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Research Article

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Exact solutions of the Laplace fractional boundary value problems via natural decomposition method

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Abstract: In this article, exact solutions of some Laplace-type fractional boundary value problems (FBVPs) are investigated via natural decomposition method. The fractional derivatives are described within Caputo operator. The natural decomposition technique is applied for the first time to boundary value problems (BVPs) and found to be an excellent tool to solve the suggested problems. The graphical representation of the exact and derived results is presented to show the reliability of the suggested technique. The present study is mainly concerned with the approximate analytical solutions of some FBVPs. Moreover, the solution graphs have shown that the actual and approximate solutions are very closed to each other. The comparison of the proposed and variational iteration methods is done for integer-order problems. The comparison, support strong relationship between the results of the suggested techniques. The overall analysis and the results obtained have confirmed the effectiveness and the simple procedure of natural decomposition technique for obtaining the solution of BVPs.

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1 Introduction

In the last few centuries, fractional partial differential equations (FPDEs) have been effectively used to model several structures and processes that can be used to develop their mathematical models. These processes are still underway, but some new correlations and insights between the various branches of mathematical reasoning have been already emerged within the structure of various models in the field of fractional calculus (FC) [1–4]. The study connected with the concept of FPDEs and their implementations is very broad. For the study of FPDEs, the books cited in refs [5,6] and [7] are suggested for readers and for implementation we refer the book given in ref. [8], which is entirely devoted to the various applications of FCs in physics, astrophysics, chemistry, etc. The book [8] describes the application of FPDEs in nuclear physics, classical mechanics, quantum mechanics, hadron spectroscopy, group theory and quantum field theory. For other important FC models and recent literature, the books in refs [9–12] are recommended for further study. In implementations, relevant processes, such as acoustic waves or anomalous convection in complex schemes, mainly run within certain bound domains in space are corresponding to the initial and boundary value problem (BVP) equations of the FPDEs that designed the procedure under evaluation [20–25]. Meanwhile, the numerical and analytical investigations are considered to be the hard topic for the solutions of FPDEs given in refs [13–19,26].

The BVPs of FPDEs play a vital and effective role in many recent processes of applied sciences. The BVPs arise in a wide range of issues, such as convection and heat transfer, the simulation of chemical processes and optimal control solution. The numerical and analytical techniques that are used for the solution of BVPs are of significant importance in different research areas of engineering and technology. For FPDEs, refs [27,28] and references therein consist of several methods that presented the approximate solution of BVPs of FPDEs. However, no universal method for handling all kinds of non-linear BVPs is available. The Adomian decomposition method (ADM) [29–32] has been recently implemented to analyze BVPs. ADM provides the solution in the form of infinite series having a quick rate of convergence toward the actual solution. Jang [33] has developed an extended ADM (EADM) to address the solution of BVPs. The key point of EADM is to develop a canonical form that contains all boundary conditions (BCs). Mohsen and El-Gamel [34] have investigated the efficiency of Galerkin and collocation techniques using sinc basis methods to solve BVPs of linear and non-linear second order. In ref. [35], the sinc-collocation approach is applied for the solution of a series of problems with second-order BVPs. Dehghan et al. [36] have applied the Adomian–Pade method for the solution to solve Volterra functional systems of equations.

In 2014, Rawashdeh and Maitama have developed a new technique which is known as natural decomposition method (NDM), which is a mixture of natural transformation (NT) and ADM. The suggested technique achieves the series form solution with infinite terms having quick convergence toward the actual solution of the problems; for example, Prakasha et al. have provided the proof of two basic results of fractional NT by using the duality of Laplace and natural transforms [37]. Many researchers investigated the solutions of various FPDEs recently, for example, Harry Dym equation [38], system Burger problems [37], hyperbolic equation [39], wave and hate equations [40] and diffusion equations [41].

In this article, the NDM is implemented to solve Laplace BVPs of FPDEs. The Caputo operator is implemented to express fractional derivatives in each problem. The NDM solutions are determined for three particular examples of fractional BVPs (FBVPs) of the Laplace equations. The results are displayed through graphs and table. The higher efficiency and accuracy of NDM is observed with the help of graphs and table (Table 1). The NDM solution for BVPs has shown the desired rate of convergence, and thus the present technique can be selected to solve other FBVPs.

2 Preliminary concept

2.1 Definition

Riemann–Liouville (R–L) definition of fractional-integral operator [5–7]:

$$I_x^{\gamma} f(x) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-\omega)^{\gamma-1} g(\omega)\mathrm{d}\omega \quad \text{if } \gamma > 0,$$
$$= f(x) \quad \text{if } \gamma = 0.$$

2.2 Definition

The Caputo operator of order γ for $m \in \mathbb{N}$, $x > 0$, $f \in \mathbb{C}_t$, $t \geq -1$ [5–7]:

$$D^{\gamma} f(x) = \frac{\partial^{\gamma} f(x)}{\partial t^{\gamma}} = I^{m-\gamma}\left[\frac{\partial^{\gamma} f(x)}{\partial t^{\gamma}}\right],$$
$$= \frac{\partial^{\gamma} f(x)}{\partial t^{\gamma}}, \quad \text{if } m-1 < \gamma \leq m, m \in \mathbb{N}.$$

