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FIXED POINT RESULTS IN $\varepsilon-$CHAINABLE COMPLETE $b-$METRIC SPACES

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Abstract. The purpose of this paper is to present some fixed point results in $\varepsilon-$chainable complete $b-$metric spaces that are inspired from famous result of Edelstein, published in 1961.
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1. Introduction and preliminaries

The method of "successive approximations" has been perfectly abstracted by Banach to express his significant fixed point theorem: Every contraction f on a complete metric space (X, d) possesses a unique fixed point. Edelstein [8] refined the contraction definition and proposed the notion of "*globally contractive*" and "*locally contractive*". In particular, we say that a self-mapping f, on a metric space (X, d), is called *globally contractive* if

$$d\left(f\left(p\right), f\left(q\right)\right) \leq \lambda d\left(p, q\right), \tag{1.1}$$

for all $p, q \in X$, where $0 \leq \lambda < 1$. In addition, f is *locally contractive* if, for every $x \in X$, there exist $\varepsilon > 0$ and $0 \leq \lambda < 1$, which may depend on x, such that

$$p, q \in S\left(x, \varepsilon\right) = \{y \in X | d\left(x, y\right) < \varepsilon\} \tag{1.2}$$

implies (1.1). Furthermore, f is $(\varepsilon, \lambda)-$ *uniformly contractive* if, it is locally contractive and both, ε and λ, are not depending on x.

The following notion is crucial for our own purposes:

Definition 1.1. ([8]) A metric space X is called ε-chainable if $\varepsilon > 0$ and for every $a, b \in X$, there exists an $\varepsilon-$chain, i.e., a finite set of points $a = x_0, x_1, ..., x_n = b$ (n may depend on both a and b) such that $d(x_{i-1}, x_i) < \varepsilon, (i = 1, 2, ..., n)$.

In what follow we recall the main result of Edelstein [8].

Theorem 1.1. ([8]) *Let f be a self-mapping on a complete ε-chainable metric space. If f is an $(\varepsilon, \lambda)-$ uniformly locally contractive mapping, then, it possesses a unique fixed point.*

One of the basic goal of this paper is to obtain a characterization of Edelstein's result in the context of $b-$metric spaces.

We, first, recollect the definition of $b-$metric that was considered by several authors, including Bakhtin [2] and Czerwik [7]. See also [17].

Definition 1.2. Let X be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to [0, \infty)$ is said to be a $b-$metric with constant s, if

(1) d is symmetric, that is, $d(x, y) = d(y, x)$ for all $x, y \in X$;
(2) d is self-distance, that is, $d(x, y) = 0$ if and only if $x = y$;
(3) d provides s-weighted triangle inequality, that is

$$d(x, z) \leq s[d(x, y) + d(y, z)], \text{ for all } x, y, z \in X.$$

In this case the triple (X, d, s) is called a $b-$metric space with constant s.

It is evident that the notions of $b-$metric and standard metric coincide in case of $s = 1$. For more details on $b-$metric spaces see e.g. [1, 3, 4, 5, 10, 11, 12] and corresponding references therein.

Example 1.1. Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$ such that

$$d(x, y) = |x - y|^p, \ p > 1.$$

It is easy to see that d is a $b-$metric with $s = 2^p$, but is not a metric.

Definition 1.3. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison function if it is increasing and $\varphi^n(t) \to 0$, as $n \to \infty$, for any $t \in [0, \infty)$.

Lemma 1.1. ([4]) *If $\varphi : [0, \infty) \to [0, \infty)$ is a comparison function, then:*

(1) *each iterate φ^k of φ, $k \geq 1$, is also a comparison function;*
(2) *φ is continuous at 0;*
(3) *$\varphi(t) < t$, for any $t > 0$.*

Definition 1.4. ([4]) A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a $c-$comparison function if

(1) φ is increasing;
(2) there exists $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi^{k+1}(t) \leq a\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

For related results see [16].

In order to give some fixed point results to the class of $b-$metric spaces, the notion of $c-$comparison function was extended to $b-$comparison function by V. Berinde [5].

Definition 1.5. ([5]) Let $s \geq 1$ be a real number. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a $b-$comparison function if the following conditions are fulfilled:

(1) φ is monotone increasing;

(2) there exist $k_0 \in \mathbb{N}$, $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi^{k+1}(t) \leq as^k\varphi^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

The following lemma is very important in the proof of our results.

