Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(5): 4280–4295.
DOI:10.3934/math.2021253
Received: 31 October 2020
Accepted: 18 January 2021
Published: 07 February 2021

Research article

Hermite-Hadamard-Fejer type inequalities via fractional integral of a function concerning another function

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Abstract: In this paper, we at first develop a generalized integral identity by associating Riemann-Liouville (RL) fractional integral of a function concerning another function. By using this identity estimates for various convexities are accomplish which are fractional integral inequalities. From our results, we obtained bounds of known fractional results which are discussed in detail. As applications of the derived results, we obtain the mid-point-type inequalities. These outcomes might be helpful in the investigation of the uniqueness of partial differential equations and fractional boundary value problems.

Keywords: Hermite-Hadamard-Fejer inequalities; convex function; generalized fractional integral; mid-point inequality; Riemann-Liouville
Mathematics Subject Classification: 26D15, 26A33, 26A51, 26D07, 26D10

1. Introduction

Fractional calculus is an important research field, not only in pure mathematics but in applied mathematics, physics, biology, engineering, economics, etc., as well. In fact, by considering derivatives and integrals of arbitrary real or complex order, we may model more efficiently certain real phenomena. Applications have been found e.g., in human body modelling [1, 2], heat conduction [3], viscoelasticity [4, 5], time series analysis [6, 7], circuits [8], material sciences [9], shear waves [10], etc. Fractional integral operators play a leading and keen role in the development of fractional calculus. The first formulation of a fractional integral operator is due to a continuous study of well-renowned mathematicians and physicists. This fractional integral is well known as RL fractional integral operator. After its existence, many other fractional integral and fractional derivative operators have been introduced. Recently, scientists in their diverse fields are working in the environment of fractional calculus in new directions of respective fields developing rapidly.

Convex functions are very close to the theory of inequalities. Many known and useful inequalities are consequences of convex functions. Some very natural inequalities for example Jensen inequality, Hadamard inequality interpret convex functions beautifully. Fractional integral inequalities are very useful in the study of fractional partial as well as ordinary differential equations. These inequalities establish the uniqueness and bounds of their solutions. In this paper, we study a general form of RL fractional integrals via convex functions. We start with the definition of a convex function.

In 1905 Jensen [11] present the definition of convex function as follows.

Definition 1.1. *A function $\psi : [a, b] \to \mathbb{R}$ is said to be convex if the inequality*

$$\psi(\theta x + (1-\theta)y) \leq \theta\psi(x) + (1-\theta)\psi(y)$$

holds for all $x, y \in [a, b]$ and all $\theta \in [0, 1]$.

Brechner introduce the definition of s-convex function in [12] which is defined as follows.

Definition 1.2. *A function $\psi : (0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if the inequality*

$$\psi(\theta x + \zeta y) \leq \theta^s\psi(x) + \zeta^s\psi(y)$$

holds for some fixed $s \in (0, 1]$ and for all $x, y \in (0, \infty), , \theta, \zeta > 0$, where $\theta + \zeta = 1$. The class of s-convex functions of the second sense usually denoted by K_s^2.

Definition 1.3. *The incomplete beta function $B_x(\theta, \zeta)$ [13] is defined by*

$$B_x(\theta, \zeta) = \int_0^x y^{\theta-1}(1-y)^{\zeta-1}dy,$$

where $0 < x < 1$, $\theta, \zeta > 0$.

Definition 1.4. *The left-sided and right-sided RL fractional integrals $I^{\varrho}_{\theta^+}\psi$ and $I^{\varrho}_{\zeta^-}\psi$ of order $\varrho > 0$ on a finite interval $[\theta, \zeta]$ are defined by*

$$I^{\varrho}_{\theta^+}\psi(x) = \frac{1}{\Gamma(\varrho)}\int_\theta^x (x-\nu)^{\varrho-1}\psi(\nu)d\nu, \quad x > \theta,$$

and

$$I^{\varrho}_{\zeta^-}\psi(x) = \frac{1}{\Gamma(\varrho)}\int_x^{\zeta}(\nu - x)^{\varrho-1}\psi(\nu)d\nu, \quad x < \zeta$$

respectively. Here, $\Gamma(.)$ *represents the usual Euler gamma function with integral representation*

$$\Gamma(x) = \int_0^{\infty} \nu^{x-1}e^{-\nu}d\nu, \ Re(x) > 0.$$

Now, we recall the definition of fractional integrals of real valued function concerning to another function [14].