2.3 Definition

The NT of $f(t)$ is given as [37,38]

$$N^{+}[f(t)] = R(s,u) = \frac{1}{u}\int_0^{\infty} e^{\frac{-st}{u}} f(t)\mathrm{d}t; \quad s, u > 0.$$

2.4 Definition

The NT inverse of $f(t)$ is given by [37,38]

$$N^{-1}[R(s,u)] = f(t) = \frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} e^{\frac{st}{u}} R(s,u)\mathrm{d}s.$$

2.5 Definition

The NT of the nth-derivative of $f(t)$ is expressed as [37,38]

$$N[f^n(t)] = R_n(s,u)$$
$$= \frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}} f^k(0), \quad n \geq 1.$$

3 Basic idea of NDM

Here, the general solution of FPDE is analyzed by using NDM [37,38].

Table 1: NDM and variational iteration method (VIM) solutions of example 1 at various fractional orders and $y = 0.8$

x	|Exact − NDM| $\gamma = 1.5$	|Exact − NDM| $\gamma = 1.8$	|Exact − VIM| $\gamma = 2$	|Exact − NDM| $\gamma = 2$
0.1	$9.4756734 \times 10^{-05}$	$2.2341275 \times 10^{-07}$	1.00×10^{-10}	1.00×10^{-10}
0.2	$2.4281834 \times 10^{-04}$	6.5405157×10^{-07}	3.00×10^{-10}	3.00×10^{-10}
0.3	4.1427333×10^{-04}	1.1973487×10^{-06}	1.00×10^{-10}	1.00×10^{-10}
0.4	6.0107726×10^{-04}	1.8174793×10^{-06}	1.00×10^{-10}	1.00×10^{-10}
0.5	8.0017319×10^{-04}	2.4965681×10^{-06}	1.00×10^{-10}	1.00×10^{-10}
0.6	1.0106683×10^{-03}	3.2257308×10^{-06}	1.10×10^{-09}	1.10×10^{-09}
0.7	1.2328432×10^{-03}	4.0014626×10^{-06}	6.00×10^{-09}	6.00×10^{-09}
0.8	1.4677104×10^{-03}	4.8238379×10^{-06}	2.11×10^{-08}	2.11×10^{-08}
0.9	1.7167937×10^{-03}	5.6955055×10^{-06}	6.90×10^{-08}	6.90×10^{-08}
1.0	1.9820106×10^{-03}	6.6210933×10^{-06}	1.99×10^{-07}	1.99×10^{-07}

$$D^{\gamma}\phi(x,t) + L\phi(x,t) + N\phi(x,t) = f(x,t),\quad x,t \geq 0,\quad m-1<\gamma<m, \tag{1}$$

where γ describes the fractional-order Caputo-type derivative, $m \in N$ and L and N are, respectively, the linear and nonlinear terms and $f(x,t)$ is the source term.

The initial known solution is

$$\phi(x,0) = k(x), \tag{2}$$

where $k(x)$ is the constant or function of x only. Using the NT in equation (1), we get

$$N^{+}[D^{\gamma}\phi(x,t)] + N^{+}[L\phi(x,t) + N\phi(x,t)] = N^{+}[f(x,t)], \tag{3}$$

and applying NT property of derivative, we get

$$\frac{s^{\gamma}}{u^{\gamma}}N^{+}[\phi(x,t)] - \frac{s^{\gamma-1}}{u^{\gamma}}\phi(x,0) = N^{+}[f(x,t)] - N^{+}[L\phi(x,t) + N\phi(x,t)],$$

$$N^{+}[\phi(x,t)] = \frac{1}{s}\phi(x,0) + \frac{u^{\gamma}}{s^{\gamma}}N^{+}[f(x,t)] - \frac{u^{\gamma}}{s^{\gamma}}N^{+}[L\phi(x,t) + N\phi(x,t)].$$

Now $\phi(x,0) = k(x)$,

$$N^{+}[\phi(x,t)] = \frac{k(x)}{s} + \frac{u^{\gamma}}{s^{\gamma}}N^{+}[f(x,t)] - \frac{u^{\gamma}}{s^{\gamma}}N^{+}[L\phi(x,t) + N\phi(x,t)]. \tag{4}$$

The ADM solution for $\phi(x,t)$ is

$$\phi(x,t) = \sum_{j=0}^{\infty}\phi_j(x,t), \tag{5}$$

and the Adomian polynomial is

$$N\phi(x,t) = \sum_{j=0}^{\infty}A_j, \tag{6}$$

$$A_j = \frac{1}{j!}\left[\frac{d^j}{d\lambda^j}\left[N\sum_{j=0}^{\infty}(\lambda^j\phi_j)\right]\right]_{\lambda=0},\quad j = 0,1,2\cdots \tag{7}$$

using equations (5) and (6) in equation (4), we get

$$N^{+}\left[\sum_{j=0}^{\infty}\phi_j(x,t)\right] = \frac{k(x)}{s} + \frac{u^{\gamma}}{s^{\gamma}}N^{+}[f(x,t)] - \frac{u^{\gamma}}{s^{\gamma}}N^{+}\left[L\sum_{j=0}^{\infty}\phi_j(x,t) + \sum_{j=0}^{\infty}A_j\right]. \tag{8}$$