Lemma 1.2. ([5]) *If $\varphi : [0, \infty) \to [0, \infty)$ is a $b-$comparison function, then we have the following conclusions:*

(1) *the series $\sum_{k=0}^{\infty} s^k\varphi^k(t)$ converges for any $t \in [0, \infty)$;*

(2) *the function $S_b : [0, \infty) \to [0, \infty)$ defined by $S_b(t) = \sum_{k=0}^{\infty} s^k\varphi^k(t)$, $t \in [0, \infty)$, is increasing and continuous at 0.*

Remark 1.1. Due to the Lemma 1.2., any $b-$comparison function is a comparison function.

2. $\varepsilon-$UNIFORMLY LOCAL $\alpha - \varphi-$CONTRACTIVE MAPPINGS

In this section, we will consider the $\alpha-$admissible mapping on $\varepsilon-$chainable $b-$metric spaces.

Definition 2.1. ([18]) Let X be a nonempty set, $f : X \to X$ be an operator and $\alpha : X \times X \to [0, \infty)$. We say that f is $\alpha-$admissible if

$$x, y \in X, \alpha(x, y) \geq 1 \Rightarrow \alpha(f(x), f(y)) \geq 1.$$

Definition 2.2. Let (X, d) be a $b-$metric space with constant $s \geq 1$, $\varphi : [0, \infty) \to [0, \infty)$ be a $b-$comparison function and $\alpha : X \times X \to [0, \infty)$ be an operator. A mapping $f : X \to X$ is said to be locally $\alpha - \varphi-$contractive if for every $x \in X$, there exists $\varepsilon > 0$, which may depend on x, such that

$$p, q \in S(x, \varepsilon) = \{y \in X | d(x, y) < \varepsilon\} \tag{2.1}$$

implies that

$$\alpha(p, q)d(f(p), f(q)) \leq \varphi(d(p, q)), \text{ for every } p, q \in X.$$

Definition 2.3. In the above context, a mapping $f : X \to X$ is said to be $\varepsilon-$uniformly local $\alpha - \varphi-$contractive mapping if it is locally $\alpha - \varphi-$contractive mapping and ε do not depend on x.

Remark 2.1. If $f : X \to X$ satisfies the Banach contraction principle, then f is a locally $\alpha - \varphi-$contractive mapping, where $\alpha(x, y) = 1$, for all $x, y \in X$ and $\varphi(t) = kt$, for all $t \geq 0$ and some $k \in [0, 1)$.

Theorem 2.1. *Let (X, d) be a complete $\varepsilon-$chainable $b-$metric space with constant $s \geq 1$, $\varphi : [0, \infty) \to [0, \infty)$ be a $b-$comparison function and $\alpha : X \times X \to [0, \infty)$. Let $f : X \to X$ be an $\alpha-$admissible mapping which has closed graph with respect to d. Suppose that* *(i) there exists an element $x_0 \in X$ such that there exists an $\varepsilon-$chain $x_1, \cdots, x_{n-1}$ from x_0 to $x_n = f(x_0)$ with $\alpha(x_i, x_{i+1}) \geq 1$, for $i \in \{0, \ldots, n-1\}$;*

(ii) f is $\varepsilon-$uniformly local $\alpha - \varphi-$contractive mapping.

Then f has at least one fixed point.

Proof. Due to the statement (i) of the theorem, there exists an element $x_0 \in X$ for which there exists an $\varepsilon-$chain $x_1, \cdots, x_{n-1}$ from x_0 to $x_n = f(x_0)$ with $\alpha(x_i, x_{i+1}) \geq 1$ for $i \in \{0, \ldots, n-1\}$. Since f is $\alpha-$admissible, we have that $\alpha(f(x_i), f(x_{i+1})) \geq 1$ for $i \in \{0, \ldots, n-1\}$.

Regarding that the space is $\varepsilon-$chainable, we observe that

$$d(x_{i-1}, x_i) < \varepsilon, \text{for all } i \in \{1, 2, \cdots, n\}.$$

Taking into account that φ is non-decreasing, we find that

$$\varphi(d(x_{i-1}, x_i)) \leq \varphi(\varepsilon), \text{ for all } i \in \{1, 2, \cdots, n\}.$$

On the other hand, since f is $\alpha-$admissible, we can easily derive that

$$\alpha(f^m(x_i), f^m(x_{i+1})) \geq 1, \text{ for all } m \in \mathbb{N}, \text{ and for all } i \in \{0, 1, \cdots, n-1\}.$$

Furthermore, keeping in mind that f is a $\varepsilon-$uniformly local $\alpha - \varphi-$contractive mapping, for each $i \in \{1, 2, \cdots, n\}$, we have

$$\begin{aligned} d(f(x_{i-1}), f(x_i)) &\leq \alpha(x_{i-1}, x_i) d(f(x_{i-1}), f(x_i)) \\ &\leq \varphi(d(x_{i-1}, x_i)) \leq \varphi(\varepsilon), \text{ for all } i \in \{0, 1, \cdots, n-1\}. \end{aligned}$$