Definition 1.5. *Let* $p : [\theta, \zeta] \to \mathbb{R}$ *be an increasing and positive function on* $(\theta, \zeta]$*, having a continuous derivative* p' *on* (θ, ζ)*. The left and right-sided RL fractional integrals of* ψ *with respect to the function* p *on* $[\theta, \zeta]$ *of order* $\varrho > 0$ *are defined respectively by*

$$\Im^{\varrho}_{\theta^+;p}\psi(x) = \frac{1}{\Gamma(\varrho)}\int_{\theta}^{x}\frac{p'(\nu)}{[p(x) - p(\nu)]^{1-\varrho}}\psi(\nu)d\nu, \ x > \theta$$

$$\Im^{\varrho}_{\zeta^-;p}\psi(x) = \frac{1}{\Gamma(\varrho)}\int_{x}^{\zeta}\frac{p'(\nu)}{[p(\nu) - p(x)]^{1-\varrho}}\psi(\nu)d\nu, \ x < \zeta$$

provided that the integrals exists.

Note that
(i) If we take $p(x) = x$, then we get the Definition 1.4 of RL fractional integrals.
(ii) If we take $p(x) = \frac{x^{\nu}}{\nu}$, $\nu > 0$, then we get the definition of Katugampola fractional integrals given in [15].
(iii) If we take $p(x) = \frac{x^{\nu+s}}{\nu+s}$, then we get the definition of generalized conformable fractional integrals defined by Khan *et al.* in [16].
(iv) If we take $p(x) = \frac{(x-\theta)^s}{s}$, $s > 0$ in left and $p(x) = \frac{(\zeta-x)^s}{s}$, $s > 0$ in right, then we get the definition of conformable fractional integrals defined by Jarad *et al.* in [17].

2. Main results

In this section, we develop an integral identity in the form of the left-sided and right-sided RL fractional integrals of a function concerning to another function. With the help of this identity, we further establish some new inequalities for classical convex and s-convex function.

Lemma 2.1. *Let* $p : [\theta, \zeta] \to \mathbb{R}$ *be an increasing and positive function on* $(\theta, \zeta]$*, having a continuous derivative* p' *on* (θ, ζ)*. Let* $\psi : [\theta, \zeta]$ *be a differentiable mapping on* (θ, ζ) *with* $\theta < \zeta$ *and* $g : [\theta, \zeta] \to \mathbb{R}$ *be bounded. If* $\psi', g \in L[\theta, \zeta]$*, then following identity holds:*

$$\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g(\theta)\Big] - \Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g\psi(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g\psi(\theta)\Big]$$

$$= \frac{1}{\Gamma(\varrho)} \int_{\theta}^{\zeta} \kappa(\nu)\psi'(\nu)d\nu,$$

where

$$\kappa(t) = \begin{cases} \int_{\theta}^{t} \frac{p'(u)}{[p(u)-p(\theta)]^{1-\varrho}} g(u)du, & t \in [\theta, \frac{\theta+\zeta}{2}], \\ \int_{\zeta}^{t} \frac{p'(u)}{[p(\zeta)-p(u)]^{1-\varrho}} g(u)du, & t \in [\frac{\theta+\zeta}{2}, \zeta]. \end{cases}$$

Proof. Consider

$$\begin{aligned} I &= \int_{\theta}^{\zeta} \kappa(\nu)\psi'(\nu)d\nu \\ &= \int_{\theta}^{\frac{\theta+\zeta}{2}} \kappa(\nu)\psi'(\nu)d\nu + \int_{\frac{\theta+\zeta}{2}}^{\zeta} \kappa(\nu)\psi'(\nu)d\nu \\ &= I_1 + I_2. \end{aligned} \tag{2.1}$$