Using the NT,

$$N^{+}[\phi_0(x,t)] = \frac{k(x)}{s} + \frac{u^{\gamma}}{s^{\gamma}}N^{+}[f(x,t)],\quad N^{+}[\phi_1(x,t)] = -\frac{u^{\gamma}}{s^{\gamma}}N^{+}[L\phi_0(x,t) + A_0], \tag{9}$$

Using the generalization, we can write

$$N^{+}[\phi_{j+1}(x,t)] = -\frac{u^{\gamma}}{s^{\gamma}}N^{+}[L\phi_j(x,t) + A_j],\quad j \geq 1. \tag{10}$$

The inverse NT of equations (9) and (10) implies that

$$\phi_0(x,t) = k(x) + N^{-1}\left[\frac{u^{\gamma}}{s^{\gamma}}N^{+}[f(x,t)]\right],\quad \phi_{j+1}(x,t) = -N^{-1}\left[\frac{u^{\gamma}}{s^{\gamma}}N^{+}[L\phi_j(x,t) + A_j]\right]. \tag{11}$$

4 Numerical implementation

Example 1. The BVP of fractional order in ref. [42] is

$$\frac{\partial^{\gamma}\phi}{\partial x^{\gamma}} + \frac{\partial^2\phi}{\partial y^2} = 0,\quad 0 < x,\quad y < \pi,\quad 1 < \gamma \leq 2, \tag{12}$$

with BCs

$$\begin{aligned}&\phi(x,0)=0, && \phi(x,\pi)=0,\\ &\phi(0,y)=\sin y, && \phi(\pi,y)=\cosh\pi\sin y.\end{aligned}\tag{13}$$

Taking the NT of equation (12),

$$\frac{s^\gamma}{u^\gamma}N^+[\phi(x,y)]-\frac{s^{\gamma-1}}{u^\gamma}A-\frac{s^{\gamma-2}}{u^\gamma}B=-N^+\left[\frac{\partial^2\phi}{\partial y^2}\right],$$

where $A=\phi(0,y)$ and $B=\phi'(\pi,y)$. Using inverse NT, we have

$$\phi(x,y)=N^{-1}\left[\frac{1}{s}A+\frac{1}{s^2}B-\frac{u^\gamma}{s^\gamma}N^+\left[\frac{\partial^2\phi}{\partial y^2}\right]\right].$$

Applying the ADM, we have

$$\phi_0(x,y)=A+Bx.\tag{14}$$

Using the BC in equation (14), we get $A=\sin y$ and $B=\sin y\left(\frac{\cosh\pi-1}{\pi}\right)$

$$\phi_0(x,y)=\sin y\left\{1+\left(\frac{\cosh\pi-1}{\pi}\right)x\right\},$$

$$\phi_{j+1}(x,y)=-N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_j}{\partial y^2}\right\}\right],\quad j=0,1,\ldots$$

for $j=0$

$$\begin{aligned}\phi_1(x,y)&=-N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_0}{\partial y^2}\right\}\right]\\&=N^{-1}\left[\sin y\left\{\frac{u^\gamma}{s^{\gamma+1}}+\left(\frac{\cosh\pi-1}{\pi}\right)\frac{u^\gamma}{s^{\gamma+2}}\right\}\right],\end{aligned}\tag{15}$$

$$\phi_1(x,y)=\sin y\left\{\frac{x^\gamma}{\Gamma(\gamma+1)}+\left(\frac{\cosh\pi-1}{\pi}\right)\frac{x^{\gamma+1}}{\Gamma(\gamma+2)}\right\}.$$

The subsequent terms are as follows:

$$\begin{aligned}\phi_2(x,y)&=-N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_1}{\partial y^2}\right\}\right]\\&=\sin y\left\{\frac{x^{\gamma+1}}{\Gamma(\gamma+2)}+\left(\frac{\cosh\pi-1}{\pi}\right)\frac{x^{\gamma+2}}{\Gamma(\gamma+3)}\right\},\\ \phi_3(x,y)&=-N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_2}{\partial y^2}\right\}\right]\\&=\sin y\left\{\frac{x^{\gamma+2}}{\Gamma(\gamma+3)}+\left(\frac{\cosh\pi-1}{\pi}\right)\frac{x^{\gamma+3}}{\Gamma(\gamma+4)}\right\},\\ \phi_4(x,y)&=-N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_3}{\partial y^2}\right\}\right]\\&=\sin y\left\{\frac{x^{\gamma+3}}{\Gamma(\gamma+4)}+\left(\frac{\cosh\pi-1}{\pi}\right)\frac{x^{\gamma+4}}{\Gamma(\gamma+5)}\right\},\\&\vdots\end{aligned}\tag{16}$$

