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GENERALIZED FRACTIONAL DERIVATIVES AND LAPLACE TRANSFORM

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Abstract. In this article, we study generalized fractional derivatives that contain kernels depending on a function on the space of absolute continuous functions. We generalize the Laplace transform in order to be applicable for the generalized fractional integrals and derivatives and apply this transform to solve some ordinary differential equations in the frame of the fractional derivatives under discussion.

1. **Introduction.** The fractional calculus studies the integration and differentiation of real or complex orders and thus it is a generalization of the usual calculus. Despite of the fact that this calculus is old, it has gained popularity in the few decades because of the interesting results obtained when this calculus was applied to model some real world problems [25, 19, 24, 22, 13, 21]. What makes fractional calculus special is the fact that there are various fractional operators. So that, any scientist working on modelling real world phenomena can choose the operator that fits the model the best.

The classical method to obtain fractional operators depended on iterating an integral to find the n^{th} order integral and then exchange n by any number. After then, the corresponding derivatives are defined (see for example [20, 17, 18, 14, 12, 15, 16]). Applications on such derivatives can be seen in [10, 11, 1, 2, 5, 6] and the references therein. Purposing a better models for real world problems, scientists obtained new fractional operators with nonlocal and nonsingular kernels using the limiting process with the asistance of the Dirac delta function. For such operators, we refer to [9, 8, 29, 3, 4]. Other types of new fractional derivatives can be found in [26, 27, 28, 23]

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In this paper, we study first some properties of the generalized fractional operators and then consider a corresponding Laplace transform which will be called generalized Laplace transform through this work. Before we start, let us recall some definitions from the fractional calculus [25, 19]. Let $g(t)$ be a strictly increasing function with continuous derivative g' on the interval (a,b). The left Riemann-Liouville fractional integral of f with respect to the function g of order $\alpha, \Re(\alpha)>0$ is defined by

$$({}_aI_g^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t \big(g(t)-g(u)\big)^{\alpha-1}f(u)g'(u)du. \tag{1}$$

It is obvious that when $g(t)=t$, (1) is the classical Riemann-Liouville fractional integral and when $g(t)=\ln t$, (1) is the Hadamarad fractional integral [25, 19, 24]. Hence (1) can be treated as the generalized Riemann-Liouville fractional integral. The left Riemann-Liouville fractional derivative of a function f of order $\alpha, \Re(\alpha)\geq 0$ with respect to g is given as

$$\begin{aligned}({}_aD_g^\alpha f)(t) &= \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^n({}_aI_g^{n-\alpha}f)(t)\\ &= \frac{\Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^n}{\Gamma(n-\alpha)}\int_a^t \big(g(t)-g(u)\big)^{n-\alpha-1}f(u)g'(u)du, \end{aligned}\tag{2}$$

where $n=[\Re(\alpha)]+1, g^{(i)}\neq 0, i=2,\dots,n$. It can be easily noticed that when $g(t)=t$, (2) is the classical Riemann-Liouville fractional derivative and when $g(t)=\ln t$, (2) is the Hadamarad fractional integral [25, 19, 24]. In [7] and [26], Caputo and Hilfer fractional derivatives of functions with respect to another functions were defined on the set of continuous functions on some interval $[a,b]$. Below, we will present the definitions of these derivatives on the set of absolute continuous functions. But, before that we need the following definitions.

Definition 1.1. Let $g\in C^n[a,b]$ such that $g'(t)>0$ on $[a,b]$. Then

$$AC_g^n[a,b]=\Big\{f:[a,b]\to\mathbb{C} \text{ and } f^{[n-1]}\in AC[a,b],\ f^{[n-1]}=\Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^{n-1}f\Big\}.$$

Definition 1.2.

$$C_{\epsilon,g}[a,b]=\Big\{f:(a,b]\to\mathbb{R} \text{ such that } \Big(g(t)-g(a)\Big)^\epsilon f(t)\in C[a,b]\Big\},$$

where $C_{0,g}[a,b]=C[a,b]$.

Definition 1.3.

$$C_{\epsilon,g}^n[a,b]=\Big\{f:(a,b]\to\mathbb{R} \text{ such that } f^{[n-1]}\in C[a,b] \text{ and } f^{[n]}\in C_{\epsilon,g}[a,b]\Big\},$$

where $C_{0,g}^n[a,b]=C^n[a,b]$.

