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Research Article

Generalized Darbo-Type F-Contraction and F-Expansion and Its Applications to a Nonlinear Fractional-Order Differential Equation

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In this work, we introduce various Darbo-type $F_{\pounds}$-contractions, and utilizing these contractions, we present some fixed point theorems. Moreover, we introduce a Darbo-type $F_{\pounds}$-expanding mapping and prove fixed point theorems under the Darbo-type $F_{\pounds}$-expanding mapping. Employing our results, we check the existence of a solution to the nonlinear fractional-order differential equation under the integral type boundary conditions. For its validity, an appropriate example is given.

1. Introduction and Preliminaries

For the sake of completeness, we provide a brief introduction and recollect basic notions, definitions, and fundamental results. In the sequel, we symbolize by $\mathbb{R}$ the set of all real numbers, by $\mathbb{N}$ the set of all positive integers, by $\bar{M}$ the closure of M and by $\overline{co}M$ the convex hull closure of M. Additionally, Ξ denotes a Banach space $B(\Xi)=\{\Lambda \neq 0 : \Lambda$ is bounded subsets of $\Xi\}$, $\ker \pounds = \{\Lambda \in \mathfrak{B}(\Xi) : \pounds(\Lambda)=0\}$ the kernel of function $\pounds : \mathfrak{B}(\Xi) \longrightarrow [0,\infty)$ and $\Omega=\{\Lambda : \Lambda \neq 0$, convex, bounded, and closed subset of $\Xi\}$.

Many researchers have been interested in the fixed point theory. This theory is branched into two notable areas. One deals with contraction mappings on metric spaces. In this area, the first important result is the Banach contraction principle. The second deals with continuous operators on convex and compact subsets of a Banach space. In this area, two important results are Brouwer's fixed point theorem and its infinite dimensional form, Schauder's fixed point theorem.

Theorem 1. *Every continuous mapping from the unit ball of $\mathbb{R}^n$ into itself has a fixed point.*

Theorem 2 (see [1]). *Let $\Lambda \in \Omega$. Then, a compact continuous operator $\Upsilon : \Lambda \longrightarrow \Lambda$ has a fixed point.*

In both theorems, compactness plays a crucial role. To overcome such hurdle, one of the techniques is to use the notion of a measure of noncompactness (in short MNC). The axiomatic definition of an MNC is as below.

Definition 3 (see [2]). A map $\pounds$: $\mathfrak{B}(\Xi) \longrightarrow [0,\infty)$ is an MNC in Ξ if for all $\Lambda, \Lambda_1, \Lambda_2 \in \mathfrak{B}(\Xi)$ it satisfies the following axioms:

(i) $\ker \pounds \neq 0$ and relatively compact in Ξ

(ii) $\Lambda_1 \subset \Lambda_2 \Rightarrow \pounds(\Lambda_1) \leq \pounds(\Lambda_2)$

(iii) $\pounds(\bar{\Lambda}) = \pounds(\Lambda)$

(iv) $\pounds(\overline{co}\Lambda) = \pounds(\Lambda)$

(v) $\pounds(\eta\Lambda_1 + (1-\eta)\Lambda_2) \le \eta\pounds(\Lambda_1) + (1-\eta)\pounds(\Lambda_2)$, $\forall \eta \in [0, 1]$

(vi) If $\{\Lambda_n\}$ is a sequence of closed sets in $\mathfrak{B}(\Xi)$such that $\Lambda_{n+1} \subset \Lambda_n$, $\forall n \in \mathbb{N}$, $\lim_{s\longrightarrow+\infty} \pounds(\Lambda_n) = 0$, then $\Lambda_\infty = \cap_{n=1}^{+\infty}\Lambda_n \neq 0$.

The Kuratowski MNC [3] is the function $\pounds : \mathfrak{B}(\Xi) \longrightarrow [0,\infty)$ defined by

$$\pounds(\mathscr{K}) = \inf\left\{\varepsilon > 0 : \mathscr{K} \subset \bigcup_{i=1}^{n} \mathscr{S}_i, \mathscr{S}_i \subset \Xi, \operatorname{diam}(\mathscr{S}_i) < \varepsilon\right\}, \quad (1)$$

where $\operatorname{diam}(S)$ is the diameter of a set S. Using the notion of an MNC, Darbo [4] published a fixed point result, which determines the existence of a fixed point.

Theorem 4 (see [4]). *Let $\Lambda \in \Omega$ and $\Upsilon : \Lambda \longrightarrow \Lambda$ be a continuous function. If there exists $k \in [0, 1)$ such that*

$$\pounds(\Upsilon(\Lambda_0)) \le k\,\pounds(\Lambda_0), \quad (2)$$

where $\Lambda_0 \subset \Lambda$ and $\pounds$ is an MNC defined on Ξ. Then, Υ has a fixed point in Λ.

It generalizes the well-known Schauder fixed point result and includes the existence part of the Banach contraction principle. Many extensions and generalizations of the Darbo fixed point result can be noticed in the existing literature.

The contraction on underlying mappings plays a central role for finding the fixed point. Inspired from this natural idea, the Banach contraction has been improved and extended by several researchers [5–9]. Wardowski [10] proposed a new contraction, called the $\mathscr{F}$-contraction and established fixed point theorems.

Definition 5 (see [10]). Let $\mathscr{F} : (0,\infty) \longrightarrow \mathbb{R}$ be a map such that

(F_1) $\mathscr{F}$ is nondecreasing

(F_2) $\lim_{n\longrightarrow+\infty} \delta_n = 0 \Leftrightarrow \lim_{n\longrightarrow+\infty} \mathscr{F}(\delta_n) = -\infty$, for any sequence $\{\delta_n\} \subset (0,\infty)$

(F_3) one can find $k \in (0, 1)$ such that $\lim_{s\longrightarrow0^+} \delta^k \mathscr{F}(\delta) = 0$.

Symbolized by $\mathbb{F}$, the family of all maps $\mathscr{F} : (0,\infty) \longrightarrow \mathbb{R}$ which fulfill the axioms (F_1) and (F_2), and by $\mathbb{S}$, the family of functions $\tau : (0,\infty) \longrightarrow \mathbb{R}$ such that $\lim_{t\longrightarrow s^+} \inf \tau(t) > 0$, $\forall s \in [0,\infty)$.

Using the specific form of $\mathscr{F}$, we deduce other known existing contractions. Many articles concerning $\mathscr{F}$-contractions and its generalizations have come into view (see, e.g., [11, 12] and the references cited therein). In particular, Jleli et al. [13] generalized $\mathscr{F}$-contraction such that $\mathscr{F}_{\mathfrak{L}}$-contraction is an established Darbo-type fixed point result.

Definition 6 (see [13]). Let $\Lambda \in \Omega$. Then, the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is $\mathscr{F}_{\pounds}$-contraction if there exists $\tau \in S$ and $F \in F$ such that

$$\tau(\pounds(\Lambda_0)) + \mathscr{F}(\pounds(\Upsilon(\Lambda_0))) \le \mathscr{F}(\pounds(\Lambda_0)), \quad (3)$$

where Λ_0 is a subset of $\Lambda, \pounds(\Lambda_0), \pounds(T(\Lambda_0)) > 0$, and $\pounds$ is an MNC defined in Ξ.

Theorem 7 (see [13]). *Let $\Lambda \in \Omega$. If the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous and an $\mathscr{F}_{\pounds}$-contraction, then Υ has a fixed point in Λ.*

Gillespie et al. [14] introduced the concept of expanding mapping. Gòrnicki [15] introduced the idea of $\mathscr{F}$-expanding mappings and presented some fixed point results. To find the fixed point of expanding mappings, one needs the following lemma.