Iteratively, we obtain

$$\begin{aligned} d\left(f^2(x_{i-1}), f^2(x_i)\right) &\leq \alpha(f(x_{i-1}), f(x_i)) d\left(\left(f^2(x_{i-1})\right), f^2(x_i)\right) \\ &\leq \varphi(d(f(x_{i-1}), f(x_i))) \leq \varphi^2(\varepsilon). \end{aligned}$$

Consequently, we derive that

$$d(f^m(x_{i-1}), f^m(x_i)) \leq \varphi^m(\varepsilon), \text{ for each } m \in \mathbb{N}.$$

On account of the axiom of $s-$weighted triangle inequality, we have

$$\begin{aligned} d\left(f^m(x_0), f^{m+1}(x_0)\right) &= d(f^m(x_0), f^m(x_n)) \\ &\leq s d(f^m(x_0), f^m(x_1)) + \ldots + s^n d(f^m(x_{n-1}), f^m(x_n)) \\ &\leq \left(s + s^2 + \ldots + s^n\right) \varphi^m(\varepsilon) \leq \gamma_s \varphi^m(\varepsilon), \end{aligned}$$

where $\gamma_s = \left(s + s^2 + \ldots + s^n\right)$.

We shall prove that $\left(f^i(x_0)\right)_{i\in\mathbb{N}}$ is a Cauchy sequence. Let j and k, with $j<k$, positive integers. Then, we have:

$$\begin{aligned} d\left(f^j(x_0),f^k(x_0)\right) &\leq sd\left(f^j(x_0),f^{j+1}(x_0)\right)+...+s^{k-j}d\left(f^{k-1}(x_0),f^k(x_0)\right)\\ &\leq \left(s\gamma_s\varphi^j(\varepsilon)+s^2\gamma_s\varphi^{j+1}(\varepsilon)+...+s^{k-j}\gamma_s\varphi^{k-1}(\varepsilon)\right)\\ &\leq \gamma_s\left(s\varphi^j(\varepsilon)+s^2\varphi^{j+1}(\varepsilon)+...+s^{k-j}\varphi^{k-1}(\varepsilon)\right)\\ &= \gamma_s\frac{1}{s^{j-1}}\sum_{i=j}^{k-1}s^i\varphi^i(\varepsilon)=\gamma_s\frac{1}{s^{j-1}}\left(S_{k-1}-S_{j-1}\right)\\ &\leq \gamma_s\frac{1}{s^{j-1}}\sum_{i=0}^{\infty}s^i\varphi^i(\varepsilon), \end{aligned}$$

where $S_k=\sum\limits_{i=0}^{k}s^i\varphi^i(\varepsilon)$. Hence, we have

$$d\left(f^j(x_0),f^k(x_0)\right)\leq\gamma_s\frac{1}{s^{j-1}}\sum_{i=0}^{\infty}s^i\varphi^i(\varepsilon)\to 0,\ \text{as } j\to\infty.$$

Finally, we conclude that $\left(f^i(x_0)\right)_{i\in\mathbb{N}}$ is a Cauchy sequence and by the completeness of the space we have that there exists $x^*(x_0)\in X$ such that $x^*(x_0)=\lim\limits_{i\to\infty}f^i(x_0)$.

Since f has a closed graph, we have that $x^*(x_0)$ is a fixed point for f. □

Remark 2.2. If we suppose, in the above theorem, that for every $x^*,y^*\in Fix(f)$ we have that $\alpha(x^*,y^*)\geq 1$, then $x^*=y^*$.

Proof. Suppose that there exists $y^*\in X$ with $x^*\neq y^*$, such that $f(y^*)=y^*$ and $\alpha(x^*,y^*)\geq 1$. Let us consider $x^*=x_0,x_1,...,x_k=y^*$ an ε-chain. We have

$$\begin{aligned} 0 &< d(x^*,y^*)=d(f(x^*),f(y^*))=d(f^m(x^*),f^m(y^*))\\ &=d(f^m(x_0),f^m(x_k))\leq\gamma_s\varphi^m(\varepsilon)\to 0,\ \text{as } m\to\infty. \end{aligned}$$

Thus we have a contradiction and hence $x^*=y^*$. □

Example 2.1. Let $X=[0,\infty]$ and $d(x,y)=(x-y)^2$. Then (X,d) is a $b-$metric space with the constant $s=4$.

