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General Raina fractional integral inequalities on coordinates of convex functions

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Abstract

Integral inequality is an interesting mathematical model due to its wide and significant applications in mathematical analysis and fractional calculus. In this study, authors have established some generalized Raina fractional integral inequalities using an (l_1,h_1)-(l_2,h_2)-convex function on coordinates. Also, we obtain an integral identity for partial differentiable functions. As an effect of this result, two interesting integral inequalities for the (l_1,h_1)-(l_2,h_2)-convex function on coordinates are given. Finally, we can say that our findings recapture some recent results as special cases.

MSC: 26D07; 26D10; 26D15

Keywords: Hermite–Hadamard inequality; Raina fractional integral operators; Coordinated convex function

1 Introduction

In the past two decades, fractional calculus has received much attention. The fast interest in the topic is due to its extensive applications in various fields such as biochemistry, physics, viscoelasticity, fluid mechanics, computer modeling, and engineering, see [1–3] for further detail. Most of the studies have been devoted to the existence and uniqueness of solutions for fractional differential equations (FDEs); see e.g. [4–9]. A fractional differential equation needs a certain inequality to be existent and unique for solution. For this reason, a huge number of mathematicians have competed to seek such inequalities; see e.g. [10–29].

Always, it is important and necessary to specify which model or definition is being used because there are many different ways of defining fractional integrals and derivatives. To further facilitate the discussion of this model, we present here the definition which is most commonly used for fractional integrals and derivatives, namely the Riemann–Liouville (RL) definition.

Definition 1.1 ([1, 2]) For any L^1 function $f(x)$ on an interval $[\chi_1,\chi_2]$ with $x\in[\chi_1,\chi_2]$, the ηth left-RL fractional integral of $f(x)$ is defined as follows:

$$ {}^{\mathrm{RL}}\mathrm{J}^{\eta}_{\chi_1+}f(x) := \frac{1}{\Gamma(\eta)}\int_{\chi_1}^{x}(x-\xi)^{\eta-1}f(\xi)\,d\xi, \quad \chi_1<x, \tag{1.1}$$

for $\mathrm{Re}(\eta) > 0$. Also, the ηth right-RL fractional integral of $f(x)$ is defined as follows:

$$ {}^{\mathrm{RL}}\mathfrak{J}^{\eta}_{\chi_2^-} f(x) := \frac{1}{\Gamma(\eta)} \int_x^{\chi_2} (\xi - x)^{\eta-1} f(\xi)\,d\xi, \quad x < \chi_2. \tag{1.2}$$

In the recent decades, a strong modern direction of research in fractional calculus has brought the attention of interested researchers in various disciplines to investigate various possible ways to define fractional integrals and derivatives, often with different properties from the classical RL in Definition 1.1. In 2005, Raina [30] introduced the new fractional integrals, often called the Raina fractional integrals, corresponding to the classical RL integrals (1.1) and (1.2).

Definition 1.2 ([30]) For any L^1 function $f(x)$ on an interval $[\chi_1, \chi_2]$ with $x \in [\chi_1, \chi_2]$, the ηth left Raina fractional integral of $f(x)$ is defined as follows:

$$ \mathfrak{I}^{\sigma}_{\rho,\eta,\chi_1^+,\omega} \varphi(x) = \int_{\chi_1}^{x} (x-t)^{\eta-1} \mathfrak{F}^{\sigma}_{\rho,\eta}\big[\omega(x-t)^{\rho}\big] \varphi(t)\,dt, \quad \chi_1 < x, \tag{1.3}$$

and the ηth right Raina fractional integral of $f(x)$ is defined as follows:

$$ \mathfrak{I}^{\sigma}_{\rho,\eta,\chi_2^-,\omega} \varphi(x) = \int_{x}^{\chi_2} (t-x)^{\eta-1} \mathfrak{F}^{\sigma}_{\rho,\eta}\big[\omega(t-x)^{\rho}\big] \varphi(t)\,dt, \quad x < \chi_2, \tag{1.4}$$

where $\mathfrak{F}^{\sigma}_{\rho,\eta}(x)$ is the generalization of Mittag-Leffler (ML) function defined as follows: For a bounded arbitrary sequence $\sigma(k)$ of real or complex numbers, we define the function $\mathfrak{F}^{\sigma}_{\rho,\eta}(x)$ by

$$ \mathfrak{F}^{\sigma}_{\rho,\eta}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \eta)} x^k, \tag{1.5}$$

where $\rho, \eta \in \mathbb{C}$ with $\mathrm{Re}(\rho) > 0$, $x \in \mathbb{R}$, and $\Gamma(\cdot)$ denotes the classical gamma function.

Remark 1.1 By making use of $\eta = \alpha$, $\sigma(0) = 1$, and $\omega = 0$ in both (1.3) and (1.4), we obtain the classical left and right-RL fractional integrals (1.1) and (1.2), respectively.

2 Literature results

Before we pass to the main findings, we review and introduce some definitions, notations, theorems which will be necessary later to proceed.

Definition 2.1 ([31]) A function $f : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex on $\mathcal{I}$ if

$$ f\big((1-\xi)\chi_1 + \xi\chi_2\big) \leq (1-\xi) f(\chi_1) + \xi f(\chi_2) \tag{2.1}$$

holds for every $\chi_1, \chi_2 \in \mathcal{I}$ and $\xi \in [0, 1]$.

Definition 2.2 ([32]) Denote $\Delta := [\chi_1, \chi_2] \times [\chi_3, \chi_4]$, where $0 < \chi_1 < \chi_2$ and $0 < \chi_3 < \chi_4$. For a function $f : \Delta \to \mathbb{R}$, the coordinated convex function on Δ is defined as follows:

$$
\begin{aligned}
& f\big((\xi_1\chi_1 + (1-\xi_1)\chi_2), (\xi_2\chi_3 + (1-\xi_2)\chi_4)\big) \\
&\quad \le \xi_1\xi_2 f(\chi_1, \chi_3) + \xi_2(1-\xi_1) f(\chi_2, \chi_3) \\
&\qquad + \xi_1(1-\xi_2) f(\chi_1, \chi_4) + (1-\xi_1)(1-\xi_2) f(\chi_2, \chi_4)
\end{aligned} \tag{2.2}
$$

for every $\xi_1, \xi_2 \in [0,1]$ and $(\chi_1, \chi_2), (\chi_3, \chi_4) \in \Delta$.

The well-known integral inequality of Hermite–Hadamard type (HH-type) for such a convex function (2.1) is given by

$$
f\left(\frac{\chi_1 + \chi_2}{2}\right) \le \frac{1}{\chi_2 - \chi_1} \int_{\chi_1}^{\chi_2} f(x)\,dx \le \frac{f(\chi_1) + f(\chi_2)}{2}. \tag{2.3}
$$

In 2001, HH-inequality (2.3) was established on the bidimensional plane Δ for such a coordinated convex function (2.2) by Dragomir [32], his result is as follows.

Theorem 2.1 *Let $f : \Delta \to \mathbb{R}$ be a coordinated convex function on Δ, then we have*

$$
\begin{aligned}
& f\left(\frac{\chi_1 + \chi_2}{2}, \frac{\chi_3 + \chi_4}{2}\right) \\
&\quad \le \frac{1}{2}\left(\frac{1}{\chi_2 - \chi_1} \int_{\chi_1}^{\chi_2} f\left(x, \frac{\chi_3 + \chi_4}{2}\right) dx + \frac{1}{\chi_4 - \chi_3} \int_{\chi_3}^{\chi_4} f\left(\frac{\chi_1 + \chi_2}{2}, y\right) dy\right) \\
&\quad \le \frac{1}{(\chi_2 - \chi_1)(\chi_4 - \chi_3)} \int_{\chi_1}^{\chi_2} \int_{\chi_3}^{\chi_4} f(x, y)\,dy\,dx \\
&\quad \le \frac{1}{4}\left(\frac{1}{\chi_2 - \chi_1} \int_{\chi_1}^{\chi_2} \big(f(x, \chi_3) + f(x, \chi_4)\big)\,dx \frac{1}{\chi_4 - \chi_3} \int_{\chi_3}^{\chi_4} \big(f(\chi_1, y) + f(\chi_2, y)\big)\,dy\right) \\
&\quad \le \frac{f(\chi_1, \chi_3) + f(\chi_1, \chi_4) + f(\chi_2, \chi_3) + f(\chi_2, \chi_4)}{4}.
\end{aligned} \tag{2.4}
$$

In 2014, HH-inequality (2.4) was generalized to fractional integrals of RL type by Sarikaya [33], which is as follows.