Lemma 8 (see [15]). *For surjective map $\mathfrak{f} : \mathscr{X} \longrightarrow \mathscr{X}$, there exists a map $\mathfrak{f}^* : \mathscr{X} \longrightarrow \mathscr{X}$ such that $\mathfrak{f} \circ \mathfrak{f}^* : \mathscr{X} \longrightarrow \mathscr{X}$ is an identity map.*

This manuscript has two aims. Firstly, we prove various fixed point theorems: the $\mathscr{F}_{\pounds}$-weak contraction, $\mathscr{F}_{\pounds}$-weak Suzuki contraction, almost $\mathscr{F}_{\pounds}$-contraction, Hardy-Rogers-type $\mathscr{F}_{\pounds}$-contraction and Reich-type $\mathscr{F}_{\pounds}$-contraction. Secondly, we prove fixed point results under the Darbo-type $\mathscr{F}_{\pounds}$-expanding mapping. We also observe that several existing results can be concluded from our main results. Furthermore, we check the existence of a solution to the nonlinear fractional-order differential equation via integral type boundary conditions, and for its validity we construct an example.

2. Generalization of Darbo-Type Results via F-Contractions

In this section, we introduce various types of $\mathscr{F}_{\pounds}$-contractions of Darbo type, and then, we prove fixed point results for mappings satisfying such contractive condition in the Banach space endowed with an MNC. We first give the definition of the $\mathscr{F}_{\pounds}$-weak contraction.

Definition 9. Let $\Lambda \in \Omega$. Then, the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is a weak contraction if there exists $\tau \in \mathbb{S}$ and $\mathscr{F} \in \mathbb{F}$ such that

$$\tau(\pounds(\Lambda_1)) + \mathscr{F}(\pounds(\Upsilon(\Lambda_1))) \le \mathscr{F}(\Delta(\Lambda_1, \Lambda_2)), \quad (4)$$

where Λ_1 and Λ_2 are subsets of $\Lambda, \pounds(\Lambda_1)$, $\pounds(\Upsilon(\Lambda_1))$, $\pounds(\Upsilon(\Lambda_2)) > 0$ and $\pounds$ is an MNC defined in Ξ and

$$\Delta(\Lambda_1, \Lambda_2) = \max\left\{\pounds(\Lambda_1), \pounds(\Upsilon(\Lambda_1)), \pounds(\Upsilon(\Lambda_2)), \frac{1}{2}\pounds(\Upsilon(\Lambda_1) \cup \Upsilon(\Lambda_2))\right\}. \quad (5)$$

In the light of $\mathscr{F}_{\pounds}$-weak contraction, we present the first result.

Theorem 10. *Let $\Lambda \in \Omega$. If the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous and an $\mathcal{F}_{£}$-weak contraction, then Υ has a fixed point in Λ.*

Proof. Define a sequence $\{\Lambda_n\}_{n=0}^{\infty}$ such that

$$\Lambda_0 = \Lambda \text{ and } \Lambda_n = \overline{co}(\Upsilon\Lambda_{n-1}), \forall n \in \mathbb{N}. \tag{6}$$

We need to prove that $\Lambda_{n+1} \subset \Lambda_n$ and $\Upsilon\Lambda_n \subset \Lambda_n, \forall n \in \mathbb{N}$. For the first inclusion, we use induction. If $n = 1$, then by (6), we get $\Lambda_0 = \Lambda$ and $\Lambda_1 = \overline{co}(\Upsilon\Lambda_0) \subset \Lambda_0$. Next, for $n > 1$, we assume that

$$\Lambda_n \subset \Lambda_{n-1}. \tag{7}$$

Then

$$\overline{co}(\Upsilon(\Lambda_n)) \subset \overline{co}(\Upsilon(\Lambda_{n-1})). \tag{8}$$

Using (6), we get the first inclusion

$$\Lambda_{n+1} \subset \Lambda_n. \tag{9}$$

With the help of inclusion (9), we obtained the second inclusion as

$$\Upsilon\Lambda_n \subset \overline{co}(\Upsilon\Lambda_n) = \Lambda_{n+1} \subset \Lambda_n. \tag{10}$$

Now, we discuss two cases subject to £. If we can find an integer $m \geq 0$ such that $£(\Lambda_m) = 0$, then Λ_m is compact. But since $\Upsilon : \Lambda_m \longrightarrow \Lambda_m$, by Theorem 2, Υ has a fixed point in $\Lambda_m \subset \Lambda$. Instead, if we take $£(\Lambda_n) > 0, \forall n \in \mathbb{N}$, then we have to testify that $\Lambda_\infty \subset \Lambda_n \in \Omega$. First, we need to show that $£(\Lambda_n) \longrightarrow 0$ as $n \longrightarrow +\infty$. From inclusion (9), we write $£(\Lambda_{n+1}) < £(\Lambda_n)$, that is $\{£(\Lambda_n)\}$ is a decreasing sequence and hence converges to $s \in \mathbb{R}$ with $s \geq 0$. Now, since $£(\Lambda_n) \in (0,\infty)$ and $s \in [0,\infty)$, so by assumption on τ, $\liminf_{t \to s^+} \tau(t) > 0$, we can find $r > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(£(\Lambda_n)) \geq r, \quad \forall n \geq n_0$. Using contraction condition (4) with $\Lambda_1 = \Lambda_n$ and $\Lambda_2 = \Lambda_{n+1}$, we write

$$\begin{aligned}\tau(£(\Lambda_n)) + \mathcal{F}(£(\Lambda_{n+1})) &= \tau(£(\Lambda_n)) + \mathcal{F}(£(\overline{co}(\Upsilon(\Lambda_n))))\\ &= \tau(£(\Lambda_n)) + \mathcal{F}(£(\Upsilon(\Lambda_n)))\\ &\leq \mathcal{F}(\Delta(\Delta_{n,}\Delta_{n+1})),\end{aligned} \tag{11}$$

where

$$\begin{aligned}\Delta(\Lambda_{n,}\Lambda_{n+1}) &= \max\left\{£(\Lambda_n), £(\Upsilon(\Lambda_n)), £(\Upsilon(\Lambda_{n+1})), \frac{1}{2}£(\Upsilon(\Lambda_n) \cup \Upsilon(\Lambda_{n+1}))\right\}\\ &\leq \max\left\{£(\Lambda_n), £(\Lambda_n), £(\Lambda_{n+1}), \frac{1}{2}£(\Lambda_n \cup \Lambda_{n+1})\right\}\\ &= \max\left\{£(\Lambda_n), £(\Lambda_n), £(\Lambda_{n+1}), \frac{1}{2}£(\Lambda_n)\right\}\\ &= £(\Lambda_n).\end{aligned} \tag{12}$$

Thus, from (11), we obtain

$$\tau(£(\Lambda_n)) + \mathcal{F}(£(\Lambda_{n+1})) \leq \mathcal{F}(£(\Lambda_n)). \tag{13}$$

From here, we write

$$\mathcal{F}(£(\Lambda_{n+1})) \leq \mathcal{F}(£(\Lambda_n)) - \tau(£(\Lambda_n)) \leq \mathcal{F}(£(\Lambda_n)) - r. \tag{14}$$

Consequently

$$\mathcal{F}(£(\Lambda_n)) \leq \mathcal{F}(£(\Lambda_{n-1})) - r, \ \forall n. \tag{15}$$

Thus

$$\begin{aligned}\mathcal{F}(£(\Lambda_n)) &\leq \mathcal{F}(£(\Lambda_{n-1})) - r \leq \mathcal{F}(£(\Lambda_{n-2})) - 2r \leq \mathcal{F}(£(\Lambda_{n-3})) - 3r\\ &\vdots\\ &\leq \mathcal{F}(£(\Lambda_{n0})) - (n - n_0)r, \ \forall n > n_0.\end{aligned} \tag{16}$$

Clearly, $\lim_{n \to +\infty} \mathcal{F}(£(\Lambda_n)) = -\infty$ and using property (F_2), we can write $\lim_{n \to +\infty} £(\Lambda_n) = 0$. Thus, by Definition 3 (vi), $\Lambda_\infty = \cap_{n=1}^{+\infty} \neq 0$ and $\Upsilon\Lambda_\infty \subset \Lambda_\infty$ as $\Upsilon\Lambda_n \subset \Lambda_n$. Also, since $\Lambda_\infty \subset \Lambda_n$ for all $n \in \mathbb{N}$, so by Definition 3(ii), $£(\Lambda_\infty) \leq £(\Lambda_n), \forall n \in \mathbb{N}$. Thus, $£(\Lambda_\infty) = 0$, that is $\Lambda_\infty \in \ker £$, and hence, Λ_∞ is bounded. But Λ_∞ is closed such that Λ_∞ is compact. Therefore, by Theorem 2, Υ has a fixed point in $\Lambda_\infty \subset \Lambda$.

