Advances in Difference Equations
a SpringerOpen Journal

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Entire solutions of a class of algebraic Briot–Bouquet differential equations utilizing majority concept

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Abstract

In this effort, the analytic solution of a class of algebraic Briot–Bouquet differential equations (ABBDE) in the open unit disk is investigated by making use of a major theory. The class is presented by the formula

$$\alpha_1 \varphi'^3(z) + \alpha_2 \varphi'^2(z)\varphi(z) + \alpha_3 \varphi'(z)\varphi^2(z) + \aleph_\varphi^k(z) = 0,$$

$$\aleph_\varphi^k(z) := a_k \varphi^k(z) + a_{k-1}\varphi^{k-1}(z) + \cdots + a_1\varphi(z) + a_0.$$

The conformal analysis (angle-preserving) of the ABBDEs is considered. Analytic outcomes of the ABBDEs are indicated by employing the major method. Some special cases are investigated.

MSC: 30C55; 30C45

Keywords: Analytic function; Subordination; Univalent function; Open unit disk; Algebraic differential equations; Majorization method

1 Introduction

The algebraic Briot–Bouquet differential equation (ABBDE) is a significant nonlinear differential equation which occurs in many applications such as neuroscience, geometry, economy, banking studies, and physics. The most important property of ABBDE is that it shows that every meromorphic solution of ABBDE belongs to the class of meromorphic functions containing elliptic functions and their degenerates, i.e., the rational functions of one exponential $e^{\xi z}$, $\xi \in \mathbb{C}$, $z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and rational functions. Many consequences for higher-order ABBDE are also presented in a sequence of documents (see [1–12]).

The majorization inequality theory (real and complex) has many applications in our real life [13]. For example, they have measured the economy, specifically in the income-distribution (economic inequality), that is, the distribution of income between the primary factors of production. New economists have, furthermore, addressed issues of income distribution, but have absorbed further on the distribution of income persons and

households. Important theoretical and policy concerns consist of the balance between income majorization inequality and economic growth, and their often-inverse association. Moreover, the Gini coefficient measures the inequality among the values of a frequency distribution (for example, levels of income) [14]. A Gini coefficient of zero states perfect equality, where all values are the same (for example, where everyone has the same income) [15]. Therefore, the solution of any class of differential equations majoring by a convex function or by special function is very attractive, especially in wealth and health distribution [16].

In this paper, we investigate a class of ABBDEs in the open unit disk. We study the geometric representation for this class and majorize them by employing special function in $\mathbb{U}$ including the exponential function.

2 Complex ABBDEs

A special class of ABBDEs is studied in [7] taking the structure

$$\alpha_1 \varphi'^3(z) + \alpha_2 \varphi'^2(z)\varphi(z) + \alpha_3 \varphi'(z)\varphi^2(z) + \aleph_\varphi^k(z) = 0, \tag{2.1}$$

where $\alpha_\iota \in \mathbb{C}$, $\iota = 1, 2, 3$, and

$$\aleph_\varphi^k(z) := a_k \varphi^k(z) + a_{k-1}\varphi^{k-1}(z) + \cdots + a_1\varphi(z) + a_0,$$

where $a_\jmath$, $\jmath = 0, 1, \ldots, k$, are constants (real or complex).

Now, we reorganize (2.1) and consider the geometric possessions by designing some classes of normalized analytic functions in $\mathbb{U}$. Then the resolution is majorized by engaging special functions in $\mathbb{U}$. Equation (2.1) suggests the homogeneous form when $\alpha_1 \neq 0$,

$$\left(\frac{z\varphi'(z)}{\varphi(z)} \right)^2 + b_2 z\left(\frac{z\varphi'(z)}{\varphi(z)} \right) + b_3 z^2 = 0, \quad z \in \mathbb{U}, b_2 = \frac{\alpha_2}{\alpha_1}, b_3 = \frac{\alpha_3}{\alpha_1}. \tag{2.2}$$

To study Eq. (2.2) geometrically, we request the following theory.