2. Auxiliary results.

Lemma 2.1. [25] *Let $\Re(\alpha)>0$ and $\Re(\beta)>0$, then*

$${}_aI_g^\alpha\Big(g(x)-g(a)\Big)^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\Big(g(t)-g(a)\Big)^{\beta+\alpha-1}. \tag{3}$$

Lemma 2.2. [25] *Let $\Re(\alpha)\geq 0$ and $\Re(\beta)>0$, then*

$${}_aD_g^\alpha\Big(g(x)-g(a)\Big)^{\beta-1}(t)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\Big(g(t)-g(a)\Big)^{\beta-\alpha-1}. \tag{4}$$

Lemma 2.3. *Let $g \in C^n[a,b]$ such that $g'(t) > 0$ on $[a,b]$. Then $f \in AC_g^n$ if and only if it can be written as*

$$f(t) = \frac{1}{(n-1)!}\int_a^t \Big(g(t)-g(s)\Big)^{n-1} f^{[n]}(s)g'(s)ds + \sum_{k=0}^{n-1}\frac{f^{[k]}(a)}{k!}\Big(g(t)-g(a)\Big)^k. \quad (5)$$

Proof. To prove necessity, let $f \in AC_g^n[a,b]$. Then, by Definition 1.1, $f^{[n-1]} \in AC[a,b]$. Thus, there exists a function $\psi \in L_1[a,b]$, such that

$$f^{[n-1]}(t) = C_{n-1} + \int_a^t \psi(s)ds. \quad (6)$$

Upon multiplying both sides of $g'(t)$ and then integration, equation (6) becomes

$$f^{[n-2]}(t) = C_{n-1}\Big(g(t)-g(a)\Big) + C_{n-2} + \int_a^t g'(\tau)\int_a^\tau \psi(s)dsd\tau. \quad (7)$$

Changing the order of integration in the double integral on the right hand side of (7) and then integrating once gives

$$f^{[n-2]}(t) = C_{n-1}\Big(g(t)-g(a)\Big) + C_{n-2} + \int_a^t \Big(g(t)-g(s)\Big)\psi(s)ds. \quad (8)$$

Performing the same procedure we reach at

$$f(t) = \sum_{k=0}^{n-1}\frac{c_k\Big(g(t)-g(a)\Big)^k}{k!} + \int_a^t \frac{\Big(g(t)-g(s)\Big)^{n-1}}{(n-1)!}\psi(s)ds. \quad (9)$$

It is obvious that $c_k = f^{[k]}(a^+), k = 0,\dots,n-1$. If one differentiates (6) and divides by $g'(t)$, one finds that $\psi(s) = g'(s)f^{[n]}(s)$. To prove the sufficiency, it enough to apply the operator $\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^{n-1}$ to the left and right hand sides of (9). □

The following theorem gives the fractional derivative of a function with respect to another function in an absolutely continuous function type space.

Theorem 2.4. *Let $\Re(\alpha) > 0, n = [\Re(\alpha)]+1$ and $f \in AC_g^n[a,b]$. Then the fractional derivative of f with respect to g exists almost everywhere and*

$$\begin{aligned}\left({}_aD_g^\alpha f\right)(t) &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \Big(g(t)-g(s)\Big)^{n-\alpha-1} f^{[n]}(s)g'(s)ds \\ &+ \sum_{k=0}^{n-1}\frac{f^{[k]}(a^+)}{\Gamma(k-\alpha+1)}\Big(g(t)-g(a)\Big)^{k-\alpha}. \quad (10)\end{aligned}$$

Proof. Performing ${}_aD_g^\alpha$ to both sides of (5) and using Lemma 2.2, one obtains

$$\begin{aligned}{}_aD_g^\alpha f(t) &= \frac{\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n}{(n-1)!\Gamma(n-\alpha)}\Big[\int_a^t\int_a^s \Big(g(t)-g(\tau)\Big)^{n-\alpha-1}\Big(g(\tau)-g(s)\Big)^{n-1} \\ &\times f^{[n]}(s)g'(s)g'(\tau)dsd\tau\Big] + \sum_{k=0}^{n-1}\frac{f^{[k]}(a^+)}{\Gamma(k-\alpha+1)}\Big(g(t)-g(a)\Big)^{k-\alpha}. \quad (11)\end{aligned}$$

If one reverses the order of integration in Equation (11), one gets

$$\begin{aligned} {}_aD_g^\alpha f(t) &= \frac{\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n}{(n-1)!\Gamma(n-\alpha)}\Big[\int_a^t\int_s^t \Big(g(t)-g(\tau)\Big)^{n-\alpha-1}\Big(g(\tau)-g(s)\Big)^{n-1} \\ &\times \quad f^{[n]}(s)g'(\tau)g'(s)d\tau ds\Big] + \sum_{k=0}^{n-1}\frac{f^{[k]}(a^+)}{\Gamma(k-\alpha+1)}\Big(g(t)-g(a)\Big)^{k-\alpha}. \end{aligned} \tag{12}$$

Setting $u = \dfrac{g(\tau)-g(s)}{g(t)-g(s)}$, Equation (12) becomes

$$\begin{aligned} {}_aD_g^\alpha f(t) &= \frac{\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n}{(n-1)!\Gamma(n-\alpha)}\Big[\int_a^t \Big(g(t)-g(s)\Big)^{2n-\alpha-1} f^{[n]}(s)g'(s)ds\Big] \\ &\times \quad \int_0^1 (1-u)^{n-\alpha-1}u^{n-1}du + \sum_{k=0}^{n-1}\frac{f^{[k]}(a^+)}{\Gamma(k-\alpha+1)}\Big(g(t)-g(a)\Big)^{k-\alpha}. \end{aligned} \tag{13}$$