For the support of Theorem 10, we construct the following example.

Example 1. Let $\Lambda = [-8, 9]$ be a subset of a Banach space $\mathbb{R}$. Then, clearly, $\Lambda \in \Omega$. Define $\Upsilon : \Lambda \longrightarrow \Lambda$, $F : (0,\infty) \longrightarrow \mathbb{R}$, $\tau : (0,\infty) \longrightarrow \mathbb{R}$ by $\Upsilon(x) = 1 - x$, $\tau(x) + 1/2$ and $F(x) = \ln x$, respectively. One can easily check that Υ is continuous, $\tau \in \mathbb{S}$ and $F \in \mathbb{F}$. Also, define an MNC, $£ : B(\Xi) \to [0,\infty)$ by $£(\Lambda) = \operatorname{diam}(\Lambda) = \sup_{x,y \in \Lambda} \|x - y\|$.

Now, let $\Lambda_1 = [0,1]$ and $\Lambda_2 = [2,7]$ be two subsets of Λ. Then, $£(\Lambda_1) = £(\Upsilon(\Lambda_1)) = 1$, $£(\Upsilon(\Lambda_1)) = 6$, $£(\Upsilon(\Lambda_1) \cup (\Lambda_1)) = 7$, and hence

$$\begin{aligned}\Delta(\Lambda_1, \Lambda_2) &= \max\left\{£(\Lambda_1), £(\Upsilon(\Lambda_1)), £(\Upsilon(\Lambda_2)), \frac{1}{2}£(\Upsilon(\Lambda_1) \cup \Upsilon(\Lambda_2))\right\} \\ &= \max\left\{1, 1, 6, \frac{7}{2}\right\} = 6. \end{aligned} \tag{17}$$

Thus, from (4), we write

$$\begin{aligned}\tau(£(\Lambda_1)) + \mathscr{F}(£(\Upsilon(\Lambda_1))) &= \tau(1) + \mathscr{F}(1) = 1.5 < \mathrm{In}(6) \\ &= \mathscr{F}(\Delta(\Lambda_1 . \Lambda_2)).\end{aligned} \tag{18}$$

That is Υ is an $\mathscr{F}_£$-weak contraction. Hence, by Theorem 10, Υ has a fixed point $1/2 \in \Lambda$.

From Theorem 10, we can deduce several pivotal results. We demonstrate some preferable corollaries that cover and extend several well-known results in the existing literature. The special case, if we take $£(\Upsilon(\Lambda_n)) > 0$, we deduce the following corollary.

Corollary 11. *Let $\Lambda \in \Omega$. If the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous such that*

$$£(\Upsilon(\Lambda_1)) \le e^{-\tau(£(\Lambda_1))}\Delta(\Lambda_1 . \Lambda_2), \tag{19}$$

where Λ_1 and Λ_2 are subsets of Λ, then Υ has a fixed point in Λ.

If we take $\mathscr{F}(x) = \ln x + x, x > 0$, in Theorem 10, we deduce the following corollary.

Corollary 12. *Let $\Lambda \in \Omega$. If the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous such that*

$$£(\Upsilon(\Lambda_1)) \le e^{\Delta(\Lambda_1, \Lambda_2) - \tau(£(\Lambda_1) - £(\Lambda_1))}\Delta(\Lambda_1, \Lambda_2), \tag{20}$$

where Λ_1 and Λ_2 are subsets of Λ, then Υ has a fixed point in Λ.

If we take $\mathscr{F}(x) = \ln(x^2 + x), x > 0$, in Theorem 10, we deduce the following corollary.

Corollary 14. *Let $\Lambda \in \Omega$. If the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous such that*

$$\begin{aligned}(£(\Upsilon(\Lambda_1)))^2 + £(\Upsilon(\Lambda_1)) \le e^{-\tau(£(\Lambda_1))}\big[(\Delta(\Lambda_1, \Lambda_2))^2 \\ + \Delta(\Lambda_1, \Lambda_2)\big],\end{aligned} \tag{21}$$

where Λ_1 and Λ_2 are subsets of Λ, then Υ has a fixed point in Λ.

Definition 13. Let $\Lambda \in \Omega$. Then, the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is an $\mathscr{F}_£$-weak Suzuki contraction if there exist $\tau \in \mathbb{S}$ and $\mathscr{F} \in \mathbb{F}$ such that

$$\tau(£(\Lambda_1)) + \mathscr{F}(£(\Upsilon(\Lambda_1))) \le \mathscr{F}(\Delta(\Lambda_1, \Lambda_2)), \tag{22}$$

where Λ_1 and Λ_2 are subsets of Λ, $(\Upsilon(\Lambda_1)) < 2£(\Lambda_1)$, $£(\Lambda_1)$, $£(\Upsilon(\Lambda_1)), £(\Upsilon(\Lambda_2)) > 0$, and £ is an MNC defined on Ξ and

$$\Delta(\Lambda_1, \Lambda_2) = \max\left\{£(\Lambda_1), £(\Upsilon(\Lambda_1)), £(\Upsilon(\Lambda_2)), \frac{1}{2}£(\Upsilon(\Lambda_1) \cup \Upsilon(\Lambda_2))\right\}. \tag{23}$$

In the light of the $\mathscr{F}_£$-weak Suzuki contraction, we provide the following result. Since the proof is very easy, we omit it.

Theorem 14. *Let $\Lambda \in \Omega$. If the map $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous and an $\mathscr{F}_£$-weak Suzuki contraction, then Υ has a fixed point in Λ.*

Definition 15. Let $\Lambda \in \Omega$. Then, the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is almost an $\mathscr{F}_£$-contraction if there exist $\tau \in \mathbb{S}$ and $\mathscr{F} \in \mathbb{F}$ such that

$$\begin{aligned}£(\Upsilon(N)) > 0 \Rightarrow \tau(£(N)) + \mathscr{F}(£(\Upsilon(N))) \\ \le \mathscr{F}(£(N) + L£(\Upsilon(N))).\end{aligned} \tag{24}$$

In the light of an almost $\mathscr{F}$-contraction, we present the following result.

Theorem 16. *Let $\Lambda \in \Omega$. If the function $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous and an almost $\mathscr{F}_£$-contraction, then Υ has a fixed point in Λ.*

Proof. Construct a sequence $\{\Lambda_n\}_{n=0}^{\infty}$ such that

$$\Lambda_0 = \Lambda \text{ and } \Lambda_n = \overline{co}(\Upsilon\Lambda_{n-1}), \text{for all } n \in \mathbb{N}. \tag{25}$$

Then, $\Lambda_{n+1} \subset \Lambda_n$ and $\Upsilon\Lambda_n \subset \Lambda_n, \forall n \in \mathbb{N}$.