Theorem 2.2 *Let $f : \Delta \to \mathbb{R}$ be a coordinated convex function on Δ, then we have*

$$
\begin{aligned}
& f\left(\frac{\chi_1 + \chi_2}{2}, \frac{\chi_3 + \chi_4}{2}\right) \\
&\quad \le \frac{\Gamma(\alpha + 1)}{4(\chi_2 - \chi_1)^\alpha}\left(J_{\chi_1^+}^{\alpha} f\left(\chi_2, \frac{\chi_3 + \chi_4}{2}\right) + J_{\chi_2^-}^{\alpha} f\left(\chi_1, \frac{\chi_3 + \chi_4}{2}\right)\right) \\
&\qquad + \frac{\Gamma(\beta + 1)}{4(\chi_4 - \chi_3)^\beta}\left(J_{\chi_3^+}^{\alpha} f\left(\frac{\chi_1 + \chi_2}{2}, \chi_4\right) + J_{\chi_4^-}^{\beta} f\left(\frac{\chi_1 + \chi_2}{2}, \chi_3\right)\right) \\
&\quad \le \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{(\chi_2 - \chi_1)^\alpha(\chi_4 - \chi_3)^\beta}\big(J_{\chi_1^+, \chi_3^+}^{\alpha,\beta} f(\chi_2, \chi_4) + J_{\chi_1^+, \chi_4^-}^{\alpha,\beta} f(\chi_2, \chi_3) \\
&\qquad + J_{\chi_2^-, \chi_3^+}^{\alpha,\beta} f(\chi_1, \chi_4) + J_{\chi_2^-, \chi_4^-}^{\alpha,\beta} f(\chi_1, \chi_3)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\Gamma(\alpha+1)}{4(\chi_2-\chi_1)^{\alpha}}\left(J^{\alpha}_{\chi_2^-}f(\chi_1,\chi_4)+J^{\alpha}_{\chi_2^-}f(\chi_1,\chi_3)+J^{\alpha}_{\chi_1^+}f(\chi_2,\chi_4)+J^{\alpha}_{\chi_1^+}f(\chi_2,\chi_3)\right)\\
&\quad+\frac{\Gamma(\beta+1)}{4(\chi_4-\chi_3)^{\beta}}\left(J^{\beta}_{\chi_4^-}f(\chi_2,\chi_3)+J^{\beta}_{\chi_4^-}f(\chi_1,\chi_3)+J^{\alpha}_{\chi_3^+}f(\chi_2,\chi_4)+J^{\alpha}_{\chi_3^+}f(\chi_1,\chi_4)\right)\\
&\leq \frac{f(\chi_1,\chi_3)+f(\chi_1,\chi_4)+f(\chi_2,\chi_3)+f(\chi_2,\chi_4)}{4}.
\end{aligned}
$$

The above inequalities have attracted many researchers in the recent years, see e.g. [34–38].

Definition 2.3 ([39]) Let $f \in L(\Delta)$. The fractional integral operators for two variable functions, where $(\rho,\eta,\omega) \in [0,+\infty)^2 \times [0,+\infty)^2 \times \mathbb{R}^2$ with $\rho=(\rho_1,\rho_2)$, $\eta=(\eta_1,\eta_2)$, $\omega=(\omega_1,\omega_2)$, and $\sigma=(\sigma_1,\sigma_2)$, are given as follows:

$$
\begin{aligned}
\Im^{\sigma}_{\rho,\eta,\chi_1^+,\chi_3^+,\omega}\varphi(x,y) &= \int_{\chi_1}^{x}\int_{\chi_3}^{y}(x-\xi_1)^{\eta_1-1}(y-\xi_2)^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\left[\omega_1(x-\xi_1)^{\rho_1}\right]\\
&\quad\times \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2(y-\xi_2)^{\rho_1}\right]\varphi(\xi_1,\xi_2)\,d\xi_2\,d\xi_1,\\
\Im^{\sigma}_{\rho,\eta,\chi_1^+,\chi_4^-,\omega}\varphi(x,y) &= \int_{\chi_1}^{x}\int_{y}^{\chi_4}(x-\xi_1)^{\eta_1-1}(\xi_2-y)^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\left[\omega_1(x-\xi_1)^{\rho_1}\right]\\
&\quad\times \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2(\xi_2-y)^{\rho_1}\right]\varphi(\xi_1,\xi_2)\,d\xi_2\,d\xi_1,\\
\Im^{\sigma}_{\rho,\eta,\chi_2^-,\chi_3^+,\omega}\varphi(x,y) &= \int_{x}^{\chi_2}\int_{\chi_3}^{y}(\xi_1-x)^{\eta_1-1}(y-\xi_2)^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\left[\omega_1(\xi_1-x)^{\rho_1}\right]\\
&\quad\times \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2(y-\xi_2)^{\rho_1}\right]\varphi(\xi_1,\xi_2)\,d\xi_2\,d\xi_1,
\end{aligned}
$$

and

$$
\begin{aligned}
\Im^{\sigma}_{\rho,\eta,\chi_2^-,\chi_4^-,\omega}\varphi(x,y) &= \int_{x}^{\chi_2}\int_{y}^{\chi_4}(\xi_1-x)^{\eta_1-1}(\xi_2-y)^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\left[\omega_1(\xi_1-x)^{\rho_1}\right]\\
&\quad\times \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2(\xi_2-y)^{\rho_1}\right]\varphi(\xi_1,\xi_2)\,d\xi_2\,d\xi_1.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\Im^{\sigma_1}_{\rho_1,\eta_1,\chi_1^+,\omega_1}\varphi\left(x,\frac{\chi_3+\chi_4}{2}\right) &= \int_{\chi_1}^{x}(x-\xi_1)^{\eta_1-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\left[\omega_1(x-\xi_1)^{\rho_1}\right]\varphi\left(\xi_1,\frac{\chi_3+\chi_4}{2}\right)d\xi_1,\\
\Im^{\sigma_1}_{\rho_1,\eta_1,\chi_2^-,\omega_1}\varphi\left(x,\frac{\chi_3+\chi_4}{2}\right) &= \int_{x}^{\chi_2}(\xi_1-x)^{\eta_1-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\left[\omega_1(\xi_1-x)^{\rho_1}\right]\varphi\left(\xi_1,\frac{\chi_3+\chi_4}{2}\right)d\xi_1,\\
\Im^{\sigma_2}_{\rho_2,\eta_2,\chi_3^+,\omega_2}\varphi\left(\frac{\chi_1+\chi_2}{2},y\right) &= \int_{\chi_3}^{y}(y-\xi_2)^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2(y-\xi_2)^{\rho_1}\right]\varphi\left(\frac{\chi_1+\chi_2}{2},\xi_2\right)d\xi_2,
\end{aligned}
$$

and

$$
\Im^{\sigma_2}_{\rho_2,\eta_2,\chi_4^-,\omega_2}\varphi\left(\frac{\chi_1+\chi_2}{2},y\right) = \int_{y}^{\chi_4}(\xi_2-y)^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2(\xi_2-y)^{\rho_1}\right]\varphi\left(\frac{\chi_1+\chi_2}{2},\xi_2\right)d\xi_2.
$$

In [39], Tunç and Sarikaya investigate the following Hermite–Hadamard for coordinated convex functions:

$$\begin{aligned} &f\left(\frac{\chi_1+\chi_2}{2},\frac{\chi_3+\chi_4}{2}\right) \\ &\leq \frac{1}{(\chi_2-\chi_1)^{\eta_1}(\chi_4-\chi_3)^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2-\chi_1)^{\rho_1})\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4-\chi_3)^{\rho_2})} \\ &\quad\times\left\{\mathfrak{J}^{\sigma}_{\rho,\eta,\chi_1^+,\chi_3^+,\omega}f(\chi_2,\chi_4)+\mathfrak{J}^{\sigma}_{\rho,\eta,\chi_1^+,\chi_4^-,\omega}f(\chi_2,\chi_3)\right. \\ &\quad\left.+\mathfrak{J}^{\sigma}_{\rho,\eta,\chi_2^-,\chi_3^+,\omega}f(\chi_1,\chi_4)+\mathfrak{J}^{\sigma}_{\rho,\eta,\chi_2^-,\chi_4^-,\omega}f(\chi_1,\chi_3)\right\} \\ &\leq \frac{(f(\chi_1,\chi_3)+f(\chi_1,\chi_4)+f(\chi_2,\chi_3)+f(\chi_2,\chi_4))}{4}. \end{aligned}$$

Definition 2.4 ([40]) Let $h_1,h_2:J\to\mathbb{R}$ be two nonnegative and nonzero functions. A function $f:\Delta\to\mathbb{R}$ is said to be (l_1,h_1)-(l_2,h_2)-convex function on the coordinates on Δ if

$$\begin{aligned} &f\left(\left[\xi_1x^{l_1}+(1-\xi_1)u^{l_1}\right]^{\frac{1}{l_1}},\left[\xi_2y^{l_2}+(1-\xi_2)v^{l_2}\right]^{\frac{1}{l_2}}\right) \\ &\leq h_1(\xi_1)h_2(\xi_2)f(x,y)+h_1(\xi_1)h_2(1-\xi_2)f(x,v)+h_1(1-\xi_1)h_2(\xi_2)f(u,y) \\ &\quad+h_1(1-\xi_1)h_2(1-\xi_2)f(u,v) \end{aligned}$$

holds for all $(x,y),(u,v)\in\Delta$ and $\xi_1,\xi_2\in(0,1)$.

As we know, there are many results on coordinates of convex functions via other types of fractional operators and other types of convex functions, see e.g. [41–44]. Therefore, integral inequalities on coordinates of convex functions via general Raina fractional integrals open a new door in the field of mathematical analysis and theory of convexity.

Motivated by the above results, in this paper we establish some generalized integral inequalities using an (l_1,h_1)-(l_2,h_2)-convex function on coordinates. Also, we obtain an integral identity for partial differentiable functions. As an effect of this result, two interesting integral inequalities for an (l_1,h_1)-(l_2,h_2)-convex function on coordinates are given. At the end, a brief conclusion is provided as well.

3 Main results

In what follows, we assume that $h_1,h_2:J\to\mathbb{R}$ are two nonnegative and nonzero functions, with $h_1(\frac{1}{2})h_2(\frac{1}{2})\neq 0$, $\sigma=(\sigma_1,\sigma_2)$, $\rho=(\rho_1,\rho_2)$, $\eta=(\eta_1,\eta_2)$, and $\omega=(\omega_1,\omega_2)$ with $\rho_1,\rho_2,\eta_1,\eta_2\in[0,+\infty)$ and $\omega_1,\omega_2\in\mathbb{R}$.