Definition 2.1 An analytic function h is subordinated to an analytic function g, written $h \prec g$, if substitutes for an analytic function w with $|w(z)| \leq |z|$ such that $h = (g(w))$ (see [17]). The Ma-Minda classes $S^*(\rho)$ and $K(\rho)$ of starlike and convex functions respectively indicated by $(\frac{zh'(z)}{h(z)}) \prec \rho(z)$ and $(1 + \frac{zh''(z)}{h'(z)}) \prec \rho(z)$, where ρ has a positive real part in $\mathbb{U}$, $\rho(0) = 1$, $\rho'(0) > 1$ and maps $\mathbb{U}$ onto a starlike-domain.

We recognize our study by applying the above inequality to formulate the following special class.

Definition 2.2 A function of normalized expansion $\varphi(z) = z + \sum_{n=2}^\infty \varphi_n z^n$, $z \in \mathbb{U}$ is said to be in the class $\mathbf{M}(\rho)$ if and only if

$$P(z) := \left(\frac{z\varphi'(z)}{\varphi(z)} \right)^2 + b_2 \left(\frac{z\varphi'(z)}{\varphi(z)} \right) + b_3 z^2 \prec \rho(z), \tag{2.3}$$

$$\left(\rho(0) = 1, \rho'(0) > 1, z \in \mathbb{U} \right).$$

It is clear that $P(0) = 1$. For example, let $\varphi(z) = z/(1 - z)$, $z \in \mathbb{U}$, then

$$P(z) = 1 + (b_2 + 2)z + (b_2 + b_3 + 3)z^2 + (b_2 + 4)z^3 + (b_2 + 5)z^4 + (b_2 + 6)z^5 + O(z^6).$$

Also, for $\varphi(z) = z/(1 - z)^2$, $z \in \mathbb{U}$, we have

$$P(z) = 1 + (b_2 + 4)z + (2b_2 + b_3 + 8)z^2 + 2(b_2 + 6)z^3 + 2(b_2 + 8)z^4$$
$$+ 2(b_2 + 10)z^5 + O(z^6).$$

In the sequel, we theorize a starlike function with positive real part such as e^z (this function yields an oscillation solution of Eq. (2.1) [7]) and a convex function (univalent)

$$\rho_e(z) = \frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \cdots$$

as well as

$$\varrho_e(z) := 1/\rho_e(z) = 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \cdots$$

is convex univalent in $\mathbb{U}$ (see [17], p. 415). Note that the coefficients are converging to the Bernoulli numbers. Moreover, the real part of the function $\varrho_e(z) = (e^z - 1)/z$ satisfies the inequality

$$\Re\left(\frac{e^{\eta z} - 1}{\eta z}\right) \geq \frac{1}{2}, \quad 0 < \eta \leq 1.793\ldots.$$

Hence, $\Re(\frac{e^{\eta z} - 1}{\eta z}) \geq 1/\rho_e(-1) = \frac{1}{2}$.

Our calculation is established by the analytic method of Caratheodory functions which are characterized in [18]. Consequently, by making use of major theory, we approximate $\aleph_\varphi^k(z)$ by a special type of $\rho(z)$, $z \in \mathbb{U}$ denoted by $\aleph_\varphi^k(z) \ll \rho(z)$. Note that the two functions are under majorization if and only if $|a_J| \leq |\rho_J|$ for all $J = 1, 2, \ldots$, where a_J and ρ_J are the coefficients of $\aleph_\varphi^k(z)$ and $\rho(z)$ respectively. In this situation, we demonstrate adequate conditions of the coefficient constraints of $\aleph_\varphi^k(z)$, for altered values of $k = 0, 1, \ldots$, using a Caratheodory theory.