Using the properties of the Beta and Gamma functions, one obtains

$$\begin{aligned} {}_aD_g^\alpha f(t) &= \frac{\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n}{(n-1)!\Gamma(2n-\alpha)}\Big[\int_a^t \Big(g(t)-g(s)\Big)^{2n-\alpha-1} f^{[n]}(s)g'(s)ds\Big] \\ &+ \quad \sum_{k=0}^{n-1}\frac{f^{[k]}(a^+)}{\Gamma(k-\alpha-1)}\Big(g(t)-g(a)\Big)^{k-\alpha}. \end{aligned} \tag{14}$$

The result is obtained by applying the operator $\left(\dfrac{1}{g'(t)}\dfrac{d}{dt}\right)^n$ to the integral. ☐

Remark 1. Equation (10) can be written as

$$\left({}_aD_g^\alpha f\right)(t) = \left({}_aI_g^{n-\alpha} f^{[n]}\right)(t) + \sum_{k=0}^{n-1}\frac{f^{[k]}(a^+)}{\Gamma(k-\alpha+1)}\Big(g(t)-g(a)\Big)^{k-\alpha} \tag{15}$$

and thus, one can define the Caputo fractional derivative of a function with respect to another function as

$$\begin{aligned} \left({}_a^CD_g^\alpha f\right)(t) &= \left({}_aD_g^\alpha f\right)(t) - \sum_{k=0}^{n-1}\frac{f^{[k]}(a^+)}{\Gamma(k-\alpha+1)}\Big(g(t)-g(a)\Big)^{k-\alpha} \\ &= {}_aD_g^\alpha\Big(f(s) - \sum_{k=0}^{n-1}\frac{f^{[k]}(a^+)}{k!}\Big(g(s)-g(a)\Big)^k\Big)(t). \end{aligned} \tag{16}$$

Theorem 2.5. *Let $\alpha > m, m \in \mathbb{N}$. Then,*

$$\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^m {}_aI_g^\alpha f(t) = {}_aI_g^{\alpha-m} f(t). \tag{17}$$

Proof.

$$\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^m {}_aI_g^\alpha f(t) = \frac{\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^m}{\Gamma(\alpha)}\int_a^t \Big(g(t)-g(s)\Big)^{\alpha-1} f(s)g'(s)ds.$$

The result is then obtained by applying the operator $\dfrac{1}{g'(t)}\dfrac{d}{dt}$ m-times to the integral on the right hand side. ☐

Corollary 1. *Let $\alpha > \beta, m-1 < \beta < m, m \in \mathbb{N}$. Then,*

$$ {}_aD_g^{\beta}\ {}_aI_g^{\alpha} f(t) = \ {}_aI_g^{\alpha-\beta} f(t). \tag{18}$$

In particular,

$$ {}_aD_g^{\alpha}\ {}_aI_g^{\alpha} f(t) = f(t). \tag{19}$$

Proof.

$$\begin{aligned} {}_aD_g^{\beta}\ {}_aI_g^{\alpha} f(t) &= \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^m\ {}_aI_g^{m-\beta}\ {}_aI_g^{\alpha} f(t) \\ &= \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^m\ {}_aI_g^{\alpha+m-\beta} f(t) = \ {}_aI_g^{\alpha-\beta} f(t), \end{aligned}$$

where the semigroup property of integrals in Lemma 2.26 in [19] is used. $\square$

Theorem 2.6. *Let $\Re(\alpha) > 0$, $n = -[-\Re(\alpha)], f \in L[a,b]$ and ${}_aI_g^{\alpha} f \in AC_g^n[a.b]$. Then*

$$\Big(\ {}_aI_g^{\alpha}\ {}_aD_g^{\alpha}\Big) f(t) = f(t) - \sum_{k=1}^{n} \frac{{}_aI_g^{k-\alpha} f(a^+)}{\Gamma(\alpha-k+1)}\Big(g(t)-g(a)\big)^{\alpha-k}. \tag{20}$$

Proof.

