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Fractional order continuity of a time semi-linear fractional diffusion-wave system

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Abstract In this work, we consider the time-fractional diffusion equations depend on fractional orders. In more detail, we study on the initial value problems for the time semi-linear fractional diffusion-wave system and discussion about continuity with respect to the fractional derivative order. We find the answer to the question: When the fractional orders get closer, are the corresponding solutions close? To answer this question, we present some depth theories on PDEs and fractional calculus. In addition, we add an example numerical to verify the proposed theory.

1. Introduction

In the area fractional calculus, there are many applications in mechanic, biological, physical science, and applied science where the generalization of classical diffusion equation is the fractional diffusion equation. There are many methods and ideas were developed for fractional PDEs; we cite the reader to [15–18,26,27,21,3,5,6,28,8,29,38–66,40,17,41], and the references therein. Tuan et al. [23] established continuity with a fractional order of the time diffusion-wave equation. In this paper, we extend the equation and consider the following coupled semi-linear fractional diffusion-wave equations with $x \in \Omega, t \in (0,T)$

$$\begin{cases} \partial_t^{\alpha} u(x,t) + \mathscr{A} u(x,t) &= \mathscr{F}(u,v), \\ \partial_t^{\alpha} v(x,t) + \mathscr{A} v(x,t) &= \mathscr{G}(u,v), \end{cases} \tag{1.1}$$

with the boundary conditions as follows: $u(x,t) = 0$ and $v(x,t) = 0$ for $x \in \partial\Omega, t \in (0,T]$. In addition, the problem

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(1.1) satifies the following initial value conditions (for $(x,t) \in \Omega \times \{0\}$)

$$u(x,t) = u_0(x), \quad u_t(x,t) = 0,$$

and

$$v(x,t) = v_0(x), \quad v_t(x,t) = 0.$$

In (1.1), the symmetric uniformly elliptic operator $\mathscr{A}$ which is defined in [15], the constant $1 < \alpha < 2$ is the fractional-order. In addition, the Caputo fractional derivative ∂_t^α (respect to t) which is defined as follows

$$\partial_t^\alpha u(x,t) := \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\eta)^{1-\alpha} \frac{\partial^2 u}{\partial \eta}(x,\eta) d\eta,$$

here, the Gamma function is Γ. If $\alpha = 2$ then ∂_t^2 is understood as the usual time derivative. Let Ω be a open domain and bounded in $\mathbb{R}^n$ adding a smooth boundary $\partial\Omega$. The functions $\mathscr{F}, \mathscr{G}, u_0, v_0$ satisfy some given assumptions. The case $1 < \alpha < 2$ of our problem is called super-diffusive model of anomalous diffusion equation. We can see some quasi linear equations with the form of (1.1), where classical time derivative $(\alpha = 2)$.

The author El-Sayed ([30]) defines the following fractional order problem

$$\partial_t^\alpha u(x,t) + \mathscr{A} u(x,t) = 0, \quad u(0) = u_0, \quad u_t(0) = 0, \tag{1.2}$$

and he also considered the existence of mild solution of (1.2). Recently, some mathematicians, such as, Mainardi [35], Schneider and Wyss [36] have studied some various properties of the fractional diffusion/wave equation. Very recently, the semi-linear super-diffusive equations are studied by E. Alvarez and his co-authors [7], here they considered the well-posedness of these problems. According to [31], the authors H. Jafari et al., to get the approximate solution for nonlinear fractional diffusion and wave equations, they applied the modified of the homotopy perturbation method.

Recall that the problem (1.1) is associated to anomalous diffusion phenomenon in physical motivations. Especially, for $1 < \alpha < 2$, the problem (1.1) is used for the super-diffusive model in heterogeneous media. Some physical background is found in Sokolov [37], Mainardi [32], Kilbas [33], Podlubny [34]. For an example about the application of fractional calculus in physical, the authors in [20] show that the fractional calculus are effective to appreciate for the damping controllers which compared to the classical derivative. In particular, in a diffusion model we need to describe a diffusion process with a time-variable memory type.

According to our searching, there are very few works on the fractional couped diffusion systems. Almost works used this model to describe chemical, physical and biological processes [1]. In the case: direct problem for the fractional diffusion system, we can find many papers which was studied in [1,2,16]. However, there are no results about the investigating continuity on the fractional order of time semi-linear fractional diffusion-wave system. Hence, in this work, we deal on the following question

Does $u_{\alpha_n} \to u_\alpha$ in an appropriate sense as $n \to \infty$?

In order to solve the above question, we have some difficulties things for considering all constants independently on the fractional orders. This is a challenge task for us when we study (1.3).

This article is numbered into sections as follows: In the first section, we introduce the problem. In the second section, we present some preliminaries which will be usful for next sections. In third section, we study the continuity property of the solution of the initial problem (1.1). In the final section, we also give a numerical example to show the useful of method which is used in this paper.

2. Preliminaries

2.1. The Mittag-Leffler function

We consider an important function in this area which is called by the Mittag-Leffler function as follows

$$E_{\alpha,\beta}(\xi) = \sum_{n=1}^{\infty} \frac{\xi^n}{\Gamma(n\alpha + \beta)},$$

where $\alpha > 0, \beta \in \mathbb{R}$ and $\xi \in \mathbb{C}$. We call to mind the following lemmas (see for example [40,13,14]), which will be useful for the results in Sections 3 and 4.

Lemma 2.1. *For $\mathfrak{C}$ is a positive constant depending only on α. If $1 < \alpha < 2$, then for all $\xi > 0$, such that*

$$|E_\alpha(-\xi)| \leqslant \mathfrak{C}, |E_{\alpha,\alpha}(-\xi)| \leqslant \mathfrak{C}.$$

Proof. Applying Lemma 2.1 in [13]. □

Now, we consider the lemmas as follows.

Lemma 2.2. *Let $\lambda > 0$ and $1 < \alpha < 2$. For all $t > 0$, we have the following identities*

$$\partial_t E_\alpha(-\lambda t^\alpha) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha),$$
$$\partial_t(t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha).$$

Proof. Applying Lemma 2.2 in [19]. □

Lemma 2.3 (*see [25]*). *Let $1 < \alpha < 2$ if T is large enough then*

$$E_\alpha(-\lambda_j T^\alpha) \neq 0, \tag{2.3}$$

for all $j \in \mathbb{N}$ then there exists two constant $\mathfrak{m}_\alpha$ and $\mathfrak{M}_\alpha$ such that

$$\frac{\mathfrak{m}_\alpha}{1 + \lambda_j T^\alpha} \leqslant |E_\alpha(-\lambda_j T^\alpha)| \leqslant \frac{\mathfrak{M}_\alpha}{1 + \lambda_j T^\alpha}. \tag{2.4}$$

From lemma 2.3 [4], we have following lemma.

Lemma 2.4. *Let $1 < a < b < 2, \alpha \in (a,b)$ and $\mathfrak{M}_1, \mathfrak{M}_2$ which are two positive constants, and $\mathfrak{M}_3$ which only depend on a,b such that for any $z > 0$ we get*

$$\frac{\mathfrak{M}_1(a,b)}{1+z} \leqslant |E_\alpha(-z)| \leqslant \frac{\mathfrak{M}_2(a,b)}{1+z}, |E_{\alpha,\alpha}(-z)| \leqslant \frac{\mathfrak{M}_3(a,b)}{1+z}. \tag{2.5}$$

Lemma 2.5. *Let $0 < a < \alpha < \alpha\prime < b$ and $0 < t \leqslant T$. For any $\epsilon > 0$ which independent on α, there always exists $\mathfrak{M}_\epsilon$ such that*

$$|t^\alpha - t^{\alpha\prime}| \leqslant \max(T^{b+2\epsilon}, 1)\mathfrak{M}_\epsilon(\alpha\prime - \alpha)^\epsilon t^{\alpha-\epsilon}. \tag{2.6}$$

Proof. See Lemma 3.2 [19] □

Using Lemmas 3.3–3.4 from section 3 [19], we consider the lemma as follows.

Lemma 2.6. *Assume that* $1 < a < \alpha < \alpha\prime < b < 2$ *and* $\epsilon > 0$. *Then there exists* $\mathscr{B}(a, b, \epsilon, \gamma, T)$ *is a positive constant satisfies*

$$|E_\alpha(-\lambda_j t^\alpha) - E_{\alpha\prime,1}(-\lambda_j t^{\alpha\prime})| \leqslant \mathfrak{B}(a, b, \epsilon, \beta, T)\lambda_j^{\beta-1} t^{-b(1-\beta)-\epsilon}[(\alpha\prime - \alpha)^\epsilon + (\alpha\prime - \alpha)], \tag{2.7}$$

for any $0 \leqslant \beta \leq 1$ and $0 < t \leqslant T$.

Proof. See Lemma 3.3 from Section 3 in [19]. □

Lemma 2.7. *Assume that* $1 < a < \alpha < \alpha\prime < b < 2$. *For* $\epsilon > 0$ *and any* $0 \leqslant \beta \leqslant 1$, *there exists* $\mathscr{C}(a, b, \epsilon, \beta, T)$ *is a positive constant satisfies*

$$\begin{aligned} &\left|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j t^\alpha) - t^{\alpha\prime-1}E_{\alpha\prime,\alpha\prime}(-\lambda_j t^{\alpha\prime})\right| \\ &\leqslant \mathscr{C}(a, b, \epsilon, \beta, T)\lambda_j^{\beta-1} t^{a\beta-\epsilon-1}[(\alpha\prime - \alpha)^\epsilon + (\alpha\prime - \alpha)]. \end{aligned} \tag{2.8}$$

Proof. Using Lemma 3.4 from section 3 in [19]. □

2.2. Some Sobolev spaces

Note $\mathbb{L}^2(\Omega), H_0^1(\Omega), H^2(\Omega)$ denote the usual Sobolev space. Let $\mathscr{A}$ is the operator considered on $\mathbb{L}^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$, where the spectrum is a increasing sequence with positive real numbers $\{\lambda_j\}_{j\geqslant 1}$ respectively which satisfy that $\lim_{j\to\infty}\lambda_j = \infty$. Let us denote by $\{\varphi_j\}_{j\geqslant 1}$ in $H^2(\Omega) \cap H_0^1(\Omega)$ are the orthonormal eigenfunctions of $\mathscr{A}$ respectively, which mean that $\mathscr{A}\varphi_j = \lambda_j\varphi_j$. The sequence form an orthonormal basic of $\mathbb{L}^2(\Omega)$ see e.g. [24]. We define by $\mathscr{A}^\eta$ the following operator

$$\mathscr{A}^\eta v := \sum_{j=1}^{\infty}\langle v, \varphi_j\rangle\lambda_j^\eta\varphi_j, \quad v \in \mathbb{D}(\mathscr{A}^\eta) = \left\{v \in \mathbb{L}^2(\Omega) : \sum_{j=1}^{\infty}|\langle v, \varphi_j\rangle|^2\lambda_j^{2\eta} < \infty\right\}.$$

The domain $\mathbb{D}(\mathscr{A}^\eta)$ is Banach spaces equipped with the norm

$$\|v\|^2_{\mathbb{D}(\mathscr{A}^\eta)} := \sum_{j=1}^{\infty}\lambda_j^{2\eta}\left|\langle v, \varphi_j\rangle\right|^2. \tag{2.10}$$

If $\eta = 1$, we have $\mathbb{D}(\mathscr{A}^1) = H^2(\Omega)$.