3 Computations
In this position, we demonstrate our theoretical results.

Theorem 3.1 *Let the function $\varphi \in M(\rho)$ achieve the inequality*

$$1 + \Upsilon\left(\frac{zP'(z)}{[P(z)]^k}\right) \prec z + \sqrt{z^2 + 1}, \quad k = 0, 1, 2,$$

where Υ is a constant and $P(z) = (\frac{z\varphi'(z)}{\varphi(z)})^2 + b_2 z(\frac{z\varphi'(z)}{\varphi(z)}) + b_3 z^2$. Then

$$P(z) \prec \rho_e(z) = \frac{z}{e^z - 1}, \quad z \in \mathbb{U}.$$

When $\Upsilon \geq \max \Upsilon_k$,

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$$\min \Upsilon_0 = \frac{-((e-1)(-2 + \sqrt{2} + \log(2) - \log(1 + \sqrt{2})))}{(e-2)} \approx 1.8516\ldots$$

and

$$\max \Upsilon_0 = (e-1)\left(\sqrt{2} + \log(2) + \log(\sqrt{2} - 1)\right) \approx 2.106\ldots$$

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$$\min \Upsilon_1 = \frac{(2 - \sqrt{2} - \log(2) + \log(1 + \sqrt{2}))}{\log(e-1)} \approx 1.5\ldots$$

and

$$\max \Upsilon_1 = \frac{(-\sqrt{(2)} - \log(2) - \log(\sqrt{(2)} - 1))}{(\log(e-1) - 1)} \approx 2.839\ldots$$

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$$\min \Upsilon_2 = \frac{2 - \sqrt{2} + \log(1/2 + 1/\sqrt{2})}{(e-2)} \approx 1.077\ldots$$

and

$$\max \Upsilon_2 = e\left(\sqrt{2} + \log(2) - \log(1 + \sqrt{2})\right) \approx 3.33\ldots.$$

Proof Case I: $k = 0 \Rightarrow 1 + \Upsilon(zP'(z)) \prec z + \sqrt{z^2 + 1}$.

Set a function $\Gamma_\Upsilon : \mathbb{U} \to \mathbb{C}$ admitting the structure

$$\Gamma_\Upsilon(z) = 1 + \frac{1}{\Upsilon}\left(z + \sqrt{z^2 + 1} - \log\left(1 + \sqrt{z^2 + 1}\right) - 1 + \log(2)\right).$$

Clearly, $\Gamma_\Upsilon(z)$ is analytic in $\mathbb{U}$ such that $\Gamma_\Upsilon(0) = 1$ and it is the outcome of the differential equation

$$1 + \Upsilon\left(z\Gamma'_\Upsilon(z)\right) = z + \sqrt{z^2 + 1}, \quad z \in \mathbb{U}. \tag{3.1}$$

Consequently, we have $\mathfrak{Q}(z) := \Upsilon(z\Gamma'_\Upsilon(z)) = z + \sqrt{z^2 + 1} - 1$ is starlike in $\mathbb{U}$. Then for $\mathfrak{J}(z) := \mathfrak{Q}(z) + 1$, we conclude that

$$\Re\left(\frac{z\mathfrak{Q}'(z)}{\mathfrak{Q}(z)}\right) = \Re\left(\frac{z\mathfrak{J}'(z)}{\mathfrak{Q}(z)}\right) > 0.$$

Thus, the Miller–Mocanu lemma (see [17], p. 132) yields that

$$1 + \Upsilon\left(zP'(z)\right) \prec 1 + \Upsilon z\Gamma'_\Upsilon(z) \quad \Rightarrow \quad P(z) \prec \Gamma_\Upsilon(z).$$

To finish this event, we only need to prove that $\Gamma_\Upsilon(z) \prec \rho_e(z)$. Clearly, the function $\Gamma_\Upsilon(z)$ is increasing in the interval $(-1, 1)$ that has the inequality

$$\Gamma_\Upsilon(-1) \leq \Gamma_\Upsilon(1).$$

Because the function $\rho_e(z)$ achieves the inequality for real ϑ

$$\frac{1}{e-1} \leq \Re\big(\rho_e(z)\big) \approx 1 - \frac{\cos(\vartheta)}{2} + \sum_{n=1}^{\infty} \frac{\beta_{2n}\cos(2n\vartheta)}{(2n)!} \leq \frac{e}{e-1},$$

then the consequence inequality holds

$$\frac{1}{e-1} \leq \Gamma_\Upsilon(-1) \leq \Gamma_\Upsilon(1) \leq \frac{e}{e-1}$$

if Υ fulfills the upper and lower bounds

$$\min \Upsilon_0 = \frac{-((e-1)(-2+\sqrt{2}+\log(2)-\log(1+\sqrt{2})))}{(e-2)} \approx 1.8516\ldots$$

and

$$\max \Upsilon_0 = (e-1)\big(\sqrt{2}+\log(2)+\log(\sqrt{2}-1)\big) \approx 2.106\ldots.$$