Now, integrating I_1 by parts, we get

$$\begin{aligned} I_1 &= \int_{\theta}^{\frac{\theta+\zeta}{2}} \kappa(\nu)\psi'(\nu)d\nu \\ &= \kappa\Big(\frac{\theta+\zeta}{2}\Big)\psi\Big(\frac{\theta+\zeta}{2}\Big) - \int_{\theta}^{\frac{\theta+\zeta}{2}} \frac{p'(\nu)}{[p(\nu)-p(\theta)]^{1-\varrho}} g(\nu)\psi(\nu)d\nu \\ &= \psi\Big(\frac{\theta+\zeta}{2}\Big)\int_{\theta}^{\frac{\theta+\zeta}{2}} \frac{p'(u)}{[p(u)-p(\theta)]^{1-\varrho}} g(u)du - \int_{\theta}^{\frac{\theta+\zeta}{2}} \frac{p'(\nu)}{[p(\nu)-p(\theta)]^{1-\varrho}} (g\psi)(\nu)d\nu \\ &= \Gamma(\varrho)\Big[\psi\Big(\frac{\theta+\zeta}{2}\Big)\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g(\theta) - \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}(g\psi)(\theta)\Big]. \end{aligned}$$

Similarly, integrating I_2 by parts, we get

$$I_2 = \Gamma(\varrho)\Big[\psi\Big(\frac{\theta+\zeta}{2}\Big)\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g(\zeta) - \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}(g\psi)(\zeta)\Big].$$

Therefore, by substituting the values of I_1 and I_2 in (2.1), we get the desired result.

Corollary 2.2. *If we choose $p(\nu) = \nu$ in Lemma 2.1, then it reduce to [18, Lemma 4], i.e.,*

$$\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^+} g(\zeta) + I^{\varrho}_{(\frac{\theta+\zeta}{2})^-} g(\theta)\Big] - \Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^+} g\psi(\zeta) + I^{\varrho}_{(\frac{\theta+\zeta}{2})^-} g\psi(\theta)\Big]$$

$$= \frac{1}{\Gamma(\varrho)} \int_{\theta}^{\zeta} \kappa(\nu)\psi'(\nu)d\nu,$$

where

$$\kappa(t) = \begin{cases} \int_{\theta}^{t} (u-\theta)^{\varrho-1} g(u)du, & t \in [\theta, \frac{\theta+\zeta}{2}], \\ \int_{\zeta}^{t} (\zeta-u)^{\varrho-1} g(u)du, & t \in [\frac{\theta+\zeta}{2}, \zeta]. \end{cases}$$

Theorem 2.3. *Let $\varrho > 0$ and $p : [\theta, \zeta] \to \mathbb{R}$ be an increasing and positive function on $(\theta, \zeta]$, having a continuous derivative p' on (θ, ζ). Let $g : [\theta, \zeta] \to \mathbb{R}$ be nonnegative, integrable and ψ is a convex function on $[\theta, \zeta]$, then the following Hermite-Hadamard-Fejer type inequality for generalized fractional integrals holds.*

$$\begin{aligned} &\Psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g(\theta)\Big] \leq \Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g\Psi(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g\Psi(\theta)\Big] \\ &\leq \frac{\Psi(\theta)+\Psi(\zeta)}{2}\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g(\theta)\Big], \end{aligned} \tag{2.2}$$

where Ψ is defined as $\Psi(\nu) = \psi(\nu) + \tilde{\psi}(\nu)$ and $\tilde{\psi}(\nu) = \psi(\theta + \zeta - \nu)$.

