$ and in view of Definition 2.2, we obtain

$$\begin{aligned}
&{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P})(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(Qf_1g_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P}f_1g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(Q)(\varkappa)\\
&\geq {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P}g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(Qf_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P}f_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(Qg_1)(\varkappa),
\end{aligned} \tag{3.6}$$

which implies (3.1). □

Theorem 3.2. *Under the assumption of **I**, **II** and let r, s and t be three non-negative continuous functions on* $[0, \infty)$. *Then the inequality*

$$\begin{aligned}
&2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(tf_1g_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}t(\varkappa)\Big)\\
&\quad+2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}t(\varkappa)\\
&\geq {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sg_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(tf_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(tg_1)(\varkappa)\Big)\\
&\quad+{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}s(\varkappa)\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(tf_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(tg_1)(\varkappa)\Big)\\
&\quad+{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}s(\varkappa)\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sg_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\Big)
\end{aligned} \tag{3.7}$$

holds for all $\rho \in (0, 1], \alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. By means of Lemma 3.1 and setting $\mathcal{P} = r$, $\mathcal{Q} = s$, we can write

$$\begin{aligned}&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tf_1g_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} t(\varkappa)\\ &\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tf_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tg_1)(\varkappa). \qquad (3.8)\end{aligned}$$

Conducting product both sides of (3.8) by ${}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)$, we obtain

$$\begin{aligned}&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tf_1g_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} t(\varkappa)\Big)\\ &\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tf_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tg_1)(\varkappa)\Big). \qquad (3.9)\end{aligned}$$

By means of Lemma 3.1 and setting $\mathcal{P} = r$, $\mathcal{Q} = t$, we can write

$$\begin{aligned}&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tf_1g_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} t(\varkappa)\\ &\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tf_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tg_1)(\varkappa). \qquad (3.10)\end{aligned}$$

Conducting product of (3.10) by ${}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)$, we obtain

$$\begin{aligned}&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tf_1g_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} t(\varkappa)\Big)\\ &\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tf_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(tg_1)(\varkappa)\Big). \qquad (3.11)\end{aligned}$$

By similar argument as we did before, yields

$$\begin{aligned}&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} t(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1g_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} t(\varkappa)\Big)\\ &\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big). \qquad (3.12)\end{aligned}$$

Adding (3.9), (3.11) and (3.12), we get the desired inequality (3.8).

□

Lemma 3.3. *Under the assumption of* ***I***, ***II*** *and let* $\mathcal{Q}$ *and* $\mathcal{P}$ *be two non-negative continuous functions on* $[0, \infty)$. *Then the inequality*

$$\begin{aligned}&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}f_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}\mathcal{Q}(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}\mathcal{P}(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(\mathcal{Q}f_1g_1)(\varkappa)\\ &\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}f_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(\mathcal{Q}g_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(\mathcal{Q}f_1)(\varkappa),\end{aligned}$$

holds for all $\rho \in (0, 1]$, $\alpha, \beta \in C$ *with* $\Re(\alpha), \Re(\beta) > 0$.

Proof. If we multiply both sides of (3.2) by $\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]\mathcal{Q}(\nu)\omega(\nu)\Psi'(\nu)}{\rho^{\beta}\Gamma(\beta)(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}$ and integrating the resulting inequality with respect to ν from 0 to $\varkappa$, we have

$$\frac{f_1(\mu)g_1(\mu)}{\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]\mathcal{Q}(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}d\nu$$

$$+\frac{f_1(\nu)g_1(\nu)}{\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]\mathcal{Q}(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}d\nu$$
$$\geq \frac{g_1(\mu)}{\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]\mathcal{Q}(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}f_1(\nu)d\nu$$
$$+\frac{f_1(\mu)}{\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]\mathcal{Q}(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}g_1(\nu)d\nu. \tag{3.13}$$

Taking product both sides of the above equation by $\omega^{-1}(\varkappa)$ and in view of Definition (2.2), we have

$$f_1(\mu)g_1(\mu)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}\mathcal{Q}(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\mathcal{Q}f_1g_1)(\varkappa)\geq f_1(\mu)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\mathcal{Q}g_1)(\varkappa)+g_1(\mu)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\mathcal{Q}f_1)(\varkappa). \tag{3.14}$$

Again, multiplying both sides of (3.14) by $\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]\mathcal{P}(\mu)\omega(\mu)\Psi'(\mu)}{\rho^{\alpha}\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}$ and integrating the resulting inequality with respect to ν from 0 to $\varkappa$, we have

$$\frac{{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}\mathcal{Q}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]\mathcal{P}(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}f_1(\mu)g_1(\mu)d\mu$$
$$+\frac{{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\mathcal{Q}f_1g_1)(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]\mathcal{P}(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}d\mu$$
$$\geq\frac{{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\mathcal{Q}g_1)(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]\mathcal{P}(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}f_1(\mu)d\mu$$
$$+\frac{{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\mathcal{Q}f_1)(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]\mathcal{P}(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}g_1(\mu)d\mu. \tag{3.15}$$

Taking product both sides of the above equation by $\omega^{-1}(\varkappa)$ and in view of Definition (2.2), we obtain

$$\begin{aligned}&{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P}f_1g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}\mathcal{Q}(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}\mathcal{P}(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\mathcal{Q}f_1g_1)(\varkappa)\\ \geq\ &{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P}f_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\mathcal{Q}g_1)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\mathcal{P}g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\mathcal{Q}f_1)(\varkappa),\end{aligned}$$

which implies (3.13). □

Theorem 3.4. *Under the assumptions **I**, **II** and let r, s and t be three non-negative continuous functions on $[0,\infty)$. Then the inequality*

$$\begin{aligned}&{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}t(\varkappa)+2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(tf_1g_1)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}t(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1g_1)(\varkappa)\Big)\\ &+\Big({}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}t(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}s(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}t(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\Big)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa)\end{aligned}$$

$$
\begin{aligned}
&\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tg_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1)(\varkappa)\Big)\\
&+{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tg_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1)(\varkappa)\Big)\\
&+{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} t(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sg_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sf_1)(\varkappa)\Big) \qquad (3.16)
\end{aligned}
$$

holds for all $\rho \in (0,1], \alpha, \beta \in C$ *with* $\Re(\alpha), \Re(\beta) > 0$.