Theorem 3.1 *Let $f:\Delta\to\mathbb{R}$ be an integrable and (l_1,h_1)-(l_2,h_2)-convex function on coordinates on Δ. Then we have*

$$\begin{aligned} &\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\left[\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}\right]^{\frac{1}{l_1}},\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right) \\ &=\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}f_g\left(\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2},\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right) \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})} \\
&\quad\times \Big\{\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega} f_g\big(\chi_2^{l_1},\chi_4^{l_2}\big) + \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega} f_g\big(\chi_2^{l_1},\chi_3^{l_2}\big) \\
&\quad+ \Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega} f_g\big(\chi_1^{l_1},\chi_4^{l_2}\big) + \Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega} f_g\big(\chi_1^{l_1},\chi_3^{l_2}\big)\Big\} \\
&\leq \frac{(f(\chi_1,\chi_3)+f(\chi_1,\chi_4)+f(\chi_2,\chi_3)+f(\chi_2,\chi_4))}{\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})} \\
&\quad\times \int_0^1\int_0^1 \xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big] \\
&\quad\times \big(h_1(\xi_1)+h_1(1-\xi_1)\big)\big(h_2(\xi_2)+h_2(1-\xi_2)\big)\,d\xi_2\,d\xi_1,
\end{aligned}
$$

where $f_g(x,y)=f(g_1(x),g_2(y))$ *with* $g_1(x)=x^{\frac{1}{l_1}}$ *and* $g_2(y)=y^{\frac{1}{l_2}}$.

Proof It is easy to see that

$$
\left[\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}\right]^{\frac{1}{l_1}} = \left[\frac{((\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1})^{\frac{1}{l_1}})^{l_1}+(((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1})^{\frac{1}{l_1}})^{l_1}}{2}\right]^{\frac{1}{l_1}} \tag{3.1}
$$

and

$$
\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}} = \left[\frac{((\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2})^{\frac{1}{l_2}})^{l_2}+(((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2})^{\frac{1}{l_2}})^{l_2}}{2}\right]^{\frac{1}{l_2}}. \tag{3.2}
$$

Making use of (3.1) and (3.2), and the fact that f is (l_1,h_1)-(l_2,h_2)-convex on the coordinates, we have

$$
\begin{aligned}
&f\left(\left[\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}\right]^{\frac{1}{l_1}},\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right) \\
&\quad\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\Big\{f\Big(\big(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\Big) \\
&\qquad+f\Big(\big(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\Big) \\
&\qquad+f\Big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\Big) \\
&\qquad+f\Big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\Big)\Big\}.
\end{aligned} \tag{3.3}
$$

Multiplying on both sides of (3.3) by

$$
\xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big],
$$

and then integrating the resulting inequality with respect to (ξ_1,ξ_2) on $[0,1]^2$, we get

$$
\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}f\left(\left[\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}\right]^{\frac{1}{l_1}},\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right) \\
&\quad\times\int_0^1\int_0^1 \xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]\,d\xi_2\,d\xi_1
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^1\int_0^1 \xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]\\
&\quad\times f\big(\big[\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big]^{\frac{1}{l_1}},\big[\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big]^{\frac{1}{l_2}}\big)\,d\xi_2\,d\xi_1\\
&\quad+\int_0^1\int_0^1 \xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]\\
&\quad\times f\big(\big[\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big]^{\frac{1}{l_1}},\big[(1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big]^{\frac{1}{l_2}}\big)\,d\xi_2\,d\xi_1\\
&\quad+\int_0^1\int_0^1 \xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2^{\rho_2}\big(\chi_4^{l_2}-\chi_3^{l_2}\big)\xi_2^{\rho_2}\big]\\
&\quad\times f\big(\big[(1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big]^{\frac{1}{l_1}},\big[\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big]^{\frac{1}{l_2}}\big)\,d\xi_2\,d\xi_1\\
&\quad+\int_0^1\int_0^1 \xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]\\
&\quad\times f\big(\big[(1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big]^{\frac{1}{l_1}},\big[(1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big]^{\frac{1}{l_2}}\big)\,d\xi_2\,d\xi_1.
\end{aligned}
\tag{3.4}
$$

By making a change of variables in (3.4), we obtain

$$
\begin{aligned}
&\frac{1}{h_1(\frac{1}{2})h_2(\frac{1}{2})}f\bigg(\bigg[\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}\bigg]^{\frac{1}{l_1}},\bigg[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\bigg]^{\frac{1}{l_2}}\bigg)\\
&\leq \frac{1}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})}\\
&\quad\times\bigg\{\int_{\chi_1^{l_1}}^{\chi_2^{l_1}}\int_{\chi_3^{l_2}}^{\chi_4^{l_2}}\big(\chi_2^{l_1}-x\big)^{\eta_1-1}\big(\chi_4^{l_2}-y\big)^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-x\big)^{\rho_1}\big]\\
&\quad\times\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-y\big)^{\rho_2}\big]f\big(x^{\frac{1}{l_1}},y^{\frac{1}{l_2}}\big)\,dy\,dx\\
&\quad+\int_{\chi_1^{l_1}}^{\chi_2^{l_1}}\int_{\chi_3^{l_2}}^{\chi_4^{l_2}}\big(\chi_2^{l_1}-x\big)^{\eta_1-1}\big(y-\chi_3^{l_2}\big)^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-x\big)^{\rho_1}\big]\\
&\quad\times\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(y-\chi_3^{l_2}\big)^{\rho_2}\big]f\big(x^{\frac{1}{l_1}},y^{\frac{1}{l_2}}\big)\,dy\,dx\\
&\quad+\int_{\chi_1^{l_1}}^{\chi_2^{l_1}}\int_{\chi_3^{l_2}}^{\chi_4^{l_2}}\big(x-\chi_1^{l_1}\big)^{\eta_1-1}\big(\chi_4^{l_2}-y\big)^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(x-\chi_1^{l_1}\big)^{\rho_1}\big]\\
&\quad\times\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2^{\rho_2}\big(\chi_4^{l_2}-y\big)\big]f\big(x^{\frac{1}{l_1}},y^{\frac{1}{l_2}}\big)\,dy\,dx\\
&\quad+\int_{\chi_1^{l_1}}^{\chi_2^{l_1}}\int_{\chi_3^{l_2}}^{\chi_4^{l_2}}\big(x-\chi_1^{l_1}\big)^{\eta_1-1}\big(y-\chi_3^{l_2}\big)^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(x-\chi_1^{l_1}\big)^{\rho_1}\big]\\
&\quad\times\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(y-\chi_3^{l_2}\big)^{\rho_2}\big]f\big(x^{\frac{1}{l_1}},y^{\frac{1}{l_2}}\big)\,dy\,dx\bigg\}\\
&= \frac{1}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})}\\
&\quad\times\big\{\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}f_g\big(\chi_2^{l_1},\chi_4^{l_2}\big)+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}f_g\big(\chi_2^{l_1},\chi_3^{l_2}\big)\\
&\quad+\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}f_g\big(\chi_1^{l_1},\chi_4^{l_2}\big)+\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}f_g\big(\chi_1^{l_1},\chi_3^{l_2}\big)\big\}.
\end{aligned}
\tag{3.5}
$$

Since f is (l_1,h_1)-(l_2,h_2)-convex on the coordinates, we have

$$\begin{aligned} & f\bigl(\bigl[\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\bigr]^{\frac{1}{l_1}},\bigl[\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\bigr]^{\frac{1}{l_2}}\bigr) \\ &\quad \le h_1(\xi_1)h_2(\xi_2)f(\chi_1,\chi_3)+h_1(\xi_1)h_2(1-\xi_2)f(\chi_1,\chi_4)+h_1(1-\xi_1)h_2(\xi_2)f(\chi_2,\chi_3) \\ &\qquad + h_1(1-\xi_1)h_2(1-\xi_2)f(\chi_2,\chi_4), \end{aligned} \tag{3.6}$$

$$\begin{aligned} & f\bigl(\bigl[\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\bigr]^{\frac{1}{l_1}},\bigl[(1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\bigr]^{\frac{1}{l_2}}\bigr) \\ &\quad \le h_1(\xi_1)h_2(1-\xi_2)f(\chi_1,\chi_3)+h_1(\xi_1)h_2(\xi_2)f(\chi_1,\chi_4)+h_1(1-\xi_1)h_2(1-\xi_2)f(\chi_2,\chi_3) \\ &\qquad + h_1(1-\xi_1)h_2(\xi_2)f(\chi_2,\chi_4), \end{aligned} \tag{3.7}$$

$$\begin{aligned} & f\bigl(\bigl((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\bigr)^{\frac{1}{l_1}},\bigl(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\bigr)^{\frac{1}{l_2}}\bigr) \\ &\quad \le h_1(1-\xi_1)h_2(\xi_2)f(\chi_1,\chi_3)+h_1(1-\xi_1)h_2(1-\xi_2)f(\chi_1,\chi_4)+h_1(\xi_1)h_2(\xi_2)f(\chi_2,\chi_3) \\ &\qquad + h_1(\xi_1)h_2(1-\xi_2)f(\chi_2,\chi_4), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} & f\bigl(\bigl((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\bigr)^{\frac{1}{l_1}},\bigl((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\bigr)^{\frac{1}{l_2}}\bigr) \\ &\quad \le h_1(1-\xi_1)h_2(1-\xi_2)f(\chi_1,\chi_3)+h_1(1-\xi_1)h_2(\xi_2)f(\chi_1,\chi_4) \\ &\qquad + h_1(\xi_1)h_2(1-\xi_2)f(\chi_2,\chi_3)+h_1(\xi_1)h_2(\xi_2)f(\chi_2,\chi_4). \end{aligned} \tag{3.9}$$