$$\begin{aligned} \Big(\ {}_aI_g^{\alpha}\ {}_aD_g^{\alpha}\Big) f(t) &= \frac{1}{\Gamma(\alpha)}\int_a^t \Big(g(t)-g(s)\Big)^{\alpha-1}\Big[\Big(\frac{1}{g'(s)}\frac{d}{ds}\Big)^n\ {}_aI_g^{n-\alpha} f\Big](s) g'(s) ds \\ &= \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)\Big\{\frac{1}{\Gamma(\alpha+1)}\int_a^t \Big(g(t)-g(s)\Big)^{\alpha}\Big[\Big(\frac{1}{g'(s)}\frac{d}{ds}\Big)^n\ {}_aI_g^{n-\alpha} f\Big](s) g'(s) ds\Big\} \\ &= \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)\Big\{\frac{1}{\Gamma(\alpha+1)}\int_a^t \Big(g(t)-g(s)\Big)^{\alpha}\frac{d}{ds}\Big[\Big(\frac{1}{g'(s)}\frac{d}{ds}\Big)^{n-1}\ {}_aI_g^{n-\alpha} f\Big](s) ds\Big\}. \end{aligned}$$

Now, performing the integration by parts n times, we obtain

$$\begin{aligned} \Big(\ {}_aI_g^{\alpha}\ {}_aD_g^{\alpha}\Big) f(t) &= \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)\Big\{\frac{1}{\Gamma(\alpha-n+1)}\int_a^t \Big(g(t)-g(s)\Big)^{\alpha-n} \\ &\times\ {}_aI_g^{n-\alpha} f(s) g'(s) ds - \sum_{k=1}^{n}\frac{{}_aI_g^{k-\alpha} f(a^+)}{\Gamma(\alpha+2-k)}\Big(g(t)-g(a)\Big)^{\alpha-k+1}\Big\} \\ &= \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)\Big\{\ {}_aI_g^{\alpha-n+1}\ {}_aI_g^{n-\alpha} f(t) \\ &- \sum_{k=1}^{n}\frac{{}_aI_g^{k-\alpha} f(a^+)}{\Gamma(\alpha+2-k)}\Big(g(t)-g(a)\Big)^{\alpha-k+1}\Big\} \\ &= \Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)\Big\{\ {}_aI_g^{1} f(t) - \sum_{k=1}^{n}\frac{{}_aI_g^{k-\alpha} f(a^+)}{\Gamma(\alpha+2-k)}\Big(g(t)-g(a)\Big)^{\alpha-k+1}\Big\} \\ &= f(t) - \sum_{k=1}^{n}\frac{{}_aI_g^{k-\alpha} f(a^+)}{\Gamma(\alpha+2-k)}\Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)\Big(g(t)-g(a)\Big)^{\alpha-k+1} \\ &= f(t) - \sum_{k=1}^{n}\frac{{}_aI_g^{k-\alpha} f(a^+)}{\Gamma(\alpha-k+1)}\Big(g(t)-g(a)\big)^{\alpha-k}. \end{aligned}$$

$\square$

3. **The generalized Laplace transform.** In this section, we present the definition of the generalized Laplace transform and state some of its properties.

Definition 3.1. Let $f, g : [a, \infty) \to \mathbb{R}$ be real valued functions such that $g(t)$ is continuous and $g'(t) > 0$ on $[0, \infty)$. The generalized Laplace transform of f is defined by

$$\mathcal{L}_g\{f(t)\}(s) = \int_a^\infty e^{-s(g(t)-g(a))} f(t) g'(t) dt, \tag{21}$$

for all values of s, the integral is valid.

In the following theorem we represent the relation between the generalized Laplace transform and the classical one.

Theorem 3.2. *Let $f, g : [a, \infty) \to \mathbb{R}$ be real valued functions such that $g(t)$ is continuous and $g'(t) > 0$ on $[0, \infty)$ and such that the generalized Laplace transform of f exists. Then*

$$\mathcal{L}_g\{f(t)\}(s) = \mathcal{L}\{f\Big(g^{-1}(t + g(a))\Big)\}(s), \tag{22}$$

where $\mathcal{L}\{f\}$ is the usual Laplace transform of f.

Proof. The proof is straight forward if one use the change of variable $u = g(t) - g(a)$ in (22). ☐

Definition 3.3. A function $f : [0, \infty) \to \mathbb{R}$ is said to be of $g(t)$-exponential order if there exist non-negative constants M, c, T such that $|f(t)| \leq Me^{cg(t)}$ for $t \geq T$.

Now, we present the conditions for the existence of the generalized Laplace transform of a function.

Theorem 3.4. *If $f : [a, \infty) \to \mathbb{R}$ is a piecewise continuous function and is of $g(t)$-exponential order, then its generalized-Laplace transform exists for $s > c$.*

Proof. The proof is straight forward. ☐

Below we present the linearity property.

Theorem 3.5. *If the the generalized Laplace transform of $f_1 : [a, \infty) \to \mathbb{R}$ exists for $s > c_1$ and the generalized Laplace transform of $f_2 : [a, \infty) \to \mathbb{R}$ exists for $s > c_2$. Then, for any constants a_1 and a_2, the generalized Laplace transform of $a_1 f_1 + a_2 f_2$, where a_1 and a_2 are constant, exists and*

$$\mathcal{L}_g\{a_1 f_1(t) + a_2 f_2(t)\}(s) = a_1 \mathcal{L}_g\{f(t)\}(s) + a_2 \mathcal{L}_g\{f_2(t)\}(s), \text{ for } s > max\{c_1, c_2\}. \tag{23}$$

The generalized Laplace transforms of some elementary functions were given in the following lemma.