For $\eta \geqslant 0$ is a given number, we have the Hilbert space as follows

$$\mathbb{H}^\eta(\Omega) = \left\{v \in \mathbb{L}^2(\Omega) : \sum_{j=1}^{\infty}\left|\langle v, \varphi_j\rangle\right|^2\lambda_j^{2\eta} < \infty\right\}. \tag{2.11}$$

We identified a norm for $w(u, v) \in \mathscr{H}^\eta(\Omega) = \mathbb{H}^\eta(\Omega) \times \mathbb{H}^\eta(\Omega)$ as follow

$$\|w\|_{\mathscr{H}^\eta(\Omega)} = \sqrt{\|u\|^2_{\mathbb{H}^\eta(\Omega)} + \|v\|^2_{\mathbb{H}^\eta(\Omega)}}. \tag{2.12}$$

Let a space of continuous functions with map $(0, T] \to \mathbb{H}^\eta(\Omega)$ is $\mathbb{C}((0, T]; \mathbb{H}^\eta(\Omega))$ and $0 < \beta < 1$ is a given number. We define the space $\mathbb{C}^\beta((0, T]; \mathbb{H}^\eta(\Omega))$ such that

$$\sup_{0<t\leqslant T} t^\beta\|f(t)\|_{\mathbb{H}^\eta(\Omega)} < \infty, \quad f \in \mathbb{C}((0, T]; \mathbb{H}^\eta(\Omega)),$$

in which (see [4])

$$\|f\|_{\mathbb{C}^\beta((0,T];\mathbb{H}^\eta(\Omega))} := \sup_{0<t\leqslant T} t^\beta\|f\|_{\mathbb{H}^\eta(\Omega)}.$$

The product space $\mathbb{V}^\beta(0, T, \mathbb{H}^\eta(\Omega)) = \mathbb{C}^\beta(0, T, \mathbb{H}^\eta(\Omega)) \times \mathbb{C}^\beta(0, T, \mathbb{H}^\eta(\Omega))$ is also a Banach space endowed with the norm

$$\|\mathbf{w}\|_{\mathbb{V}^\beta(0,T,\mathbb{H}^\eta(\Omega))} = \sqrt{\|u\|^2_{\mathbb{C}^\beta(0,T,\mathbb{H}^\eta(\Omega))} + \|v\|^2_{\mathbb{C}^\beta(0,T,\mathbb{H}^\eta(\Omega))}},$$

for $w = (u, v) \in \mathbb{V}^\beta(0, T, \mathbb{H}^\eta(\Omega))$. Let p is a given positive real number, we have $\mathbb{L}_p^\infty(0, T, \mathbb{H}^\eta(\Omega))$ is a Banach space with norm

$$\|f\|_{\mathbb{L}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))} := ess\sup e^{-pt}\|f\|_{\mathbb{H}^\eta(\Omega)}. \tag{2.13}$$

The product space $\mathbb{W}_p^\infty(0, T, \mathbb{H}^\eta(\Omega)) = \mathbb{L}_p^\infty(0, T, \mathbb{H}^\eta(\Omega)) \times \mathbb{L}_p^\infty(0, T, \mathbb{H}^\eta(\Omega))$ is also a Banach space endowed with the norm

$$\|\mathbf{w}\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))} = \sqrt{\|u\|^2_{\mathbb{L}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))} + \|v\|^2_{\mathbb{L}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}},$$

where $w = (u, v) \in \mathbb{W}_p^\infty(0, T, \mathbb{H}^\eta(\Omega))$.

3. Continuity with respect to fractional order of initial value problem

The aim of this section, we consider the existence of a mild solution of problem (1.1) with the initial condition u_0, v_0). From [23], we have the solution in terms as

$$\begin{cases} \mathbf{u}(\cdot, t) = \mathscr{S}_{\mathscr{A},\alpha}(t)u_0 + \int_0^t \mathscr{P}_{\mathscr{A},\alpha}(t-s)\mathscr{F}(u, v)(\cdot, s)ds, \\ \mathbf{v}(\cdot, t) = \mathscr{S}_{\mathscr{A},\alpha}(t)v_0 + \int_0^t \mathscr{P}_{\mathscr{A},\alpha}(t-s)\mathscr{G}(u, v)(\cdot, s)ds, \end{cases} \tag{3.14}$$

where

$$\mathscr{S}_{\mathscr{A},\alpha}(t)w := \sum_{j=1}^{\infty}E_\alpha(-\lambda_j t^\alpha)\langle w, \varphi_j\rangle\varphi_j,$$

$$\mathscr{P}_{\mathscr{A},\alpha}(t-s)w := \sum_{j=1}^{\infty}(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-s)^\alpha)\langle w, \varphi_j\rangle\varphi_j.$$

Lemma 3.1. *Let* $1 < \alpha < 2, \gamma < \eta < \gamma + 1$ *and* $w \in \mathbb{H}^\eta(\Omega)$. *The following inequalities hold:*

$$\|\mathscr{S}_{\mathscr{A},\alpha}(t)w\|_{\mathbb{H}^\eta(\Omega)} \leqslant \mathfrak{M}_2(a, b)t^{\alpha(\gamma-\eta)}\|w\|_{\mathbb{H}^\gamma(\Omega)}, \tag{3.15}$$

$$\|\mathscr{P}_{\mathscr{A},\alpha}(t-s)w\|_{\mathbb{H}^\eta(\Omega)} \leqslant \mathfrak{M}_3(a, b)(t-s)^{\alpha-1}\|w\|_{\mathbb{H}^\eta(\Omega)}. \tag{3.16}$$

Proof. Using lemma (2.4), we get

$$\begin{aligned} \|\mathscr{S}_{\mathscr{A},\alpha}(t)w\|^2_{\mathbb{H}^\eta(\Omega)} &= \sum_{j=1}^{\infty}\lambda_j^{2\eta}E^2_{\alpha,1}(-\lambda_j t^\alpha)|\langle w, \varphi_j\rangle|^2 \leqslant \sum_{j=1}^{\infty}\lambda_j^{2\eta}\left(\frac{\mathfrak{M}_2(a,b)}{1+\lambda_j t^\alpha}\right)^2|\langle w, \varphi_j\rangle|^2 \\ &\leqslant \sum_{j=1}^{\infty}\lambda_j^{2\eta}\left(\frac{\mathfrak{M}_2(a,b)}{1+\lambda_j t^\alpha}\right)^{2(\gamma-\eta+1)}\left(\frac{\mathfrak{M}_2(a,b)}{1+\lambda_j t^\alpha}\right)^{2(\eta-\gamma)}|\langle w, \varphi_j\rangle|^2 \\ &\leqslant \mathfrak{M}_2^2(a, b)t^{2\alpha(\gamma-\eta)}\sum_{j=1}^{\infty}\lambda_j^{2\gamma}|\langle w, \varphi_j\rangle|^2 \leqslant \mathfrak{M}_2^2(a, b)t^{2\alpha(\gamma-\eta)}\|w\|^2_{\mathbb{H}^\gamma(\Omega)}. \end{aligned}$$

Therefore, we have estimate as

$$\|\mathscr{S}_{\mathscr{A},\alpha}(t)w\|_{\mathbb{H}^\eta(\Omega)} \leqslant \mathfrak{M}_2(a, b)t^{\alpha(\gamma-\eta)}\|w\|_{\mathbb{H}^\gamma(\Omega)}. \tag{3.17}$$

Similarly, using lemma (2.4), we have

$$\begin{aligned}\|\mathscr{P}_{\mathscr{A},\alpha}(t-s)w\|^2_{\mathbb{H}^\eta(\Omega)} &= \sum_{j=1}^{\infty}\lambda_j^{2\eta}(t-s)^{2(\alpha-1)}E^2_{\alpha,\alpha}(-\lambda_j(t-s)^\alpha)|\langle w,\varphi_j\rangle|^2\\ &\leqslant \sum_{j=1}^{\infty}\lambda_j^{2\eta}(t-s)^{2(\alpha-1)}\left(\frac{\mathfrak{M}_3(a,b)}{1+\lambda_j(t-s)^\alpha}\right)^2|\langle w,\varphi_j\rangle|^2\\ &\leqslant \mathfrak{M}_3^2(a,b)(t-s)^{2\alpha-2}\|w\|^2_{\mathbb{H}^\eta(\Omega)}.\end{aligned} \tag{3.18}$$

Therefore, we deduce

$$\|\mathscr{P}_{\mathscr{A},\alpha}(t-s)w\|_{\mathbb{H}^\eta(\Omega)} \leqslant \mathfrak{M}_3(a,b)(t-s)^{\alpha-1}\|w\|_{\mathbb{H}^\eta(\Omega)}.$$

Completing the proof of lemma (3.1). □

Lemma 3.2. *Let* $1<\alpha<2, \gamma<\eta<\gamma+1, \beta=\gamma-\eta+1$ *with* $0<\beta<1$ *and* $w\in\mathbb{H}^\eta(\Omega)$. *The following inequalities hold:*

$$\begin{aligned}&\left\|[\mathscr{S}_{\mathscr{A},\alpha\prime}(t)-\mathscr{S}_{\mathscr{A},\alpha}(t)]w\right\|_{\mathbb{H}^\eta(\Omega)}\\ &\quad\leqslant \mathfrak{B}(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]t^{-b(\eta-\gamma)-\epsilon}\|w\|_{\mathbb{H}^\gamma(\Omega)},\end{aligned} \tag{3.20}$$

$$\begin{aligned}&\left\|[\mathscr{P}_{\mathscr{A},\alpha\prime}(t-s)-\mathscr{P}_{\mathscr{A},\alpha}(t-s)]w\right\|_{\mathbb{H}^\eta(\Omega)}\\ &\quad\leqslant \mathscr{C}(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]\lambda_0^{\beta-1}(t-s)^{a(\gamma+1-\eta)-\epsilon-1}\|w\|_{\mathbb{H}^\eta(\Omega)}.\end{aligned} \tag{3.21}$$