Adding inequalities (3.6)–(3.9), multiplying the resulting inequality by

$$\xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\bigl[\omega_1\bigl(\chi_2^{l_1}-\chi_1^{l_1}\bigr)^{\rho_1}\xi_1^{\rho_1}\bigr]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\bigl[\omega_2\bigl(\chi_4^{l_2}-\chi_3^{l_2}\bigr)^{\rho_2}\xi_2^{\rho_2}\bigr],$$

and then integrating the result with respect to (ξ_1,ξ_2) on $[0,1]^2$, we get

$$\begin{aligned} & \int_0^1\int_0^1 \xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\bigl[\omega_1\bigl(\chi_2^{l_1}-\chi_1^{l_1}\bigr)^{\rho_1}\xi_1^{\rho_1}\bigr]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\bigl[\omega_2\bigl(\chi_4^{l_2}-\chi_3^{l_2}\bigr)^{\rho_2}\xi_2^{\rho_2}\bigr] \\ &\qquad \times \bigl\{f\bigl(\bigl[\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\bigr]^{\frac{1}{l_1}},\bigl[\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\bigr]^{\frac{1}{l_2}}\bigr) \\ &\qquad + f\bigl(\bigl[\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\bigr]^{\frac{1}{l_1}},\bigl[(1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\bigr]^{\frac{1}{l_2}}\bigr) \\ &\qquad + f\bigl(\bigl((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\bigr)^{\frac{1}{l_1}},\bigl(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\bigr)^{\frac{1}{l_2}}\bigr) \\ &\qquad + f\bigl(\bigl((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\bigr)^{\frac{1}{l_1}},\bigl((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\bigr)^{\frac{1}{l_2}}\bigr)\bigr\}\,d\xi_2\,d\xi_1 \\ &\quad \le \bigl(f(\chi_1,\chi_3)+f(\chi_1,\chi_4)+f(\chi_2,\chi_3)+f(\chi_2,\chi_4)\bigr) \\ &\qquad \times \int_0^1\int_0^1 \xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\bigl[\omega_1\bigl(\chi_2^{l_1}-\chi_1^{l_1}\bigr)^{\rho_1}\xi_1^{\rho_1}\bigr]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\bigl[\omega_2\bigl(\chi_4^{l_2}-\chi_3^{l_2}\bigr)^{\rho_2}\xi_2^{\rho_2}\bigr] \\ &\qquad \times \bigl(h_1(\xi_1)h_2(\xi_2)+h_1(\xi_1)h_2(1-\xi_2)+h_1(1-\xi_1)h_2(\xi_2) \\ &\qquad + h_1(1-\xi_1)h_2(1-\xi_2)\bigr)\,d\xi_2\,d\xi_1. \end{aligned}$$

Making use of the change of variables and multiplying the result by

$$\frac{1}{\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})},$$

we obtain

$$\begin{aligned}
&\frac{1}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})} \\
&\quad\times\big\{\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}f_g(\chi_2^{l_1},\chi_4^{l_2})+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}f_g(\chi_2^{l_1},\chi_3^{l_2}) \\
&\quad+\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}f_g(\chi_1^{l_1},\chi_4^{l_2})+\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}f_g(\chi_1^{l_1},\chi_3^{l_2})\big\} \\
&\quad\le\frac{(f(\chi_1,\chi_3)+f(\chi_1,\chi_4)+f(\chi_2,\chi_3)+f(\chi_2,\chi_4))}{\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})} \\
&\quad\times\int_0^1\int_0^1\xi_1^{\eta_1-1}\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}\xi_2^{\rho_2}\big] \\
&\quad\times\big(h_1(\xi_1)+h_1(1-\xi_1)\big)\big(h_2(\xi_2)+h_2(1-\xi_2)\big)\,d\xi_2\,d\xi_1.
\end{aligned}$$

This rearranges to the proof of Theorem 3.1. □

Remark 3.1 Theorem 3.1 with $l_1=l_2=1$ and $h_1(\xi_1)=h_2(\xi_1)=\xi_1$ becomes Theorem 2.1 in [39].

Remark 3.2 Theorem 3.1 with $l_1=l_2=1$, $\eta_1=\eta_2=\alpha$, $\sigma_1(0)=\sigma_2(0)=1$, $\omega_1=\omega_2=0$, and $h_1(\xi_1)=h_2(\xi_1)=\xi_1$ becomes Theorem 3 in [33].

Remark 3.3 Theorem 3.1 with $\eta_1=\eta_2=1$, $\sigma_1(0)=\sigma_2(0)=1$, and $\omega_1=\omega_2=0$ becomes Theorem 2.1 in [40].

Remark 3.4 Theorem 3.1 with $l_1=l_2=1$, $\eta_1=\eta_2=1$, $\sigma_1(0)=\sigma_2(0)=1$, $\omega_1=\omega_2=0$, and $h_1(\xi_1)=h_2(\xi_1)=h(\xi_1)$ becomes Theorem 7 in [35].

Theorem 3.2 *Let $f:\Delta\to\mathbb{R}$ be an integrable and (l_1,h_1)-(l_2,h_2)-convex function on coordinates on Δ. Then we have*

$$\begin{aligned}
&f\bigg(\Big[\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}\Big]^{\frac{1}{l_1}},\Big[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\Big]^{\frac{1}{l_2}}\bigg) \\
&\quad\le\frac{h_1(\frac{1}{2})(\Im^{\sigma_1}_{\rho_1,\eta_1,(\chi_1^{l_1})^+,\omega_1}f_g(\chi_2^{l_1},\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2})+\Im^{\sigma_1}_{\rho_1,\eta_1,(\chi_2^{l_1})^-,\omega_1}f_g(\chi_1^{l_1},\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}))}{2(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})} \\
&\quad+\frac{h_2(\frac{1}{2})(\Im^{\sigma_2}_{\rho_2,\eta_2,(\chi_3^{l_2})^+,\omega_2}f_g(\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2},\chi_4^{l_2})+\Im^{\sigma_2}_{\rho_2,\eta_2,(\chi_4^{l_2})^-,\omega_2}f_g(\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2},\chi_3^{l_2}))}{2(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})} \\
&\quad\le h_1\Big(\frac{1}{2}\Big)\frac{f(\chi_1,[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}]^{\frac{1}{l_2}})+f(\chi_2,[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}]^{\frac{1}{l_2}})}{2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})}
\end{aligned}$$

$$\times \int_0^1 \xi_1^{\eta_1-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\big(h_1(\xi_1)+h_1(1-\xi_1)\big)\,d\xi_1$$
$$+ h_2\left(\frac{1}{2}\right)\frac{f([\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}]^{\frac{1}{l_1}},\chi_3)+f([\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}]^{\frac{1}{l_1}},\chi_4)}{2\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})}$$
$$\times \int_0^1 \xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]\big(h_2(\xi_2)+h_2(1-\xi_2)\big)\,d\xi_2,$$

where $f_g(x,y)=f(g_1(x),g_2(y))$ *with* $g_1(x)=x^{\frac{1}{l_1}}$ *and* $g_2(y)=y^{\frac{1}{l_2}}$.

Proof Since f is an (l_1,h_1)-(l_2,h_2)-convex function on coordinates on Δ, the partial mapping $f_x:[\chi_3,\chi_4]\to\mathbb{R}$ defined by $f_x(v)=f(x,v)$ is (l_2,h_2)-convex with respect to v on $[\chi_3,\chi_4]$, and $f_y:[\chi_1,\chi_2]\to\mathbb{R}$ defined by $f_y(u)=f(u,y)$ is (l_1,h_1)-convex with respect to u on $[\chi_1,\chi_2]$. So, we have

$$\begin{aligned}
&f_x\left(\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right)\\
&=f_x\left(\left[\frac{((\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2})^{\frac{1}{l_2}})^{l_2}+(((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2})^{\frac{1}{l_2}})^{l_2}}{2}\right]^{\frac{1}{l_2}}\right)\\
&\le h_2\left(\frac{1}{2}\right)\Big(f_x\big(\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)+f_x\big(\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\Big)\\
&\le h_2\left(\frac{1}{2}\right)\big(h_2(\xi_2)f_x(\chi_3)+h_2(1-\xi_2)f_x(\chi_4)+h_2(1-\xi_2)f_x(\chi_3)+h_2(\xi_2)f_x(\chi_4)\big)\\
&=h_2\left(\frac{1}{2}\right)\big(h_2(\xi_2)+h_2(1-\xi_2)\big)\big(f_x(\chi_3)+f_x(\chi_4)\big).
\end{aligned}\tag{3.10}$$

From (3.10), we get

$$\begin{aligned}
\frac{1}{h_2(\frac{1}{2})}f_x\left(\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right)&\le f_x\big(\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)+f_x\big(\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\\
&\le \big(h_2(\xi_2)+h_2(1-\xi_2)\big)\big(f_x(\chi_3)+f_x(\chi_4)\big).
\end{aligned}\tag{3.11}$$