Proof. By means of Lemma 3.3 and setting $\mathcal{P} = s, Q = t$, we can write

$$
\begin{aligned}
&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} t(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1g_1)(\varkappa)\\
&\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tg_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1)(\varkappa). \qquad (3.17)
\end{aligned}
$$

Conducting product both sides of (3.17) by ${}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)$, we obtain

$$
\begin{aligned}
&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} t(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1g_1)(\varkappa)\Big)\\
&\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tg_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1)(\varkappa)\Big). \qquad (3.18)
\end{aligned}
$$

Again, by means of Lemma 3.3 and setting $\mathcal{P} = r, Q = t$, we can write

$$
\begin{aligned}
&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} t(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1g_1)(\varkappa)\\
&\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tg_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1)(\varkappa). \qquad (3.19)
\end{aligned}
$$

Conducting product both sides of (3.19) by ${}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)$, we obtain

$$
\begin{aligned}
&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} t(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1g_1)(\varkappa)\Big)\\
&\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tg_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(tf_1)(\varkappa)\Big). \qquad (3.20)
\end{aligned}
$$

By similar arguments as we did before, yields

$$
\begin{aligned}
&{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} t(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1g_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} r(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(rf_1g_1)(\varkappa)\Big)\\
&\geq {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} t(\varkappa)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sg_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sf_1)(\varkappa)\Big). \qquad (3.21)
\end{aligned}
$$

Adding (3.18), (3.20) and (3.21), we get the desired inequality (3.16).

□

Remark 4. Theorem 3.2 and Theorem 3.4 lead to the following conclusions:
(1) Let f_1 and g_1 are the asynchronous functions on $[0,\infty)$, then (3.8) and (3.16) are reversed.
(2) Let r, s and t are negative on $[0,\infty)$, then (3.8) and (3.16) are reversed.
(3) Let r, s are positive t is negative on $[0,\infty)$, then (3.8) and (3.16) are reversed.

In the next, we derive certain novel Grüss-type integral inequalities via weighted generalized proportional fractional integral operators.

Lemma 3.5. *Suppose an integrable function f_1 defined on $[0,\infty)$ satisfying the assertions **I**,**II** and* (1.7) *on* $[0,\infty)$ *and let a continuous function r defined on* $[0,\infty)$. *Then the inequality*

$$\begin{aligned} &{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} r(\varkappa)\, {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} (r f_1^2)(\varkappa) - \Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} (r f_1)(\varkappa)\Big)^2 \\ &\leq \Big(\Phi\, {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} x(\varkappa) - {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} (r f_1)(\varkappa)\Big)\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} (r f_1)(\varkappa) - \phi\, {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} r(\varkappa)\Big) \\ &\quad - {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} r(\varkappa)\, {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}\Big(r(\varkappa)(\Phi - f_1(\varkappa))(f_1(\varkappa) - \phi)\Big) \end{aligned} \tag{3.22}$$

holds for all $\rho \in (0,1], \alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. By the given hypothesis and utilizing (1.7). For any $\mu, \nu \in [0,\infty)$, we have

$$\begin{aligned} &(\Phi - f_1(\nu))(f_1(\mu) - \phi) + (\Phi - f_1(\mu))(f_1(\nu) - \phi) - (\Phi - f_1(\mu))(f_1(\mu) - \phi) - (\Phi - f_1(\nu))(f_1(\nu) - \phi) \\ &\leq f_1^2(\mu) + f_1^2(\nu) - 2f_1(\mu)f_1(\nu). \end{aligned} \tag{3.23}$$

Multiplying both sides of (3.23) by $\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha}}$ and integrating the resulting inequality with respect to ν from 0 to $\varkappa$, we have

$$\begin{aligned} &\frac{(f_1(\mu) - \phi)}{\rho^\alpha\Gamma(\alpha)} \int_0^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa) - \Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa) - \Psi(\nu))^{1-\alpha}}(\Phi - f_1(\nu))d\nu \\ &\quad + \frac{(\Phi - f_1(\mu))}{\rho^\alpha\Gamma(\alpha)} \int_0^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa) - \Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa) - \Psi(\nu))^{1-\alpha}}(f_1(\nu) - \phi)d\nu \\ &\quad - \frac{(\Phi - f_1(\mu))(f_1(\mu) - \phi)}{\rho^\alpha\Gamma(\alpha)} \int_0^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa) - \Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa) - \Psi(\nu))^{1-\alpha}}d\nu \\ &\quad - \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_0^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa) - \Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa) - \Psi(\nu))^{1-\alpha}}(\Phi - f_1(\nu))(f_1(\nu) - \phi)d\nu \\ &\leq \frac{f_1^2(\mu)}{\rho^\alpha\Gamma(\alpha)} \int_0^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa) - \Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa) - \Psi(\nu))^{1-\alpha}}d\nu \\ &\quad + \frac{1}{\rho^\alpha\Gamma(\alpha)} \int_0^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa) - \Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa) - \Psi(\nu))^{1-\alpha}}f_1^2(\nu)d\nu \\ &\quad - 2\frac{f_1(\mu)}{\rho^\alpha\Gamma(\alpha)} \int_0^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa) - \Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa) - \Psi(\nu))^{1-\alpha}}f_1(\nu)d\nu. \end{aligned} \tag{3.24}$$