Lemma 3.6. *1.* $\mathcal{L}_g\{1\}(s) = \dfrac{1}{s}, \ s > 0.$

2. $\mathcal{L}_g\{(g(t) - g(a))^\beta\}(s) = \dfrac{\Gamma(\beta)}{s^\beta}, \ \Re(\beta) > 0, s > 0.$

3. $\mathcal{L}_g\{e^{\lambda g(t)}\}(s) = \dfrac{e^{\lambda g(a)}}{s - \lambda}, \ s > \lambda.$

Proof. The proof is executed by using the relation (22). ☐

The generalized Laplace transforms of the generalized derivatives of integer order are presented below.

Theorem 3.7. *Let the function $f(t) \in C_g[a,T]$ and of $g(t)$- exponential order such that $f^{[1]}(t)$ is piecewise continuous over every finite interval $[a,T]$. Then generalized Laplace transform of $f^{[1]}(t)$ exists,*

$$\mathcal{L}_g\{f^{[1]}(t)\}(s) = s\mathcal{L}_g\{f(t)\}(s) - f(a). \tag{24}$$

Proof. Let $a < t_1, t_2, ..., t_n < T$ be the points in the interval $[a,T]$ where $f^{[1]}$ is discontinuous. Then we have

$$\begin{aligned} \int_a^T e^{-s(g(t)-g(a))} f^{[1]}(t)g'(t)dt &= \int_a^T e^{-s(g(t)-g(a))} f'(t)dt \\ &= \int_a^{t_1} e^{-s(g(t)-g(a))} f'(t)dt \\ &+ \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} e^{-s(g(t)-g(a))} f'(t)dt \\ &+ \int_{t_n}^T e^{-s(g(t)-g(a))} f'(t)dt. \end{aligned}$$

Integrating by parts gives

$$\begin{aligned} \int_a^T e^{-s(g(t)-g(a))} f^{[1]}(t)g'(t)dt &= e^{-s(g(t)-g(a))} f(t)\Big|_a^{t_1} + \sum_{i=1}^{n-1} e^{-s(g(t)-g(a))} f(t)\Big|_{t_i}^{t_{i+1}} \\ &+ e^{-s(g(t)-g(a))} f(t)\Big|_{t_n}^T \\ &+ s\Big[\int_0^{t_1} e^{-s(g(t)-g(a))} f(t)g'(t)dt \\ &+ \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} e^{-s(g(t)-g(a))} f(t)f(t)g'(t)dt \\ &+ \int_{t_n}^T e^{-s(g(t)-g(a))} f(t)g'(t)dt\Big]. \end{aligned}$$

Going one step further, we obtain

$$\begin{aligned} \int_a^T e^{-s(g(t)-g(a))} f^{[1]}(t)g'(t)dt &= e^{-s(g(T)-g(a))} f(T) - f(0) \\ &+ s\int_0^T e^{-s(g(t)-g(a))} f(t)g'(t)dt. \end{aligned} \tag{25}$$

The result is obtained by taking the limit as $T \to \infty$ of both sides of equation (25). $\square$

Theorem 3.7 can be generalized as follows.

Corollary 2. *Let $f \in C_g^{n-1}[a,t)$ such that $f^{[i]}, i = 0,1,2,...,n-1$ are g-exponential order. Let $f^{[n]}$ be a piecewise continuous function on the interval $[a,T]$. Then, the* *generalized Laplace transform of $f^{[n]}(t)$ exists and*

$$\mathcal{L}_g\{f^{[n]}(t)\}(s) = s^n \mathcal{L}_g\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1}(f^{[k]})(a). \tag{26}$$

Proof. The proof can be done by mathematical induction. □

To be able to find the generalized Laplace transforms of the generalized fractional operators , we need to define the generalized convolution integral.

Definition 3.8. Let f and h be two functions which are piecewise continuous at each interval $[0,T]$ and of exponential order. We define the generalized convolution of f and h by

$$(f *_g h)(t) = \int_a^t f(\tau)h\Big(g^{-1}(g(t)+g(a)-g(\tau))\Big)g'(\tau)d\tau. \tag{27}$$

The generalized convolution of two functions is commutative.

Lemma 3.9. *Let f and h be two functions which are piecewise continuous at each interval $[a,T]$ and of exponential order. Then*

$$f *_g g = h *_g f. \tag{28}$$

Proof. The proof can be easily stated once the change of variable $u = g^{-1}(g(t) + g(a) - g(\tau))$ is utilized. □

Below we present the ρ-Laplace transform of the ρ-convolution integral.