If we take an integer $m \ge 0$ such that $£(\Lambda_m) = 0$, then Λ_m is a compact, and by Theorem 2, we can find a fixed point of Υ in $\Lambda_m \subset \Lambda$. Let us take $£(\Lambda_n) > 0, \forall n \in \mathbb{N}$. Then, $\{£\ (\Lambda_n)\}$ is a decreasing sequence and hence converges to $s \in \mathbb{R}$ with $s \ge 0$. Now, since $£(\Lambda_n) \in (0,\infty)$ and $s \in [0,\infty)$, so by assumption on τ, $\liminf_{t \to s^+} \tau(t) > 0$, we can find $r > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(£(\Lambda_n)) \ge r, \forall n \ge n_0$. Now, assume that

$$£(\Upsilon(\Lambda_n)) > 0. \tag{26}$$

Then, setting $N = \Lambda_n$ in (24), we have

$$\begin{aligned}\tau(£(\Lambda_n)) + \mathscr{F}(£(\Lambda_{n+1})) &= \tau(£(\Lambda_n)) + \mathscr{F}(£(\overline{co}(\Upsilon(\Lambda_{n+1})))) \\ &= \tau(£(\Lambda_n)) + \mathscr{F}(£(\Upsilon(\Lambda_n))) \\ &\le \mathscr{F}(£(\Lambda_n) + L\ £(\Upsilon(\Lambda_n))).\end{aligned} \tag{27}$$

From here, we write

$$\begin{aligned}\mathscr{F}(\pounds(\Lambda_{n+1})) &\le \mathscr{F}(\pounds(\Lambda_n) + L\pounds(\Upsilon(\Lambda_n))) - \tau(\pounds(\Lambda_n)) \\ &< \mathscr{F}(\pounds(\Lambda_n) + L\pounds(\Lambda_n)) - r, \text{ for } all\ n.\end{aligned} \tag{28}$$

Consequently

$$\mathscr{F}(\pounds(\Lambda_n)) < \mathscr{F}((1+L)\pounds(\Lambda_{n-1})) - r, \text{ for all } n. \tag{29}$$

Thus

$$\begin{aligned}\mathscr{F}(\pounds(\Lambda_n)) &\le \mathscr{F}((1+L)\pounds(\Lambda_{n-1})) - r \\ &\le \mathscr{F}((1+L)\pounds(\Lambda_{n-2})) - 2r \\ &\le \mathscr{F}((1+L)\pounds(\Lambda_{n-3})) - 3r \\ &\le \mathscr{F}\left((1+L)\pounds\left(\Lambda_{n_0}\right)\right) - (n-n_0)r, \ \forall n > n_0.\end{aligned} \tag{30}$$

Clearly $\lim_{n\to+\infty} F(\pounds(\Lambda_n)) = -\infty$, and using the property (F_2), we can write $\lim_{n\to+\infty} F(\pounds(\Lambda_n)) = 0$.

Following the same steps as in Theorem 10, we can easily show that Υ has a fixed point in $\Lambda_\infty \subset \Lambda$.

Definition 17. Let $\Lambda \in \Omega$. Then, the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is the Hardy-Rogers $F_\pounds$-contraction, if we can find $\tau \in S$, $F \in F$, and $A_0 \subset \Xi$ such that

$$\begin{aligned}\pounds(\Upsilon(A)) > 0 \Rightarrow \tau(\pounds(A)) + \mathscr{F}(\pounds(\Upsilon(A))) \\ \le \mathscr{F}(\lambda_1\pounds(A) + \lambda_2\pounds(A_0) + \lambda_3\pounds(\Upsilon(A)) \\ + \lambda_4\pounds(\Upsilon(A_0))).\end{aligned} \tag{31}$$

for all $A \subset \Lambda$, where $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \le 1$ with $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$.

Remark 18.

(1) If $\lambda_2 = 0$, then (31) is a Reich-type $F_\pounds$-contraction

(2) If $\lambda_2 = \lambda_3 = \lambda_4 = 0$ and $\lambda_1 = 1$, then the contraction (31) becomes a Darbo-type $F_\pounds$-contraction.

Theorem 19. *Let $\Lambda \in \Omega$. If the function $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous and a Hardy-Rogers-type $F_\pounds v$-contraction, then Υ has a fixed point in Λ.*

Proof. Construct a sequence such that

$$\Lambda_0 = \Lambda \text{ and } \Lambda_n = \overline{co}(\Upsilon\Lambda_{n-1}), \text{ for all } n \in \mathbb{N}. \tag{32}$$

Then, $\Lambda_{n+1} \subset \Lambda_n$ and $\Upsilon\Lambda_n \subset \Lambda_n \quad \forall n \in \mathbb{N}$.

Now, we discuss two cases subject to £. If we consider m as a nonnegative integer with $\pounds(\Lambda m) = 0$, then Λ_m is a compact. But $\Upsilon : \Lambda_m \longrightarrow \Lambda_m$, so by Theorem 2, Υ has a fixed point in $\Lambda_m \subset \Lambda$. Instead, let us take $\pounds(\Lambda_n) > 0, \forall n \in \mathbb{N}$. From (9), we write $\pounds(\Lambda_{n+1}) < \pounds(\Lambda_n)$, that is $\{\pounds(\Lambda_n)\}$ is a decreasing sequence and hence converges $s \in \mathbb{R}$ with $s \ge 0$. Now since $\pounds(\Lambda_n) \in (0,\infty)$ and $s \in [0,\infty)$, so by assumption on τ, $\lim_{t\to s^+} \inf \tau(\pounds(\Lambda_n)) > 0$, we can find $r > 0$ and $n_0 \in \mathbb{N}$ such that $\tau(\pounds(\Lambda_n)) \ge r \quad \forall n \ge n_0$. Using (31) with $\mathbb{A} = \Lambda_n$ and $\mathbb{A}_0 = \Lambda_{n+1}$, we have

$$\begin{aligned}\tau(\pounds(\Lambda_n)) + \mathscr{F}\left(\pounds(\Lambda_{n+1}\right) &= \tau(\pounds(\Lambda_n)) + \mathscr{F}(\pounds)(\overline{co}(\Upsilon(\Lambda_n)))) \\ &= \tau(\pounds(\Lambda_n)) + \mathscr{F}\left(\pounds\left(\Upsilon(\Lambda_n + 1)\right)\right) \\ &\le \mathscr{F}(\lambda_1\pounds(\Lambda_n) + \lambda_2\pounds(\Lambda_n + 1) \\ &\quad + \lambda_3\pounds\left(\pounds\left(\Upsilon(\Lambda_n)\right)\right) + \lambda_4\pounds\left(\Upsilon(\Lambda_n + 1)\right)) \\ &< \mathscr{F}(\lambda_1\pounds(\Lambda_n) + \lambda_2\pounds(\Lambda_n + 1) + \lambda_3\pounds,\end{aligned} \tag{33}$$

but $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \le 1$ and $\mathscr{F}$ is nondecreasing; thus, we have

$$\mathscr{T}(\pounds(\Lambda_n)) + \mathscr{F}(\pounds(\Lambda_{n+1})) \le \mathscr{F}(\pounds(\Lambda_n)). \tag{34}$$

The rest of the proof is analogous to that of Theorem 10.

We provide the following result, the proof is easy, so we omit it.

Theorem 20. *Let $\Lambda \in \Omega$. If the function $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous and a Reich $F_\pounds$-contraction then Υ has a fixed point in Λ.*

3. Darbo-Type Result via F-Expansion

In this section, we introduce the Darbo-type $\mathscr{F}$-expanding mapping and establish some fixed point results.