Proof. Using lemma (2.6), we get

$$\begin{aligned}&\|[\mathscr{S}_{\mathscr{A},\alpha\prime}(t)-\mathscr{S}_{\mathscr{A},\alpha}(t)]w\|^2_{\mathbb{H}^\eta(\Omega)} = \sum_{j=1}^{\infty}\lambda_j^{2\eta}[E_{\alpha\prime,1}(-\lambda_j t^{\alpha\prime})-E_\alpha(-\lambda_j t^\alpha)]^2|\langle w,\varphi_j\rangle|^2\\ &\leqslant \sum_{j=1}^{\infty}\lambda_j^{2\eta}\mathfrak{B}^2(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]^2\lambda_j^{2(\beta-1)}t^{-2b(1-\beta)-2\epsilon}|\langle w,\varphi_j\rangle|^2\\ &\leqslant \mathfrak{B}^2(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]^2 t^{-2b(1-\beta)-2\epsilon}\sum_{j=1}^{\infty}\lambda_j^{2(\eta+\beta-1)}|\langle w,\varphi_j\rangle|^2\\ &\leqslant \mathfrak{B}^2(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]^2 t^{-2b(1-\beta)-2\epsilon}\|w\|^2_{\mathbb{H}^{\eta+\beta-1}(\Omega)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\left\|[\mathscr{S}_{\mathscr{A},\alpha\prime}(t)-\mathscr{S}_{\mathscr{A},\alpha}(t)]w\right\|_{\mathbb{H}^\eta(\Omega)}\\ &\quad\leqslant \mathfrak{B}(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]t^{-b(\eta-\gamma)-\epsilon}\|w\|_{\mathbb{H}^\gamma(\Omega)}.\end{aligned} \tag{3.22}$$

where $\gamma=\eta+\beta-1<\eta, 0<\beta<1$. Similarly, applying lemma (2.7) with $0<\beta<1$, we get

$$\begin{aligned}&\|[\mathscr{P}_{\mathscr{A},\alpha\prime}(t-s)-\mathscr{P}_{\mathscr{A},\alpha}(t-s)]w\|^2_{\mathbb{H}^\eta(\Omega)}\\ &=\sum_{j=1}^{\infty}\lambda_j^{2\eta}\left[(t-s)^{\alpha\prime-1}E_{\alpha\prime,\alpha\prime}(-\lambda_j(t-s))^{\alpha\prime}-(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_j(t-s))^\alpha\right]^2|\langle w,\varphi_j\rangle|^2\\ &\leqslant \sum_{j=1}^{\infty}\lambda_j^{2\eta}\left[\mathscr{C}(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]\lambda_j^{\beta-1}(t-s)^{a\beta-\epsilon-1}\right]^2|\langle w,\varphi_j\rangle|^2\\ &\leqslant \left[\mathscr{C}(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]\lambda_0^{\beta-1}(t-s)^{a\beta-\epsilon-1}\right]^2\sum_{j=1}^{\infty}\lambda_j^{2\eta}|\langle w,\varphi_j\rangle|^2\\ &\leqslant \left[\mathscr{C}(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]\lambda_0^{\beta-1}(t-s)^{a\beta-\epsilon-1}\right]^2\|w\|^2_{\mathbb{H}^\eta(\Omega)}.\end{aligned}$$

And so

$$\begin{aligned}&\left\|[\mathscr{P}_{\mathscr{A},\alpha\prime}(t-s)-\mathscr{P}_{\mathscr{A},\alpha}(t-s)]w\right\|_{\mathbb{H}^\eta(\Omega)}\\ &\qquad\leqslant \mathscr{C}(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^\epsilon\\ &+(\alpha\prime-\alpha)]\lambda_0^{\beta-1}(t-s)^{a(\gamma+1-\eta)-\epsilon-1}\|w\|_{\mathbb{H}^\eta(\Omega)}.\end{aligned}$$

we get all estimates of Lemma 3.2. This complete the proof. □

We assume that $\mathscr{F},\mathscr{G}$ satisfy the assumptions as follows.

(S.1)

$$\begin{aligned}\|\mathscr{F}(u,v)(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)} \leqslant C_1\left(1+\|u(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)}+\|v(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)}\right),\\ \|\mathscr{G}(u,v)(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)} \leqslant C_2\left(1+\|u(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)}+\|v(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)}\right),\end{aligned} \tag{3.23}$$

where $(u,v)\in\mathscr{H}^\eta(\Omega)=\mathbb{H}^\eta(\Omega)\times\mathbb{H}^\eta(\Omega)$.

(S.2).

$$\begin{aligned}\|\mathscr{F}(u_1,v_1)(\cdot,t)-\mathscr{F}(u_2,v_2)(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)} &\leqslant K_1\left(\|u_1-u_2\|_{\mathbb{H}^\eta(\Omega)}+\|v_1-u_2\|_{\mathbb{H}^\eta(\Omega)}\right),\\ \|\mathscr{G}(u_1,v_1)(\cdot,t)-\mathscr{G}(u_2,v_2)(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)} &\leqslant K_2\left(\|u_1-u_2\|_{\mathbb{H}^\eta(\Omega)}+\|v_1-u_2\|_{\mathbb{H}^\eta(\Omega)}\right),\end{aligned} \tag{3.24}$$

where $(u_1,v_1)\in\mathscr{H}^\eta(\Omega),(u_2,v_2)\in\mathscr{H}^\eta(\Omega)$.

Definition 3.1. If the solution $w=(u(\cdot,t),v(\cdot,t))\in\mathbb{W}_p^\infty(0,T;\mathbb{H}^\eta(\Omega))$ satisfies System (3.14), then w is called a mild solution of Problem (1.1).

Theorem 3.1. *Let* $(u_0,v_0)\in\mathscr{H}^\eta(\Omega)$. *Assume that* $1<a<\alpha<\alpha\prime<b<2$ *and* $0<\beta<1$. *The problem (1.1) has unique solution* $w(u,v)\in\mathbb{W}_p^\infty(0,T;\mathbb{H}^\eta(\Omega))$. *Let* $w_\alpha\in\mathbb{W}_p^\infty(0,T;\mathbb{H}^\eta(\Omega))$ *and* $w_{\alpha\prime}\in\mathbb{W}_p^\infty(0,T;\mathbb{H}^\eta(\Omega))$ *be two solution of (1.1) with fractional order* α *and* $\alpha\prime$ *respectively. If exist numbers* γ,ϵ *satisfy* $0<\epsilon<\left(\min a(\gamma+1-\eta)-\frac{1}{2}, b(\gamma-\eta)+\frac{1}{2}\right)$ *and* $\gamma<\eta<\gamma+1$ *then*

$$\|\mathbf{w}\|_{\mathbb{V}^{\alpha(\eta-\gamma)}((\mathbf{0},\mathbf{T}],\mathbb{H}^\eta(\Omega))} \leqslant \sqrt{\overline{\mathfrak{C}}^{\eta,\gamma}_{1,\alpha}(\mathbf{a},\mathbf{b})}\exp\left(\frac{\overline{\mathfrak{C}}^{\eta,\gamma}_{2,\alpha}(\mathbf{a},\mathbf{b},\mathbf{T})}{\mathbf{2}}\right),$$

$$\begin{aligned}&\|\mathbf{w}_{\alpha\prime}-\mathbf{w}_\alpha\|_{\mathbb{V}^{b(\eta-\gamma)+\epsilon}(0,T,\mathbb{H}^\eta(\Omega))}\\ &\leqslant [(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]\\ &\qquad\times\left[\mathscr{M}_1^{\eta,\gamma,\epsilon}\left(a,b,T,\|w_0\|_{\mathscr{H}^\gamma(\Omega)},C_1,C_2\right)\exp\right.\\ &\left.\mathscr{M}_2^{\eta,\gamma,\epsilon}(a,b,T,K_1,K_2)[(\alpha\prime-\alpha)^\epsilon+(\alpha\prime-\alpha)]^2T\right]^{1/2}.\end{aligned} \tag{3.26}$$

Proof (***Proof of Theorem 3.1***). Proof of this theorem consist some smaller parts.

Part 1. Existence and uniqueness of the solution of (1.1) For $w\in\mathbb{W}_p^\infty(0,T;\mathbb{H}^\eta(\Omega))$ We consider the following function

$$\mathfrak{D}\mathbf{w} := (\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}(\cdot,t),\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}(\cdot,t))$$

where

$$\begin{cases}\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}(\cdot,t) &= \mathscr{S}_{\mathscr{A},\alpha}(t)u_0+\int_0^t\mathscr{P}_{\mathscr{A},\alpha}(t-s)\mathscr{F}(u,v)(\cdot,s)ds,\\ \mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}(\cdot,t) &= \mathscr{S}_{\mathscr{A},\alpha}(t)v_0+\int_0^t\mathscr{P}_{\mathscr{A},\alpha}(t-s)\mathscr{G}(u,v)(\cdot,s)ds.\end{cases} \tag{3.27}$$

By applying the Banach fixed-point theorem, we obtain that the equation $\mathfrak{D}w_*=w_*$ has the unique solution $w_*\in\mathbb{W}_p^\infty(0,T;\mathbb{H}^\eta(\Omega))$.