The NDM approximation is as follows:

$$\begin{aligned}\phi(x,y)&=\phi_0(x,y)+\phi_1(x,y)+\phi_2(x,y)\\&\quad+\phi_3(x,y)+\phi_4(x,y)+\cdots\end{aligned}$$

$$\begin{aligned}\phi(x,y)=\sin y\Bigg[&1+\left(\frac{\cosh\pi-1}{\pi}\right)x+\frac{x^\gamma}{\Gamma(\gamma+1)}\\&+\left(\frac{\cosh\pi-1}{\pi}\right)\frac{x^{\gamma+1}}{\Gamma(\gamma+2)}+\frac{x^{\gamma+1}}{\Gamma(\gamma+2)}\\&+\left(\frac{\cosh\pi-1}{\pi}\right)\frac{x^{\gamma+2}}{\Gamma(\gamma+3)}+\frac{x^{\gamma+2}}{\Gamma(\gamma+3)}\\&+\left(\frac{\cosh\pi-1}{\pi}\right)\frac{x^{\gamma+3}}{\Gamma(\gamma+4)}+\frac{x^{\gamma+3}}{\Gamma(\gamma+4)}\\&+\left(\frac{\cosh\pi-1}{\pi}\right)\frac{x^{\gamma+4}}{\Gamma(\gamma+5)}+\cdots\Bigg].\end{aligned}$$

The exact result of equation (12) at $\gamma=2$

$$\phi(x,y)=\cosh x\sin y.$$

Example 2. The BVP of fractional order in ref. [42] is

$$\frac{\partial^\gamma\phi}{\partial y^\gamma}+\frac{\partial^2\phi}{\partial x^2}=0,\quad 0<x,\ \ y<\pi,\ \ 1<\gamma\le 2,\tag{17}$$

with BCs

$$\begin{aligned}&\phi_y(x,0)=0,\ \ \phi_y(x,\pi)=2\cos(2x)\sinh(2\pi),\\&\phi_x(0,y)=0,\ \ \phi_x(\pi,y)=0.\end{aligned}\tag{18}$$

Taking the NT of equation (17),

$$\frac{s^\gamma}{u^\gamma}N^+[\phi(x,y)]-\frac{s^{\gamma-1}}{u^\gamma}A-\frac{s^{\gamma-2}}{u^\gamma}B=-N^+\left[\frac{\partial^2\phi}{\partial x^2}\right],$$

where $A=\phi_y(x,0)$ and $B=\phi_x(\pi,y)$. Using inverse NT, we have

$$\phi(x,y)=N^{-1}\left[\frac{1}{s}A+\frac{1}{s^2}B-\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi}{\partial x^2}\right\}\right].$$

Applying the ADM, we have

$$\phi_0(x,y)=A+By.\tag{19}$$

Using the BC in equation (19), we get $A=$ constant and $B=2\cos(2x)\sinh(2\pi)$

$$\phi_0(x,y)=C+2\cos(2x)\sinh(2\pi)y,$$

$$\phi_{j+1}(x,y)=-N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_j}{\partial x^2}\right\}\right],\quad j=0,1,2,\ldots$$

for $j=0$:

$$\phi_1(x,y) = -N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_0}{\partial y^2}\right\}\right]$$
$$= N^{-1}\left[8\sinh(2\pi)\cos(2x)\frac{y^{\gamma+1}}{\Gamma(\gamma+2)}\frac{u^\gamma}{s^{\gamma+2}}\right], \quad (20)$$
$$\phi_1(x,y) = 8\sinh(2\pi)\cos(2x)\frac{y^{\gamma+1}}{\Gamma(\gamma+2)}.$$

The subsequent terms are as follows:

$$\phi_2(x,y) = -N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_1}{\partial x^2}\right\}\right] = 32\sinh(2\pi)\cos(2x)\frac{y^{2\gamma+1}}{\Gamma(2\gamma+2)},$$
$$\phi_3(x,y) = -N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_2}{\partial x^2}\right\}\right] = 128\sinh(2\pi)\cos(2x)\frac{y^{3\gamma+1}}{\Gamma(2\gamma+4)}, \quad (21)$$
$$\vdots$$

The NDM solution for example 2 is as follows:

$$\phi(x,y) = \phi_0(x,y) + \phi_1(x,y) + \phi_2(x,y) + \phi_3(x,y) + \cdots$$
$$\phi(x,y) = C + 2\cos(2x)\sinh(2\pi)y + 8\sinh(2\pi)\cos(2x)\frac{y^{\gamma+1}}{\Gamma(\gamma+2)} + 32\sinh(2\pi)\cos(2x)\frac{y^{2\gamma+1}}{\Gamma(2\gamma+2)} + 128\sinh(2\pi)\cos(2x)\frac{y^{3\gamma+1}}{\Gamma(2\gamma+4)} + \cdots.$$

The exact result of equation (12) at $\gamma = 2$

$$\phi(x,y) = \cos(2x)\cosh(2y).$$

Example 3. The BVP of fractional order in ref. [42] is

$$\frac{\partial^\gamma\phi}{\partial x^\gamma} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad 0 < x, \quad y < \pi, \quad 1 < \gamma \le 2, \quad (22)$$

with BCs

$$\phi(x,0) = \sinh x, \quad \phi(x,\pi) = -\sinh x, \quad \phi(0,y) = 0, \quad \phi(\pi,y) = \sinh\pi\cos y. \quad (23)$$