Taking product both sides of the above equation by $\omega^{-1}(\varkappa)$ and in view of Definition (2.2), we obtain

$$(\Phi\, {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} r(\varkappa) - {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} (r f_1)(\varkappa))(f_1(\mu) - \phi) + (\Phi - f_1(\mu))({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} (r f_1)(\varkappa) - \phi\, {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+} r(\varkappa))$$

$$-(\Phi - f_1(\mu))(f_1(\mu) - \phi)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa) - {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(r(\varkappa)(\Phi - f_1(\varkappa))(f_1(\varkappa) - \phi))$$
$$\leq f_1^2(\mu)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1^2)(\varkappa) - 2f_1(\mu)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa). \tag{3.25}$$

Multiplying both sides of (3.25) by $\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}$ and integrating the resulting inequality with respect to μ from 0 to $\varkappa$, we have

$$(\Phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa) - {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\nu))\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}(f_1(\mu)-\phi)d\mu$$
$$+({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa) - \phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa))\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}(\Phi-f_1(\mu))d\mu$$
$$-\Big(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}(\Phi-f_1(\mu))(f_1(\mu)-\phi)d\mu\Big){}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)$$
$$-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(r(\varkappa)(\Phi-f_1(\nu))(f_1(\nu)-\phi)\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}d\nu$$
$$\leq\Big(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}f_1^2(\mu)d\mu\Big){}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)$$
$$+\Big(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}d\mu\Big){}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1^2)(\varkappa)$$
$$-2\Big(\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}f_1(\mu)d\mu\Big){}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa). \tag{3.26}$$

Taking product both sides of the above equation by $\omega^{-1}(\varkappa)$ and in view of Definition (2.2), we obtain

$$\begin{aligned}
&(\Phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa) - {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa))({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa) - \phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa))\\
&\quad+(\Phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa) - {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa))({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa) - \phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa))\\
&\quad-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}\big(r(\varkappa)(\Phi-f_1(\varkappa))(f_1(\varkappa)-\phi)\big){}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\\
&\quad-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}\Big(r(\varkappa)(\Phi-f_1(\varkappa))(f_1(\varkappa)-\phi)\Big)\\
&\leq {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1^2)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa) + {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1^2)(\varkappa)\\
&\quad-2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa),
\end{aligned} \tag{3.27}$$

which gives (3.22) and proves the lemma. □

Theorem 3.6. *Suppose two integrable functions f_1 and g_1 defined on $[0,\infty)$ satisfying the assertions **I**, **II** and (1.7) on $[0,\infty)$ and let a continuous function r defined on $[0,\infty)$. Then the inequality*

$$\Big|{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa) - {}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\Big| \leq \frac{(\Phi-\phi)(\Upsilon-\gamma)}{4}\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\Big)^2 \tag{3.28}$$

holds for all $\rho \in (0,1], \alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. By the given hypothesis stated in Theorem 3.6. Also, assume that μ, ν be defined by

$$\mathfrak{T}(\mu,\nu) = (f_1(\mu) - f_1(\nu))(g_1(\mu) - g_1(\nu)), \quad \mu,\nu \in [0,\varkappa], \quad \varkappa > 0. \tag{3.29}$$

Multiplying both sides of (3.30) by $\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{\rho^\alpha\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha}}$ and integrating the resulting inequality with respect to μ and ν from 0 to $\varkappa$, we can state that

$$\begin{aligned}
&\frac{1}{\rho^{2\alpha}\Gamma^2(\alpha)}\int_0^\varkappa\int_0^\varkappa \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}} \\
&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha}}\mathfrak{T}(\mu,\nu)d\mu d\nu \\
&= \frac{1}{\rho^{2\alpha}\Gamma^2(\alpha)}\int_0^\varkappa\int_0^\varkappa \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}} \\
&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha}} \\
&\times(f_1(\mu)-f_1(\nu))(g_1(\mu)-g_1(\nu))d\mu d\nu.
\end{aligned}\tag{3.30}$$

Taking product both sides of the above equation by $\omega^{-1}(\varkappa)$ and in view of Definition (2.2), we obtain

$$\begin{aligned}
&\frac{\omega^{-2}(\varkappa)}{\rho^{2\alpha}\Gamma^2(\alpha)}\int_0^\varkappa\int_0^\varkappa \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}} \\
&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha}}\mathfrak{T}(\mu,\nu)d\mu d\nu \\
&= 2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa) - 2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa).
\end{aligned}\tag{3.31}$$

Thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we can write that

$$\begin{aligned}
&\Bigg(\frac{\omega^{-2}(\varkappa)}{\rho^{2\alpha}\Gamma^2(\alpha)}\int_0^\varkappa\int_0^\varkappa \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}} \\
&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha}}\mathfrak{T}(\mu,\nu)d\mu d\nu\Bigg)^2 \\
&\leq \Bigg(\frac{\omega^{-2}(\varkappa)}{\rho^{2\alpha}\Gamma^2(\alpha)}\int_0^\varkappa\int_0^\varkappa \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}} \\
&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha}}(f_1(\mu)-f_1(\nu))d\mu d\nu\Bigg)
\end{aligned}$$

$$\Big(\frac{\omega^{-2}(\varkappa)}{\rho^{2\alpha}\Gamma^{2}(\alpha)}\int_{0}^{\varkappa}\int_{0}^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}$$
$$\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]r(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\alpha}}(g_1(\mu)-g_1(\nu))d\mu d\nu\Big)$$
$$=4\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1^2)(\varkappa)-\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\Big)^2\Big)$$
$$\times\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1^2)(\varkappa)-\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big)^2\Big). \tag{3.32}$$