Let $w_1(u_1(\cdot,t),v_1(\cdot,t))\in\mathbb{W}_p^\infty(0,T;\mathbb{H}^\eta(\Omega)), w_2(u_2(\cdot,t),v_2(\cdot,t))\in$-$\mathbb{W}_p^\infty(0,T;\mathbb{H}^\eta(\Omega))$ we have

$$\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_1(\cdot,t)-\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_2(\cdot,t)=\int_0^t\mathscr{P}_{\mathscr{A},\alpha}(t-s)[\mathscr{F}(u_1,v_1)(\cdot,s)-\mathscr{F}(u_2,v_2)(\cdot,s)]ds, \tag{3.28}$$

$$\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}_1(\cdot,t)-\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}_2(\cdot,t)=\int_0^t\mathscr{P}_{\mathscr{A},\alpha}(t-s)[\mathscr{G}(u_1,v_1)(\cdot,s)-\mathscr{G}(u_2,v_2)(\cdot,s)]ds. \tag{3.29}$$

We get the following estimate

$$\begin{aligned}&\left\|e^{-pt}(\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_1(\cdot,t)-\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_2(\cdot,t))\right\|_{\mathbb{H}^\eta(\Omega)}\\ =\;&\left\|\int_0^t\mathscr{P}_{\mathscr{A},\alpha}(t-s)e^{-pt}[\mathscr{F}(u_1,v_1)(\cdot,s)-\mathscr{F}(u_2,v_2)(\cdot,s)]ds\right\|_{\mathbb{H}^\eta(\Omega)}.\end{aligned}$$

Applying lemma (3.1), we get the following estimate

$$\begin{aligned}&\left\|e^{-pt}(\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_1(\cdot,t)-\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_2(\cdot,t))\right\|_{\mathbb{H}^\eta(\Omega)}\\ &\leqslant\int_0^t\mathfrak{M}_3(a,b)(t-s)^{\alpha-1}e^{-p(t-s)}\Big[e^{-ps}\|\mathscr{F}(u_1,v_1)(\cdot,s)\\ &-\mathscr{F}(u_2,v_2)(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\Big]ds.\end{aligned}$$

Using (3.24) and applying Hölder inequality, we obtain

$$\begin{aligned}&\|e^{-pt}(\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_1(\cdot,t)-\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_2(\cdot,t))\|_{\mathbb{H}^\eta(\Omega)}\\ &\leqslant K_1\mathfrak{M}_3(a,b)\int_0^t e^{-ps}\Big(\|\mathbf{u}_1(\cdot,s)-\mathbf{u}_2(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\\ &+\|\mathbf{v}_1(\cdot,s)-\mathbf{v}_2(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\Big)(t-s)^{\alpha-1}e^{-p(t-s)}ds\\ &\leqslant K_1\mathfrak{M}_3(a,b)\Big(\|\mathbf{u}_1-\mathbf{u}_2\|_{\mathbb{L}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}+\|\mathbf{v}_1-\mathbf{v}_2\|_{\mathbb{L}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}\Big)\int_0^t(t-s)^{\alpha-1}e^{-p(t-s)}ds\\ &\leqslant K_1\mathfrak{M}_3(a,b)\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}\int_0^t(t-s)^{\alpha-1}e^{-p(t-s)}ds\\ &\leqslant K_1\mathfrak{M}_3(a,b)\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}\int_0^t(t-s)^{\alpha-1}ds\\ &\leqslant\frac{K_1\mathfrak{M}_3(a,b)}{\alpha}t^\alpha\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}.\end{aligned}$$

Hence

$$\begin{aligned}&\left\|\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_1-\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_2\right\|_{\mathbb{L}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}\\ &\leqslant\frac{K_1\mathfrak{M}_3(a,b)}{\alpha}T^\alpha\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}.\end{aligned} \tag{3.30}$$

We can obtain a similar estimate

$$\begin{aligned}&\left\|\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}_1-\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}_2\right\|_{\mathbb{L}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}\\ &\leqslant\frac{K_2\mathfrak{M}_3(a,b)}{\alpha}T^\alpha\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}.\end{aligned} \tag{3.31}$$

Therefore, we find that

$$\begin{aligned}&\|\mathfrak{D}\mathbf{w}_1-\mathfrak{D}\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}=\left\|\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_1-\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_2\right\|_{\mathbb{L}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}\\ &+\left\|\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}_1-\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}_2\right\|_{\mathbb{L}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}\\ &\leqslant\frac{(K_1+K_2)\mathfrak{M}_3(a,b)}{\alpha}T^\alpha\|\mathbf{w}_1-\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}.\end{aligned} \tag{3.32}$$

If $\mathbf{w}_2=0$ then for any $\mathbf{w}\in\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))$ then

$$\begin{aligned}\|\mathfrak{D}\mathbf{w}\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}&\leqslant\|\mathfrak{D}\mathbf{w}-\mathfrak{D}\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}+\|\mathfrak{D}\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}\\ &\leqslant\frac{(K_1+K_2)\mathfrak{M}_3(a,b)}{\alpha}T^\alpha\|\mathbf{w}\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}+\|\mathfrak{D}\mathbf{w}_2\|_{\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))}.\end{aligned} \tag{3.33}$$

$$\mathfrak{D}\mathbf{w}_2:=(\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_2(\cdot,t),\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}_2(\cdot,t)),$$

where $\mathbf{u}_2=\mathbf{v}_2=0$ and

$$\begin{cases}\mathfrak{D}_{\mathscr{A},\alpha}\mathbf{u}_2(\cdot,t)&=\mathscr{S}_{\mathscr{A},\alpha}(t)u_0,\\ \mathfrak{D}_{\mathscr{A},\alpha}\mathbf{v}_2(\cdot,t)&=\mathscr{S}_{\mathscr{A},\alpha}(t)v_0.\end{cases} \tag{3.34}$$

By applying Lemma 3.1, we deduce that $\mathfrak{D}\mathbf{w}_2\in\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))$. Therefore, we conclude that if any $\mathbf{w}\in\mathbb{W}_p^\infty(0,T,\mathbb{H}^\eta(\Omega))$ then $\mathfrak{D}\mathbf{w}$ is bounded.

Part 2. From (3.14) and using inequality $(u+v)^2\leqslant 2(u^2+v^2)$, we have

$$\begin{aligned}\|\mathbf{u}(\cdot,t)\|^2_{\mathbb{H}^\eta(\Omega)}&\leqslant 2\|\mathscr{S}_{\mathscr{A},\alpha}(t)u_0\|^2_{\mathbb{H}^\eta(\Omega)}\\ &+2\left\|\int_0^t\mathscr{P}_{\mathscr{A},\alpha}(t-s)\mathscr{F}(u,v)(\cdot,s)ds\right\|^2_{\mathbb{H}^\eta(\Omega)}.\end{aligned} \tag{3.35}$$

Applying Lemma 3.1, assumption (3.23), we obtain

$$\begin{aligned}\|\mathbf{u}(\cdot,t)\|^2_{\mathbb{H}^\eta(\Omega)}\leqslant\;&2\mathfrak{M}_2^2(a,b)t^{2\alpha(\gamma-\eta)}\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)}\\ +\;&2\Big(\int_0^t\mathfrak{M}_3(a,b)(t-s)^{\alpha-1}\|\mathscr{F}((u,v)(\cdot,s))\|_{\mathbb{H}^\eta(\Omega)}ds\Big)^2.\end{aligned}$$

Multiplying both side to $t^{2\alpha(\eta-\gamma)}$ and using Hölder inequality, we get

$$\begin{aligned}&\Big(t^{\alpha(\eta-\gamma)}\|\mathbf{u}(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)}\Big)^2\\ &\leqslant 2\mathfrak{M}_2^2(a,b)\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &+2\mathfrak{M}_3^2(a,b)C_1^2t^{2\alpha(\eta-\gamma)}\Big(\int_0^t\Big(1+\|\mathbf{u}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}+\|\mathbf{v}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\Big)s^{\alpha(\eta-\gamma)}s^{\alpha(\gamma-\eta)}(t-s)^{\alpha-1}ds\Big)^2\\ &\leqslant 2\mathfrak{M}_2^2(a,b)\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &+2\mathfrak{M}_3^2(a,b)C_1^2t^{2\alpha(\eta-\gamma)}\int_0^t\Big(1+\|\mathbf{u}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}+\|\mathbf{v}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\Big)^2s^{2\alpha(\eta-\gamma)}ds\int_0^t s^{2\alpha(\gamma-\eta)}(t-s)^{2\alpha-2}ds.\end{aligned}$$

Using Beta function property $\int_0^t s^{\theta-1}(t-s)^{\vartheta-1}ds=t^{\theta+\vartheta-1}\mathbf{B}(\theta,\vartheta),\theta>0,\vartheta>0$ with assumption $2\alpha(\gamma-\eta)>-1$, we obtain

$$\begin{aligned}&\Big(t^{\alpha(\eta-\gamma)}\|\mathbf{u}(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)}\Big)^2\\ &\leqslant 2\mathfrak{M}_2^2(a,b)\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &\quad+2\mathfrak{M}_3^2(a,b)C_1^2t^{2\alpha(\eta-\gamma+1)-1}\mathbf{B}(\theta,\vartheta)\int_0^t\\ &\Big(1+\|\mathbf{u}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}+\|\mathbf{v}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\Big)^2s^{2\alpha(\eta-\gamma)}ds,\end{aligned}$$

where $\theta:=2\alpha(\gamma-\eta)+1,\vartheta:=2\alpha-1$, by applying inequality $(m+n+k)^2\leqslant 3(m^2+n^2+k^2)$, we get

$$\begin{aligned}&\Big(t^{\alpha(\eta-\gamma)}\|\mathbf{u}(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)}\Big)^2\\ &\leqslant 2\mathfrak{M}_2^2(a,b)\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &+6\mathfrak{M}_3^2(a,b)C_1^2t^{2\alpha(\eta-\gamma+1)-1}\mathbf{B}(\theta,\vartheta)\int_0^t\Big(1+\|\mathbf{u}(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)}+\|\mathbf{v}(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)}\Big)s^{2\alpha(\eta-\gamma)}ds\\ &\leqslant 2\mathfrak{M}_2^2(a,b)\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)}+\frac{6\mathfrak{M}_3^2(a,b)C_1^2t^{2\alpha(2\eta-2\gamma+1)}\mathbf{B}(\theta,\vartheta)}{2\alpha(\eta-\gamma)+1}\\ &+6\mathfrak{M}_3^2(a,b)C_1^2t^{2\alpha(\eta-\gamma+1)-1}\mathbf{B}(\theta,\vartheta)\int_0^t\Big(\|\mathbf{u}(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)}+\|\mathbf{v}(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)}\Big)s^{2\alpha(\eta-\gamma)}ds.\end{aligned} \tag{3.36}$$

Similarly, we can also obtain

$$\begin{aligned}&\Big(t^{\alpha(\eta-\gamma)}\|\mathbf{v}(\cdot,t)\|_{\mathbb{H}^\eta(\Omega)}\Big)^2\leqslant 2\mathfrak{M}_2^2(a,b)\|v_0\|^2_{\mathbb{H}^\gamma(\Omega)}+\frac{6\mathfrak{M}_3^2(a,b)C_1^2t^{2\alpha(2\eta-2\gamma+1)}\mathbf{B}(\theta,\vartheta)}{2\alpha(\eta-\gamma)+1}\\ &+6\mathfrak{M}_3^2(a,b)C_1^2t^{2\alpha(\eta-\gamma+1)-1}\mathbf{B}(\theta,\vartheta)\int_0^t\Big(\|\mathbf{u}(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)}+\|\mathbf{v}(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)}\Big)s^{2\alpha(\eta-\gamma)}ds.\end{aligned} \tag{3.37}$$