Proof. Since ψ is convex on $[\theta, \zeta]$, for all $x, y \in [\theta, \zeta]$, we have

$$\psi\Big(\frac{x+y}{2}\Big) \leq \frac{\psi(x)+\psi(y)}{2}.$$

Now, for $\nu \in [0, 1]$, choose $x = \frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta$ and $y = \frac{2-\nu}{2}\theta + \frac{\nu}{2}\zeta$, we get

$$2\psi\Big(\frac{\theta+\zeta}{2}\Big) \leq \psi\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big) + \psi\Big(\frac{2-\nu}{2}\theta + \frac{\nu}{2}\zeta\Big). \tag{2.3}$$

Multiplying both sides of (2.3) by

$$\frac{\zeta-\theta}{2\Gamma(\varrho)} \frac{p'(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta)g(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta)}{(p(\zeta) - p(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta))^{1-\varrho}},$$

and integrating the resulting inequality with respect to ν over $[0, 1]$, we get

$$\begin{aligned} &\frac{\zeta-\theta}{2\Gamma(\varrho)}\Psi\Big(\frac{\theta+\zeta}{2}\Big)\int_0^1 \frac{p'\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)g\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)}{\Big(p(\zeta) - p\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)\Big)^{1-\varrho}}d\nu \\ &\leq \frac{\zeta-\theta}{2\Gamma(\varrho)}\int_0^1 \frac{p'\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)}{\Big(p(\zeta) - p\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)\Big)^{1-\varrho}} g\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)\psi\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)d\nu \\ &+ \frac{\zeta-\theta}{2\Gamma(\varrho)}\int_0^1 \frac{p'\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)}{\Big(p(\zeta) - p\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)\Big)^{1-\varrho}} g\Big(\frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta\Big)\psi\Big(\frac{2-\nu}{2}\theta + \frac{\nu}{2}\zeta\Big)d\nu. \end{aligned}$$

Substituting $u = \frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta$, we obtain

$$\Psi\Big(\frac{\theta+\zeta}{2}\Big)\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g(\zeta) \le \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g\psi(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g\tilde{\psi}(\zeta),$$

i.e.

$$\Psi\Big(\frac{\theta+\zeta}{2}\Big)\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g(\zeta) \le \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g\Psi(\zeta). \tag{2.4}$$

Similarly, multiplying both sides of (2.3) by

$$\frac{\zeta-\theta}{2\Gamma(\varrho)}\frac{p'(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta)g(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta)}{\big(p(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta)-p(\theta)\big)^{1-\varrho}},$$

and integrating the resulting inequality with respect to ν over $[0,1]$, we obtain

$$\Psi\Big(\frac{\theta+\zeta}{2}\Big)\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g(\theta) \le \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g\Psi(\theta). \tag{2.5}$$

By adding the inequalities (2.4) and (2.5), we have

$$\Psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g(\theta)\Big] \le \Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g\Psi(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g\Psi(\theta)\Big], \tag{2.6}$$

which prove the left half part of inequality (2.2).
For the proof of second half, since ψ is convex on $[\theta,\zeta]$, we have

$$\psi\Big(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta\Big)+\psi\Big(\frac{2-\nu}{2}\theta+\frac{\nu}{2}\zeta\Big) \le \psi(\theta)+\psi(\zeta). \tag{2.7}$$

Multiplying both sides of (2.7) by

$$\frac{\zeta-\theta}{2\Gamma(\varrho)}\frac{p'(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta)g(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta)}{\big(p(\zeta)-p(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta)\big)^{1-\varrho}},$$

and integrating the resulting inequality with respect to ν over $[0,1]$, we get

$$\begin{aligned}
&\frac{\zeta-\theta}{2\Gamma(\varrho)}\int_0^1 \frac{p'\Big(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta\Big)}{\Big(p(\zeta)-p\Big(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta\Big)\Big)^{1-\varrho}}g\Big(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta\Big)\psi\Big(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta\Big)d\nu\\
&+\frac{\zeta-\theta}{2\Gamma(\varrho)}\int_0^1 \frac{p'\Big(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta\Big)}{\Big(p(\zeta)-p\Big(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta\Big)\Big)^{1-\varrho}}g\Big(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta\Big)\psi\Big(\frac{2-\nu}{2}\theta+\frac{\nu}{2}\zeta\Big)d\nu\\
&\le [\psi(\theta)+\psi(\zeta)]\frac{\zeta-\theta}{2\Gamma(\varrho)}\int_0^1 \frac{p'(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta)g(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta)}{(p(\zeta)-p(\frac{\nu}{2}\theta+\frac{2-\nu}{2}\zeta))^{1-\varrho}}d\nu.
\end{aligned}$$