Since $(\Phi-f_1(\mu))(f_1(\mu)-\phi)\geq 0$ and $(\Upsilon-g_1(\mu))(g_1(\mu)-\gamma)\geq 0$, we have

$${}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}\Big(r(\varkappa)(\Phi-f_1(\mu))(f_1(\mu)-\phi)\Big)\geq 0, \tag{3.33}$$

and

$${}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}\Big(r(\varkappa)(\Upsilon-g_1(\mu))(g_1(\mu)-\gamma)\Big)\geq 0. \tag{3.34}$$

Therefore, from (3.33), (3.34) and Lemma 3.5, we get

$${}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1^2)(\varkappa)-\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\Big)^2$$
$$\leq\Big(\Phi\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)-{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\Big)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)-\phi\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\Big) \tag{3.35}$$

and

$${}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1^2)(\varkappa)-\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big)^2$$
$$\leq\Big(\Upsilon\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)-{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)-\gamma\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\Big). \tag{3.36}$$

Combining (3.30), (3.31), (3.35) and (3.36), we deduce that

$$\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(xf_1g_1)(\varkappa)-{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big)^2$$
$$\leq\Big(\Phi\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)-{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\Big)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf)(\varkappa)-\phi\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\Big)$$
$$\times\Big(\Upsilon\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)-{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)-\gamma\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\Big). \tag{3.37}$$

Taking into consideration the elementary inequality $4a_1a_2\leq(a_1+a_2)^2$, $a_1,a_2\in\mathbb{R}$, we can state that

$$4\Big(\Phi\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)-{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\Big)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)-\phi\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\Big)\leq\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)(\Phi-\phi)\Big)^2 \tag{3.38}$$

and

$$4\Big(\Upsilon\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)-{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big)\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)-\gamma\,{}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\Big)\leq\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)(\Upsilon-\gamma)\Big)^2. \tag{3.39}$$

From (3.37)-(3.39), we obtain (3.28). This completes the proof of Theorem 3.6. □

Lemma 3.7. *Suppose two integrable functions f_1 and g_1 defined on $[0,\infty)$ satisfying the assertions **I**, **II** and* (1.7) *on* $[0,\infty)$ *and let two continuous function r and s defined on* $[0,\infty)$. *Then the inequality*

$$\begin{aligned}&\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1g_1)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa)\\&-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\Big)^2\\&\leq\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1^2)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1^2)(\varkappa)\\&-2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1)(\varkappa)\Big)\\&\times\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1^2)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1^2)(\varkappa)\\&-2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)\Big)\end{aligned}\tag{3.40}$$

holds for all $\rho\in(0,1],\alpha,\beta\in C$ *with* $\Re(\alpha),\Re(\beta)>0$.

Proof. Taking product (3.30) by $\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{\rho^{\alpha}\Gamma(\alpha)(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{\rho^{\beta}\Gamma(\beta)(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}$ and integrating the resulting inequality with respect to μ and ν from 0 to $\varkappa$, we can state that

$$\begin{aligned}&\frac{1}{\rho^{\alpha}\Gamma(\alpha)\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\\&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}\mathfrak{T}(\mu,\nu)d\mu d\nu\\&=\frac{1}{\rho^{\alpha}\Gamma(\alpha)\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\\&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}\\&\times(f_1(\mu)-f_1(\nu))(g_1(\mu)-g_1(\nu))d\mu d\nu.\end{aligned}\tag{3.41}$$

Taking product both sides of the above equation by $\omega^{-2}(\varkappa)$ and utilizing Definition (2.2), we have

$$\begin{aligned}&\frac{\omega^{-2}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\\&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}\mathfrak{T}(\mu,\nu)d\mu d\nu\\&={}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1g_1)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa)\\&-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa).\end{aligned}\tag{3.42}$$

Then, thanks to the weighted Cauchy-Schwartz integral inequality for double integrals, we conclude (3.40). □

Lemma 3.8. *Suppose an integrable function f_1 defined on $[0,\infty)$ satisfying the assertions **I** and **II** on $[0,\infty)$ and let two continuous function r and s defined on $[0,\infty)$. Then the inequality*

$$\begin{aligned}
&{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1^2)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1^2)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)-2\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\\
&\le(\Phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa))({}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1)(\varkappa)-\phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa))\\
&\quad+(\Phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1)(\varkappa))({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)-\phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa))\\
&\quad-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}\Big(s(\varkappa)(\Phi-f_1(\varkappa))(f_1(\varkappa)-\phi)\Big)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\\
&\quad-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}\Big(r(\varkappa)(\Phi-f_1(\varkappa))(f_1(\varkappa)-\phi)\Big)
\end{aligned}\tag{3.43}$$

holds for all $\rho\in(0,1]$, $\alpha,\beta\in C$ with $\Re(\alpha),\Re(\beta)>0$.

Proof. Multiplying both sides of (3.25) by $\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{\rho^{\beta}\Gamma(\beta)(\Psi(\varkappa)-\Psi(\mu))^{1-\beta}}$ and integrating the resulting inequality with respect to μ from 0 to $\varkappa$. Then, by multiplying with $\omega^{-1}(\varkappa)$ and in view of Definition 2.2, concludes

$$\begin{aligned}
&(\Phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa))({}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1)(\varkappa)-\phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa))\\
&\quad+(\Phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1)(\varkappa))({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)-\phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa))\\
&\quad-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}\Big(s(\varkappa)(\Phi-f_1(\varkappa))(f_1(\varkappa)-\phi)\Big)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\\
&\quad-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}\Big(r(\varkappa)(\Phi-f_1(\varkappa))(f_1(\varkappa)-\phi)\Big)\\
&\le{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1^2)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1^2)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\\
&\quad-2\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa),
\end{aligned}\tag{3.44}$$

which gives (3.43) and proves the lemma. □

Theorem 3.9. *Suppose two integrable functions f_1 and g_1 defined on $[0,\infty)$ satisfying the assertions **I**,**II** and (1.7) on $[0,\infty)$ and let two continuous function r and s defined on $[0,\infty)$. Then the inequality*

$$\begin{aligned}
&\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1g_1)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa)\\
&\quad-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\Big)^2\\
&\le\Big\{(\Phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa))({}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1)(\varkappa)-\phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa))\\
&\quad+({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)-\phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa))(\Phi\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1)(\varkappa))\Big\}\\
&\quad\times\Big\{(\Upsilon\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa))({}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)-\gamma\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa))\\
&\quad+({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)-\gamma\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa))(\Upsilon\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa))\Big\}
\end{aligned}\tag{3.45}$$

holds for all $\rho\in(0,1]$, $\alpha,\beta\in C$ with $\Re(\alpha),\Re(\beta)>0$.