Definition 21. Let $\Lambda \in \Omega$. Then the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is $F_\pounds$-expanding if there exist $\tau \in S$ and $F \in F$ such that

$$\pounds(\Lambda_0) > 0 \Rightarrow \mathscr{F}(\pounds(\Upsilon(\Lambda_0))) \ge \mathscr{F}(\pounds(\Lambda_0)) + \tau(\pounds(\Lambda_0)), \tag{35}$$

where $\Lambda_0 \subset \Lambda$.

In the light of $\mathscr{F}_\pounds$-expanding mapping, we present the following result.

Theorem 22. *Let $\Lambda \in \Omega$. If the mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is continuous, surjective, and $\mathscr{F}_\pounds$-expanding, then Υ has a fixed point in Λ.*

Proof. Since $\Upsilon : \Lambda \longrightarrow \Lambda$ is surjective, so by Lemma 8, we can find a function $\Upsilon^* : \Lambda \longrightarrow \Lambda$ such that $\Upsilon \circ \Upsilon^*$ is the identity function on Λ. Let Λ_1 and Λ_2 be any subsets of Λ such that $\Lambda_2 = \Upsilon^*(\Lambda_1)$. Assume that $\pounds(\Lambda_2) > 0$; then on using (35), we can write

$$\mathscr{F}(\pounds(\Upsilon(\Lambda_2))) \ge \mathscr{F}(\pounds(\Lambda_2)) + \tau(\pounds(\Lambda_2)). \tag{36}$$

Since $\Upsilon(\Lambda_2) = \Upsilon(\Upsilon^*(\Lambda_1)) = (\Upsilon \circ \Upsilon^*)(\Lambda_1) = \Lambda_1$, then (36) becomes

$$\mathscr{F}(\pounds(\Lambda_1)) \geq \mathscr{F}(\pounds(\Upsilon^*(\Lambda_1))) + \tau(\pounds(\Upsilon^*(\Lambda_1))). \quad (37)$$

Now, if u is fixed point of Υ^*, then $\Upsilon u = \Upsilon(\Upsilon^* u) = u$. Thus, to show that Υ has a fixed point, it is sufficient to show that Υ^* has a fixed point. To do this, construct a sequence such that

$$\Lambda_0 = \Lambda \text{ and } \Lambda_n = \overline{co}(\Upsilon^* \Lambda_{n-1}), \text{ for all } n \in \mathbb{N}. \quad (38)$$

Then, we can easily show that $\Lambda_{n+1} \subset \Lambda_n$ and $\Upsilon^* \Lambda_n \subset \Lambda_n$.

Next, if we take an integer $m \geq 0$ with $\pounds(\Lambda_m) = 0$, then Λ_m is compact. So, by Theorem 2, Υ^* has a fixed point in $\Lambda_m \subset \Lambda$. Now, let us take $\pounds(\Lambda_n) > 0$, for all $n \in N$. We have to prove that $\Lambda_\infty \subset \Lambda_n$ is a nonempty, bounded, closed, and convex subset of Ξ. For this, since the sequence $\{\pounds(\Lambda_n)\}$ is decreasing, it converges to $s \geq 0$. Now, assume that $\pounds(\Lambda_{n+1}) > 0$. Then

$$0 < \pounds(\Lambda_{n+1}) = \pounds(\overline{co}(\Upsilon^*(\Lambda_n))) = \pounds(\Upsilon^*(\Lambda_n)), \quad (39)$$

that is, $\pounds(\Upsilon^*(\Lambda_n)) \in (0,\infty)$ and $s \in [0,\infty)$, so by assumption on τ, $\liminf_{t \to s^+} \tau(\pounds(\Upsilon^*(\Lambda_n))) > 0$, we can find $r > 0$ and $n_0 \in N$ such that $\tau(\pounds(\Upsilon^*(\Lambda_n))) \geq r$ for all $n \geq n_0$. Using (37) with $\Lambda = \Lambda_n$, we write

$$\begin{aligned}\tau(\pounds(\Upsilon^*(\Lambda_n))) + \mathscr{F}(\pounds(\Lambda_{n+1})) &= \tau(\pounds(\Upsilon^*(\Lambda_n))) + \mathscr{F}(\pounds(\overline{co}(\Upsilon^*(\Lambda_n)))) \\ &= \tau(\pounds(\Upsilon^*(\Lambda_n))) + \mathscr{F}(\pounds(\Upsilon^*(\Lambda_n))) \\ &\leq \mathscr{F}(\pounds(\Lambda_n)).\end{aligned} \quad (40)$$

From here, we write

$$\mathscr{F}(\pounds(\Lambda_{n+1})) \leq \mathscr{F}(\pounds(\Lambda_n)) - \tau(\pounds(\Upsilon^*(\Lambda_n))) \leq \mathscr{F}(\pounds(\Lambda_n)) - r. \quad (41)$$

Consequently,

$$\mathscr{F}(\pounds(\Lambda_n)) \leq \mathscr{F}(\pounds(\Lambda_{n-1})) - r. \quad (42)$$

By routine calculation, one can easily obtain

$$\mathscr{F}(\pounds(\Lambda_n)) \leq \mathscr{F}(\pounds(\Lambda_{n0})) - (n - n_0)r, \ \forall n > n_0. \quad (43)$$

Clearly, $\lim_{n \to +\infty} \mathscr{F}(\pounds(\Lambda_n)) = -\infty$ and using the property (F_2), we can write $\lim_{n \to +\infty} \pounds(\Lambda_n) = 0$. Thus, by Definition 3$(vi)$, $\Lambda_\infty = \cap_{n=1}^{+\infty}$ is nonempty and $\Upsilon^* \Lambda_\infty \subset \Lambda_\infty$ as Υ^* $\Lambda_n \subset \Lambda_n$. Also, since $\Lambda_\infty \subset \Lambda_n$, for all $n \in \mathbb{N}$, so by Definition 3(ii), $\pounds(\Lambda_\infty) \leq \pounds(\Lambda_n)$, for all $n \in \mathbb{N}$. Thus, $\pounds(\Lambda_\infty) = 0$, and hence, $\Lambda_\infty \in \ker\pounds$, that is Λ_∞ is bounded. But $\Lambda_\infty$is closed such that Λ_∞ is compact. Therefore, by Theorem 2, Υ^* has a fixed point in $\Lambda_\infty \subset \Lambda$. Consequently, Υ has a fixed point in $\Lambda_\infty \subset \Lambda$.

From Theorem 22, we can deduce several pivotal results. We demonstrate some preferable corollaries that cover and extend several known theorems in the literature. The special case, if we take $\mathscr{F}(x) = \ln x, x > 0$, in Theorem 22, we deduce the following corollary.

Corollary 23. *Let $\Lambda \in \Omega$ and $\Upsilon : \Lambda \longrightarrow \Lambda$ be a surjective and continuous mapping such that*

$$\pounds(\Upsilon(\Lambda)) \geq \pounds(\Lambda) e^{\tau(\pounds(\Lambda))}, \text{ for all } \Lambda \subset \Lambda.lim \quad (44)$$

Then Υ has a fixed point in Λ.

If we take $\mathscr{F}(x) = \ln x + x, x > 0$, in Theorem 22, we get the following corollary.

Corollary 24. *Let $\Lambda \in \Omega$ and $\Upsilon : \Lambda \longrightarrow \Lambda$ be a surjective and continuous mapping such that*

$$\pounds(\Upsilon(\Lambda)) \geq \pounds(\Lambda) e^{2\tau(\pounds(\Lambda)) - \pounds(\Upsilon(\Lambda))}, \text{ for all } \Lambda \subset \Lambda. \quad (45)$$

Then, Υ has a fixed point in Λ.

If we take $\mathscr{F}(x) = \ln(x^2 + x), x > 0$, in Theorem 22, we deduce the following corollary.