From (3.36) and (3.37), we arrive at

$$\begin{aligned}&\Big(\|\mathbf{u}(\cdot,t)\|^2_{\mathbb{H}^\eta(\Omega)}+\|\mathbf{v}(\cdot,t)\|^2_{\mathbb{H}^\eta(\Omega)}\Big)t^{2\alpha(\eta-\gamma)}\\ &\leqslant\overline{\mathfrak{C}}^{\eta,\gamma}_{1,\alpha}(a,b)+\overline{\mathfrak{C}}^{\eta,\gamma}_{2,\alpha}(a,b)\int_0^t\Big(\|\mathbf{u}(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)}+\|\mathbf{v}(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)}\Big)s^{2\alpha(\eta-\gamma)}ds,\end{aligned} \tag{3.38}$$

where

$$\overline{\mathfrak{C}}^{\eta,\gamma}_{1,\alpha}(a,b) := 2\mathfrak{M}_2^2(a,b)\left(\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)} + \|v_0\|^2_{\mathbb{H}^\gamma(\Omega)}\right) + \frac{6\mathfrak{M}_3^2(a,b)\left(C_1^2+C_2^2\right)T^{2\alpha(2\eta-2\gamma+1)}\mathbf{B}(\theta,\vartheta)}{2\alpha(\eta-\gamma)+1},$$

$$\overline{\mathfrak{C}}^{\eta,\gamma}_{2,\alpha}(a,b) := 6\mathfrak{M}_3^2(a,b)\left(C_1^2+C_2^2\right)T^{2\alpha(\eta-\gamma+1)-1}\mathbf{B}(\theta,\vartheta).$$

Applying Gronwall's inequality, we obtain

$$\|\mathbf{u}\|^2_{\mathbb{C}^{\alpha(\eta-\gamma)}((\mathbf{0},\mathbf{T}],\mathbb{H}^\eta(\Omega))} + \|\mathbf{v}\|^2_{\mathbb{C}^{\alpha(\eta-\gamma)}((\mathbf{0},\mathbf{T}],\mathbb{H}^\eta(\Omega))} \leqslant \overline{\mathfrak{C}}^{\eta,\gamma}_{1,\alpha}(\mathbf{a},\mathbf{b})\exp\left(\overline{\mathfrak{C}}^{\eta,\gamma}_{2,\alpha}(\mathbf{a},\mathbf{b})\mathbf{T}\right). \tag{3.39}$$

Therefore

$$\|\mathbf{w}\|_{\mathbb{V}^{\alpha(\eta-\gamma)}((\mathbf{0},\mathbf{T}],\mathbb{H}^\eta(\Omega))} \leqslant \sqrt{\overline{\mathfrak{C}}^{\eta,\gamma}_{1,\alpha}(\mathbf{a},\mathbf{b})}\exp\left(\frac{\overline{\mathfrak{C}}^{\eta,\gamma}_{2,\alpha}(\mathbf{a},\mathbf{b})\mathbf{T}}{\mathbf{2}}\right). \tag{3.40}$$

Part 3. From Eq. (3.14), we then obtain

$$\begin{cases} \mathbf{u}_\alpha(\cdot,t) &= \mathscr{S}_{\mathscr{A},\alpha}(t)u_0 + \int_0^t \mathscr{P}_{\mathscr{A},\alpha}(t-s)\mathscr{F}(u_\alpha,v_\alpha)(\cdot,s)ds,\\ \mathbf{v}_\alpha(\cdot,t) &= \mathscr{S}_{\mathscr{A},\alpha}(t)v_0 + \int_0^t \mathscr{P}_{\mathscr{A},\alpha}(t-s)\mathscr{G}(u_\alpha,v_\alpha)(\cdot,s)ds, \end{cases} \tag{3.41}$$

and

$$\begin{cases} \mathbf{u}_{\alpha'}(\cdot,t) &= \mathscr{S}_{\mathscr{A},\alpha'}(t)u_0 + \int_0^t \mathscr{P}_{\mathscr{A},\alpha'}(t-s)\mathscr{F}(u_{\alpha'},v_{\alpha'})(\cdot,s)ds,\\ \mathbf{v}_{\alpha'}(\cdot,t) &= \mathscr{S}_{\mathscr{A},\alpha'}(t)v_0 + \int_0^t \mathscr{P}_{\mathscr{A},\alpha'}(t-s)\mathscr{G}(u_{\alpha'},v_{\alpha'})(\cdot,s)ds. \end{cases} \tag{3.42}$$

Since (3.41) and (3.42), we get

$$\begin{aligned}\mathbf{u}_{\alpha'}(\cdot,t) - \mathbf{u}_\alpha(\cdot,t) =& [\mathscr{S}_{\mathscr{A},\alpha'}(t) - \mathscr{S}_{\mathscr{A},\alpha}(t)]u_0\\ &+ \int_0^t [\mathscr{P}_{\mathscr{A},\alpha'}(t-s) - \mathscr{P}_{\mathscr{A},\alpha}(t-s)]\mathscr{F}(u_{\alpha'},v_{\alpha'})(\cdot,s)ds\\ &+ \int_0^t \mathscr{P}_{\mathscr{A},\alpha}(t-s)[\mathscr{F}(u_{\alpha'},v_{\alpha'})(\cdot,s) - \mathscr{F}(u_\alpha,v_\alpha)(\cdot,s)]ds,\end{aligned} \tag{3.43}$$

$$\begin{aligned}\mathbf{v}_{\alpha'}(\cdot,t) - \mathbf{v}_\alpha(\cdot,t) =& [\mathscr{S}_{\mathscr{A},\alpha}(t) - \mathscr{S}_{\mathscr{A},\alpha'}(t)]v_0\\ &+ \int_0^t [\mathscr{P}_{\mathscr{A},\alpha'}(t-s) - \mathscr{P}_{\mathscr{A},\alpha}(t-s)]\mathscr{G}(u_{\alpha'},v_{\alpha'})(\cdot,s)ds\\ &+ \int_0^t \mathscr{P}_{\mathscr{A},\alpha}(t-s)[\mathscr{G}(u_{\alpha'},v_{\alpha'})(\cdot,s) - \mathscr{G}(u_\alpha,v_\alpha)(\cdot,s)]ds.\end{aligned} \tag{3.44}$$

Using Lemmas 3.1 and 3.2, we have estimate as follows

$$\begin{aligned}&\|\mathbf{u}_{\alpha'}(\cdot,t) - \mathbf{u}_\alpha(\cdot,t)\|^2_{\mathbb{H}^\eta(\Omega)} \leqslant 3\left\|[\mathscr{S}_{\mathscr{A},\alpha'}(t) - \mathscr{S}_{\mathscr{A},\alpha}(t)]u_0\right\|^2_{\mathbb{H}^\eta(\Omega)}\\ &+ 3\left\|\int_0^t [\mathscr{P}_{\mathscr{A},\alpha'}(t-s) - \mathscr{P}_{\mathscr{A},\alpha}(t-s)]\mathscr{F}(u_{\alpha'},v_{\alpha'})(\cdot,s)ds\right\|^2_{\mathbb{H}^\eta(\Omega)}\\ &+ 3\left\|\int_0^t \mathscr{P}_{\mathscr{A},\alpha}(t-s)[\mathscr{F}(u_{\alpha'},v_{\alpha'})(\cdot,s) - \mathscr{F}(u_\alpha,v_\alpha)(\cdot,s)]ds\right\|^2_{\mathbb{H}^\eta(\Omega)}\\ &\leqslant 3\mathfrak{B}^2(a,b,\epsilon,\beta,T)[(\alpha'-\alpha)^\epsilon + (\alpha'-\alpha)]^2 t^{-2b(\eta-\gamma)-2\epsilon}\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &+ 3\mathscr{C}^2(a,b,\epsilon,\beta,T)[(\alpha'-\alpha)^\epsilon + (\alpha'-\alpha)]^2\lambda_0^{2\beta-2}\\ &\left(\int_0^t (t-s)^{a(\gamma+1-\eta)-\epsilon-1}\|\mathscr{F}(u_{\alpha'},v_{\alpha'})(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}ds\right)^2\\ &+ \mathfrak{M}_3^2(a,b)\left(\int_0^t (t-s)^{\alpha-1}\|\mathscr{F}(u_{\alpha'},v_{\alpha'})(\cdot,s) - \mathscr{F}(u_\alpha,v_\alpha)(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}ds\right)^2.\end{aligned} \tag{3.45}$$

Using assumption (3.23) and (3.24), we obtain

$$\begin{aligned}\|\mathbf{u}_{\alpha'}(\cdot,t) &- \mathbf{u}_\alpha(\cdot,t)\|^2_{\mathbb{H}^\eta(\Omega)}\\ &\leqslant 3\mathfrak{B}^2(a,b,\epsilon,\beta,T)[(\alpha'-\alpha)^\epsilon + (\alpha'-\alpha)]^2 t^{-2b(\eta-\gamma)-2\epsilon}\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &+3K_1^2\mathfrak{M}_3^2(a,b)\mathscr{I}\\ &+3C_1^2\mathscr{C}^2(a,b,\epsilon,\beta,T)[(\alpha'-\alpha)^\epsilon + (\alpha'-\alpha)]^2\lambda_0^{2\beta-2}\mathscr{J},\end{aligned}$$

where

$$\mathscr{I} := \left(\int_0^t (t-s)^{(\gamma-\eta+1)\alpha-1}\left(\|\mathbf{u}_{\alpha'}(\cdot,s) - \mathbf{u}_\alpha(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)} + \|\mathbf{v}_{\alpha'}(\cdot,s) - \mathbf{v}_\alpha(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\right)ds\right)^2, \tag{3.46}$$

$$\mathscr{J} := \left(\int_0^t (t-s)^{a(\gamma+1-\eta)-\epsilon-1}\left(1 + \|\mathbf{u}_{\alpha'}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)} + \|\mathbf{v}_{\alpha'}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\right)ds\right)^2. \tag{3.47}$$

Multiplying both side to $t^{2b(\eta-\gamma)+2\epsilon}$, we get

$$\begin{aligned}t^{2b(\eta-\gamma)+2\epsilon}\|\mathbf{u}_{\alpha'}(\cdot,t) &- \mathbf{u}_\alpha(\cdot,t)\|^2_{\mathbb{H}^\eta(\Omega)}\\ &\leqslant 3\mathfrak{B}^2(a,b,\epsilon,\beta,T)[(\alpha'-\alpha)^\epsilon + (\alpha'-\alpha)]^2\|u_0\|^2_{\mathbb{H}^\gamma(\Omega)}\\ &+3K_1^2\mathfrak{M}_3^2(a,b)t^{2b(\eta-\gamma)+2\epsilon}\mathscr{I}\\ &+3C_1^2\mathscr{C}^2(a,b,\epsilon,\beta,T)[(\alpha'-\alpha)^\epsilon + (\alpha'-\alpha)]^2\lambda_0^{2\beta-2}t^{2b(\eta-\gamma)+2\epsilon}\mathscr{J}.\end{aligned} \tag{3.48}$$