Proof. Since $(\Phi-f_1(\mu))(f_1(\mu)-\phi)\ge0$ and $(\Upsilon-g_1(\mu))(g_1(\mu)-\gamma)\ge0$, we have

$$-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}\Big(s(\varkappa)(\Phi-f_1(\varkappa))(f_1(\varkappa)-\phi)\Big)-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}\Big(r(\varkappa)(\Phi-f_1(\varkappa))(f_1(\varkappa)-\phi)\Big)\le0\tag{3.46}$$

and

$$-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}\Big(s(\varkappa)(\Upsilon-g_1(\varkappa))(g_1(\varkappa)-\gamma)\Big)-{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}\Big(r(\varkappa)(\Upsilon-g_1(\varkappa))(g_1(\varkappa)-\gamma)\Big)\leq 0. \tag{3.47}$$

Utilizing Lemma 3.8 to f_1 and g_1, and utilizing Lemma 3.7 and the inequalities (3.46) and (3.47), yields (3.45). □

Theorem 3.10. *Suppose two integrable functions f_1 and g_1 defined on $[0,\infty)$ satisfying the assertions **I**, **II** and (1.7) on $[0,\infty)$ and let two continuous function r and s defined on $[0,\infty)$. Then the inequality*

$$\begin{aligned}&\Big|{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1g_1)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa)\\&\quad-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\Big|\\&\leq{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)(\Phi-\phi)(\Upsilon-\gamma)\end{aligned} \tag{3.48}$$

holds for all $\rho\in(0,1]$, $\alpha,\beta\in C$ with $\Re(\alpha),\Re(\beta)>0$.

Proof. Taking into consideration the assumption (1.7), we have

$$\Big|f_1(\mu)-f_1(\nu)\Big|\leq\Phi-\phi,\qquad\Big|g_1(\mu)-g_1(\nu)\Big|\leq\Upsilon-\gamma,\quad\mu,\nu\in[0,\infty), \tag{3.49}$$

which implies that

$$\Big|\mathfrak{T}(\mu,\nu)\Big|=\Big|f_1(\mu)-f_1(\nu)\Big|\Big|g_1(\mu)-g_1(\nu)\Big|\leq(\Phi-\phi)(\Upsilon-\gamma). \tag{3.50}$$

From (3.42) and (3.50), we obtain that

$$\begin{aligned}&\Big|{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1g_1)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa)\\&\quad-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\Big|\\&\leq\frac{\omega^{-2}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\\&\quad\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}\mathfrak{T}(\mu,\nu)d\mu d\nu\\&\leq\frac{\omega^{-2}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\\&\quad\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}\Big((\Phi-\phi)(\Upsilon-\gamma)\Big)d\mu d\nu\\&={}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)(\Phi-\phi)(\Upsilon-\gamma).\end{aligned} \tag{3.51}$$

This ends the proof.

□

Theorem 3.11. *Suppose two integrable functions f_1 and g_1 defined on $[0,\infty)$ satisfying the assertions **I**, **II** and (1.7) on $[0,\infty)$ and let two continuous function r and s defined on $[0,\infty)$. Then the inequality*

$$\begin{aligned}
&\Big| {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sf_1g_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} s(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1g_1)(\varkappa) \\
&\quad - {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sg_1)(\varkappa) - {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big| \\
&\leq L\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sg_1^2)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} s(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1^2)(\varkappa) \\
&\quad -2\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sg_1)(\varkappa)\Big)
\end{aligned} \tag{3.52}$$

holds for all $\rho \in (0,1]$, $\alpha,\beta \in C$ with $\Re(\alpha), \Re(\beta) > 0$.

Proof. Taking into consideration the assumption (1.12), we have

$$\Big|f_1(\mu) - f_1(\nu)\Big| \leq L\Big|g_1(\mu) - g_1(\nu)\Big| \quad \mu,\nu \in [0,\infty), \tag{3.53}$$

which implies that

$$\Big|\mathfrak{T}(\mu,\nu)\Big| = \Big|f_1(\mu) - f_1(\nu)\Big|\Big|g_1(\mu) - g_1(\nu)\Big| \leq L\big(g_1(\mu) - g_1(\nu)\big)^2. \tag{3.54}$$

From (3.42) and (3.54), we obtain that

$$\begin{aligned}
&\Big| {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sf_1g_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} s(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1g_1)(\varkappa) \\
&\quad - {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sg_1)(\varkappa) - {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big| \\
&\leq \frac{\omega^{-2}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)\rho^{\beta}\Gamma(\beta)} \int_0^{\varkappa}\int_0^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}} \\
&\quad \times \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}} \mathfrak{T}(\mu,\nu)\,d\mu d\nu \\
&\leq L\frac{\omega^{-2}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)\rho^{\beta}\Gamma(\beta)} \int_0^{\varkappa}\int_0^{\varkappa} \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}} \\
&\quad \times \frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}} \big(g_1(\mu)-g_1(\nu)\big)^2 d\mu d\nu \\
&= L\Big({}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sg_1^2)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} s(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1^2)(\varkappa) \\
&\quad -2\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sg_1)(\varkappa)\Big).
\end{aligned} \tag{3.55}$$