Multiplying (3.11) by $\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}\xi_2^{\rho_2}]$ and then integrating the resulting inequalities with respect to ξ_2 on $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{h_2(\frac{1}{2})}f_x\left(\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right)\int_0^1 \xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]\,d\xi_2\\
&=\frac{1}{h_2(\frac{1}{2})}f_x\left(\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right)\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}\big(\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\big)\\
&\le \int_0^1 \xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]f_x\big(\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_2\\
&\quad+\int_0^1 \xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]f_x\big(\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_2\\
&=\frac{1}{(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}}\int_{\chi_3^{l_2}}^{\chi_4^{l_2}}\big(\chi_4^{l_2}-w\big)^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\big[\omega_2\big(\chi_4^{l_2}-w\big)^{\rho_2}\big]f_x(w)\,dw
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}}\int_{\chi_3^{l_2}}^{\chi_4^{l_2}}\left(w-\chi_3^{l_2}\right)^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2\left(w-\chi_3^{l_2}\right)^{\rho_2}\right]f_x(w)\,dw\\
&=\frac{1}{(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}}\left(\mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_3^{l_2})^+,\omega_2}f_x\left(\chi_4^{l_2}\right)+\mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_4^{l_2})^-,\omega_2}f_x\left(\chi_3^{l_2}\right)\right)\\
&\le\left(f_x(\chi_3)+f_x(\chi_4)\right)\int_0^1\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2\left(\chi_4^{l_2}-\chi_3^{l_2}\right)^{\rho_2}\xi_2^{\rho_2}\right]\\
&\quad\times\left(h_2(\xi_2)+h_2(1-\xi_2)\right)d\xi_2.
\end{aligned}
\tag{3.12}
$$

This implies that

$$
\begin{aligned}
&\frac{1}{h_2(\frac{1}{2})}f\left(x,\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right)\\
&\le\frac{1}{(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})}\\
&\quad\times\left(\mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_3^{l_2})^+,\omega_2}f_g\left(x^{l_1},\chi_4^{l_2}\right)+\mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_4^{l_2})^-,\omega_2}f_g\left(x^{l_1},\chi_3^{l_2}\right)\right)\\
&\le\frac{f(x,\chi_3)+f(x,\chi_4)}{\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})}\int_0^1\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2\left(\chi_4^{l_2}-\chi_3^{l_2}\right)^{\rho_2}\xi_2^{\rho_2}\right]\\
&\quad\times\left(h_2(\xi_2)+h_2(1-\xi_2)\right)d\xi_2.
\end{aligned}
\tag{3.13}
$$

Put $x=[\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}]^{\frac{1}{l_1}}$ into (3.13) to get

$$
\begin{aligned}
&f\left(\left[\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}\right]^{\frac{1}{l_1}},\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right)\\
&\le\frac{h_2(\frac{1}{2})}{(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})}\\
&\quad\times\left(\mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_3^{l_2})^+,\omega_2}f_g\left(\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2},\chi_4^{l_2}\right)+\mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_4^{l_2})^-,\omega_2}f_g\left(\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2},\chi_3^{l_2}\right)\right)\\
&\le h_2\left(\frac{1}{2}\right)\frac{f([\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}]^{\frac{1}{l_1}},\chi_3)+f([\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}]^{\frac{1}{l_1}},\chi_4)}{\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}(\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2})}\\
&\quad\times\int_0^1\xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\left[\omega_2\left(\chi_4^{l_2}-\chi_3^{l_2}\right)^{\rho_2}\xi_2^{\rho_2}\right]\left(h_2(\xi_2)+h_2(1-\xi_2)\right)d\xi_2.
\end{aligned}
\tag{3.14}
$$

Similarly, we can deduce

$$
\begin{aligned}
&f\left(\left[\frac{\chi_1^{l_1}+\chi_2^{l_1}}{2}\right]^{\frac{1}{l_1}},\left[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right]^{\frac{1}{l_2}}\right)\\
&\le\frac{h_1(\frac{1}{2})}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})}\\
&\quad\times\left(\mathfrak{I}^{\sigma_1}_{\rho_1,\eta_1,(\chi_1^{l_1})^+,\omega_1}f_g\left(\chi_2^{l_1},\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right)+\mathfrak{I}^{\sigma_1}_{\rho_1,\eta_1,(\chi_2^{l_1})^-,\omega_1}f_g\left(\chi_1^{l_1},\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}\right)\right)
\end{aligned}
$$

$$\leq h_1\left(\frac{1}{2}\right)\frac{f(\chi_1,[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}]^{\frac{1}{l_2}})+f(\chi_2,[\frac{\chi_3^{l_2}+\chi_4^{l_2}}{2}]^{\frac{1}{l_2}})}{\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}(\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1})}$$
$$\times\int_0^1 \xi_1^{\eta_1-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho 1}\big]\big(h_1(\xi_1)+h_1(1-\xi_1)\big)\,d\xi_1. \tag{3.15}$$

By adding (3.14) and (3.15) together, and then multiplying the result by $\frac{1}{2}$, we get the desired result. Thus we get the proof of Theorem 3.2. □

Remark 3.5 Theorem 3.2 with $l_1 = l_2 = 1$ and $h_1(\xi_1) = h_2(\xi_1) = \xi_1$ becomes Theorem 2.2 in [39].

Lemma 3.1 *Let $f:\Delta\to\mathbb{R}$ be a partial differentiable function on Δ. If $\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\in L(\Delta)$, then we have*

$$\frac{f(\chi_1,\chi_3)}{4\chi_1^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_1,\chi_4)}{4\chi_1^{1-l_1}\chi_4^{1-l_2}}+\frac{f(\chi_2,\chi_3)}{4\chi_2^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_2,\chi_4)}{4\chi_2^{1-l_1}\chi_4^{1-l_2}}-A$$
$$+\frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}$$
$$\times\Big(\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}\chi_1^{l_1-1}\chi_4^{l_2-1}f_g\big(\chi_1^{l_1},\chi_4^{l_2}\big)+\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}\chi_1^{l_1-1}\chi_3^{l_2-1}f_g\big(\chi_1^{l_1},\chi_3^{l_2}\big)$$
$$+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}\chi_2^{l_1-1}\chi_3^{l_2-1}f_g\big(\chi_2^{l_1},\chi_3^{l_2}\big)+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}\chi_2^{l_1-1}\chi_4^{l_2-1}f_g\big(\chi_2^{l_1},\chi_4^{l_2}\big)\Big)$$
$$=\frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}$$
$$\times\left(\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_1\,d\xi_2\right.$$
$$+\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_1\,d\xi_2$$
$$+\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_1\,d\xi_2$$
$$\left.+\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_1\,d\xi_2\right),$$

where

$$\mathcal{B}(\xi_1,\xi_2)=\xi_1^{\eta_1}\xi_2^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big] \tag{3.16}$$

and

$$A=\frac{1}{4(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}\left(\frac{\Im^{\sigma_2}_{\rho_2,\eta_2,(\chi_4^{l_2})^-,\omega_2}\chi_3^{l_2-1}f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_1^{1-l_1}}\right.$$
$$+\frac{\Im^{\sigma_2}_{\rho_2,\eta_2,(\chi_3^{l_2})^+,\omega_2}\chi_4^{l_2-1}f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_1^{1-l_1}}+\frac{\Im^{\sigma_2}_{\rho_2,\eta_2,(\chi_4^{l_2})^-,\omega_2}\chi_3^{l_2-1}f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_2^{1-l_1}}$$

$$+ \frac{\Im^{\sigma_2}_{\rho_2,\eta_2,(\chi_3^{l_2})^+,\omega_2} \chi_4^{l_2-1} f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_2^{1-l_1}}\Bigg) + \frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}$$
$$\times \Bigg(\frac{\Im^{\sigma_1}_{\rho_1,\eta_1,(\chi_2^{l_1})^-,\omega_1} \chi_1^{l_1-1} f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_3^{1-l_2}} + \frac{\Im^{\sigma_1}_{\rho_1,\eta_1,(\chi_2^{l_1})^-,\omega_1} \chi_1^{l_1-1} f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_4^{1-l_2}}$$
$$+ \frac{\Im^{\sigma_1}_{\rho_1,\eta_1,(\chi_1^{l_1})^+,\omega_1} \chi_2^{l_1-1} f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_3^{l_2-1}} + \frac{\Im^{\sigma_1}_{\rho_1,\eta_1,(\chi_1^{l_1})^+,\omega_1} \chi_2^{l_1-1} f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_4^{1-l_2}}\Bigg), \tag{3.17}$$

and $f_g(x,y) = f(g_1(x), g_2(y))$ *with* $g_1(x) = x^{\frac{1}{l_1}}$ *and* $g_2(y) = y^{\frac{1}{l_2}}$.