This yields the subordination inequalities

$$\Gamma_\Upsilon(z) \prec \frac{z}{e^z-1} \quad \Rightarrow \quad P(z) \prec \frac{z}{e^z-1}, \quad z \in \mathbb{U}.$$

Case II: $k = 1 \Rightarrow 1 + \Upsilon\big(\frac{zP'(z)}{P(z)}\big) \prec z + \sqrt{z^2+1}$.
Now we present a function $\Pi_\Upsilon : \mathbb{U} \to \mathbb{C}$ structuring by the formula

$$\Pi_\Upsilon(z) = \exp\left(\frac{1}{\Upsilon}\big(z+\sqrt{z^2+1}-\log\big(1+\sqrt{z^2+1}\big)-1+\log(2)\big)\right).$$

Obviously, $\Pi_\Upsilon(z)$ is analytic in $\mathbb{U}$ having $\Pi_\Upsilon(0) = 1$ and it is an outcome of the differential equation

$$1 + \Upsilon\left(\frac{z\Pi'_\Upsilon(z)}{\Pi_\Upsilon(z)}\right) = z + \sqrt{z^2+1}, \quad z \in \mathbb{U}. \tag{3.2}$$

By using $\mathfrak{Q}(z) = z + \sqrt{z^2+1} - 1$, which is starlike in $\mathbb{U}$ and $\mathfrak{J}(z) = \mathfrak{Q}(z) + 1$, we get

$$\Re\left(\frac{z\mathfrak{Q}'(z)}{\mathfrak{Q}(z)}\right) = \Re\left(\frac{z\mathfrak{J}'(z)}{\mathfrak{Q}(z)}\right) > 0, \quad z \in \mathbb{U}.$$

Then again, by Miller–Mocanu lemma, we obtain

$$1 + \Upsilon\left(\frac{zP'(z)}{P(z)}\right) \prec 1 + \Upsilon\left(\frac{z\Pi'_\Upsilon(z)}{\Pi_\Upsilon(z)}\right) \quad \Rightarrow \quad P(z) \prec \Pi_\Upsilon(z).$$

Consequently, the following inequality holds:

$$\frac{1}{e-1} \leq \Pi_\Upsilon(-1) \leq \Pi_\Upsilon(1) \leq \frac{e}{e-1}$$

if Υ has the upper and lower bounds

$$\min \Upsilon_1 = \frac{(2 - \sqrt{2} - \log(2) + \log(1 + \sqrt{2}))}{\log(e-1)} \approx 1.5\ldots$$

and

$$\max \Upsilon_1 = \frac{(-\sqrt{2} - \log(2) - \log(\sqrt{2} - 1))}{(\log(e-1) - 1)} \approx 2.839\ldots.$$

This indicates the subordination inequalities

$$\Pi_\Upsilon(z) \prec \frac{z}{e^z - 1} \quad \Rightarrow \quad P(z) \prec \frac{z}{e^z - 1}, \quad z \in \mathbb{U}.$$

Case III: $k = 2 \Rightarrow 1 + \Upsilon(\frac{zP'(z)}{P^2(z)}) \prec z + \sqrt{z^2 + 1}$.