Theorem 3.10. *Let f and h be two functions which are piecewise continuous at each interval $[a,T]$ and of exponential order. Then*

$$\mathcal{L}_g\{f *_g h\} = \mathcal{L}_g\{f\}\mathcal{L}_g\{h\}. \tag{29}$$

Proof.

$$\begin{aligned}\mathcal{L}_g\{f\}\mathcal{L}_g\{h\} &= \int_a^\infty e^{-s(g(t)-g(a))}f(t)g'(t)dt \int_a^\infty e^{-s(g(u)-g(a))}h(u)g'(u)du \\ &= \int_a^\infty \int_a^\infty e^{-s(g(t)+g(u)-2g(a))}f(t)h(u)g'(t)g'(u)dtdu\end{aligned}$$

Now, setting choosing τ satisfying $g(\tau) = g(t) + g(u) - g(a)$, we get

$$\begin{aligned}\mathcal{L}_g\{f\}\mathcal{L}_g\{h\} &= \int_0^\infty \int_u^\infty e^{-s(g(\tau)-g(a))} f\Big(g^{-1}(g(\tau)-g(u)+g(a))\Big) \times \\ &\quad h(u)g'(\tau)g'(u)d\tau du \\ &= \int_a^\infty e^{-s(g(\tau)-g(a))}\Big[\int_a^\tau f\Big(g^{-1}(g(\tau)-g(u)+g(a))\Big) \times \\ &\quad h(u)g'(u)du\Big]g'(\tau)d\tau \\ &= \mathcal{L}_g\{f *_g h\}.\end{aligned}$$

□

Now, we can find the generalized Laplace transform of the generalized fractional operators.

4. **The generalized Laplace transforms of the generalized fractional integrals and derivatives.** In the following theorem, we present the generalized Laplace transform of the left generalized fractional integral starting at a.

Theorem 4.1. *Let $\alpha > 0$ and f be a piecewise continuous function on each interval $[a, t]$ and of g(t)-exponential order. Then*

$$\mathcal{L}_g\{({}_aI^{\alpha,\rho} f)(t)\} = \frac{\mathcal{L}_g\{f(t)\}}{s^\alpha}. \tag{30}$$

Proof. The proof can be done using the definition of the generalized fractional integral (1), Theorem 3.5, Theorem 3.10 and Lemma 3.6. Actually,

$$\begin{aligned}
\mathcal{L}_g\{({}_aI_g^{\alpha} f)(t)\}(s) &= \frac{1}{\Gamma(\alpha)}\mathcal{L}_g\{(g(t)-g(a))^{\alpha-1} *_g f(t)\}(s) \\
&= \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{s^\alpha}\mathcal{L}_g\{f(t)\} \\
&= \frac{\mathcal{L}_g\{f(t)\}}{s^\alpha}.
\end{aligned}$$

□

Now we can present the generalized Laplace transform of the left generalized fractional derivative.

Corollary 3. *Let $\alpha > 0$ and $f \in AC_g^n[a, b]$ for any $b > a$, $g \in C^n[a, b]$ such that $g'(t) > 0$ and ${}_aI^{n-k-\alpha,\rho} f, k = 0, 1, ..., n-1$ be of $g(t)$-exponential order. Then*

$$\mathcal{L}_\rho\{({}_aD_g^\alpha f)(t)\}(s) = s^\alpha \mathcal{L}_g\{f(t)\} - \sum_{k=0}^{n-1} s^{n-k-1}\Big({}_aI_g^{n-k-\alpha} f\Big)(a^+). \tag{31}$$

Proof.

$$\begin{aligned}
\mathcal{L}_g\{({}_aD_g^\alpha f)(t)\}(s) &= \mathcal{L}_g\{\Big({}_aI^{n-\alpha} f\Big)^{[n]}(t)\} \\
&= s^n\mathcal{L}_g\{ {}_aI_g^{n-\alpha} f)(t)\} - \sum_{k=0}^{n-1} s^{n-k-1}\Big(\Big({}_aI_g^{n-\alpha}\Big)^{[k]} f\Big)(a^+) \\
&= s^n\frac{\mathcal{L}_g\{f(t)\}}{s^{n-\alpha}} - \sum_{k=0}^{n-1} s^{n-k-1}\Big(\Big({}_aI^{n-\alpha}\Big)^{[k]} f\Big)(a^+) \\
&= s^\alpha\mathcal{L}_g\{f(t)\} - \sum_{k=0}^{n-1} s^{n-k-1}\Big(\Big({}_aI^{n-\alpha}\Big)^{[k]} f\Big)(a^+).
\end{aligned}$$

Now, the proof is completed by using Theorem 2.5. □

The following corollary states the generalized Laplace transform of the left generalized Caputo fractional derivative.