This ends the proof. □

Theorem 3.12. *Suppose two integrable functions f_1 and g_1 defined on $[0,\infty)$ satisfying the assertions **I**, **II** and the lipschitzian condition with the constants $\mathcal{M}_1$ and $\mathcal{M}_2$ and let two continuous function r and s defined on $[0,\infty)$. Then the inequality*

$$\Big| {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta}(sf_1g_1)(\varkappa) + {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\beta} s(\varkappa)\, {}_{\omega}^{\Psi}\Omega_{0^+}^{\rho;\alpha}(rf_1g_1)(\varkappa)$$

$$
\begin{aligned}
&-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\Big|\\
&\leq \mathcal{M}_1\mathcal{M}_2\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\varkappa^2 s(\varkappa))+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\varkappa^2 r(\varkappa))\\
&-2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\varkappa r(\varkappa))\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\varkappa s(\varkappa))\Big)
\end{aligned}
\tag{3.56}
$$

holds for all $\rho\in(0,1],\alpha,\beta\in C$ *with* $\Re(\alpha),\Re(\beta)>0$.

Proof. By the given hypothesis, we have

$$
\Big|f_1(\mu)-f_1(\nu)\Big|\leq \mathcal{M}_1\Big|\mu-\nu\Big| \quad \Big|g_1(\mu)-g_1(\nu)\Big|\leq \mathcal{M}_2\Big|\mu-\nu\Big| \quad \mu,\nu\in[0,\infty), \tag{3.57}
$$

which implies that

$$
\Big|\mathfrak{T}(\mu,\nu)\Big|=\Big|f_1(\mu)-f_1(\nu)\Big|\Big|g_1(\mu)-g_1(\nu)\Big|\leq \mathcal{M}_1\mathcal{M}_2(\mu-\nu)^2. \tag{3.58}
$$

From (3.42) and (3.58), we obtain that

$$
\begin{aligned}
&\Big|{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1g_1)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa)\\
&-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\Big|\\
&\leq \frac{\omega^{-2}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\\
&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}\mathfrak{T}(\mu,\nu)d\mu d\nu\\
&\leq L\frac{\omega^{-2}(\varkappa)}{\rho^{\alpha}\Gamma(\alpha)\rho^{\beta}\Gamma(\beta)}\int_0^{\varkappa}\int_0^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]r(\mu)\omega(\mu)\Psi'(\mu)}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\\
&\times\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\nu))]s(\nu)\omega(\nu)\Psi'(\nu)}{(\Psi(\varkappa)-\Psi(\nu))^{1-\beta}}(\mu-\nu)^2 d\mu d\nu\\
&=\mathcal{M}_1\mathcal{M}_2\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\varkappa^2 s(\varkappa))+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\varkappa^2 r(\varkappa))\\
&-2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\varkappa r(\varkappa))\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\varkappa s(\varkappa))\Big).
\end{aligned}
\tag{3.59}
$$

This ends the proof.

□

Corollary 1. *Let* f_1 *and* g_1 *be two differentiable functions on* $[0,\infty)$ *and let* r *and* s *be two non-negative continuous functions on* $[0,\infty)$. *Then the inequality*

$$
\begin{aligned}
&\Big|{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sf_1g_1)(\varkappa)+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1g_1)(\varkappa)\\
&-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(sg_1)(\varkappa)-{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(sf_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(rg_1)(\varkappa)\Big|\\
&\leq \|f_1'\|_{\infty}\|g_1'\|_{\infty}\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}r(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\varkappa^2 s(\varkappa))+{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}s(\varkappa)\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\varkappa^2 r(\varkappa))
\end{aligned}
$$

$$-2\,{}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{0^+}(\varkappa r(\varkappa))\,{}^{\Psi}_{\omega}\Omega^{\rho;\beta}_{0^+}(\varkappa s(\varkappa))\Big) \tag{3.60}$$

holds for all $\rho \in (0,1], \alpha, \beta \in C$ *with* $\Re(\alpha), \Re(\beta) > 0$.

Proof. We have $f_1(\mu) - f_1(\nu) = \int_\nu^\mu f_1'(\varkappa)d\varkappa$ and $g_1(\mu) - g_1(\nu) = \int_\nu^\mu g_1'(\varkappa)d\varkappa$. That is, $\big|f_1(\mu) - f_1(\nu)\big| \leq \|f_1'\|_\infty\big|\mu-\nu\big|$, $\big|g_1(\mu) - g_1(\nu)\big| \leq \|g_1'\|_\infty\big|\mu-\nu\big|, \mu, \nu \in [0,\infty)$, and the immediate consequence follows from Theorem 3.12. This completes the proof. □

Example 3.13. Let $\rho,\ \alpha > 0,\ q_1, q_2 > 1$ with $q_1^{-1} + q_2^{-1} = 1$, and $\omega \neq 0$ be a function on $[0,\infty)$. Let f_1 be an integrable function defined on $[0,\infty)$ and ${}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1^+} f_1$ be the weighted generalized proportional fractional integral operator satisfying assumption **II**. Then we have

$$\Big|\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1^+} f_1\Big)(\varkappa)\Big| \leq \Theta\|(f_1 \circ \omega)(\mu)\|_{L_1(a_1,\varkappa)},$$

where

$$\Theta = \frac{\omega^{-1}(\varkappa)(-1)^{\alpha-1}}{\Gamma(\alpha)}\Big\{\Big(\frac{\rho}{q_1(\rho-1)}\Big)^{\alpha-1+1/q_1}\Big\}^{1/q_1}\Phi^{1/q_1}\Big(q_1(\alpha-1)+1, \frac{q_1(\rho-1)}{\rho}(\Psi(\varkappa)-\Psi(a_1))\Big)$$

and

$$\Phi(\alpha,\varkappa) = \int_0^\varkappa e^{-\nu}\nu^{\alpha-1}d\nu$$

is the incomplete gamma function [52, 53].