By using Hölder's inequality, we get

$$\begin{aligned}\mathscr{I} &= \left(\int_0^t (t-s)^{(\gamma-\eta+1)\alpha-1}s^{-b(\eta-\gamma)-\epsilon}s^{b(\eta-\gamma)+\epsilon}\right.\\ &\left.\left(\|\mathbf{u}_{\alpha'}(\cdot,s) - \mathbf{u}_\alpha(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)} + \|\mathbf{v}_{\alpha'}(\cdot,s) - \mathbf{v}_\alpha(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\right)ds\right)^2\\ &\leqslant \int_0^t (t-s)^{2(\gamma-\eta+1)\alpha-2}s^{-2b(\eta-\gamma)-2\epsilon}ds\\ &\times\int_0^t s^{2b(\eta-\gamma)+2\epsilon}\left(\|\mathbf{u}_{\alpha'}(\cdot,s) - \mathbf{u}_\alpha(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)} + \|\mathbf{v}_{\alpha'}(\cdot,s) - \mathbf{v}_\alpha(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\right)^2 ds.\end{aligned}$$

Applying Beta function property with assumption $0 < \epsilon < \min a(\gamma+1-\eta) - \frac{1}{2}, b(\gamma-\eta) + \frac{1}{2}$ then $2a(\gamma+1-\eta) - 2\epsilon - 2 > -1$, we obtain

$$\begin{aligned}\mathscr{I} \leqslant\ & t^{2(\gamma+1-\eta)\alpha-1-2b(\eta-\gamma)-2\epsilon}\mathbf{B}(\theta_1,\vartheta_1)\\ \times\ & \int_0^t s^{2b(\eta-\gamma)+2\epsilon}\left(\|\mathbf{u}_{\alpha'}(\cdot,s) - \mathbf{u}_\alpha(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)} + \|\mathbf{v}_{\alpha'}(\cdot,s) - \mathbf{v}_\alpha(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\right)^2 ds,\end{aligned} \tag{3.49}$$

where

$$\theta_1 := -2b(\eta-\gamma) - 2\epsilon - 1, \vartheta_1 := 2(\gamma+1-\eta)\alpha - 1.$$

Therefore

$$\begin{aligned}\mathscr{I} \leqslant\ & 2t^{2(\gamma+1-\eta)\alpha-1-2b(\eta-\gamma)-2\epsilon}\mathbf{B}(\theta_1,\vartheta_1)\\ & \int_0^t s^{2b(\eta-\gamma)+2\epsilon}\left(\|\mathbf{u}_{\alpha'}(\cdot,s) - \mathbf{u}_\alpha(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)} + \|\mathbf{v}_{\alpha'}(\cdot,s) - \mathbf{v}_\alpha(\cdot,s)\|^2_{\mathbb{H}^\eta(\Omega)}\right)ds.\end{aligned} \tag{3.50}$$

Similarly, using (3.40) we have inequality as follow with $t \in [0,T]$

$$\begin{aligned}\mathscr{J} &= \left(\int_0^t (t-s)^{a(\gamma+1-\eta)-\epsilon-1}s^{\alpha(\gamma-\eta)}s^{\alpha(\eta-\gamma)}\left(1 + \|\mathbf{u}_{\alpha'}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)} + \|\mathbf{v}_{\alpha'}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\right)ds\right)^2\\ &\leqslant \int_0^t (t-s)^{2a(\gamma+1-\eta)-2\epsilon-2}s^{2\alpha(\gamma-\eta)}ds\int_0^t s^{2\alpha(\eta-\gamma)}\left(1 + \|\mathbf{u}_{\alpha'}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)} + \|\mathbf{v}_{\alpha'}(\cdot,s)\|_{\mathbb{H}^\eta(\Omega)}\right)ds\Big)^2 ds\\ &\leqslant 3t^{2a(\gamma+1-\eta)-2\epsilon+2\alpha(\gamma-\eta)-1}\mathbf{B}(\theta_2,\vartheta_2)\left(1 + \|\mathbf{u}_{\alpha'}\|^2_{\mathbb{C}^{\alpha(\eta-\gamma)}((\mathbf{0},\mathbf{T}],\mathbb{H}^\eta(\Omega))} + \|\mathbf{v}_{\alpha'}\|^2_{\mathbb{C}^{\alpha(\eta-\gamma)}((\mathbf{0},\mathbf{T}],\mathbb{H}^\eta(\Omega))}\right),\end{aligned} \tag{3.51}$$

where

$$\theta_2 := 2\alpha(\gamma-\eta) + 1, \vartheta_2 := 2a(\gamma+1-\eta) - 2\epsilon - 1.$$

To facilitate the calculation, we set

$$\overline{\mathfrak{C}}^{\eta,\gamma}_{\alpha,\alpha'}(a,b,T) = 3T^{2(\gamma-\eta)(a+\alpha)-2\epsilon+2a-1}\mathbf{B}(\theta_2,\vartheta_2)\left(1 + \|\mathbf{u}_{\alpha'}\|^2_{\mathbb{C}^{\alpha(\eta-\gamma)}((\mathbf{0},\mathbf{T}],\mathbb{H}^\eta(\Omega))} + \|\mathbf{v}_{\alpha'}\|^2_{\mathbb{C}^{\alpha(\eta-\gamma)}((\mathbf{0},\mathbf{T}],\mathbb{H}^\eta(\Omega))}\right).$$

From (3.48)–(3.50), we get

$$\begin{aligned} & t^{2b(\eta-\gamma)+2\epsilon}\|\mathbf{u}_{\alpha\prime}(\cdot,t)-\mathbf{u}_{\alpha}(\cdot,t)\|^2_{\mathbb{H}^{\eta}(\Omega)} \\ & \leqslant 3\mathfrak{B}^2(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2\|u_0\|^2_{\mathbb{H}^{\gamma}(\Omega)} \\ & +3C_1^2\mathscr{C}^2(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2\lambda_0^{2\beta-2}T^{2b(\eta-\gamma)+2\epsilon}\overline{\mathfrak{C}}^{\eta,\gamma}_{\alpha,\alpha\prime}(a,b,T) \\ & +3K_1^2\mathfrak{M}_3^2(a,b)t^{2(\gamma+1-\eta)\alpha-1}\mathbf{B}(\theta_1,\vartheta_1) \\ & \times\int_0^t s^{2b(\eta-\gamma)+2\epsilon}\Big(\|\mathbf{u}_{\alpha\prime}(\cdot,s)-\mathbf{u}_{\alpha}(\cdot,s)\|^2_{\mathbb{H}^{\eta}(\Omega)}+\|\mathbf{v}_{\alpha\prime}(\cdot,s)-\mathbf{v}_{\alpha}(\cdot,s)\|^2_{\mathbb{H}^{\eta}(\Omega)}\Big)ds. \end{aligned} \tag{3.52}$$

Similarly, we also deduce

$$\begin{aligned} & t^{2b(\eta-\gamma)+2\epsilon}\|\mathbf{v}_{\alpha\prime}(\cdot,t)-\mathbf{v}_{\alpha}(\cdot,t)\|^2_{\mathbb{H}^{\eta}(\Omega)} \\ & \leqslant 3\mathfrak{B}^2(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2\|v_0\|^2_{\mathbb{H}^{\gamma}(\Omega)} \\ & +3C_2^2\mathscr{C}^2(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2\lambda_0^{2\beta-2}T^{2b(\eta-\gamma)+2\epsilon}\overline{\mathfrak{C}}^{\eta,\gamma}_{\alpha,\alpha\prime}(a,b,T) \\ & +3K_2^2\mathfrak{M}_3^2(a,b)t^{2(\gamma+1-\eta)\alpha-1}\mathbf{B}(\theta_1,\vartheta_1) \\ & \times\int_0^t s^{2b(\eta-\gamma)+2\epsilon}\Big(\|\mathbf{u}_{\alpha\prime}(\cdot,s)-\mathbf{u}_{\alpha}(\cdot,s)\|^2_{\mathbb{H}^{\eta}(\Omega)}+\|\mathbf{v}_{\alpha\prime}(\cdot,s)-\mathbf{v}_{\alpha}(\cdot,s)\|^2_{\mathbb{H}^{\eta}(\Omega)}\Big)ds. \end{aligned} \tag{3.53}$$

Compose (3.52) and (3.53) we have

$$\begin{aligned} & t^{2b(\eta-\gamma)+2\epsilon}\Big(\|\mathbf{u}_{\alpha\prime}(\cdot,t)-\mathbf{u}_{\alpha}(\cdot,t)\|^2_{\mathbb{H}^{\eta}(\Omega)}+\|\mathbf{v}_{\alpha\prime}(\cdot,t)-\mathbf{v}_{\alpha}(\cdot,t)\|^2_{\mathbb{H}^{\eta}(\Omega)}\Big) \\ & \leqslant 3\mathfrak{B}^2(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2\Big(\|u_0\|^2_{\mathbb{H}^{\gamma}(\Omega)}+\|v_0\|^2_{\mathbb{H}^{\gamma}(\Omega)}\Big) \\ & +3\big(C_1^2+C_2^2\big)\mathscr{C}^2(a,b,\epsilon,\beta,T)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2\lambda_0^{2\beta-2}T^{2b(\eta-\gamma)+2\epsilon}\overline{\mathfrak{C}}^{\eta,\gamma}_{\alpha,\alpha\prime}(a,b,T) \\ & +3\big(K_1^2+K_2^2\big)\mathfrak{M}_3^2(a,b)T^{2(\gamma+1-\eta)\alpha-1}\mathbf{B}(\theta_1,\vartheta_1) \\ & \int_0^t s^{2b(\eta-\gamma)+2\epsilon}\Big(\|\mathbf{u}_{\alpha\prime}(\cdot,s)-\mathbf{u}_{\alpha}(\cdot,s)\|^2_{\mathbb{H}^{\eta}(\Omega)}+\|\mathbf{v}_{\alpha\prime}(\cdot,s)-\mathbf{v}_{\alpha}(\cdot,s)\|^2_{\mathbb{H}^{\eta}(\Omega)}\Big)ds \\ & \leqslant \mathscr{M}_1^{\eta,\gamma,\epsilon}\Big(a,b,T,\|w_0\|_{\mathscr{H}^{\gamma}(\Omega)},C_1,C_2\Big)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2 \\ & +\mathscr{M}_2^{\eta,\gamma,\epsilon}(a,b,T,K_1,K_2)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2 \\ & \times\int_0^t s^{2b(\eta-\gamma)+2\epsilon}\Big(\|\mathbf{u}_{\alpha\prime}(\cdot,s)-\mathbf{u}_{\alpha}(\cdot,s)\|^2_{\mathbb{H}^{\eta}(\Omega)}+\|\mathbf{v}_{\alpha\prime}(\cdot,s)-\mathbf{v}_{\alpha}(\cdot,s)\|^2_{\mathbb{H}^{\eta}(\Omega)}\Big)ds. \end{aligned} \tag{3.54}$$