Now, we introduce a function $\Pi_\Upsilon : \mathbb{U} \to \mathbb{C}$ having the formal

$$\Theta_\Upsilon(z) = \left(1 - \frac{1}{\Upsilon}\left(z + \sqrt{z^2 + 1} - \log\left(1 + \sqrt{z^2 + 1}\right) - 1 + \log(2)\right) \right)^{-1}.$$

Clearly, $\Pi_\Upsilon(z)$ is analytic in $\mathbb{U}$ admitting $\Pi_\Upsilon(0) = 1$, and it is the outcome of the differential equation

$$1 + \Upsilon\left(\frac{z\Theta'_\Upsilon(z)}{\Theta_\Upsilon(z)}\right) = z + \sqrt{z^2 + 1}, \quad z \in \mathbb{U}. \tag{3.3}$$

By utilizing $\mathfrak{Q}(z) = z + \sqrt{z^2 + 1} - 1$, which is starlike in $\mathbb{U}$ and $\mathfrak{J}(z) = \mathfrak{Q}(z) + 1$, we get

$$\Re\left(\frac{z\mathfrak{Q}'(z)}{\mathfrak{Q}(z)}\right) = \Re\left(\frac{z\mathfrak{J}'(z)}{\mathfrak{J}(z)}\right) > 0, \quad z \in \mathbb{U}.$$

Consequently, by Miller–Mocanu lemma, we get

$$1 + \Upsilon\left(\frac{zP'(z)}{P^2(z)}\right) \prec 1 + \Upsilon\left(\frac{z\Theta'_\Upsilon(z)}{\Theta^2_\Upsilon(z)}\right) \quad \Rightarrow \quad P(z) \prec \Theta_\Upsilon(z).$$

Accordingly, we attain

$$\frac{1}{e-1} \leq \Theta_\Upsilon(-1) \leq \Theta_\Upsilon(1) \leq \frac{e}{e-1}$$

if Υ_2 admits the upper and lower bounds

$$\min \Upsilon_2 = \frac{2 - \sqrt{2} + \log(1/2 + 1/\sqrt{2})}{(e-2)} \approx 1.077\ldots$$

and

$$\max \Upsilon_2 = e\left(\sqrt{2} + \log(2) - \log(1 + \sqrt{2})\right) \approx 3.33\ldots.$$

This yields the subordination inequalities

$$\Theta_\Upsilon(z) \prec \frac{z}{e^z - 1} \quad \Rightarrow \quad P(z) \prec \frac{z}{e^z - 1}, \quad z \in \mathbb{U}. \qquad \square$$

The next result studies the subordination with respect to the function

$$\varrho_e(z) = \frac{e^z - 1}{z}, \quad z \in \mathbb{U}.$$

Theorem 3.2 *Let the assumptions of Theorem* 3.1 *hold for a constant λ where*

$$1 + \lambda\left(\frac{zP'(z)}{[P(z)]^k}\right) \prec z + \sqrt{z^2 + 1}, \quad k = 0, 1, 2.$$

Then

$$P(z) \prec \varrho_e(z) = \frac{e^z - 1}{z}, \quad z \in \mathbb{U}.$$

When $\lambda \geq \max \lambda_k$,

- $$\min \lambda_0 = \frac{(\sqrt{2} + \log(2) + \log(\sqrt{2} - 1))}{(e - 2)} \approx 1.706\ldots$$

  and

  $$\max \lambda_0 = -e\left(-2 + \sqrt{2} + \log(2) - \log\left(1 + \sqrt{(2)}\right)\right) \approx 2.10399\ldots.$$

- $$\min \lambda_1 = \frac{(-2 + \sqrt{2} + \log(2) + \log(\sqrt{2} - 1))}{(\log(e - 1) - 1)} \approx 1.70\ldots$$

  and

  $$\max \lambda_1 = \left(\sqrt{2} + \log(2) + \log(\sqrt{2} - 1)\right)/\log(e - 1) \approx 2.2\ldots.$$

- $$\min \lambda_2 = -(e - 1)\left(-2 + \sqrt{2} + \log(2) - \log(1 + \sqrt{2})\right) \approx 1.329\ldots$$

  and

  $$\max \lambda_2 = \frac{((e - 1)(\sqrt{2} + \log(2) - \log(1 + \sqrt{2})))}{(e - 2)} \approx 2.932\ldots.$$