Taking the NT of equation (22),

$$\frac{s^\gamma}{u^\gamma}N^+[\phi(x,y)] - \frac{s^{\gamma-1}}{u^\gamma}A - \frac{s^{\gamma-2}}{u^\gamma}B = -N^+\left[\frac{\partial^2\phi}{\partial y^2}\right],$$

where $A = \phi(0,y)$ and $B = \phi(\pi,y)$. Using inverse NT, we have

$$\phi(x,y) = N^{-1}\left[\frac{1}{s}A + \frac{1}{s^2}B - \frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi}{\partial y^2}\right\}\right].$$

Using the ADM, we have

$$\phi_0(x,y) = A + Bx. \quad (24)$$

Using the BC in equation (24), we get $A = 0$ and $B = \cos y$

$$\phi_0(x,y) = x\cos y,$$
$$\phi_{j+1}(x,y) = -N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_j}{\partial y^2}\right\}\right], \quad j = 0,1,2,\ldots$$

for $j = 0$:

$$\phi_1(x,y) = -N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_0}{\partial y^2}\right\}\right] = N^{-1}\left[\frac{\cos y u^\gamma}{s^{\gamma+2}}\right],$$
$$\phi_1(x,y) = \cos y\frac{x^{\gamma+1}}{\Gamma(\gamma+2)}. \quad (25)$$

The subsequent terms are as follows:

$$\phi_2(x,y) = -N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_1}{\partial y^2}\right\}\right] = \cos y\frac{x^{2\gamma+1}}{\Gamma(2\gamma+2)},$$
$$\phi_3(x,y) = -N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_2}{\partial y^2}\right\}\right] = \cos y\frac{x^{3\gamma+1}}{\Gamma(2\gamma+4)}, \quad (26)$$
$$\phi_4(x,y) = -N^{-1}\left[\frac{u^\gamma}{s^\gamma}N^+\left\{\frac{\partial^2\phi_3}{\partial y^2}\right\}\right] = \cos y\frac{x^{4\gamma+1}}{\Gamma(2\gamma+6)},$$
$$\vdots$$

Example 3 has the following NDM solution

$$\phi(x,y) = \phi_0(x,y) + \phi_1(x,y) + \phi_2(x,y) + \phi_3(x,y) + \phi_4(x,y) + \cdots$$
$$\phi(x,y) = \cos y\left(x + \frac{x^{\gamma+1}}{\Gamma(\gamma+2)} + \frac{x^{2\gamma+1}}{\Gamma(2\gamma+2)} + \frac{x^{3\gamma+1}}{\Gamma(2\gamma+4)} + \frac{x^{4\gamma+1}}{\Gamma(2\gamma+6)} + \cdots\right).$$

The exact result of equation (22) at $\gamma = 2$

$$\phi(x,y) = \sinh x\cos y.$$

5 Results and graph discussion

The actual and NDM solutions of example 1 are plotted in Figures 1 and 2, respectively, at $\gamma = 2$, and the closed

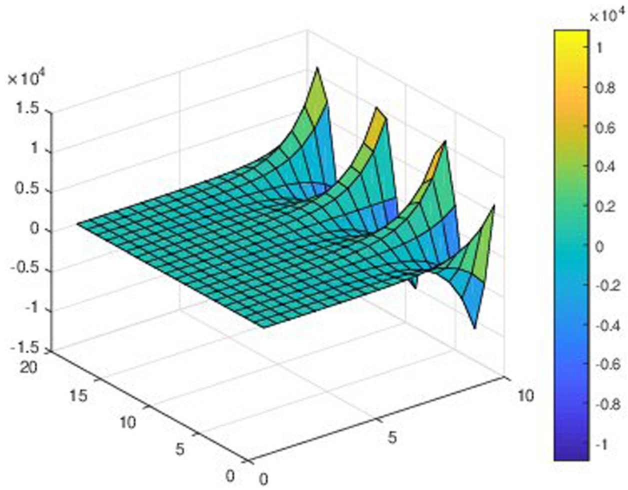

Figure 1: The exact solution of example 1 at $\gamma = 2$.

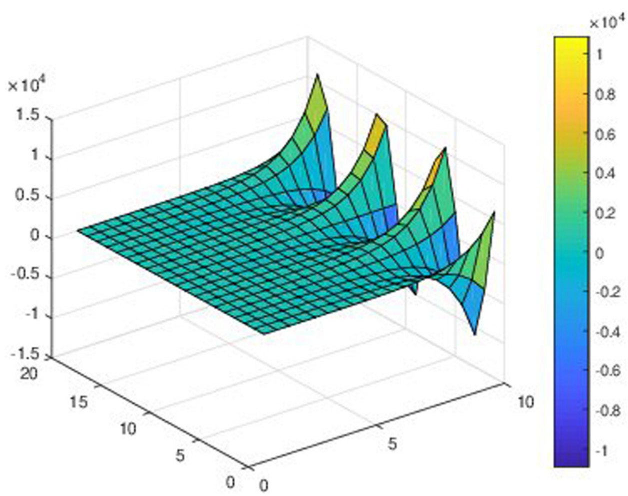

Figure 2: NDM solution graph of example 1 at $\gamma = 2$.