Corollary 25. *Let $\Lambda \in \Omega$ and $\Upsilon : \Lambda \longrightarrow \Lambda$ be a surjective and continuous mapping such that*

$$\pounds(\Upsilon(\Lambda))(\pounds(\Upsilon(\Lambda)) + 1) \geq \pounds(\Lambda)(\pounds(\Lambda) + 1) e^{\tau(\pounds(\Lambda))}, \text{ for all } \Lambda \subset \Lambda. \quad (46)$$

Then Υ has a fixed point in the set Λ.

If we take $\mathscr{F}(x) = \arctan(-1/x)$ with $x > 0$, in Theorem 22, we deduce the following corollary.

Corollary 26. *Let $\Lambda \in \Omega$ and $\Upsilon : \Lambda \longrightarrow \Lambda$ be a surjective and continuous mapping such that*

$$\pounds(\Upsilon(\Lambda)) \geq \frac{\pounds(\Lambda) + tan\tau(\pounds(\Lambda))}{1 - tan\tau(\pounds(\Lambda)).\pounds(\Lambda)}, \text{ for all } \Lambda \subset \Lambda. \quad (47)$$

Then, Υ has a fixed point in Λ.

4. Applications

This section deals with some practicing of our fixed point results. Let $(\mathfrak{E}, \|.\|)$ be a Banach space having the zero element 0. Let $\boldsymbol{B}(a, r)$ be the closed ball with center a and radius r and $\boldsymbol{B}_r$ be the ball $\boldsymbol{B}(0, r)$. Our aim is to illustrate sufficient conditions for the existence of a solution of a nonlinear fractional-order differential equation:

$${}^cD^{\wp} u(t) = \psi(t, u(t)), t \in [0, \Delta], \Delta \geq 1, \quad (48)$$

under the integral type boundary conditions:

$$p_1 u(0) + q_1 u(\Delta) = I^{\wp} \phi_1(\Delta, u(\Delta)), \quad (49)$$

$$p_2 u'(0) + q_2 u'(\Delta) = I^{\wp} \phi_2(\Delta, u(\Delta)), \quad (50)$$

where $\wp\in(1,2]$, ${}^cD$ is the Caputo fractional derivative, ψ, $\varphi_1,\varphi_2:[0,\triangle]\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions, and p_k and q_k $(k=1,2)$ are positive real numbers.

Lemma 27 (see [16]). *For $p_j\in\mathbb{R}$, $j=0,1,2,\cdots,r-1$, we have*

$$I^{\wp}[{}^cD^{\wp}h(t)]=h(t)+p_0+p_1t+p_2t^2+\ldots\cdots+p_{r-1}t^{r-1}. \quad (51)$$

To proceed further, we convert the nonlinear fractional-order differential equation (48) to an integral equation. For this, we prove the following lemma.

Lemma 28. *A solution of the fractional-order boundary value problem (48) is*

$$\begin{aligned}u(t)&=I^{\wp}\psi(t,u(t))-\frac{q_1}{p_1+q_1}I^{\wp}\psi(\Delta,u(\Delta))+\frac{q_2}{p_2+q_2}\\&\quad\cdot\left(\frac{q_1\Delta}{(p_1+q_1)}-t\right)I^{\wp-1}\psi(\Delta,u(\Delta))+\frac{1}{p_1+q_1}I^{\wp}\psi(\Delta,u(\Delta))\\&\quad+\frac{1}{p_2+q_2}\left(t-\frac{q_1\Delta}{(p_1+q_1)}\right)I^{\wp}\varphi_2(\Delta,u(\Delta)).\end{aligned} \quad (52)$$

Proof. First of all, apply the Riemma-Liouville fractional integrable operator $I^{\wp}$ of order $\wp$ to equation (48), and using Lemma 27, we can easily deduce that

$$u(t)=C_0+C_1+I^{\wp}\psi(t,u(t)), \quad (53)$$

and by differentiating (53), we get

$$u'(t)=C_0+C_1t+I^{\wp-1}\psi(t,u(t)). \quad (54)$$

But $u(0)=C_0$, $u'(0)=C_1$, $u(\Delta)=C_0+C_1\Delta+I^{\wp}\psi(\Delta,u(\Delta))$, and $u'(\Delta)=C_1+I^{\wp-1}\psi(\Delta,u(\Delta))$.

Substituting the values of u (0) and u (Δ) in (49), we get

$$\begin{aligned}C_0&=-\frac{q_1\Delta}{p_1+q_1}C_1-\frac{q_1}{p_1+q_1}I^{\wp}\psi(\Delta,u(\Delta))\\&\quad+\frac{1}{p_1+q_1}I^{\wp}\varphi_1(\Delta,u(\Delta)).\end{aligned} \quad (55)$$

Similarly, substituting the values of $u'(0)$ and $u'(\Delta)$ in (50), we get

$$C_1=-\frac{q_2}{p_2+q_2}I^{\wp-1}\psi(\Delta,u(\Delta))+\frac{1}{p_2+q_2}I^{\wp}\phi_2(\Delta,u(\Delta)). \quad (56)$$

Putting the value of C_1 in (55), we deduce

$$\begin{aligned}C_0&=-\frac{q_1q_2\Delta}{(p_1+q_1)(p_2+q_2)}I^{\wp-1}\psi(\Delta,u(\Delta))\\&\quad-\frac{q_1\Delta}{(p_1+q_1)(p_2+q_2)}I^{\wp}\phi_2(\Delta,u(\Delta))\\&\quad-\frac{q_1}{p_1+q_1}I^{\wp}\psi(\Delta,u(\Delta))+\frac{1}{p_1+q_1}I^{\wp}\phi_2(\Delta,u(\Delta)).\end{aligned} \quad (57)$$

Thus, by switching the values of C_0 and C_1 in (53) and by routine calculations, we get equation (52).

Notice that Lemma (49) indicates that the solution of differential equation (48) is equivalent to the solution of integral equation (52). Now, we are in a position to present the existence result.

Theorem 29. *Let $u,v\in\boldsymbol{B}_r$ and $\psi,\varphi_1,\varphi_2:[0,\Delta]\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous mapping such that*

$$|\psi(t,u(t))-\psi(t,u(t))|\le\frac{e^{-(2r+(1/k))}}{3\Delta^{\wp}(\wp\Delta+2)}\|u-v\|, \quad (58)$$

$$|\phi_1(t,u(t))-\phi_1(t,u(t))|\le\frac{e^{-(2r+(1/k))}}{3\Delta\wp}\|u-v\|, \quad (59)$$

$$|\phi_2(t,u(t))-\phi_2(t,u(t))|\le\frac{e^{-(2r+(1/k))}}{3\Delta\wp+1}\|u-v\|, \quad (60)$$

$$\left(\left(2+\frac{1}{r(\wp)}\right)\mathfrak{M}_1+\mathfrak{M}_2+\mathfrak{M}_3\right)\Delta^{\wp+1}\le r, \quad (61)$$

where $\wp\in(1,2]$, $\Bbbk>1$, $\Delta\ge1$, $\sup_{s\in[0,\Delta]}|\psi(s,u(s))|=\mathfrak{M}_1<\infty$, $\sup_{s\in[0,\Delta]}|\phi_1(s,u(s))|=\mathfrak{M}_2<\infty$, and $\sup_{s\in[0,\Delta]}\|\phi_1(s,u(s))\|=\mathfrak{M}_3<\infty$. Then, there exists a solution of the fractional-order integral equation (52) in $\boldsymbol{B}_r$. Accordingly, there exists a solution of the nonlinear fractional-order differential equation (48) in $\boldsymbol{B}_r$.