In there, we applied the norm of element belong $\mathscr{H}^{\gamma}(\Omega)$ as follow

$$\|w_0\|^2_{\mathscr{H}^{\gamma}(\Omega)}=\|u_0\|^2_{\mathbb{H}^{\gamma}(\Omega)}+\|v_0\|^2_{\mathbb{H}^{\gamma}(\Omega)}.$$

By applying inequality Gronwall, we get the following estimate

$$\begin{aligned} & t^{2b(\eta-\gamma)+2\epsilon}\Big(\|\mathbf{u}_{\alpha\prime}(\cdot,t)-\mathbf{u}_{\alpha}(\cdot,t)\|^2_{\mathbb{H}^{\eta}(\Omega)}+\|\mathbf{v}_{\alpha\prime}(\cdot,t)-\mathbf{v}_{\alpha}(\cdot,t)\|^2_{\mathbb{H}^{\eta}(\Omega)}\Big) \\ & \quad\leqslant \mathscr{M}_1^{\eta,\gamma,\epsilon}\Big(a,b,T,\|w_0\|_{\mathscr{H}^{\gamma}(\Omega)},C_1,C_2\Big)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2 \\ & \qquad\exp\Big(\mathscr{M}_2^{\eta,\gamma,\epsilon}(a,b,T,K_1,K_2)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2t\Big). \end{aligned} \tag{3.56}$$

Therefore,

$$\begin{aligned} & \|\mathbf{u}_{\alpha\prime}-\mathbf{u}_{\alpha}\|^2_{\mathbb{C}^{b(\eta-\gamma)+\epsilon}(0,T,\mathbb{H}^{\eta}(\Omega))}+\|\mathbf{v}_{\alpha\prime}-\mathbf{v}_{\alpha}\|^2_{\mathbb{C}^{b(\eta-\gamma)+\epsilon}(0,T,\mathbb{H}^{\eta}(\Omega))} \\ & \leqslant \mathscr{M}_1^{\eta,\gamma,\epsilon}\Big(a,b,T,\|w_0\|_{\mathscr{H}^{\gamma}(\Omega)},C_1,C_2\Big)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2 \\ & \quad\exp\Big(\mathscr{M}_2^{\eta,\gamma,\epsilon}(a,b,T,K_1,K_2)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2T\Big). \end{aligned} \tag{3.57}$$

Finally, we obtain

$$\begin{aligned} & \|\mathbf{w}_{\alpha\prime}-\mathbf{w}_{\alpha}\|_{\mathbb{V}^{b(\eta-\gamma)+\epsilon}(0,T,\mathbb{H}^{\eta}(\Omega))} \\ & \leqslant[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)] \\ & \qquad\Big[\mathscr{M}_1^{\eta,\gamma,\epsilon}\Big(a,b,T,\|w_0\|_{\mathscr{H}^{\gamma}(\Omega)},C_1,C_2\Big) \\ & \Big\{\exp\mathscr{M}_2^{\eta,\gamma,\epsilon}(a,b,T,K_1,K_2)[(\alpha\prime-\alpha)^{\epsilon}+(\alpha\prime-\alpha)]^2T\Big]^{1/2}. \end{aligned} \tag{3.58}$$

This complete the proof. □

4. Numerical test

The goal of this section is presenting an example to verify the results which are shown in our theory. At the benginning, we setup some tools to support our calculations. The examples are involved with the Laplace operator $-\frac{\partial^2}{\partial x^2}$, the domain $\Omega=[0,\pi]$, and $T=1$. Then we study the problem as follows

$$\begin{cases} \partial_t^{\alpha}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t) &=\mathscr{F}(u,v),\quad x\in(0,\pi),t\in(0,1),\\ \partial_t^{\alpha}v(x,t)-\frac{\partial^2}{\partial x^2}v(x,t) &=\mathscr{G}(u,v),\quad x\in(0,\pi),t\in(0,1), \end{cases} \tag{4.59}$$

which satifies the boundary condition

$$u(x,t)=v(x,t)=0,\quad (x,t)\in\{0,\pi\}\times(0,1], \tag{4.60}$$

and the intial conditions

$$u(x,0)=u_0(x)\text{ and }v(x,0)=v_0(x),x\in(0,\pi), \tag{4.61}$$

where the Caputo fractional derivative as follows

$$\partial_t^{\alpha}w(\cdot,t):=\frac{1}{\Gamma(2-\alpha)}\int_0^t(t-\eta)^{1-\alpha}\frac{\partial^2w(\cdot,\eta)}{\partial\eta}d\eta,$$

where $\alpha\in(1,2)$ and Γ is the Gamma function which was calculated by using a function `Gamma(·)` in Matlab software. To calculate the value of Mittag-Leffler function, we run the code Matlab which was written by Igor Podlubny [18].

In $L^2(0,\pi)$ space, we have the orthogonal basis and eigenvalues are $\varphi_j(x)=\sqrt{\frac{2}{\pi}}\sin(jx)$ and $\lambda_j=j^2$, $j=1,2,\ldots,$ respectively.

Next, we partition the time variable $t\in(0,1)$ and spatial variable $x\in(0,\pi)$ as follows

$$x_i=(i-1)\frac{\pi}{H_x},\quad t_k=(k-1)\frac{1}{H_t},\quad i=1,\cdots,H_x+1,$$
$$k=1,\cdots,H_t+1,$$

where $H_x,H_t>0$are two given integer numbers.

To compare the regularity of the solution, we consider the following estimations with fractional derivative order α

$$\mathrm{E}_u^{\alpha,\alpha^*}(t)=\sqrt{\frac{\sum_{i=1}^{H_x+1}|u^{\alpha}(x_i,t)-u^{\alpha^*}(x_i,t)|^2}{H_x+1}}, \tag{4.62}$$

$$\mathrm{E}_v^{\alpha,\alpha^*}(t)=\sqrt{\frac{\sum_{i=1}^{H_x+1}|v^{\alpha}(x_i,t)-v^{\alpha^*}(x_i,t)|^2}{H_x+1}}, \tag{4.63}$$

where (u^{α},v^{α}) and $(u^{\alpha^*},v^{\alpha^*})$ are the solutions in case α and α^*, respectively, which satisfies α^* approachs to α. Namely, $\alpha^*=\alpha+\varepsilon,\varepsilon\longrightarrow0^+$.

Then the solution of problem (4.59), (4.60) and (4.61) as follows

$$\begin{cases} u(x,t) &=\sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}E_{\alpha}\big(-j^2t^{\alpha}\big)u_{0,j}\sin(jx)\\ &\quad+\sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-j^2(t-s)^{\alpha}\big)\\ &\quad\mathscr{F}_j(u,v)(s)ds\sin(jx),\\ v(x,t) &=\sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}E_{\alpha}\big(-j^2t^{\alpha}\big)v_{0,j}\sin(jx)\\ &\quad+\sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(-j^2(t-s)^{\alpha}\big)\\ &\quad\mathscr{G}_j(u,v)(s)ds\sin(jx), \end{cases} \tag{4.64}$$

where

$$\begin{cases} u_{0,j} = \langle u_0, \varphi_j \rangle_{L^2(0,\pi)} = \sqrt{\frac{2}{\pi}} \int_0^\pi u_0(x)\sin(jx)dx, \\ v_{0,j} = \langle v_0, \varphi_j \rangle_{L^2(0,\pi)} = \sqrt{\frac{2}{\pi}} \int_0^\pi v_0(x)\sin(jx)dx, \\ \mathscr{F}_j(u,v) = \langle \mathscr{F}(u,v), \varphi_j \rangle_{L^2(0,\pi)} = \sqrt{\frac{2}{\pi}} \int_0^\pi \mathscr{F}(u,v)(x)\sin(jx)dx, \\ \mathscr{G}_j(u,v) = \langle \mathscr{G}(u,v), \varphi_j \rangle_{L^2(0,\pi)} = \sqrt{\frac{2}{\pi}} \int_0^\pi \mathscr{G}(u,v)(x)\sin(jx)dx. \end{cases} \tag{4.65}$$

In code Matlab, we have the solution (4.64) which can be writen in form matrix as follows

$$\begin{bmatrix} u(x_1,t_1) & u(x_2,t_1) & u(x_3,t_1) & \cdots & u(x_{H_x+1},t_1) \\ u(x_1,t_2) & u(x_2,t_2) & u(x_3,t_2) & \cdots & u(x_{H_x+1},t_2) \\ u(x_1,t_3) & u(x_2,t_3) & u(x_3,t_3) & \cdots & u(x_{H_x+1},t_3) \\ \vdots & \vdots & \vdots & \vdots & \\ u(x_1,t_{H_t+1}) & u(x_2,t_{H_t+1}) & u(x_3,t_{H_t+1}) & \cdots & u(x_{H_x+1},t_{H_t+1}) \end{bmatrix}_{(H_t+1)\times(H_x+1)},$$

and

$$\begin{bmatrix} v(x_1,t_1) & v(x_2,t_1) & v(x_3,t_1) & \cdots & v(x_{H_x+1},t_1) \\ v(x_1,t_2) & v(x_2,t_2) & v(x_3,t_2) & \cdots & v(x_{H_x+1},t_2) \\ v(x_1,t_3) & v(x_2,t_3) & v(x_3,t_3) & \cdots & v(x_{H_x+1},t_3) \\ \vdots & \vdots & \vdots & \vdots & \\ v(x_1,t_{H_t+1}) & v(x_2,t_{H_t+1}) & v(x_3,t_{H_t+1}) & \cdots & v(x_{H_x+1},t_{H_t+1}) \end{bmatrix}_{(H_t+1)\times(H_x+1)},$$

where, let N_j a truncated parameters, we have

$$\begin{cases} \mathbf{u}(x_i,t_k) &= \sqrt{\frac{2}{\pi}}\sum_{j=1}^{N(j)} E_\alpha(-j^2 t_k^\alpha) u_{0,j}\sin(jx_i) \\ & + \sqrt{\frac{2}{\pi}}\sum_{j=1}^{N(j)} \int_0^{t_k} (t_k-s)^{\alpha-1} E_{\alpha,\alpha}(-j^2(t_k-s)^\alpha)\mathscr{F}_j(u,v)(s)ds\sin(jx_i), \\ \mathbf{v}(x_i,t_k) &= \sqrt{\frac{2}{\pi}}\sum_{j=1}^{N(j)} E_\alpha(-j^2 t_k^\alpha) v_{0,j}\sin(jx_i) \\ & + \sqrt{\frac{2}{\pi}}\sum_{j=1}^{N(j)} \int_0^{t_k} (t_k-s)^{\alpha-1} E_{\alpha,\alpha}(-j^2(t_k-s)^\alpha)\mathscr{G}_j(u,v)(s)ds\sin(jx_i), \end{cases} \tag{4.66}$$