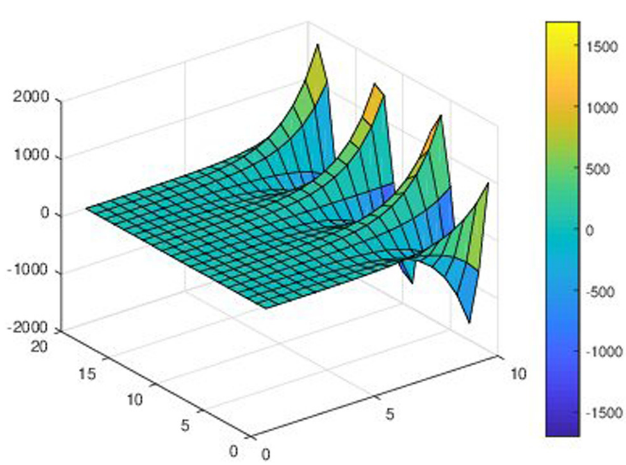

Figure 3: NDM solution graph of example 1 at $\gamma = 1.8$.

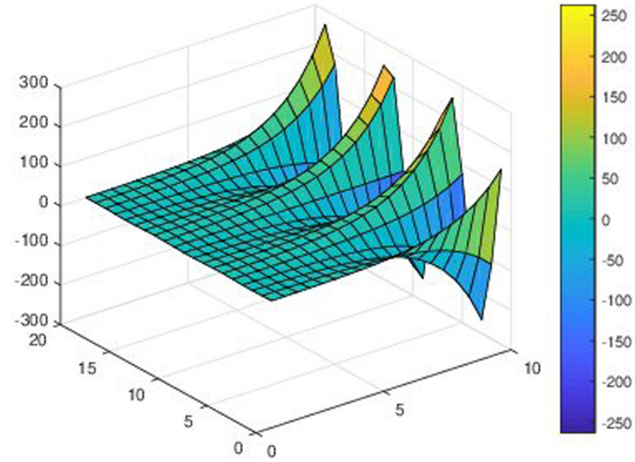

Figure 4: NDM solution graph of example 1 at $\gamma = 1.6$.

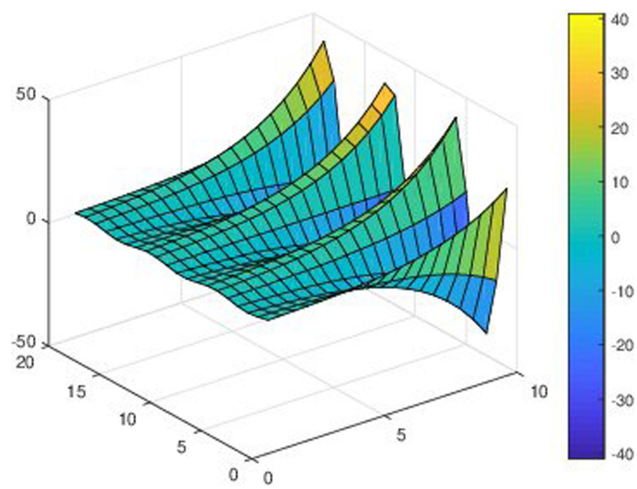

Figure 5: NDM solution graph of example 1 at $\gamma = 1.4$.

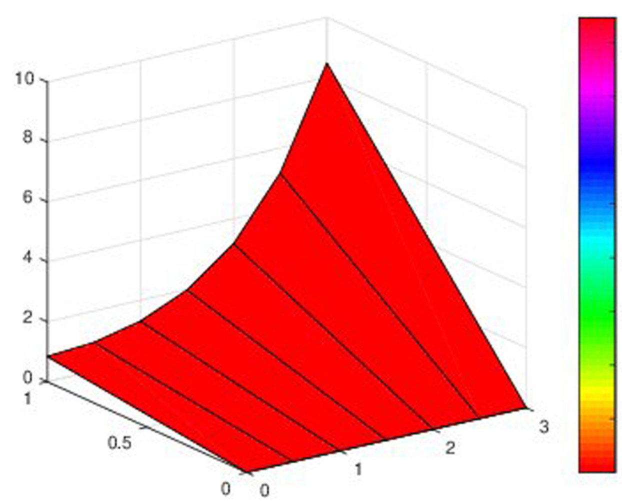

Figure 6: The exact solution of example 2 at $\gamma = 2$.

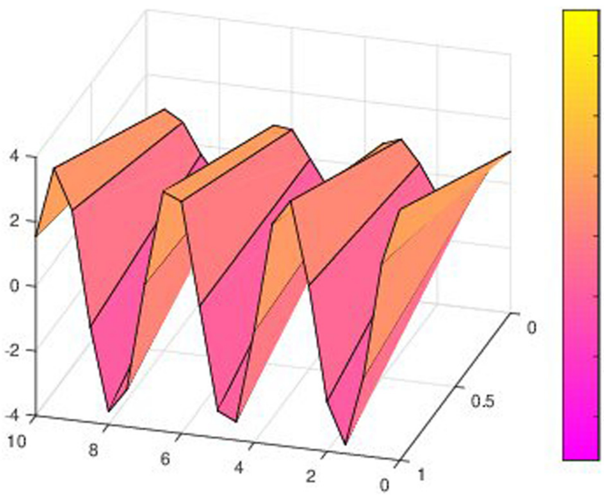

Figure 7: NDM solution graph of example 2 at $\gamma = 2$.