Proof State the convex univalent function $\varrho_e(z) = \frac{e^z - 1}{z}$. It is clear that $\varrho(0) = 1$ with a positive real part. Moreover, it satisfies the inequality

$$\frac{e-1}{e} \leq \Re\big(\varrho_e(z)\big) \leq e - 1, \quad z \in \mathbb{U}.$$

By the conversation of Theorem 3.1, we indicate the following inequality:

$$\frac{e-1}{e} \leq \Gamma_\lambda(-1) \leq \Gamma_\lambda(1) \leq e - 1$$

if λ has the upper and lower bounds

$$\min \lambda_0 = \frac{(\sqrt{2} + \log(2) + \log(\sqrt{2} - 1))}{(e - 2)} \approx 1.706\ldots$$

and

$$\max \lambda_0 = -e\big(-2 + \sqrt{2} + \log(2) - \log\big(1 + \sqrt{(2)}\big)\big) \approx 2.10399\ldots.$$

This establishes the subordination inequalities

$$\Gamma_\lambda(z) \prec \frac{e^z - 1}{z} \quad \Rightarrow \quad P(z) \prec \frac{e^z - 1}{z}, \quad z \in \mathbb{U}.$$

Similarly, we have

$$\min \lambda_1 = \frac{(-2 + \sqrt{2} + \log(2) + \log(\sqrt{2} - 1))}{(\log(e - 1) - 1)} \approx 1.70\ldots$$

and

$$\max \lambda_1 = \big(\sqrt{2} + \log(2) + \log(\sqrt{2} - 1)\big)/\log(e - 1) \approx 2.2\ldots.$$

This yields the subordination inequalities

$$\Pi_\lambda(z) \prec \frac{e^z - 1}{z} \quad \Rightarrow \quad P(z) \prec \frac{e^z - 1}{z}, \quad z \in \mathbb{U}.$$

Lastly, we get the upper and lower bounds

$$\min \lambda_2 = -(e - 1)\big(-2 + \sqrt{2} + \log(2) - \log(1 + \sqrt{2})\big) \approx 1.329\ldots$$

and

$$\max \lambda_2 = \frac{((e - 1)(\sqrt{2} + \log(2) - \log(1 + \sqrt{2})))}{(e - 2)} \approx 2.932\ldots.$$

This admits the subordination inequalities

$$\Theta_\lambda(z) \prec \frac{e^z - 1}{z} \quad \Rightarrow \quad P(z) \prec \frac{e^z - 1}{z}, \quad z \in \mathbb{U}. \qquad \square$$

4 Majorization

We progress to involve the formula

$$\aleph_\varphi^k(z) = a_k\varphi^k(z) + a_{k-1}\varphi^{k-1}(z) + \cdots + a_1\varphi(z) + a_0$$

for some k to investigate the geometric solutions of Eq. (2.1). Dividing Eq. (2.1) by $\alpha_1 \neq 0$, we receive

$$\varphi'^3(z) - \frac{\alpha_2}{\alpha_1}\varphi'^2(z)\varphi(z) - \frac{\alpha_3}{\alpha_1}\varphi'(z)\varphi^2(z) = -\frac{\aleph_\varphi^k(z)}{\alpha_1}, \quad z \in \mathbb{C}. \tag{4.1}$$

We have the following result.

Theorem 4.1 *Consider CADE* (4.1) *with* $\alpha_1 = -1$ *and* $a_0 = 1$. *If* $\varphi \in M(\rho)$ *is a convex univalent function in* $\mathbb{U}$ *fulfilling the restriction of Theorem* 3.1, *then the communication values* a_ι *have the following construction*:

$$a_1 = -\frac{1}{2}, \qquad a_2 = \frac{7}{12}, \qquad a_3 = -\frac{8}{12}, \qquad a_4 = \frac{74}{100}, \qquad a_5 = -\frac{79}{100}. \tag{4.2}$$