Let $f:X\to X$ be given by

$$f(x)=\begin{cases}\frac{7}{24}, & x\in\left[0,\frac{1}{2}\right)\\ \frac{x}{x+1}, & x\in\left[\frac{1}{2},1\right]\\ \frac{5}{4}, & x>1\end{cases},\quad \varphi(t)=\begin{cases}\frac{t}{2}, & t\in[0,1]\\ \frac{1}{2}, & t>1\end{cases}$$

and $\alpha:X\times X\to[0,\infty),\alpha(x,y)=\begin{cases}1, & x\in[0,1]\\ 0, & \text{otherwise.}\end{cases}$

We have that:

- It is obvious that $f:X\to X$ is an $\alpha-$admissible mapping which has closed graph with respect to d.

- There exists an element $x_0 \in X$ such that there exists an $\varepsilon-$chain $x_1, \cdots, x_{n-1}$ from x_0 to $x_n = f(x_0)$ with $\alpha(x_i, x_{i+1}) \geq 1$, for $i \in \{0, \ldots, n-1\}$.
  Case 1. $x \in \left[0, \frac{1}{2}\right)$.
  Let $x_0 = \frac{1}{3}, x_1 = ... = x_{n-1} = \frac{1}{6}, x_n = f(x_0) = \frac{7}{24}$ and let $\varepsilon = \frac{1}{2}$.
  It is obvious that $d(x_i, x_{i+1}) < \frac{1}{2}$ and $\alpha(x_i, x_{i+1}) \geq 1$, for $i \in \{0, \ldots, n-1\}$.
  Case 2. $x \in \left[\frac{1}{2}, 1\right]$.
  Let $x_0 = 1, x_1 = ... = x_{n-1} = \frac{2}{3}, x_n = f(x_0) = \frac{1}{2}$ and let $\varepsilon = \frac{1}{2}$.
  It is obvious that $d(x_i, x_{i+1}) < \frac{1}{2}$ and $\alpha(x_i, x_{i+1}) \geq 1$, for $i \in \{0, \ldots, n-1\}$.
  Case 3. $x > 1$.
  Let $x_0 = \frac{3}{2}, x_1 = ... = x_{n-1} = \frac{4}{3}, x_n = f(x_0) = \frac{5}{4}$ and let $\varepsilon = \frac{1}{2}$.
  It is obvious that $d(x_i, x_{i+1}) < \frac{1}{2}$ and $\alpha(x_i, x_{i+1}) \geq 1$, for $i \in \{0, \ldots, n-1\}$.
- f is $\varepsilon-$uniformly local $\alpha - \varphi-$contractive mapping.
  Since $\alpha(x, y) = 1$, for all $x \in [0, 1]$, we have to prove that

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \text{ for all } x, y \in [0, 1].$$

Case 1. $x \in \left[0, \frac{1}{2}\right)$.

$$d(f(x), f(y)) = 0 \leq \varphi(d(x, y)), \text{ for all } x, y \in \left[0, \frac{1}{2}\right).$$

Case 2. $x \in \left[\frac{1}{2}, 1\right]$.

$$d(f(x), f(y)) = \frac{d(x, y)}{(x+1)^2 (y+1)^2} \leq \frac{d(x, y)}{\frac{81}{16}} \leq \varphi(d(x, y)), \text{ for all } x, y \in \left[\frac{1}{2}, 1\right].$$

Case 3. $x > 1$.

$$d(f(x), f(y)) = 0 \leq \varphi(d(x, y)), \text{ for all } x, y > 1.$$

3. $(\varepsilon, \lambda)-$uniformly locally contractive mappings

Definition 3.1. Let (X, d) be a $b-$metric space with constant $s \geq 1$ and $f : X \to X$. We say that f is globally contractive with constant λ, if $0 \leq \lambda < \frac{1}{s}$ and the condition

$$d(f(p), f(q)) \leq \lambda d(p, q), \tag{3.1}$$

holds for every $p, q \in X$.

Definition 3.2. Let (X, d) be a $b-$metric space with constant $s \geq 1$ and $f : X \to X$. We say that f is locally contractive if for every $x \in X$, there exist $\varepsilon > 0$ and $0 \leq \lambda < \frac{1}{s}$, which may depend on x, such that

$$p, q \in S(x, \varepsilon) = \{y \in X | d(x, y) < \varepsilon\} \tag{3.2}$$

implies (3.1).

Definition 3.3. Let (X, d) be a $b-$metric space with constant $s \geq 1$ and $f : X \to X$. We say that f is $(\varepsilon, \lambda)-$ uniformly locally contractive if it is locally contractive and both ε and λ are not depending on x.