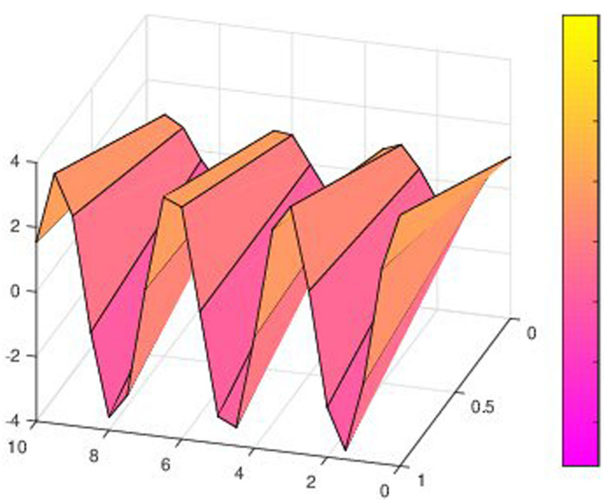

Figure 8: NDM solution graph of example 2 at $\gamma = 1.8$.

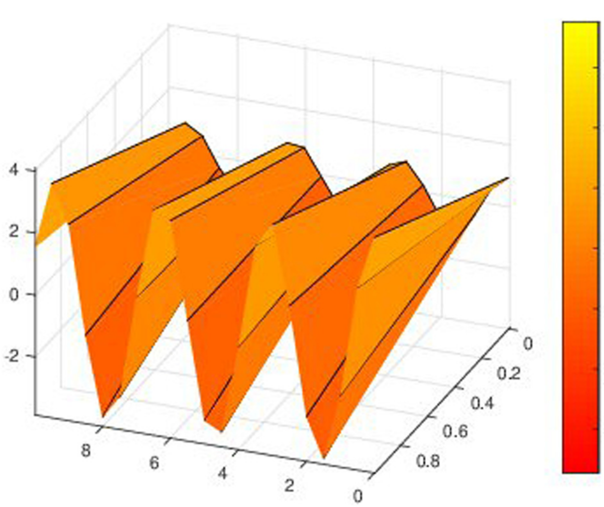

Figure 9: NDM solution graph of example 2 at $\gamma = 1.6$.

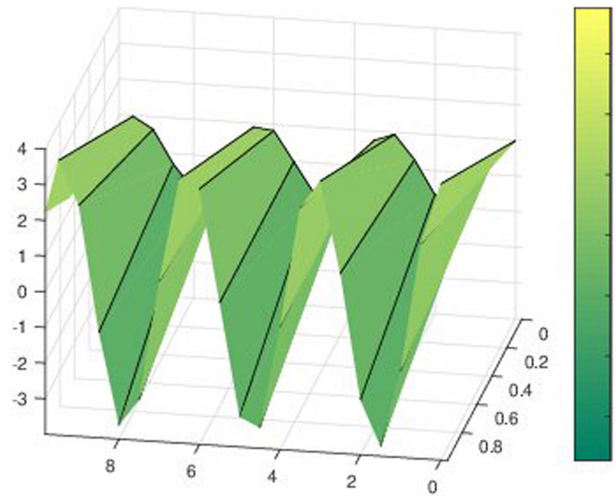

Figure 10: NDM solution graph of example 2 at $\gamma = 1.4$.

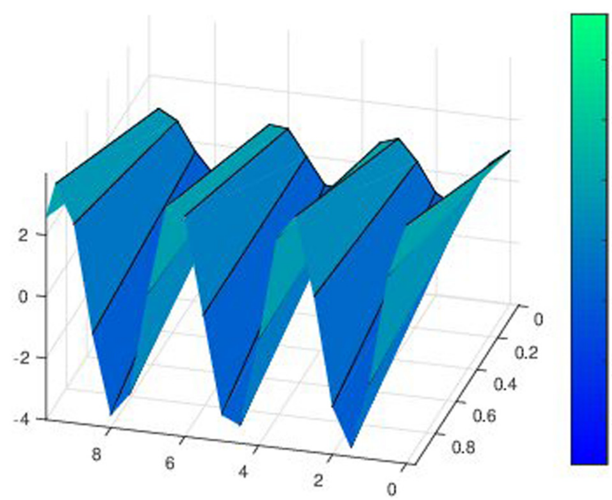

Figure 11: The exact solution of example 3 at $\gamma = 2$.

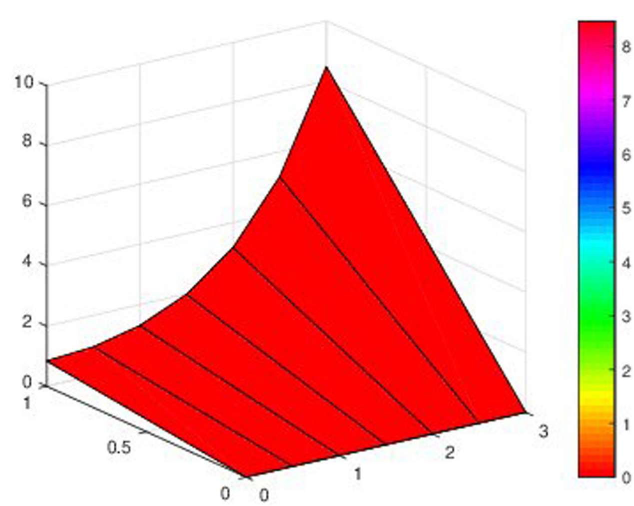

Figure 12: NDM solution graph of example 3 at $\gamma = 2$.