Proof Set

$$\hbar := \hbar_1 - \hbar_2 - \hbar_3 + \hbar_4, \tag{3.18}$$

where

$$\hbar_1 := \int_0^1\int_0^1 \xi_1^{\eta_1}\xi_2^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]$$
$$\times \frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}}, \big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_1\,d\xi_2;$$
$$\hbar_2 := \int_0^1\int_0^1 \xi_1^{\eta_1}\xi_2^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]$$
$$\times \frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}}, \big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_1\,d\xi_2;$$
$$\hbar_3 := \int_0^1\int_0^1 \xi_1^{\eta_1}\xi_2^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]$$
$$\times \frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}}, \big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_1\,d\xi_2;$$
$$\hbar_4 := \int_0^1\int_0^1 \xi_1^{\eta_1}\xi_2^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]$$
$$\times \frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}}, \big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_1\,d\xi_2.$$

Integrating by parts $\hbar_1$, we have

$$\hbar_1 = \int_0^1 \xi_2^{\eta_2}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]\bigg(\int_0^1 \xi_1^{\eta_1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}\big[\omega_1\big(\chi_2^{l_1}-\chi_1^{l_1}\big)^{\rho_1}\xi_1^{\rho_1}\big]$$
$$\times \frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}}, \big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_1\bigg)\,d\xi_2$$
$$= \frac{l_1\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_1^{l_1}-\chi_2^{l_1})\chi_1^{1-l_1}}\int_0^1 \xi_2^{\eta_2}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}\big[\omega_2\big(\chi_4^{l_2}-\chi_3^{l_2}\big)^{\rho_2}\xi_2^{\rho_2}\big]$$
$$\times \frac{\partial f}{\partial\xi_2}\big(\chi_1, \big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\,d\xi_2$$

$$
\begin{aligned}
&-\int_0^1 \frac{l_1\xi_1^{\eta_1-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}\xi_1^{\rho_1}]}{(\chi_1^{l_1}-\chi_2^{l_1})(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1})^{\frac{1}{l_1}-1}}\biggl(\int_0^1 \xi_2^{\eta_2}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}\bigl[\omega_2\bigl(\chi_4^{l_2}-\chi_3^{l_2}\bigr)^{\rho_2}\xi_2^{\rho_2}\bigr] \\
&\times \frac{\partial f}{\partial \xi_2}\bigl(\bigl(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\bigr)^{\frac{1}{l_1}},\bigl(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\bigr)^{\frac{1}{l_2}}\bigr)\,d\xi_2\biggr)\,d\xi_1 \\
&= \frac{l_1\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})}\frac{l_2\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_4^{l_2}-\chi_3^{l_2})}\frac{f(\chi_1,\chi_3)}{\chi_1^{1-l_1}\chi_3^{1-l_2}} \\
&\quad-\frac{l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})\chi_1^{1-l_1}}\int_0^1 \xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\bigl[\omega_2\bigl(\chi_4^{l_2}-\chi_3^{l_2}\bigr)^{\rho_2}\xi_2^{\rho_2}\bigr] \\
&\quad\times \bigl(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\bigr)^{1-\frac{1}{l_2}}f\bigl(\chi_1,\bigl(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\bigr)^{\frac{1}{l_2}}\bigr)\,d\xi_2 \\
&\quad-\frac{l_1l_2\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})\chi_3^{1-l_2}}\int_0^1 \xi_1^{\eta_1-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\bigl[\omega_1\bigl(\chi_2^{l_1}-\chi_1^{l_1}\bigr)^{\rho_1}\xi_1^{\rho_1}\bigr] \\
&\quad\times \bigl(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\bigr)^{1-\frac{1}{l_1}}f\bigl(\bigl(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\bigr)^{\frac{1}{l_1}},\chi_3\bigr)\,d\xi_1 \\
&\quad+\frac{l_1l_2}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}\int_0^1\int_0^1 \xi_1^{\eta_1-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\bigl[\omega_1\bigl(\chi_2^{l_1}-\chi_1^{l_1}\bigr)^{\rho_1}\xi_1^{\rho_1}\bigr] \\
&\quad\times \xi_2^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\bigl[\omega_2\bigl(\chi_4^{l_2}-\chi_3^{l_2}\bigr)^{\rho_2}\xi_2^{\rho_2}\bigr] \\
&\quad\times \bigl(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\bigr)^{1-\frac{1}{l_1}}\bigl(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\bigr)^{1-\frac{1}{l_2}} \\
&\quad\times f\bigl(\bigl(\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\bigr)^{\frac{1}{l_1}},\bigl(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\bigr)^{\frac{1}{l_2}}\bigr)\,d\xi_2\,d\xi_1.
\end{aligned}
$$

By the change of variables, we get

$$
\begin{aligned}
\hbar_1 &= \frac{l_1\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})}\frac{l_2\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_4^{l_2}-\chi_3^{l_2})}\frac{f(\chi_1,\chi_3)}{\chi_1^{1-l_1}\chi_3^{1-l_2}} \\
&\quad-\frac{l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}\chi_1^{1-l_1}} \\
&\quad\times\int_{\chi_3^{l_2}}^{\chi_4^{l_2}}\bigl(\chi_4^{l_2}-y\bigr)^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\bigl[\omega_2\bigl(\chi_4^{l_2}-y\bigr)^{\rho_2}\bigr]y^{1-\frac{1}{l_2}}f_g\bigl(\chi_1^{l_1},y\bigr)\,dy \\
&\quad-\frac{l_1l_2\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})\chi_3^{1-l_2}} \\
&\quad\times\int_{\chi_1^{l_1}}^{\chi_2^{l_1}}\bigl(\chi_2^{l_1}-x\bigr)^{\eta_1-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\bigl[\omega_1\bigl(\chi_2^{l_1}-x\bigr)^{\rho_1}\bigr]x^{1-\frac{1}{l_1}}f_g\bigl(x,\chi_3^{l_2}\bigr)\,dx \\
&\quad+\frac{l_1l_2}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}}\int_{\chi_1^{l_1}}^{\chi_2^{l_1}}\int_{\chi_3^{l_2}}^{\chi_4^{l_2}}\bigl(\chi_2^{l_1}-x\bigr)^{\eta_1-1}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1}\bigl[\omega_1\bigl(\chi_2^{l_1}-x\bigr)^{\rho_1}\bigr] \\
&\quad\times\bigl(\chi_4^{l_2}-y\bigr)^{\eta_2-1}\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2}\bigl[\omega_2\bigl(\chi_4^{l_2}-y\bigr)^{\rho_2}\bigr]x^{1-\frac{1}{l_1}}y^{1-\frac{1}{l_2}}f_g(x,y)\,dy\,dx.
\end{aligned}
\tag{3.19}
$$

Making use of Definition 2.3 in (3.19), we get

$$
\hbar_1 = \frac{l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}\frac{f(\chi_1,\chi_3)}{\chi_1^{1-l_1}\chi_3^{1-l_2}}
$$

$$
\begin{aligned}
&- \frac{l_1 l_2 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}} \chi_1^{l_1-1} \mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_3^{l_2})^+,\omega_2} \chi_4^{l_2-1} f_g(\chi_1^{l_1},\chi_4^{l_2}) \\
&- \frac{l_1 l_2 \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})} \chi_3^{l_2-1} \mathfrak{I}^{\sigma_1}_{\rho_1,\eta_1,(\chi_1^{l_1})^+,\omega_1} \chi_2^{l_1-1} f_g(\chi_2^{l_1},\chi_3^{l_2}) \\
&+ \frac{l_1 l_2}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}} \mathfrak{I}^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega} \chi_2^{l_1-1}\chi_4^{l_2-1} f_g(\chi_2^{l_1},\chi_4^{l_2}).
\end{aligned} \tag{3.20}
$$

Likewise, we can deduce

$$
\begin{aligned}
\hbar_2 = &- \frac{l_1 l_2 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}] \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})} \frac{f(\chi_1,\chi_4)}{\chi_1^{1-l_1}\chi_4^{1-l_2}} \\
&+ \frac{l_1 l_2 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}} \chi_1^{l_1-1} \mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_4^{l_2})^-,\omega_2} \chi_3^{l_2-1} f_g(\chi_1^{l_1},\chi_3^{l_2}) \\
&+ \frac{l_1 l_2 \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})} \chi_4^{l_2-1} \mathfrak{I}^{\sigma_1}_{\rho_1,\eta_1,a(\chi_1^{l_1})^+,\omega_1} \chi_2^{l_1-1} f_g(\chi_2^{l_1},\chi_4^{l_2}) \\
&- \frac{l_1 l_2}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}} \mathfrak{I}^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega} \chi_2^{l_1-1}\chi_3^{l_2-1} f_g(\chi_2^{l_1},\chi_3^{l_2}),
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
\hbar_3 = &- \frac{l_1 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})} \frac{l_2 \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_4^{l_2}-\chi_3^{l_2})} \frac{f(\chi_2,\chi_3)}{\chi_2^{1-l_1}\chi_3^{1-l_2}} \\
&+ \frac{l_1 l_2 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}} \chi_2^{l_1-1} \mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_3^{l_2})^+,\omega_2} \chi_4^{l_2-1} f_g(\chi_2^{l_1},\chi_4^{l_2}) \\
&+ \frac{l_1 l_2 \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})} \chi_3^{l_2-1} \mathfrak{I}^{\sigma_1}_{\rho_1,\eta_1,(\chi_2^{l_1})^-,\omega_1} \chi_1^{l_1-1} f_g(\chi_1^{l_1},\chi_3^{l_2}) \\
&- \frac{l_1 l_2}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}} \mathfrak{I}^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega} \chi_1^{l_1-1}\chi_4^{l_2-1} f_g(\chi_1^{l_1},\chi_4^{l_2}),
\end{aligned} \tag{3.22}
$$

and finally

$$
\begin{aligned}
\hbar_4 = &\frac{l_1 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})} \frac{l_2 \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_4^{l_2}-\chi_3^{l_2})} \frac{f(\chi_2,\chi_4)}{\chi_2^{1-l_1}\chi_4^{1-l_2}} \\
&- \frac{l_1 l_2 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]}{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}} \chi_2^{l_1-1} \mathfrak{I}^{\sigma_2}_{\rho_2,\eta_2,(\chi_4^{l_2})^-,\omega_2} \chi_3^{l_2-1} f_g(\chi_2^{l_1},\chi_3^{l_2}) \\
&- \frac{l_1 l_2 \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})} \chi_4^{l_2-1} \mathfrak{I}^{\sigma_1}_{\rho_1,\eta_1,(\chi_2^{l_1})^-,\omega_1} \chi_1^{l_1-1} f_g(\chi_1^{l_1},\chi_4^{l_2}) \\
&+ \frac{l_1 l_2}{(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1+1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2+1}} \mathfrak{I}^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega} \chi_1^{l_1-1}\chi_3^{l_2-1} f_g(\chi_1^{l_1},\chi_3^{l_2}).
\end{aligned} \tag{3.23}
$$