Proof From Eq. (4.1) together with Theorem 3.1, we have $\aleph_\varphi^k(z) \prec \rho_e(z)$. Since φ is convex univalent in $\cup$, then it indicates the extreme function construction

$$\varphi(z) = \frac{z}{(1-z)} = z + z^2 + \cdots.$$

Thus, we obtain

$$\aleph_\varphi^0(z) = 1,$$

$$\aleph_\varphi^1(z) = 1 + a_1z + a_1z^2 + a_1z^3 + a_1z^4 + a_1z^5 + O(z^6),$$

$$\aleph_\varphi^2(z) = 1 + a_1z + (a_1 + a_2)z^2 + (a_1 + 2a_2)z^3 + (a_1 + 3a_2)z^4 + (a_1 + 4a_2)z^5 + O(z^6),$$

$$\aleph_\varphi^3(z) = 1 + a_1z + (a_1 + a_2)z^2 + (a_1 + 2a_2 + a_3)z^3 + \big(a_1 + 3(a_2 + a_3)\big)z^4$$
$$\qquad + (a_1 + 4a_2 + 6a_3)z^5 + O(z^6),$$

$$\aleph_\varphi^4(z) = 1 + a_1z + (a_1 + a_2)z^2 + (a_1 + 2a_2 + a_3)z^3 + (a_1 + 3a_2 + 3a_3 + a_4)z^4$$
$$\qquad + (a_1 + 4a_2 + 6a_3 + 4a_4)z^5 + O(z^6),$$

$$\aleph_\varphi^5(z) = 1 + a_1z + (a_1 + a_2)z^2 + (a_1 + 2a_2 + a_3)z^3 + (a_1 + 3a_2 + 3a_3 + a_4)z^4$$
$$\qquad + (a_1 + 4a_2 + 6a_3 + 4a_4 + a_5)z^5 + O(z^6),$$

$$\aleph_\varphi^6(z) = 1 + a_1z + (a_1 + a_2)z^2 + (a_1 + 2a_2 + a_3)z^3 + (a_1 + 3a_2 + 3a_3 + a_4)z^4$$
$$\qquad + (a_1 + 4a_2 + 6a_3 + 4a_4 + a_5)z^5 + O(z^6),$$

$$\vdots$$

Moreover, we attain that

$$\rho_e(z) = \frac{z}{e^z - 1} = \sum_{n=0}^\infty \frac{B_n z^n}{n!},$$

where B_n is the Bernoulli numbers having the inequality

$$|B_n| \ll 4\sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2n}, \qquad B_{2n+1} = 0,$$

$$(B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30, B_6 = 1/42).$$

Comparing the coefficients of $\aleph_\varphi^k(z)$ and $\rho_e(z)$, we have

$$a_1 = \frac{B_1}{1!} = -\frac{1}{2},$$

$$a_2 = -a_1 + \frac{B_2}{2!} = \frac{7}{12},$$

$$a_3 = -a_1 - 2a_2 + \frac{B_3}{3!} = -\frac{8}{12},$$

$$a_4 = -a_1 - 3a_2 - 3a_3 + \frac{B_4}{4!} = \frac{74}{100},$$

$$a_5 = -a_1 - 4a_2 - 6a_3 - 4a_4 + \frac{B_5}{5!} = -\frac{79}{100}. \qquad \square$$

The next result indicates the value of constant coefficients of $\aleph_\varphi^k$ when φ is starlike in $\mathbb{U}$.

Theorem 4.2 *Consider CADE* (4.1) *with* $\alpha_1 = -1$ *and* $a_0 = 1$. *If* $\varphi \in M(\rho)$ *is a starlike function in* $\mathbb{U}$ *fulfilling the construction of Theorem* 3.1, *then the communication values* a_ι *have the construction*

$$a_1 = -\frac{1}{2}, \qquad a_2 = \frac{13}{12}, \qquad a_3 = -\frac{28}{10}, \qquad a_4 = \frac{795}{100}, \qquad a_5 = -24. \qquad (4.3)$$