Remark 3.1. If $f : X \to X$ is an $(\varepsilon, \lambda)-$ uniformly locally contractive mapping, then f is continuous.

Theorem 3.1. *Let X be a complete ε-chainable $b-$metric space with constant $s \geq 1$ and $f : X \to X$ be a $(\varepsilon, \lambda)-$uniformly locally contractive mapping.*

Then, the following conclusions hold:

(i) f is a Picard operator, i.e., there exists a unique fixed point $x^* \in X$ of f and, for every $x \in X$ the sequence $(f^j(x))_{j\in\mathbb{N}}$ converges to x^*, as $j \to \infty$;

(ii) for every $x \in X$ we have the following estimation

$$d\left(f^j(x), x^*\right) \leq \frac{s^3\varepsilon\lambda^j}{(s-1)(1-s\lambda)}, \text{ for each } j \in \mathbb{N}.$$

Proof. (i) According to Remark 3.1, f is continuous so, it is enough to consider in Theorem 2.1. the particular expressions $\alpha(x, y) = 1$, for all $x, y \in X$ and $\varphi(t) = \lambda t$, $t \in [0, \infty)$. Thus f is a Picard operator.

(ii) Let $x \in X$ be arbitrary chosen and let us consider the $\varepsilon-$chain

$$x = x_0 \cdot x_1, ..., x_n = f(x).$$

$$d(x, f(x)) \leq d(x_0, x_n) \leq sd(x_0, x_1) + s^2 d(x_1, x_2) + ... + s^n d(x_{n-1}, x_n).$$

Now, for every pair of consecutive points in the $\varepsilon-$chain, we have $d(x_{i-1}, x_i) < \varepsilon$ and hence

$$d(x, f(x)) < \left(s + s^2 + ... + s^n\right)\varepsilon = \gamma_s\varepsilon.$$

Since f is (ε, λ)-uniformly locally contractive, we have

$$d(f(x_{i-1}), f(x_i)) \leq \lambda d(x_{i-1}, x_i) < \lambda\varepsilon.$$

By induction, we obtain

$$d(f^m(x_{i-1}), f^m(x_i)) < \lambda^m\varepsilon, \text{ for every } m \in \mathbb{N}^*.$$

We have

$$d\left(f^m(x), f^{m+1}(x)\right) = d(f^m(x_0), f^m(x_n)) < \gamma_s\lambda^m\varepsilon.$$

Let j and k with $j < k$ be positive integers.

$$d\left(f^j(x), f^k(x)\right) < \gamma_s\lambda^j\varepsilon\left(1 + s\lambda + ... + (s\lambda)^{k-j-1}\right).$$

If we take $k = j + p$, with $p \in \mathbb{N}^*$, then, for every $j \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we get that

$$d\left(f^j(x), f^{j+p}(x)\right) < \gamma_s\frac{\lambda^j\varepsilon s}{1-s\lambda} \leq \frac{s^2\varepsilon\lambda^j}{(s-1)(1-s\lambda)}.$$

Then

$$\begin{aligned} d\left(f^j(x), x^*\right) &\leq s\left(d\left(f^j(x), f^{j+p}(x)\right) + d\left(f^{j+p}(x), x^*\right)\right) \\ &\leq \frac{s^3\varepsilon\lambda^j}{(s-1)(1-s\lambda)} + sd\left(f^{j+p}(x), x^*\right) \end{aligned}$$

Letting $p \to \infty$ we get

$$d\left(f^j(x), x^*\right) \leq \frac{s^3\varepsilon\lambda^j}{(s-1)(1-s\lambda)}, \text{ for each} j \in \mathbb{N}.$$ □

Concerning the data dependence problem for the fixed point problem with $(\varepsilon, \lambda)-$uniformly locally contractive mappings, we can make the following remark.

Remark 3.2. Consider $g : X \to X$ a mapping having at least one fixed point y^* and there exists $\eta > 0$ such that $d(f(x), g(x)) \leq \eta$, for every $x \in X$. Then, by Theorem 3.1., we have

$$d\left(f^j(y^*), x^*\right) \leq \frac{s^3 \varepsilon \lambda^j}{(s-1)(1-s\lambda)}, \text{ for each } j \in \mathbb{N}.$$

Thus

$$d(y^*, x^*) \leq s\left(d(g(y^*), f(y^*)) + d(f(y^*), x^*)\right) \leq s\eta + \frac{s^4 \varepsilon \lambda}{(s-1)(1-s\lambda)}.$$

We notice that, when $\eta \searrow 0$ (i.e. g tends to f), then

$$d(y^*, x^*) \leq \frac{s^4 \varepsilon \lambda}{(s-1)(1-s\lambda)},$$

which shows that we cannot get (at least by this method) data dependence for the unique fixed point of an $(\varepsilon, \lambda)-$uniformly locally contractive mapping on a complete $\varepsilon-$chainable $b-$metric space.