Corollary 4. *Let $\alpha > 0$ and $f \in AC_\gamma^n[a, b]$ for any $b > a$ and $f^{[k]}, k = 0, 1, ..., n$ be of $g(t)$-exponential order. Then*

$$\mathcal{L}_g\{({}_a^CD_g^\alpha f)(t)\}(s) = s^\alpha\Big[\mathcal{L}_g\{f(t)\} - \sum_{k=0}^{n-1} s^{-k-1}(f^{[k]})(a^+)\Big]. \tag{32}$$

Proof.

$$\begin{aligned}\mathcal{L}_g\{({}_a^C D_g^\alpha f)(t)\}(s) &= \mathcal{L}_g\{({}_a I_g^{n-\alpha} f^{[n]})(t)\} \\ &= s^{\alpha-n}\mathcal{L}_g\{f^{[n]})(t)\} \\ &= s^{\alpha-n}\Big[s^n\mathcal{L}_g\{f(t)\}(s) - \sum_{k=0}^{n-1} s^{n-k-1}(f^{[k]})(a^+)\Big] \\ &= s^\alpha\Big[\mathcal{L}_g\{f(t)\} - \sum_{k=0}^{n-1} s^{-k-1}(f^{[k]})(a^+)\Big].\end{aligned}$$

□

The Mittag-Leffler functions have important roles in the theory of fractional calculus [25, 19, 24]. The Mittag-Leffler function is given by [25, 19, 24]

$$E_\alpha(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)},\ z\in\mathbb{C}, \Re(\alpha)>0. \tag{33}$$

The Mittag-Leffler function involving two parameters is given by [25, 19, 24]

$$E_{\alpha,\beta}(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)},\ z\in\mathbb{C}, \Re(\alpha)>0, \Re(\beta)>0. \tag{34}$$

In the following lemma, we present the generalized Laplace transforms of some specified Mittag-Leffler functions.

Lemma 4.2. *Let* $\Re(\alpha)>0$ *and* $\left|\frac{\lambda}{s^\alpha}\right|<1$. *Then*

$$\mathcal{L}_g\Big\{E_\alpha\Big(\lambda(g(t)-g(a))^\alpha\Big)\Big\} = \frac{s^{\alpha-1}}{s^\alpha-\lambda}, \tag{35}$$

and

$$\mathcal{L}_g\{(g(t)-g(a))^{\beta-1}E_{\alpha,\beta}\Big(\lambda(g(t)-g(a))^\alpha\Big)\Big\} = \frac{s^{\alpha-\beta}}{s^\alpha-\lambda}. \tag{36}$$

Proof.

$$\begin{aligned}\mathcal{L}_g\Big\{E_\alpha\Big(\lambda(g(t)-g(a))^\alpha\Big)\Big\} &= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+1)}\mathcal{L}_g\{(g(t)-g(a))^{k\alpha}\} \\ &= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+1)}\frac{\Gamma(k\alpha+1)}{s^{k\alpha+1}} \\ &= \frac{1}{s}\sum_{k=0}^{\infty}\Big(\frac{\lambda}{s^\alpha}\Big)^k \\ &= \frac{s^{\alpha-1}}{s^\alpha-\lambda}.\end{aligned}$$

This was the proof of (35). Now,

$$\begin{aligned}\mathcal{L}_g\{(g(t)-g(a))^{\beta-1}E_{\alpha,\beta}\Big(\lambda(g(t)-g(a))^\alpha\Big)\Big\} &= \sum_{k=0}^{\infty}\frac{\lambda^k\mathcal{L}_g\{(g(t)-g(a))^{k\alpha+\beta-1}\}}{\Gamma(k\alpha+\beta)} \\ &= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\beta)}\frac{\Gamma(k\alpha+\beta)}{s^{k\alpha+\beta}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{s^\beta}\sum_{k=0}^{\infty}\left(\frac{\lambda}{s^\alpha}\right)^k \\
&= \frac{1}{s^\beta}\frac{1}{1-\frac{\lambda}{s^\alpha}} \\
&= \frac{s^{\alpha-\beta}}{s^\alpha-\lambda}.
\end{aligned}
$$

This proves (36). □

5. **Solution of some generalized fractional differential equations by the generalized Laplace transforms.** In this section, we are considering two Cauchy problems in the frame of the generalized Riemann-Liouville fractional derivatives and the generalized Caputo fractional derivative.

Theorem 5.1. *The Cauchy problem*

$$
\begin{gathered}
{}_aD_g^\alpha y(t)-\lambda y(t)=f(t),\ t>a,\ 0<\alpha\leq 1,\ \lambda\in\mathbb{R}, \\
({}_aI_g^{1-\alpha}y)(a^+)=c,\ c\in\mathbb{R},
\end{gathered}
\tag{37}
$$

has the solution

$$
\begin{aligned}
y(t) &= c(g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^\alpha\Big) \\
&+ \int_a^t (g(t)-g(\tau))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(\tau))^\alpha\Big)f(\tau)g'(\tau)d\tau.
\end{aligned}
\tag{38}
$$