Making use of (3.20)–(3.23) in (3.18) and then multiplying by

$$
\frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4 l_1 l_2 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}] \mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]},
$$

we arrive at the desired result. Thus we get the proof of Lemma 3.1. □

Theorem 3.3 *Let $f:\Delta\to\mathbb{R}$ be a partial differentiable function on Δ. If $|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}|$ is an (l_1,h_1)-(l_2,h_2)-convex function on coordinates on Δ, then we have*

$$\begin{aligned}&\bigg|\frac{f(\chi_1,\chi_3)}{4\chi_1^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_1,\chi_4)}{4\chi_1^{1-l_1}\chi_4^{1-l_2}}+\frac{f(\chi_2,\chi_3)}{4\chi_2^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_2,\chi_4)}{4\chi_2^{1-l_1}\chi_4^{1-l_2}}-A\\&\quad+\frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}\\&\quad\times\bigg(\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}\bigg(\frac{f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_1^{1-l_1}\chi_4^{1-l_2}}\bigg)+\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}\bigg(\frac{f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_1^{1-l_1}\chi_3^{1-l_2}}\bigg)\\&\quad+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}\bigg(\frac{f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_2^{1-l_1}\chi_3^{1-l_2}}\bigg)+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}\bigg(\frac{f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_2^{1-l_1}\chi_4^{1-l_2}}\bigg)\bigg)\bigg|\\&\le\frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}\\&\quad\times\bigg(\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\big(h_1(1-\xi_1)+h_1(\xi_1)\big)\big(h_2(1-\xi_2)+h_2(\xi_2)\big)\,d\xi_1\,d\xi_2\bigg)\\&\quad\times\bigg(\bigg|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\bigg|+\bigg|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\bigg|+\bigg|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\bigg|+\bigg|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\bigg|\bigg),\end{aligned}$$

where $\mathcal{B}(\xi_1,\xi_2)$ and A are as in (3.16) and (3.17), respectively, and $f_g(x,y)=f(g_1(x),g_2(y))$ with $g_1(x)=x^{\frac{1}{l_1}}$ and $g_2(y)=y^{\frac{1}{l_2}}$.

Proof By Lemma 3.1 and the properties of modulus, we have

$$\begin{aligned}&\bigg|\frac{f(\chi_1,\chi_3)}{4\chi_1^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_1,\chi_4)}{4\chi_1^{1-l_1}\chi_4^{1-l_2}}+\frac{f(\chi_2,\chi_3)}{4\chi_2^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_2,\chi_4)}{4\chi_2^{1-l_1}\chi_4^{1-l_2}}-A\\&\quad+\frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}\\&\quad\times\bigg(\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}\bigg(\frac{f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_1^{1-l_1}\chi_4^{1-l_2}}\bigg)+\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}\bigg(\frac{f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_1^{1-l_1}\chi_3^{1-l_2}}\bigg)\\&\quad+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}\bigg(\frac{f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_2^{1-l_1}\chi_3^{1-l_2}}\bigg)+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}\bigg(\frac{f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_2^{1-l_1}\chi_4^{1-l_2}}\bigg)\bigg)\bigg|\\&\le\frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}\\&\quad\times\bigg(\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\bigg|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big((\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1})^{\frac{1}{l_1}},\\&\quad\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\bigg|\,d\xi_1\,d\xi_2\\&\quad+\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\bigg|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big((\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1})^{\frac{1}{l_1}},\\&\quad\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\bigg|\,d\xi_1\,d\xi_2\end{aligned}$$

$$+\int_0^1\int_0^1 \mathcal{B}(\xi_1,\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\Big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\right.$$
$$\left.\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\Big)\right|d\xi_1\,d\xi_2$$
$$+\int_0^1\int_0^1 \mathcal{B}(\xi_1,\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\Big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\right.$$
$$\left.\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\Big)\right|d\xi_1\,d\xi_2\Bigg).$$

Using the (l_1,h_1)-(l_2,h_2)-convexity of $|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}|$ on coordinates, we obtain

$$\Bigg|\frac{f(\chi_1,\chi_3)}{4\chi_1^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_1,\chi_4)}{4\chi_1^{1-l_1}\chi_4^{1-l_2}}+\frac{f(\chi_2,\chi_3)}{4\chi_2^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_2,\chi_4)}{4\chi_2^{1-l_1}\chi_4^{1-l_2}}-A$$
$$+\frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}$$
$$\times\Bigg(\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}\bigg(\frac{f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_1^{1-l_1}\chi_4^{1-l_2}}\bigg)+\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}\bigg(\frac{f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_1^{1-l_1}\chi_3^{1-l_2}}\bigg)$$
$$+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}\bigg(\frac{f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_2^{1-l_1}\chi_3^{1-l_2}}\bigg)+\Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}\bigg(\frac{f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_2^{1-l_1}\chi_4^{1-l_2}}\bigg)\Bigg)\Bigg|$$
$$\le\frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}$$
$$\times\Bigg(\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\bigg(h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|$$
$$+h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|+h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|$$
$$+h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|\bigg)d\xi_1\,d\xi_2$$
$$+\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\bigg(h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|$$
$$+h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|+h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|$$
$$+h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|\bigg)d\xi_1\,d\xi_2$$
$$+\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\bigg(h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|$$
$$+h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|+h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|$$
$$+h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|\bigg)d\xi_1\,d\xi_2$$
$$+\int_0^1\int_0^1\mathcal{B}(\xi_1,\xi_2)\bigg(h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|$$

$$
\begin{aligned}
&+ h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right| + h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right| \\
&+ h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|\Bigg) d\xi_1\, d\xi_2\Bigg) \\
= &\frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]} \\
&\times \left(\int_0^1\int_0^1 \mathcal{B}(\xi_1,\xi_2)\big(h_1(1-\xi_1)+h_1(\xi_1)\big)\big(\big(h_2(1-\xi_2)+h_2(\xi_2)\big)\big)\, d\xi_1\, d\xi_2\right) \\
&\times \left(\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right| + \left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right| + \left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right| + \left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|\right).
\end{aligned}
$$

This completely ends the proof of Theorem 3.3. □

Corollary 3.1 *Theorem 3.3 with $|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}| \le K$ gives the new inequality*:

$$
\begin{aligned}
&\Bigg|\frac{f(\chi_1,\chi_3)}{4\chi_1^{1-l_1}\chi_3^{1-l_2}} + \frac{f(\chi_1,\chi_4)}{4\chi_1^{1-l_1}\chi_4^{1-l_2}} + \frac{f(\chi_2,\chi_3)}{4\chi_2^{1-l_1}\chi_3^{1-l_2}} + \frac{f(\chi_2,\chi_4)}{4\chi_2^{1-l_1}\chi_4^{1-l_2}} - A \\
&+ \frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]} \\
&\times \left(\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_1^{1-l_1}\chi_4^{1-l_2}}\right) + \Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_1^{1-l_1}\chi_3^{1-l_2}}\right)\right. \\
&\left.+ \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_2^{1-l_1}\chi_3^{1-l_2}}\right) + \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_2^{1-l_1}\chi_4^{1-l_2}}\right)\right)\Bigg| \\
\le &\frac{K(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]} \\
&\times \left(\int_0^1\int_0^1 \mathcal{B}(\xi_1,\xi_2)\big(h_1(1-\xi_1)+h_1(\xi_1)\big)\big(h_2(1-\xi_2)+h_2(\xi_2)\big)\, d\xi_1\, d\xi_2\right).
\end{aligned}
$$

Remark 3.6 Theorem 3.3 with $l_1 = l_2 = 1$ and $h_1(\xi_1) = h_2(\xi_1) = \xi_1$ becomes Theorem 3.2 in [39].

Remark 3.7 Theorem 3.3 with $l_1 = l_2 = 1$, $\eta_1 = \eta_2 = \alpha$, $\sigma_1(0) = \sigma_2(0) = 1$, $\omega_1 = \omega_2 = 0$, and $h_1(\xi_1) = h_2(\xi_1) = \xi_1$ becomes Theorem 3 in [33].

Theorem 3.4 *Let $f : \Delta \to \mathbb{R}$ be a partial differentiable function on Δ. If $|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}|^q$ is an (l_1,h_1)-(l_2,h_2)-convex function on coordinates on Δ, then for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$
\begin{aligned}
&\Bigg|\frac{f(\chi_1,\chi_3)}{4\chi_1^{1-l_1}\chi_3^{1-l_2}} + \frac{f(\chi_1,\chi_4)}{4\chi_1^{1-l_1}\chi_4^{1-l_2}} + \frac{f(\chi_2,\chi_3)}{4\chi_2^{1-l_1}\chi_3^{1-l_2}} + \frac{f(\chi_2,\chi_4)}{4\chi_2^{1-l_1}\chi_4^{1-l_2}} - A \\
&+ \frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]} \\
&\times \left(\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_1^{1-l_1}\chi_4^{1-l_2}}\right) + \Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_1^{1-l_1}\chi_3^{1-l_2}}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_2^{1-l_1}\chi_3^{1-l_2}}\right) + \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_2^{1-l_1}\chi_4^{1-l_2}}\right)\Bigg)\Bigg| \\
\le{}& \frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]} \\
&\times \left(\int_0^1\int_0^1 \left[\mathcal{B}(\xi_1,\xi_2)\right]^p d\xi_1\,d\xi_2\right)^{\frac{1}{p}} \Bigg[\Bigg\{\int_0^1\int_0^1 \Bigg(h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|^q \\
&+ h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|^q + h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|^q \\
&+ h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|^q\Bigg)\,d\xi_1\,d\xi_2\Bigg\}^{\frac{1}{q}} \\
&+ \Bigg\{\int_0^1\int_0^1 \Bigg(h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|^q + h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|^q \\
&+ h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|^q \\
&+ h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|^q\Bigg)\,d\xi_1\,d\xi_2\Bigg\}^{\frac{1}{q}} \\
&+ \Bigg\{\int_0^1\int_0^1 \Bigg(h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|^q \\
&+ h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|^q + h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|^q \\
&+ h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|^q\Bigg)\,d\xi_1\,d\xi_2\Bigg\}^{\frac{1}{q}} \\
&+ \Bigg\{\int_0^1\int_0^1 \Bigg(h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|^q \\
&+ h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|^q + h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|^q \\
&+ h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|^q\Bigg)\,d\xi_1\,d\xi_2\Bigg\}^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

where $\mathcal{B}(\xi_1,\xi_2)$ *and* A *are defined as in* (3.16) *and* (3.17), *respectively, and* $f_g(x,y) = f(g_1(x),g_2(y))$ *with* $g_1(x) = x^{\frac{1}{l_1}}$ *and* $g_2(y) = y^{\frac{1}{l_2}}$.