Proof Plainly, from the conditions, we get $\aleph_\varphi^k(z) \prec \rho_e(z)$. Since φ is starlike in $\cup$, then it has the extreme function formula

$$\varphi(z) = \frac{z}{(1-z)^2} = z + 2z^2 + \cdots.$$

Thus, we get

$$\aleph_\varphi^0(z) = 1,$$

$$\aleph_\varphi^1(z) = 1 + a_1 z + 2a_1 z^2 + 3a_1 z^3 + 4a_1 z^4 + 5a_1 z^5 + O(z^6),$$

$$\aleph_\varphi^2(z) = 1 + a_1 z + (2a_1 + a_2)z^2 + (3a_1 + 4a_2)z^3 + (4a_1 + 10a_2)z^4$$
$$+ 5(a_1 + 4a_2)z^5 + O(z^6),$$

$$\aleph_\varphi^3(z) = 1 + a_1 z + (2a_1 + a_2)z^2 + (3a_1 + 4a_2 + a_3)z^3 + (4a_1 + 10a_2 + 6a_3)z^4$$
$$+ (5a_1 + 20a_2 + 21a_3)z^5 + O(z^6),$$

$$\aleph_\varphi^4(z) = 1 + a_1 z + (2a_1 + a_2)z^2 + (3a_1 + 4a_2 + a_3)z^3 + (4a_1 + 10a_2 + 6a_3 + a_4)z^4$$
$$+ (5a_1 + 20a_2 + 21a_3 + 8a_4)z^5 + O(z^6),$$

$$\aleph_\varphi^5(z) = 1 + a_1 z + (2a_1 + a_2)z^2 + (3a_1 + 4a_2 + a_3)z^3 + (4a_1 + 10a_2 + 6a_3 + a_4)z^4$$

$$+ (5a_1 + 20a_2 + 21a_3 + 8a_4 + a_5)z^5 + O(z^6),$$

$$\aleph_\varphi^6(z) = 1 + a_1 z + (2a_1 + a_2)z^2 + (3a_1 + 4a_2 + a_3)z^3 + (4a_1 + 10a_2 + 6a_3 + a_4)z^4$$

$$+ (5a_1 + 20a_2 + 21a_3 + 8a_4 + a_5)z^5 + O(z^6),$$

$$\vdots$$

Contrasting the connections of $\aleph_\varphi^m(z)$ and $\rho_e(z)$, we have

$$a_1 = \frac{B_1}{1!} = -\frac{1}{2},$$

$$a_2 = -2a_1 + \frac{B_2}{2!} = \frac{13}{12},$$

$$a_3 = -3a_1 - 4a_2 + \frac{B_3}{3!} = -\frac{28}{10},$$

$$a_4 = -4a_1 - 10a_2 - 6a_3 + \frac{B_4}{4!} = \frac{795}{100},$$

$$a_5 = -5a_1 - 20a_2 - 21a_3 - 8a_4 + \frac{B_5}{5!} = -24. \qquad \square$$

Remark 4.3

- Note that Theorems 4.1 and 4.2 indicate that $\aleph_\varphi^k(z)$ accumulates at $k = 5$, which yields the expansion formulas

$$\aleph_{z/(1-z)}^5 = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{100} + O(z^6)$$

and

$$\aleph_{z/(1-z)^2}^5 = 1 - \frac{z}{2} + \frac{8z^2}{100} - \frac{2z^4}{100} + O(z^6).$$

- One can extend Theorems 4.1 and 4.2 in expressions of α for all estimates. In this suggestion, we get the constant connections $\Bbbk_l = \frac{a_l}{-\alpha_1}$ provided $\alpha \neq 0$

$$\Bbbk_1 = -\frac{1}{2}, \qquad \Bbbk_2 = \frac{7}{12}, \qquad \Bbbk_3 = -\frac{8}{12}, \qquad \Bbbk_4 = \frac{74}{100}, \qquad \Bbbk_5 = -\frac{79}{100}$$

and

$$\Bbbk_1 = -\frac{1}{2}, \qquad \Bbbk_2 = \frac{13}{12}, \qquad \Bbbk_3 = -\frac{28}{10}, \qquad \Bbbk_4 = \frac{795}{100}, \qquad \Bbbk_5 = -24,$$

respectively.

Abbreviations
ABBDE, Algebraic Briot–Bouquet differential equations.

Availability of data and materials
Not applicable.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Consent for publication
Not applicable.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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