We consider the problem (4.59), (4.60) and (4.61) with the following input data

$$\begin{cases} u_0(x) = \sin(x), v_0(x) = -\sin(2x) & x \in (0,\pi), \\ \mathscr{F} = \frac{1}{2}u(x,t) + v(x,t) - 2\Gamma(\alpha-2)f(x,t), & x \in (0,\pi), t \in (0,1), \\ \mathscr{G} = u(x,t) + \frac{1}{3}v(x,t) - 3(\alpha-3)\Gamma(\alpha-2)g(x,t), & x \in (0,\pi), t \in (0,1), \end{cases}$$

with

$$\begin{aligned} f(x,t) = {} & 2\Gamma(\alpha-2)\sin(2x)\alpha t^3 - \Gamma(\alpha-2)\sin(x)\alpha t^2 - 6\Gamma(\alpha-2)\sin(2x)\alpha t^2 \\ & -4\Gamma(\alpha-2)\sin(2x)t^3 + 2\Gamma(\alpha-2)\sin(x)\alpha t + 2\Gamma(\alpha-2)\sin(x)t^2 \\ & +6\Gamma(\alpha-2)\sin(2x)\alpha t + 12\Gamma(\alpha-2)\sin(2x)t^2 - \Gamma(\alpha-2)\sin(x)\alpha \\ & -4\Gamma(\alpha-2)\sin(x)t - 2\Gamma(\alpha-2)\alpha\sin(2x) - 12\Gamma(\alpha-2)\sin(2x)t \\ & +4\sin(x)t(2-\alpha) + 2\Gamma(\alpha-2)\sin(x) + 4\Gamma(\alpha-2)\sin(2x), \end{aligned}$$

$$\begin{aligned} g(x,t) = {} & -11\Gamma(\alpha-2)\sin(2x)\alpha^2t^3 + 3\Gamma(\alpha-2)\sin(x)\alpha^2t^2 + 33\Gamma(\alpha-2)\sin(2x)\alpha^2t^2 \\ & +55\Gamma(\alpha-2)\sin(2x)\alpha t^3 - 6\Gamma(\alpha-2)\sin(x)\alpha^2 t - 15\Gamma(\alpha-2)\sin(x)\alpha t^2 \\ & -33\Gamma(\alpha-2)\sin(2x)\alpha^2 t - 165\Gamma(\alpha-2)\sin(2x)\alpha t^2 - 66\Gamma(\alpha-2)\sin(2x)t^3 \\ & +3\Gamma(\alpha-2)\sin(x)\alpha^2 + 30\Gamma(\alpha-2)\sin(x)\alpha t \\ & +18\Gamma(\alpha-2)\sin(x)t^2 + 11\Gamma(\alpha-2)\sin(2x)\alpha^2 \\ & +165\Gamma(\alpha-2)\sin(2x)\alpha t + 198\Gamma(\alpha-2)\sin(2x)t^2 - 15\Gamma(\alpha-2)\sin(x)\alpha \\ & -36\Gamma(\alpha-2)\sin(x)t - 18\sin(2x)t(2-\alpha)\alpha + 18\sin(2x)t^{2-\alpha}t - 55\Gamma(\alpha-2)\sin(2x)\alpha \\ & -198\Gamma(\alpha-2)\sin(2x)t + 18\Gamma(\alpha-2)\sin(x) + 54\sin(2x)t^{2-a} + 66\Gamma(\alpha-2)\sin(2x). \end{aligned}$$

Table 1 The estimation for $t = 0.1$ and $\alpha = 1.5$.

$\{\alpha+\varepsilon, \alpha\}$	$H_x = 40, H_t = 40$		
	ε	E_u^{α,α^*}	E_v^{α,α^*}
$\{1.51, 1.5\}$	10^{-1}	0.016630876412637	0.029022995052921
$\{1.501, 1.5\}$	10^{-2}	0.010125012722883	0.011941963325382
$\{1.5001, 1.5\}$	10^{-3}	0.006892326925566	0.008276288565691

In Table 1, we can see the results of regularity by derivative order. Moreover, in Figs. 1 and 2, we also show the graph of the solution in the case $\alpha = 1.5$. When fixing the variable t, we have the error estimation between (u_α, v^α) and $(u_{\alpha^*}, v^{\alpha^*})$.

By choosing $H_x = H_t = 40, \alpha = 1.5$ and $\varepsilon = 10^{-a}$, where $a \in \{1,2,3\}$. We conclude that if ε goes to 0, then the errors

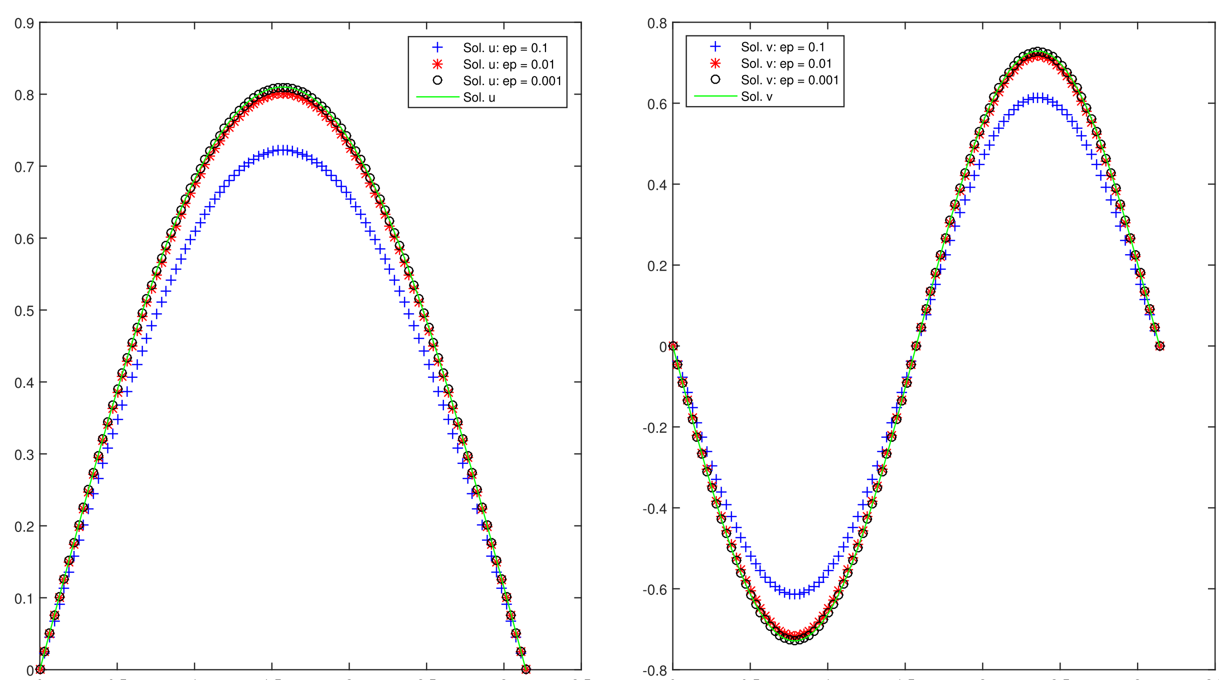

Fig. 1 Comparison the regularity for α-derivative order of u, v at $t = 0.1, x \in (0,\pi)$.

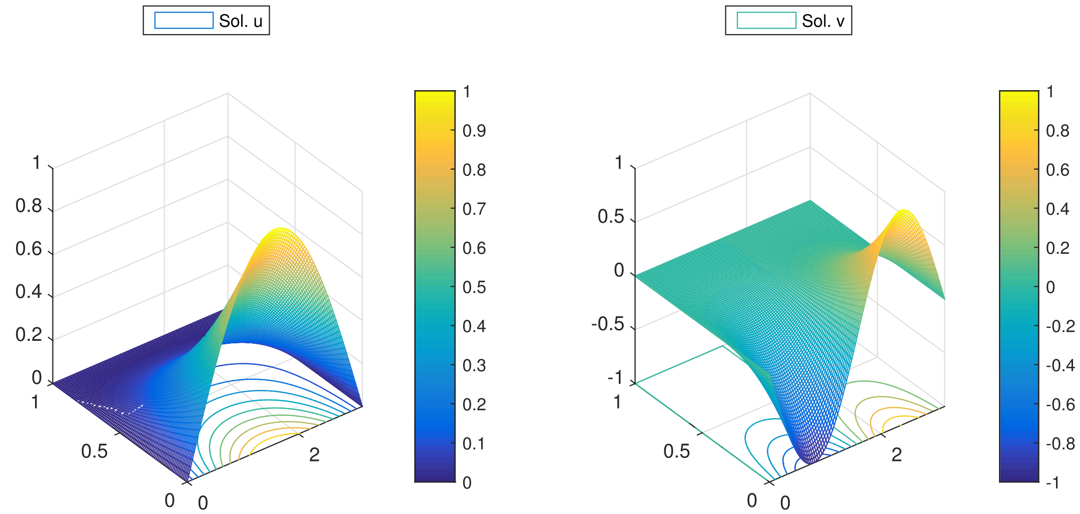

Fig. 2 The solutions (u, v) on $(x, t) \in (0, \pi) \times (0, 1)$.

also convergence to 0. It means that the smaller the order, the smaller the error between the solutions. This shows that the proposed method is used in this work which is effective.

5. Conclusion

In this paper, we study on the coupled of semi-linear fractional diffusion-wave equations. We present the results about uniqueness, existence and regularity of the solution. According to our search results, this is considered to be the first result in this area. Moreover, we give an example numerical to illustrate our method.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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