4. $\varepsilon-$uniformly ordered locally $\varphi-$contractive mappings

In this section, we will consider the case of ordered ε-chainable b-metric spaces.

Definition 4.1. Let (X, d) be a $b-$metric space with constant $s \geq 1$ and "$\preceq$" be a partial order on X. A mapping $f : X \to X$ is said to be ordered locally $\varphi-$contractive if, for every $x \in X$, there exists $\varepsilon > 0$, which may depend on x, such that

$$p, q \in S(x, \varepsilon) = \{y \in X | d(x, y) < \varepsilon\} \tag{4.1}$$

implies that

$$d(f(p), f(q)) \leq \varphi(d(p, q)), \text{ for every } p, q \in X \text{ with } p \preceq q \text{ or } q \preceq p.$$

Definition 4.2. In the above context, a mapping $f : X \to X$ is said to be $\varepsilon-$uniformly ordered locally $\varphi-$contractive if it is ordered locally φ-contractive and ε does not depend on x.

In the case of a $\varepsilon-$chainable $b-$metric space endowed with a partially order "$\preceq$", we can prove the following Ran-Reurings type theorem.

Theorem 4.1. *Let (X, d) be a complete $\varepsilon-$chainable $b-$metric space with constant $s \geq 1$. Suppose that X is endowed with a partial order "$\preceq$". Let $f : X \to X$ be a mapping which has closed graph with respect to d and it is increasing with respect to "$\preceq$". Suppose that there exist a b-comparison function $\varphi : [0.\infty) \to [0.\infty)$ and an element $x_0 \in X$ such that:*

(i) $x_0 \preceq f(x_0)$ or $f(x_0) \preceq x_0$ and there exists an $\varepsilon-$chain $x_1, \cdots, x_{n-1}$ from x_0 to $x_n = f(x_0)$ such that every two consecutive elements of the chain are comparable with respect to "$\preceq$";

(ii) f is $\varepsilon-$uniformly ordered locally $\varphi-$contractive.

Then f has at least one fixed point.

Proof. Define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x \preceq y \text{ or } y \preceq x, \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, f is a $\varepsilon-$uniformly local $\alpha - \varphi-$contractive mapping, that is,

$$\alpha(p,q)d\left(f\left(p\right), f\left(q\right)\right) \leq \varphi\left(d\left(p,q\right)\right), \text{ for every } p, q \in X.$$

From condition (i), we have $\alpha(x_0, f(x_0)) \geq 1$. Moreover, for all $x, y \in X$, from the monotone property of f, we have

$$\alpha(x,y) \geq 1 \Longrightarrow x \preceq y \text{ or } y \preceq x \Longrightarrow fx \preceq fy \text{ or } fy \preceq fx \Longrightarrow \alpha(fx, fy) \geq 1.$$

Thus f is $\alpha-$admissible and we can apply Theorem 2.1. □

Remark 4.1. If, in the above theorem, additionally, we assume that for every elements $x, y \in X$ there exists an $\varepsilon-$chain such that every two consecutive elements are comparable, then the fixed point is unique. Indeed, suppose that there exists $y^* \in X$ with $x^* \neq y^*$, such that $f(y^*) = y^*$. Let us consider $x^* = x_0, x_1, ..., x_k = y^*$ be an $\varepsilon-$chain, such that x_{i-1} and x_i are comparable, for $i \in \{1, 2, \cdots, k\}$.. Then, $d(x_{i-1}, x_i) < \varepsilon$ and

$$d(f^m(x_{i-1}), f^m(x_i)) \leq \varphi^m(\varepsilon), \text{ for every } i \in \{1, 2, \cdots, k\} \text{ and } m \in \mathbb{N}.$$

Hence, we have

$$\begin{aligned} 0 &< d(x^*, y^*) = d\left(f\left(x^*\right), f\left(y^*\right)\right) = d\left(f^m\left(x^*\right), f^m\left(y^*\right)\right) \\ &= d\left(f^m\left(x_0\right), f^m\left(x_k\right)\right) < \left(s + s^2 + ... + s^k\right)\varphi^m\left(\varepsilon\right) \to 0, \text{ as } m \to \infty. \end{aligned}$$

Thus, we have a contradiction. Hence $x^* = y^*$.

5. Applications to the Coupled Fixed Point Problem

In this section, we'll give an application of Theorem 4.1. for coupled fixed points. Our result extends some results given in [14, 13, 15]. In this respect we need several auxiliary notions.