Proof. Let $\boldsymbol{B}_r=\{u\in C([0,\mathfrak{T}],\mathbb{R}):\|u\|\le r\}$. Then, $\boldsymbol{B}_r$ is a nonempty, closed, bounded, and convex subset of $(\mathfrak{E},\|.\|)$. Define $\tau,\mathscr{F}:(0,\infty)\longrightarrow\mathbb{R}$ by $\tau(x)=x+(1/\Bbbk)$ with $\Bbbk>1$ and $\mathscr{F}(x)=\ln x$ and the operator $\Upsilon:\boldsymbol{B}_r\rightarrow\boldsymbol{B}_r$ by

$$\begin{aligned}\Upsilon u(t)&=I^{\wp}\psi(t,u(t))-\frac{q_1}{p_1+q_1}I^{\wp}\psi(\Delta,u(\Delta))-\frac{q_2}{p_2+q_2}\\&\quad\cdot\left(\frac{q_1\Delta}{p_1+q_1}-t\right)I^{\wp-1}\psi(\Delta,u(\Delta))\\&=\frac{1}{p_1+q_1}I^{\wp}\phi_1(\Delta,u(\Delta))+\frac{1}{p_2+q_2}\\&\quad\cdot\left(t-\frac{q_1\Delta}{p_1+q_1}\right)I^{\wp}\phi_2(\Delta,u(\Delta)).\end{aligned} \quad (62)$$

First, we will show that $\Upsilon : \boldsymbol{Br} \longrightarrow \boldsymbol{Br}$ is well defined. Let $u \in \boldsymbol{Br}$, for some r. Then, for all $x \in [0, 4]$, we have

$$\begin{aligned}
|\Upsilon u(t)| \le{}& \frac{1}{\Gamma(\wp)}\int_0^t (t-s)^{\wp-1}|\psi(s,u(s))|ds \\
&+ \frac{q_1}{p_1+q_1}\frac{1}{\Gamma(\wp)}\int_0^\Delta (\Delta-s)^{\wp-1}|\psi(s,u(s))|ds \\
&+ \frac{q_2}{p_2+q_2}\left|\frac{q_1\Delta}{p_1+q_1}-t\right|\frac{1}{\Gamma(\wp-1)}\int_0^\Delta (\Delta-s)^{\wp-2}|\psi(s,u(s))|ds \\
&+ \frac{1}{p_1+q_1}\frac{1}{\Gamma(\wp)}\int_0^\Delta (\Delta-s)^{\wp-1}|\varphi_1(s,u(s))|ds \\
&+ \frac{1}{p_2+q_2}\left|t-\frac{q_1\Delta}{p_1+q_1}\right|\frac{1}{\Gamma(\wp)}\int_0^\Delta (\Delta-s)\wp^{\wp-1}|\varphi_1(s,u(s))|ds \\
\le{}& \frac{\|\psi(s,u(s))\|}{\Gamma(\wp)}\int_0^t (t-s)^{\wp-1}ds + \frac{\|\psi(s,u(s))\|}{\Gamma(\wp)}\int_0^\Delta (\Delta-s)^{\wp-1}ds \\
&+ \frac{\|\psi(s,u(s))\|\Delta}{\Gamma(\wp-1)}\int_0^\Delta (\Delta-s)^{\wp-2}ds + \frac{\|\varphi_1(s,u(s))\|}{\Gamma(\wp)}\int_0^\Delta (\Delta-s)^{\wp-1}ds \\
&+ \frac{\|\varphi_2(s,u(s))\|\Delta}{\Gamma(\wp)}\int_0^\Delta (\Delta-s)^{\wp-1}ds \le \frac{\mathfrak{M}_1}{\Gamma(\wp)}\frac{t^\wp}{\wp} + \frac{\mathfrak{M}_1}{\Gamma(\wp)}\frac{\Delta^\wp}{\wp} \\
&+ \frac{\mathfrak{M}_1\Delta}{\Gamma(\wp-1)}\frac{\Delta^{\wp-1}}{\wp-1} + \frac{\mathfrak{M}_2}{\Gamma(\wp)}\frac{\Delta^\wp}{\wp} + \frac{\mathfrak{M}_3\Delta}{\Gamma(\wp)}\frac{\Delta^\wp}{\wp} \\
\le{}& \left(\left(2+\frac{1}{\Gamma(\wp)}\right)\mathfrak{M}_1 + \mathfrak{M}_2 + \mathfrak{M}_3\right)\Delta^{\wp+1} \le r.
\end{aligned} \tag{63}$$

That is $\|\Upsilon(\mathrm{u})\| \le r$ for all $u \in \boldsymbol{B}_r$, which implies that $\Upsilon(\mathrm{u}) \in \boldsymbol{B}_r$, and hence, $\Upsilon : \boldsymbol{B}_r \longrightarrow \boldsymbol{B}_r$ is well defined.

Now, we have to show that $\Upsilon : \boldsymbol{B}_r \longrightarrow \boldsymbol{B}_r$ is continuous. For this, consider

$$\begin{aligned}
&\|\Upsilon u(t) - \Upsilon v(t)\| \\
&\le \frac{1}{\Gamma(\wp)}\int_0^t (t-s)^{\wp-1}\|\psi(s,u(s))-\psi(s,v(s))\|ds \\
&\quad+ \frac{q_1}{(p_1+q_1)\Gamma(\wp)}\int_0^\Delta (\Delta-s)^{\wp-1}\|\psi(s,u(s))-\psi(s,v(s))\|ds \\
&\quad+ \frac{q_2}{p_2+q_2}\left\|\frac{q_1}{p_1+q_1}-t\right\|\int_0^\Delta \frac{(\Delta-s)^{\wp-2}\|\psi(s,u(s))-\psi(s,v(s))\|}{\Gamma(\wp-1)}ds \\
&\quad+ \frac{1}{(p_1+q_1)\Gamma(\wp)}\int_0^\Delta (\Delta-s)^{\wp-1}\|\phi_1(s,u(s))-\phi_1(s,v(s))\|ds \\
&\quad+ \frac{1}{p_2+q_2}\left\|t-\frac{q_1\Delta}{p_1+q_1}\right\|\int_0^\Delta \frac{(\Delta-s)^{\wp-2}\|\phi_2(s,u(s))-\phi_2(s,v(s))\|}{\Gamma(\wp)}ds \\
&\le \frac{\|\psi(s,u(s))-\psi(s,v(s))\|\Delta^\wp}{\Gamma(\wp+1)} + \frac{\|\psi(s,u(s))-\psi(s,v(s))\|\Delta^\wp}{\Gamma(\wp+1)} \\
&\quad+ \frac{\|\psi(s,u(s))-\psi(s,v(s))\|\Delta^{\wp+1}}{\Gamma(\wp)} + \frac{\|\phi_1(s,u(s))-\phi_1(s,v(s))\|\Delta^\wp}{\Gamma(\wp+1)} \\
&\quad+ \frac{\|\varphi_2(s,u(s))-\phi_2(s,v(s))\|\Delta^{\wp+1}}{\Gamma(\wp+1)} \\
&\le \Delta^{\wp+1}\|\varphi_2(s,u(s))-\phi_2(s,v(s))\|.
\end{aligned} \tag{64}$$

Since ψ, φ_1, and φ_2 are continuous, so for $\epsilon_1 = 1/\Delta^\wp(\wp\Delta+2)$, $\epsilon_2 = 1/\Delta^\wp$, $\epsilon_3 = 1/\Delta^{\wp+1} > 0$, there exist $\delta_1, \delta_2, \delta_3 > 0$ such that

$$\begin{aligned}
&\|\psi(s,u(s))-\psi(s,v(s))\| < \frac{\epsilon_1}{3}, \text{whenever } \|u-v\| < \delta_1, \\
&\|\phi_1(s,u(s))-\phi_1(s,v(s))\| < \frac{\epsilon_2}{3}, \text{whenever } \|u-v\| < \delta_2, \\
&\|\phi_2(s,u(s))-\phi_2(s,v(s))\| < \frac{\epsilon_3}{3}, \text{whenever } \|u-v\| < \delta_3.
\end{aligned} \tag{65}$$

If $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$ and $\delta = \min\{\delta 1, \delta 2, \delta 3\}$, then we conclude that $\|\Upsilon v(t)\Upsilon v(t)\| < \epsilon$, whenever $\|u-v\| < \delta$.