Proof Making use of Lemma 3.1 and the properties of modulus, we get

$$
\begin{aligned}
&\Bigg|\frac{f(\chi_1,\chi_3)}{4\chi_1^{1-l_1}\chi_3^{1-l_2}} + \frac{f(\chi_1,\chi_4)}{4\chi_1^{1-l_1}\chi_4^{1-l_2}} + \frac{f(\chi_2,\chi_3)}{4\chi_2^{1-l_1}\chi_3^{1-l_2}} + \frac{f(\chi_2,\chi_4)}{4\chi_2^{1-l_1}\chi_4^{1-l_2}} - A \\
&+ \frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]} \\
&\times \left(\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_1^{1-l_1}\chi_4^{1-l_2}}\right) + \Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_1^{1-l_1}\chi_3^{1-l_2}}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_2^{1-l_1}\chi_3^{1-l_2}}\right) + \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_2^{1-l_1}\chi_4^{1-l_2}}\right)\Bigg)\Bigg| \\
\le{}& \frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4l_1 l_2 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]} \\
&\times \left(\int_0^1\int_0^1 \mathcal{B}(\xi_1,\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big((\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}},\right.\right. \\
&\left.\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\right| d\xi_1\,d\xi_2 \\
&+ \int_0^1\int_0^1 \mathcal{B}(\xi_1,\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big((\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}},\right. \\
&\left.\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\right| d\xi_1\,d\xi_2 \\
&+ \int_0^1\int_0^1 \mathcal{B}(\xi_1,\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\right. \\
&\left.\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\right| d\xi_1\,d\xi_2 \\
&+ \int_0^1\int_0^1 \mathcal{B}(\xi_1,\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\right. \\
&\left.\left.\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\right| d\xi_1\,d\xi_2\right).
\end{aligned}
$$

Making use of the (l_1,h_1)-(l_2,h_2)-convexity of $|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}|^q$ on coordinates and Hölder's inequality, we obtain

$$
\begin{aligned}
&\Bigg|\frac{f(\chi_1,\chi_3)}{4\chi_1^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_1,\chi_4)}{4\chi_1^{1-l_1}\chi_4^{1-l_2}}+\frac{f(\chi_2,\chi_3)}{4\chi_2^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_2,\chi_4)}{4\chi_2^{1-l_1}\chi_4^{1-l_2}}-A \\
&+ \frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]} \\
&\times\Bigg(\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_1^{1-l_1}\chi_4^{1-l_2}}\right)+\Im^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_1^{1-l_1}\chi_3^{1-l_2}}\right) \\
&+ \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_2^{1-l_1}\chi_3^{1-l_2}}\right) + \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_2^{1-l_1}\chi_4^{1-l_2}}\right)\Bigg)\Bigg| \\
\le{}& \frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4l_1 l_2 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]} \\
&\times\left(\int_0^1\int_0^1\big[\mathcal{B}(\xi_1,\xi_2)\big]^p\,d\xi_1\,d\xi_2\right)^{\frac{1}{p}} \\
&\times\left(\left\{\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big((\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\right|^q d\xi_1\,d\xi_2\right\}^{\frac{1}{q}}\right. \\
&+\left\{\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big((\xi_1\chi_1^{l_1}+(1-\xi_1)\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\right|^q d\xi_1\,d\xi_2\right\}^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&+\left\{\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big(\xi_2\chi_3^{l_2}+(1-\xi_2)\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\right|^q d\xi_1\,d\xi_2\right\}^{\frac{1}{q}}\\
&+\left\{\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}\big(\big((1-\xi_1)\chi_1^{l_1}+\xi_1\chi_2^{l_1}\big)^{\frac{1}{l_1}},\big((1-\xi_2)\chi_3^{l_2}+\xi_2\chi_4^{l_2}\big)^{\frac{1}{l_2}}\big)\right|^q d\xi_1\,d\xi_2\right\}^{\frac{1}{q}}\Bigg)\\
&\le\frac{(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{4l_1l_2\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}\\
&\times\left(\int_0^1\int_0^1\big[\mathcal{B}(\xi_1,\xi_2)\big]^p d\xi_1\,d\xi_2\right)^{\frac{1}{p}}\Bigg[\Bigg\{\int_0^1\int_0^1\Bigg(h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|^q\\
&+h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|^q+h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|^q\\
&+h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|^q\Bigg)d\xi_1\,d\xi_2\Bigg\}^{\frac{1}{q}}\\
&+\Bigg\{\int_0^1\int_0^1\Bigg(h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|^q+h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|^q\\
&+h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|^q\\
&+h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|^q\Bigg)d\xi_1\,d\xi_2\Bigg\}^{\frac{1}{q}}\\
&+\Bigg\{\int_0^1\int_0^1\Bigg(h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|^q\\
&+h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|^q\\
&+h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|^q+h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|^q\Bigg)d\xi_1\,d\xi_2\Bigg\}^{\frac{1}{q}}\\
&+\Bigg\{\int_0^1\int_0^1\Bigg(h_1(1-\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_3)\right|^q\\
&+h_1(1-\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_1,\chi_4)\right|^q+h_1(\xi_1)h_2(1-\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_3)\right|^q\\
&+h_1(\xi_1)h_2(\xi_2)\left|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}(\chi_2,\chi_4)\right|^q\Bigg)d\xi_1\,d\xi_2\Bigg\}^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

This rearranges to the proof of Theorem 3.4 □

Corollary 3.2 *Theorem 3.4 with* $|\frac{\partial^2 f}{\partial\xi_1\partial\xi_2}|^q\le K$, *give the new inequality*:

$$
\begin{aligned}
&\Bigg|\frac{f(\chi_1,\chi_3)}{4\chi_1^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_1,\chi_4)}{4\chi_1^{1-l_1}\chi_4^{1-l_2}}+\frac{f(\chi_2,\chi_3)}{4\chi_2^{1-l_1}\chi_3^{1-l_2}}+\frac{f(\chi_2,\chi_4)}{4\chi_2^{1-l_1}\chi_4^{1-l_2}}-A\\
&+\frac{1}{4(\chi_2^{l_1}-\chi_1^{l_1})^{\eta_1}(\chi_4^{l_2}-\chi_3^{l_2})^{\eta_2}\mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}\\
&\times\left(\mathfrak{I}^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_4^{l_2})}{\chi_1^{1-l_1}\chi_4^{1-l_2}}\right)+\mathfrak{I}^{\sigma}_{\rho,\eta,(\chi_2^{l_1})^-,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_1^{l_1},\chi_3^{l_2})}{\chi_1^{1-l_1}\chi_3^{1-l_2}}\right)\right.
\end{aligned}
$$

$$+ \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_4^{l_2})^-,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_3^{l_2})}{\chi_2^{1-l_1}\chi_3^{1-l_2}}\right) + \Im^{\sigma}_{\rho,\eta,(\chi_1^{l_1})^+,(\chi_3^{l_2})^+,\omega}\left(\frac{f_g(\chi_2^{l_1},\chi_4^{l_2})}{\chi_2^{1-l_1}\chi_4^{1-l_2}}\right)\bigg)\bigg|$$

$$\leq \frac{K(\chi_2^{l_1}-\chi_1^{l_1})(\chi_4^{l_2}-\chi_3^{l_2})}{l_1 l_2 \mathfrak{F}^{\sigma_1}_{\rho_1,\eta_1+1}[\omega_1(\chi_2^{l_1}-\chi_1^{l_1})^{\rho_1}]\mathfrak{F}^{\sigma_2}_{\rho_2,\eta_2+1}[\omega_2(\chi_4^{l_2}-\chi_3^{l_2})^{\rho_2}]}$$

$$\times \left(\int_0^1\int_0^1 \big[\mathcal{B}(\xi_1,\xi_2)\big]^p\,d\xi_1\,d\xi_2\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1\int_0^1 \big(h_1(1-\xi_1)+h_1(\xi_1)\big)\big(h_2(1-\xi_2)+h_2(\xi_2)\big)\,d\xi_1\,d\xi_2\right)^{\frac{1}{q}}.$$

4 Conclusion

Since convexity has wide applications in many mathematical areas, the general class of (l_1,h_1)-(l_2,h_2)-convex functions on coordinates can be applied to obtain several results in convex analysis, special functions, related optimization theory, mathematical inequalities and may stimulate further research in different areas of pure and applied sciences.

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