Definition 5.1. ([9]) Let $(X, \preceq)$ be a partially ordered set and let $T : X \times X \to X$ be a mapping. We say that T has the mixed monotone property if $T(\cdot, y)$ is monotone increasing for any $y \in X$ and $T(x, \cdot)$ is monotone decreasing for any $x \in X$.

Definition 5.2. If (X, d) is a $b-$metric space and $T : X \times X \to X$ is an operator, then by definition, a coupled fixed point for T is a pair $(x^*, y^*) \in X \times X$ satisfying

$$\begin{cases} x^* = T\left(x^*, y^*\right) \\ y^* = T\left(y^*, x^*\right) \end{cases} \tag{5.1}$$

Let us define

$$\widetilde{d}\left(\left(x,y\right),\left(u,v\right)\right) = \max\left\{d\left(x,u\right), d\left(y,v\right)\right\}. \tag{5.2}$$

Remark 5.1. It is easy to see that if (X, d) is a $b-$metric space with constant $s \geq 1$, then $\widetilde{d}$ is a $b-$metric on $X \times X$, with the same constant $s \geq 1$ and $\left(X \times X, \widetilde{d}\right)$ is a $b-$metric space.

Lemma 5.1. *If (X,d) is an ε-chainable $b-$metric space, the $\left(X\times X,\widetilde{d}\right)$ is an ε-chainable $b-$metric space, too.*

Proof. From Remark 5.1 we have that $\left(X\times X,\widetilde{d}\right)$ is a $b-$metric space.
Let $(x,y),(u,v)\in X\times X$. We must show that there exists an $\varepsilon-$chain

$$(x,y)=(x_0,y_0),(x_1,y_1),...,(x_n,y_n)=(u,v)$$

such that $\widetilde{d}((x_{i-1},y_{i-1}),(x_i,y_i))<\varepsilon$, for all $i\in\{1,...,n\}$.

For x and u, since the space X is ε-chainable, there exist $x=x_0,x_1,...,x_n=u$, such that $d(x_{i-1},x_i)<\varepsilon$, for all $i\in\{1,...,n\}$.

For y and v, since the space X is ε-chainable, there exist $y=y_0,y_1,...,y_n=v$, such that $d(y_{i-1},y_i)<\varepsilon$, for all $i\in\{1,...,n\}$.

Suppose $n\geq m$. We have the following two cases:

Case 1. For $i\in\{1,...,m\}$, we have

$$\widetilde{d}((x_{i-1},y_{i-1}),(x_i,y_i))=\max\{d(x_{i-1},x_i),d(y_{i-1},y_i)\}<\varepsilon.$$

Case 2. For $j\in\{m+1,...,n-1\}$ and $y_{m+1}=y_{m+2}=...=y_n=v$, we consider

$$\widetilde{d}((x_j,y_j),(x_{j+1},y_{j+1}))=\max\{d(x_j,x_{j+1}),d(y_j,y_{j+1})\}<\max\{\varepsilon,0\}=\varepsilon.$$

It follows that $\left(X\times X,\widetilde{d}\right)$ is an ε-chainable $b-$metric space. □

Definition 5.3. Let (X,d) be a $b-$metric space, $\varphi:[0,\infty)\to[0,\infty)$ be a a $b-$comparison function and $T:X\times X\to X$ be a given operator. We say that T is globally $\varphi-$contractive, if f

$$d(T(x,y),T(u,v))\leq\varphi\left(\widetilde{d}((x,y),(u,v))\right),\ \text{for all }(x,y),(u,v)\in X\times X. \quad (5.3)$$

Definition 5.4. Let (X,d) be a $b-$metric space, $\varphi:[0,\infty)\to[0,\infty)$ be a $b-$comparison function and "$\preceq$" be a partial order on X. A mapping $T:X\times X\to X$ is said to be *order locally* $\varphi-$contractive, if for every $(x,y)\in X\times X$, there exists $\varepsilon>0$, which may depend on x and y, such that

$$(s,t),(u,v)\in S((x,y),\varepsilon)=\left\{(p,q)\in X\times X|\,\widetilde{d}((x,y),(p,q))<\varepsilon\right\} \quad (5.4)$$

implies that

$$d(T(s,t),T(u,v))\leq\varphi\left(\widetilde{d}((s,t),(u,v))\right),\ \text{for every } s\preceq u \text{ and } v\preceq t. \quad (5.5)$$

Definition 5.5. In the above context, a mapping $T:X\times X\to X$ is said to be $\varepsilon-$uniformly ordered locally $\varphi-$contractive if it is ordered locally φcontractive and ε does not depend on x and y.