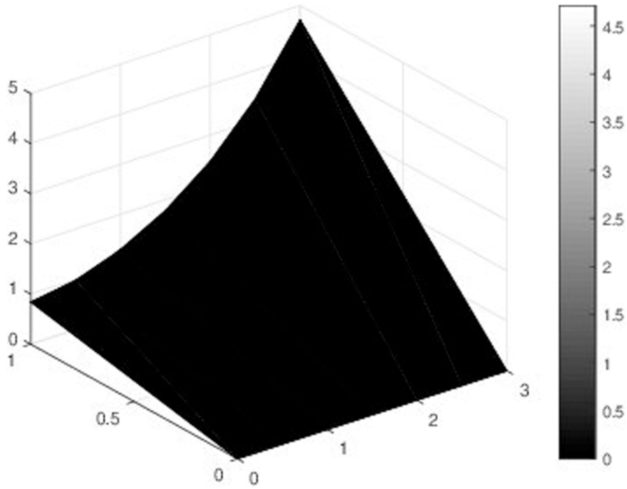

Figure 13: NDM solution graph of example 3 at $\gamma = 1.8$.

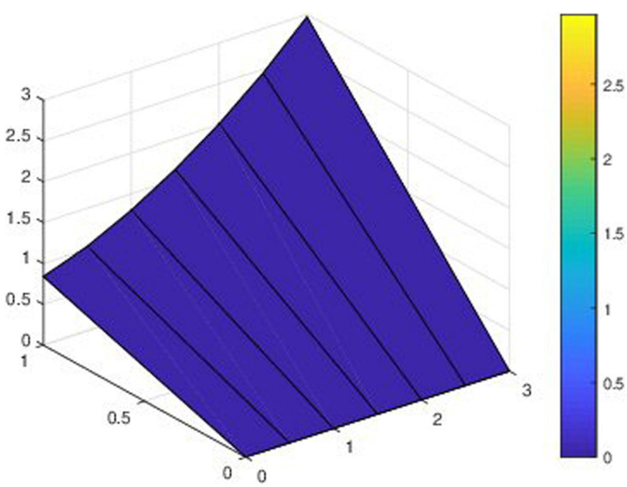

Figure 14: NDM solution graph of example 3 at $\gamma = 1.6$.

contact of the actual and NDM solutions is analyzed. The graphs in Figures 3–5 represent the NDM solutions at $\gamma = 1.8, \ 1.6$ and 1.4 of example 1, respectively. The fractional results are very accurate and found to be convergent to an integer order solution of each problem. In Figures 6 and 7, the plots expressed the NDM and actual solutions of example 2 at $\gamma = 2$, respectively. The closed relation of NDM and actual solutions is established in Figures 1–5. In Figures 8–10, the graphs are drawn to verify the results at $\gamma = 1.8, \ 1.6$ and 1.4 of example 2, respectively. The convergence of fractional to integer-order solutions is investigated. In Figures 11 and 12, the graphs represent the actual and NDM solutions, respectively, at $\gamma = 2$ of example 3. A closed resemblance is found between NDM and actual solutions.

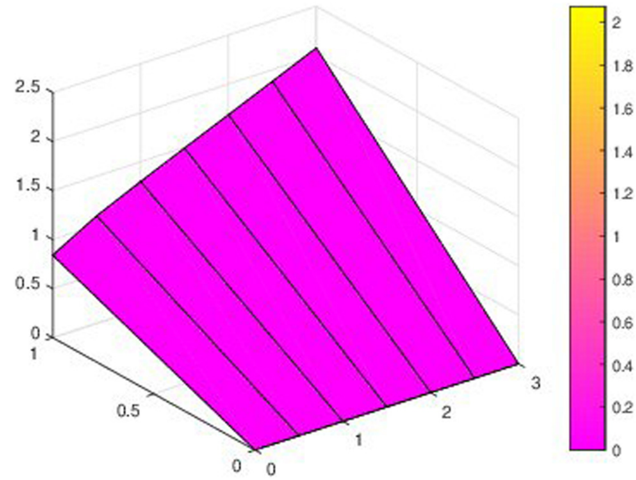

Figure 15: NDM solution graph of example 3 at $\gamma = 1.4$.

In Figures 13–15, the solutions at $\gamma = 1.8, \ 1.6$ and 1.4 are determined, respectively. It is analyzed that for various values of γ, the solutions have the sufficient convergence to the solution of the problem at integer order of example 3.

6 Conclusions

In this article, the natural decomposition technique is extended to solve FBVPs of the Laplace equations. The NDM solutions are compared with the exact and variational iteration method solutions. The comparison has also been done by using various solution graphs of different problems. It is confirmed that the proposed technique is in closed agreement with the actual and variational iteration solutions. Table is constructed to verify the accuracy of the suggested technique. It is confirmed from the table that the present method has the sufficient degree of accuracy. The convergence of the fractional-order solutions toward integer-order solution is observed. Furthermore, the present method is simple and straight forward and therefore can be modified for the solutions of BVPs of FPDEs.

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