Proof. It follows from Definition 2.2 and the modulus property that

$$\Big|\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1^+} f_1\Big)(\varkappa)\Big| \leq \frac{\omega^{-1}(\varkappa)}{\rho^\alpha\Gamma(\rho)}\int_{a_1}^{\varkappa}\frac{\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]}{(\Psi(\varkappa)-\Psi(\mu))^{1-\alpha}}\Psi'(\mu)\big|f_1(\mu)\omega(\mu)\big|d\mu$$

for $\varkappa > a_1$.

Making use of the well-known Hölder inequality, we obtain

$$\Big|\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1^+} f_1\Big)(\varkappa)\Big| \leq \frac{\omega^{-1}(\varkappa)}{\rho^\alpha\Gamma(\rho)}\Big(\int_{a_1}^{\varkappa}\frac{q_1\exp[\frac{\rho-1}{\rho}(\Psi(\varkappa)-\Psi(\mu))]}{(\Psi(\varkappa)-\Psi(\mu))^{q_1(1-\alpha)}}\Psi'(\mu)d\mu\Big)^{1/q_1}\|f_1 \circ \omega(\mu)\|_{L_1(a_1,\varkappa)}.$$

Let $\theta = \Psi(\varkappa) - \Psi(\mu)$. Then elaborated computations lead to

$$\begin{aligned}\Big|\Big({}^{\Psi}_{\omega}\Omega^{\rho;\alpha}_{a_1^+} f_1\Big)(\varkappa)\Big| \quad &\leq \frac{(-1)^{\alpha-1}\omega^{-1}(\varkappa)}{\rho^\alpha\Gamma(\alpha)}\Big\{\Big(\frac{\rho}{q_1(\rho-1)}\Big)^{\alpha-1+1/q_1}\Big\}^{1/q_1}\\ &\times\Phi^{1/q_1}\Big(q_1(\alpha-1)+1, \frac{q_1(\rho-1)}{\rho}(\Psi(\varkappa)-\Psi(a_1))\Big)\|f_1 \circ \omega(\mu)\|_{L_1(a_1,\varkappa)}.\end{aligned}$$

□

4. Special cases

Here, we aim at present some new generalizations via weighted generalized proportional fractional, weighted generalized Riemann-Liouville and weighted Riemann-Liouville fractional integral operators, which are the new estimates of the main consequences.

Lemma 4.1. *Let f_1 and g_1 be two synchronous functions on* $[0,\infty)$. *Assume that Q and $\mathcal{P}$ be two non-negative continuous mappings on* $[0,\infty)$. *Then the inequality*

$$\begin{aligned} &{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P})(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Qf_1g_1)(\varkappa) + {}_{\omega}\Omega\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}f_1g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Q)(\varkappa) \\ &\geq {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}g_1)(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Qf_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}f_1)(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Qg_1)(\varkappa), \end{aligned}$$

holds for all $\rho \in (0,1], \alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. Letting $\Psi(\varkappa) = \varkappa$ and Lemma 3.1 yields the proof of Lemma 4.1. □

Lemma 4.2. *Let f_1 and g_1 be two synchronous functions on* $[0,\infty)$. *Assume that Q and $\mathcal{P}$ be two non-negative continuous mappings on* $[0,\infty)$. *Then the inequality*

$$\begin{aligned} &{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P})(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Qf_1g_1)(\varkappa) + {}_{\omega}\Omega\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}f_1g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Q)(\varkappa) \\ &\geq {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}g_1)(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Qf_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}f_1)(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Qg_1)(\varkappa), \end{aligned}$$

holds for all $\rho \in (0,1], \alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. Letting $\Psi(\varkappa) = \varkappa$ and Lemma 3.1 yields the proof of Lemma 4.2. □

Lemma 4.3. *Under the assumption of Lemma 3.1, then the inequality*

$$\begin{aligned} &{}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(\mathcal{P})(\varkappa)\,{}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(Qf_1g_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(\mathcal{P}f_1g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Q)(\varkappa) \\ &\geq {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(\mathcal{P}g_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(Qf_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}f_1)(\varkappa)\,{}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(Qg_1)(\varkappa), \end{aligned}$$

holds for all $\alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. Letting $\rho = 1$ and Lemma 3.1 yields the proof of Lemma 4.3. □

Lemma 4.4. *Under the assumption of Lemma 4.2, then the inequality*

$$\begin{aligned} &{}_{\omega}\Omega_{0^+}^{\alpha}(\mathcal{P})(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\alpha}(Qf_1g_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\alpha}(\mathcal{P}f_1g_1)(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(Q)(\varkappa) \\ &\geq {}_{\omega}\Omega_{0^+}^{\alpha}(\mathcal{P}g_1)(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\alpha}(Qf_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(\mathcal{P}f_1)(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\alpha}(Qg_1)(\varkappa), \end{aligned}$$

holds for all $\alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. Letting $\rho = 1$, $\Psi(\varkappa) = \varkappa$ and Lemma 3.1 yields the proof of Lemma 4.4. □