Theorem 5.1. *Let (X,d) be a complete $\varepsilon-$chainable $b-$metric space with constant $s\geq 1$. suppose that X is endowed with a partial order "$\preceq$". Let $T:X\times X\to X$ be an operator with closed graph which has the mixed monotone property on $X\times X$. Assume that the following conditions are satisfied:*

(i) there exists $(x_0,y_0)\in X\times X$ with $x_0\preceq T(x_0,y_0)$ and $T(y_0,x_0)\preceq y_0$ such that there exists an $\varepsilon-$chain $x_0,x_1,...,x_n=T(x_0,y_0)$, such that every two consecutive

elements of the chain are comparable with respect to "$\preceq$", and there exists an $\varepsilon-$chain $y_0, y_1, ..., y_n = T(y_0, x_0)$, such that every two consecutive elements of the chain are comparable with respect to "$\preceq$";

(ii) T is $\varepsilon-$uniformly ordered locally $\varphi-$contractive.

Then, there exists $(x^*(x_0, y_0), y^*(x_0, y_0)) \in X \times X$ a solution of the coupled fixed point problem (5.1) such that the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in X defined by

$$\begin{cases} x_{n+1} = T(x_n, y_n) \\ y_{n+1} = T(y_n, x_n) \end{cases}, \text{ for } n \in \mathbb{N}.$$

have the property that $x_n \to x^*(x_0, y_0), y_n \to y^*(x_0, y_0)$, as $n \to \infty$.

Moreover, for every pair $(x, y) \in X \times X$ with $x \preceq x_0, y_0 \preceq y$, we have that $T^n(x, y) \to x^*(x_0, y_0)$ and $T^n(y, x) \to y^*(x_0, y_0)$, as $n \to \infty$.

Proof. We denote $Z = X \times X$ and consider the functional $\widetilde{d} : Z \times Z \to [0, \infty)$, defined by

$$\widetilde{d}((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}.$$

Let $F_T : Z \to Z$ be an operator given by

$$F_T(x, y) = (T(x, y), T(y, x)), \text{ for all } (x, y) \in Z.$$

We shall prove that F verifies the conditions of Theorem 5.1.

By (i) and Lemma 5.1. we have that $(x_0, y_0) \preceq (T(x_0, y_0), T(y_0, x_0))$ and there exists an $\varepsilon-$chain $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n) = (T(x_0, y_0), T(y_0, x_0))$ such that $x_{i-1} \preceq x_i, y_i \preceq y_{i-1}$ (or reversely).

We shall prove that F_T is $\varepsilon-$uniformly ordered locally $\varphi-$contractive

Let $(x, y), (u, v) \in Z$ with $x \preceq u, v \preceq y$ (or reversely).

$$\begin{aligned} \widetilde{d}(F_T(x, y), F_T(u, v)) &= \widetilde{d}((T(x, y), T(y, x)), (T(u, v), T(v, u))) \\ &= \max\{d(T(x, y), T(u, v)), d(T(y, x), T(v, u))\}. \end{aligned}$$

Since T is $\varepsilon-$uniformly ordered locally $\varphi-$contractive, we have

$$\begin{aligned} \widetilde{d}(F_T(x, y), F_T(u, v)) &\leq \max\{\varphi(\max\{d(x, u), d(y, v)\}), \varphi(\max\{d(y, v), d(x, u)\})\} \\ &= \varphi(\max\{d(x, u), d(y, v)\}) = \varphi\left(\widetilde{d}((x, y), (u, v))\right). \end{aligned}$$

By Theorem 4.1 we obtain that there exists $(x^*(x_0, y_0), y^*(x_0, y_0)) \in Z$ such that

$$F_T(x^*(x_0, y_0), y^*(x_0, y_0)) = (x^*(x_0, y_0), y^*(x_0, y_0)),$$

and

$$F_T^n(x_0, y_0) \to (x^*(x_0, y_0), y^*(x_0, y_0)), \text{ as } n \to \infty.$$

We have

$$\begin{cases} x^*(x_0, y_0) = T(x^*(x_0, y_0), y^*(x_0, y_0)) \\ y^*(x_0, y_0) = T(y^*(x_0, y_0), x^*(x_0, y_0)) \end{cases}$$

and because

$$F_T^n(x_0, y_0) = (T^n(x_0, y_0), T^n(y_0, x_0)),$$

we obtain that $T^n(x_0, y_0) \to x^*(x_0, y_0)$ and $T^n(y_0, x_0) \to y^*(x_0, y_0)$, as $n \to \infty$. □

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