Proof. Applying the generalized Laplace transform to both sides of the equation (37) and then using Corollary 3 with $n=1$, one gets

$$
\begin{aligned}
\mathcal{L}_{\{}y(t)\} &= c\,\frac{1}{s^\alpha-\lambda}+\frac{1}{s^\alpha-\lambda}\mathcal{L}_g\{f(t)\} \\
&= c\mathcal{L}_g\Big\{(g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^\alpha\Big)\Big\} \\
&+ \mathcal{L}_g\Big\{(g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^\alpha\Big)\Big\}\mathcal{L}_g\{f(t)\} \\
&= \mathcal{L}_\rho\Big\{c(g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^\alpha\Big) \\
&+ (g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^\alpha\Big)*_g f(t)\Big\}.
\end{aligned}
$$

Hence we obtain,

$$
\begin{aligned}
y(t) &= c(g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^\alpha\Big) \\
&+ (g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^\alpha\Big)*_g f(t) \\
&= c(g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^\alpha\Big) \\
&+ \int_a^t (g(t)-g(\tau))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(\tau))^\alpha\Big)f(\tau)g'(\tau)d\tau.
\end{aligned}
$$

□

Remark 2. If $g(t)=t$, the solution (38) coincides with the solution given in equation (4.1.14) in [19] and if $g(t)=\log t$, (38) coincides with equation (4.1.99) in [19].

Next, we consider a Cauchy problem in the frame of generalized Caputo fractional derivatives.

Theorem 5.2. *The Cauchy problem*

$$\begin{aligned} {}^{C}_{a}D^{\alpha}_{g}y(t) - \lambda y(t) = f(t),\ t > a,\ 0 < \alpha \leq 1,\ \lambda \in \mathbb{R}, \\ y(a^{+}) = c,\ c \in \mathbb{R}, \end{aligned} \tag{39}$$

has the solution

$$y(t) = cE_{\alpha}\Big(\lambda(g(t)-g(a))^{\alpha}\Big)+\int_{a}^{t}(g(t)-g(\tau))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(\tau))^{\alpha}\Big)f(\tau)g'(\tau)d\tau. \tag{40}$$

Proof. Applying the generalized Laplace transform to both sides of the equation (39) and then using Corollary 4 with $n = 1$, one gets

$$\begin{aligned} \mathcal{L}_{g}\{y(t)\} &= c\,\frac{s^{\alpha-1}}{s^{\alpha}-\lambda} + \frac{1}{s^{\alpha}-\lambda}\mathcal{L}_{g}\{f(t)\} \\ &= c\,\mathcal{L}_{g}\Big\{E_{\alpha}\Big(\lambda(g(t)-g(a))^{\alpha}\Big)\Big\} \\ &+ \mathcal{L}_{g}\Big\{(g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^{\alpha}\Big)\Big\}\mathcal{L}_{g}\{f(t)\} \\ &= \mathcal{L}_{g}\Big\{cE_{\alpha}\Big(\lambda(g(t)-g(a))^{\alpha}\Big) \\ &+ (g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^{\alpha}\Big) *_{g} f(t)\Big\} \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= cE_{\alpha}\Big(\lambda(g(t)-g(a))^{\alpha}\Big) + (g(t)-g(a))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(a))^{\alpha}\Big) *_{g} f(t) \\ &= cE_{\alpha}\Big(\lambda(g(t)-g(a))^{\alpha}\Big) \\ &+ \int_{a}^{t}(g(t)-g(\tau))^{\alpha-1}E_{\alpha,\alpha}\Big(\lambda(g(t)-g(\tau))^{\alpha}\Big)f(\tau)g'(\tau)d\tau. \end{aligned}$$

□

Remark 3. If $g(t) = t$, (40) coincides with (4.1.66) in [19].

6. **Conclusion.** The classical Laplace transform played an important role to solve classical differential equations with integer orders and fractional Riemann-Liouville and Caputo derivatives. The integral and the derivatives utilized in those equations are classical. When the integrals and derivatives are general, the need of a more general integral transform arises. For the sake of this, we employed a modified Laplace transform that can used to solve differential equation involving a wider class of derivatives and their fractional versions. We studied the basic theory of the Laplace transform under consideration and proved its convolution theorem to find the Laplace transform of the generalized fractional integrals and derivatives. After obtaining the generalized Laplace transforms for certain weighted Mittag-Leffler functions, we were able to solve nonhomogeneous linear fractional dynamic equations in the frame of generalized fractional Riemann-Liouville and Caputo type operators. We have shown that the solution representations are expressible by means of the Mittag-Leffler functions and coincide with the results found in the literature for special cases of $g(t)$.

The generalized Laplace transform which we have discussed will serve as an effective tool to solve dynamical systems depending on generalized fractional operators, whose kernel is singular. The question arises here is the possibility of developing a new integral transform that can be easily applied to solve dynamical systems in the frame of fractional operators with nonsingular kernels [8], [3], [4].

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