Theorem 4.5. *Let f_1 and g_1 be two synchronous functions on* $[0,\infty)$. *Assume that r, s and t be three non-negative continuous functions on* $[0,\infty)$. *Then the inequality*

$$\begin{aligned} &2\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}r(\varkappa)\Big({}_{\omega}\Omega_{0^+}^{\rho;\alpha}s(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(tf_1g_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(sf_1g_1)(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}t(\varkappa)\Big) \\ &+2\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}(rf_1g_1)(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}s(\varkappa)\,{}_{\omega}\Omega_{0^+}^{\rho;\alpha}t(\varkappa) \end{aligned}$$

$$
\begin{aligned}
&\geq {}_{\omega}\Omega_{0^+}^{\rho;\alpha} r(\varkappa)\Big({}_{\omega}\Omega_{0^+}^{\rho;\alpha}(sg_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(tf_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(tg_1)(\varkappa)\Big)\\
&\quad + {}_{\omega}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\Big({}_{\omega}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(tf_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(tg_1)(\varkappa)\Big)\\
&\quad + {}_{\omega}\Omega_{0^+}^{\rho;\alpha} s(\varkappa)\Big({}_{\omega}\Omega_{0^+}^{\rho;\alpha}(sg_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(rf_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(sf_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\rho;\alpha}(rg_1)(\varkappa)\Big)
\end{aligned}
$$

holds for all $\rho \in (0,1], \alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. Letting $\Psi(\varkappa) = \varkappa$ and Theorem 3.2 yields the proof of Theorem 4.5. □

Theorem 4.6. *Under the assumption of **I**, **II** and let* r, s *and* t *be three non-negative continuous functions on* $[0,\infty)$. *Then the inequality*

$$
\begin{aligned}
&2\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha} r(\varkappa)\Big({}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha} s(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(tf_1g_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(sf_1g_1)(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha} t(\varkappa)\Big)\\
&\quad +2\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(rf_1g_1)(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha} s(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha} t(\varkappa)\\
&\geq {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha} r(\varkappa)\Big({}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(sg_1)(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(tf_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(sf_1)(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(tg_1)(\varkappa)\Big)\\
&\quad + {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha} s(\varkappa)\Big({}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(rg_1)(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(tf_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(rf_1)(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(tg_1)(\varkappa)\Big)\\
&\quad + {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha} s(\varkappa)\Big({}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(sg_1)(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(rf_1)(\varkappa) + {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(sf_1)(\varkappa)\, {}^{\Psi}_{\omega}\Omega_{0^+}^{\alpha}(rg_1)(\varkappa)\Big)
\end{aligned}
$$

holds for all $\alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. Letting $\rho = 1$ and Theorem 3.2 yields the proof of Theorem 4.6. □

Theorem 4.7. *Under the assumption of Theorem 4.5, then the inequality*

$$
\begin{aligned}
&2\, {}_{\omega}\Omega_{0^+}^{\alpha} r(\varkappa)\Big({}_{\omega}\Omega_{0^+}^{\alpha} s(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha}(tf_1g_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\alpha}(sf_1g_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha} t(\varkappa)\Big)\\
&\quad +2\, {}_{\omega}\Omega_{0^+}^{\alpha}(rf_1g_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha} s(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha} t(\varkappa)\\
&\geq {}_{\omega}\Omega_{0^+}^{\alpha} r(\varkappa)\Big({}_{\omega}\Omega_{0^+}^{\alpha}(sg_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha}(tf_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\alpha}(sf_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha}(tg_1)(\varkappa)\Big)\\
&\quad + {}_{\omega}\Omega_{0^+}^{\alpha} s(\varkappa)\Big({}_{\omega}\Omega_{0^+}^{\alpha}(rg_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha}(tf_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\alpha}(rf_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha}(tg_1)(\varkappa)\Big)\\
&\quad + {}_{\omega}\Omega_{0^+}^{\alpha} s(\varkappa)\Big({}_{\omega}\Omega_{0^+}^{\alpha}(sg_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha}(rf_1)(\varkappa) + {}_{\omega}\Omega_{0^+}^{\alpha}(sf_1)(\varkappa)\, {}_{\omega}\Omega_{0^+}^{\alpha}(rg_1)(\varkappa)\Big)
\end{aligned}
$$

holds for all $\alpha \in C$ *with* $\Re(\alpha) > 0$.

Proof. Letting $\rho = 1,\ \Psi(\varkappa) = \varkappa$ and Theorem 3.2 yields the proof of Theorem 4.7. □

Remark 5. The computed results lead to the following conclusion:
(1) Setting $\rho = 1, \Psi(\varkappa) = \varkappa$ and $r(\varkappa) = s(\varkappa) = 1$, and using the relation (2.7), (2.8) and the assumption $\omega(\varkappa) = 1$, then Theorem 3.6 and Theorem 3.9 reduces to the known results due to Dahmani et al. [38].
(2) Setting $\rho = 1, \Psi(\varkappa) = \varkappa$ and using the relation (2.7), (2.8) and the assumption $\omega(\varkappa) = 1$, then Theorem 3.10–3.12, and Corollary 1 reduces to the known results due to Dahmani et al. [38] and Dahmani [40], respectively.

5. Conclusions

A new generalized fractional integral operator is proposed in this paper. The novel investigation is used to generate novel weighted fractional operators in the Riemann-Liouville, generalized Riemann-Liouville, Hadamard, Katugampola, Generalized proportional fractional, generalized Hadamard proportional fractional and henceforth, which effectively alleviates the adverse effect of another function Ψ and proportionality index ρ. Utilizing the weighted generalized proportional fractional operator technique, we derived the analogous versions of the extended Chebyshev and Grüss type inequalities that improve the accuracy and efficiency of the proposed technique. Contemplating the Remark 2 and 3, several existing results can be identified in the literature. Some innovative particular cases constructed by this method are tested and analyzed for statistical theory, fractional Schrödinger equation [20, 21]. The results show that the method proposed in this paper can stably and efficiently generate integral inequalities for convexity with better operators performance, thus providing a reliable guarantee for its application in control theory [54].

Conflict of interest

The authors declare that they have no competing interests.

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