Hence, $\Upsilon : \mathbf{B}_r \longrightarrow \mathbf{B}_r$ is continuous. Next, we have to show that $\Upsilon : \mathbf{B}_r \longrightarrow \mathbf{B}_r$ is an $F_\pounds$-weak contraction. Let Λ_1 and Λ_2 be any two subsets of $\mathbf{B}_r$ and $u, v \in \Lambda_1$. Then, by conditions (58), (59), and (60), we have

$$\begin{aligned}
\mathrm{diam}(\Upsilon\Lambda_1) &= \|\Upsilon u(t) - \Upsilon v(t)\| \\
&\le \Delta^\wp(\wp\Delta+2)\|\psi(s,u(s))-\psi(s,v(s))\| \\
&\quad+ \Delta^\wp\|\varphi_1(s,u(s))-\varphi_1(s,v(s))\| \\
&\quad+ \Delta^{\wp+1}\|\varphi_2(s,u(s))-\varphi_2(s,v(s))\| \\
&\le e^{-(2r+1/k)}\|u-v\| \\
&= e - \left(\mathrm{diam}(\Lambda_1) + \frac{1}{k}\right)\mathrm{diam}(\Lambda_1) \\
&\le e^{-(\mathrm{diam}(\Lambda_1)+1/k}\max\{\mathrm{diam}(\Lambda_1), \\
&\quad \mathrm{diam}(\Upsilon(\Lambda_1)), \mathrm{diam}(\Upsilon(\Lambda_2)), \\
&\quad \mathrm{diam}(\Upsilon(\Lambda_1)) \cup (\Upsilon(\Lambda_2))\}.
\end{aligned} \tag{66}$$

From here, we write

$$\begin{aligned}
\pounds(\Upsilon\Lambda_1) = e^{-(\pounds(\Lambda_1)+1/k)}\max\Big\{&\pounds(\Lambda_1), \pounds(\Upsilon(\Lambda_1)), \pounds(\Upsilon(\Lambda_2)), \frac{1}{2}\pounds(\Upsilon\Lambda_1) \\
&\cup (\Upsilon(\Lambda_2))\Big\}.
\end{aligned} \tag{67}$$

Consequently,

$$\mathrm{In}(\pounds(\Upsilon\Lambda_1)) \le -\left(\pounds(\Lambda_1) + \frac{1}{\Bbbk}\right) + \mathrm{In}(\Delta(\Lambda_1, \Lambda_2)), \tag{68}$$

which implies that

$$\tau(\pounds(\Lambda_1)) + \mathscr{F}(\pounds(\Upsilon\Lambda_1)) \le \mathscr{F}(\Delta(\Lambda_1, \Lambda_2)). \tag{69}$$

That is $\Upsilon : \mathbf{B}_r \longrightarrow \mathbf{B}_r$ is an $F_\pounds$-weak contraction. Thus, by Theorem 10, Υ has a fixed point in $\mathbf{B}_r$. Consequently, equation (52) has a solution in $\mathbf{B}_r$.

To illustrate the existence result (Theorem 29), we present an example.

Example 2. Consider the nonlinear fractional-order differential equation

$$\begin{cases} {}^{c}D^{3/4}u(t) = \dfrac{e^{-t-2}\sin(u(t))}{(t+7)^3}, t \in [0,3], \\ 2u(0) + 3u(3) = \displaystyle\int_0^3 \dfrac{(3-s)^{1/2}\cos(u(s))}{\Gamma 3/2(s+4)^4} ds, \\ \dfrac{1}{3}u'(0) + \dfrac{1}{2}u'(3) = \displaystyle\int_0^3 \dfrac{(3-s)^{1/2}e^{-(u(s)+6)}}{\Gamma 3/2} ds, \end{cases} \quad (70)$$

where ${}^{c}D$ is the Caputo fractional-order derivative and ψ, $\varphi_1, \varphi_2 : [0,3] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions defined as

$$\psi(t,u(t)) = \frac{e^{-t-2}\sin(u(t))}{(t+7)^3}, \phi_1(t,\mathrm{u}(t)) = \frac{\cos(u(t))}{(t+4)^4}, \text{and } \phi_2(t,u(t))e^{-(u(t)+6)}. \quad (71)$$

Now, we have to verify the conditions of Theorem 29. Let $\Lambda 1 \subset \boldsymbol{B}1$ and $u, v \in \Lambda_1$; then, for$k = 10, t \in [0,3], \Delta = 3,$ and $\wp = 3/2$, we first show condition (58) of Theorem 29. To do this, we have

$$\begin{aligned} \|\psi(t,u(t)) - \psi(t,v(t))\| &= \frac{e^{-t-2}}{(t+7)^3}\|\sin(u(t)) - \sin(v(t))\| \\ &\le \frac{e^{-2-(1/10)}}{3 \times 3^{(3/2)}((3/2)\times 3 + 2)}\|u-v\| \\ &= \frac{e^{-(2r+(1/k))}}{3\Delta^{\wp}(\wp\Delta + 2)}\|u-v\|. \end{aligned} \quad (72)$$

Next, to show condition (59) of Theorem 29, we have

$$\begin{aligned} \|\phi_1(t,u(t)) - \phi_2(t,v(t))\| &= \left\|\frac{\cos(u(t))}{(t+4)^4} - \frac{\cos(v(t))}{(t+4)^4}\right\| \\ &\le \frac{1}{(t+4)^4}\|u-v\| \\ &\le \frac{e{-2-(1/10)}}{3 \times 3^{3/2}}\|u-v\| \\ &= \frac{e^{-(2r+(1/k))}}{3\Delta^{\wp}}\|u-v\|. \end{aligned} \quad (73)$$

Also, to show condition (60) of Theorem 29, we have

$$\begin{aligned} \|\phi_2(t,u(\mathrm{t})) - \phi_2(t,v(\mathrm{t}))\| &= \left\|e^{-(u(t)+6)} - e^{-(v(t)+6)}\right\| \\ &\le e^{-6}\|u\text{-}v\| \le \frac{e^{-2-(1/10)}}{3 \times 3^{5/2}}\|u\text{-}v\| \\ &= \frac{e^{-2r+(1/k)}}{3\Delta^{\wp+1}}\|u\text{-}v\|. \end{aligned} \quad (74)$$

Finally, we have to verify condition (75) of Theorem 29. For this, since $M_1 < 0.0003945, M_2 < 0.003906,$ and $M_3 \approx 0.0024787$, we get

$$\left(\left(2 + \frac{1}{\Gamma(\wp)}\right)M_{\mathbf{1}} + M_2 + M_3\right)\Delta^{\wp+1} < 0.117977 < 1 = r. \quad (75)$$

Thus, all the conditions of Theorem 29 are satisfied. Hence the nonlinear fractional-order differential equation (70) has a solution in $\boldsymbol{B}_1$.

5. Conclusion

Through measure of noncampactness, various new F-contraction and F-expanding mappings have been presented. In the Banach spaces, fixed point results were established, from which several existing results can be extracted. For the accuracy of our results, we have checked the existence of a solution to the nonlinear fractional-order differential equation under the integral-type boundary conditions with an example.

Data Availability

Not applicable.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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