Using the change of variable $u = \frac{\nu}{2}\theta + \frac{2-\nu}{2}\zeta$, we get

$$\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g\psi(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g\tilde{\psi}(\zeta) \le [\psi(\theta)+\psi(\zeta)]\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g(\zeta),$$

that is

$$\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g\Psi(\zeta) \leq \frac{\Psi(\theta)+\Psi(\zeta)}{2} \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g(\zeta). \tag{2.8}$$

Similarly, multiplying both sides of (2.7) by

$$\frac{\zeta-\theta}{2\Gamma(\varrho)} \frac{p'(\frac{2-\nu}{2}\theta+\frac{\nu}{2}\zeta)g(\frac{2-\nu}{2}\theta+\frac{\nu}{2}\zeta)}{(p(\zeta)-p(\frac{2-\nu}{2}\theta+\frac{\nu}{2}\zeta))^{1-\varrho}},$$

and integrating the resulting inequality with respect to ν over $[0,1]$, we get

$$\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g\Psi(\theta) \leq \frac{\Psi(\theta)+\Psi(\zeta)}{2} \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g(\theta). \tag{2.9}$$

By adding (2.8) and (2.9), we obtain

$$\left[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g\Psi(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g\Psi(\theta)\right] \leq \frac{\Psi(\theta)+\Psi(\zeta)}{2} \left[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g(\theta)\right]. \tag{2.10}$$

From (2.6) and (2.10), we get inequality (2.2).

Corollary 2.4. *If we take $p(\nu)=\nu$ and g is symmetric to $\frac{\zeta+\theta}{2}$ in Theorem 2.3, then it reduces to the following inequality*

$$\begin{aligned}
&\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^+} g(\zeta) + I^{\varrho}_{(\frac{\theta+\zeta}{2})^-} g(\theta)\Big] \leq \Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^+} g\psi(\zeta) + I^{\varrho}_{(\frac{\theta+\zeta}{2})^-} g\psi(\theta)\Big] \\
&\leq \frac{\psi(\theta)+\psi(\zeta)}{2}\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^+} g(\zeta) + I^{\varrho}_{(\frac{\theta+\zeta}{2})^-} g(\theta)\Big],
\end{aligned}$$

for RL fractional integral.

Theorem 2.5. *Let $p:[\theta,\zeta]\to\mathbb{R}$ be an increasing and positive function on $(\theta,\zeta]$, having a continuous derivative p' on (θ,ζ). Let $\psi: I\to\mathbb{R}$ be a differentiable function on I^0 and $\psi'\in L[\theta,\zeta]$ and $g:[\theta,\zeta]\to\mathbb{R}$ is continuous function. If $|\psi'|\in K_s^2$ on $[\theta,\zeta]$ for some fixed $s\in(0,1]$, then the following inequality for fractional integrals holds.*

$$\begin{aligned}
&\Big|\Psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g(\theta)\Big] \\
&-\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p} g\Psi(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p} g\Psi(\theta)\Big]\Big| \\
&\leq \frac{\|g\|_{\infty,[\theta,\zeta]}\Big(|\psi'(\theta)|+|\psi'(\zeta)|\Big)}{(\zeta-\theta)^s\Gamma(\varrho+1)}\Big[\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho} + \int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho}\Big) \\
&\times\Big((\zeta-\nu)^s+(\nu-\theta)^s\Big)d\nu\Big],
\end{aligned}$$

where Ψ is defined as $\Psi(\nu)=\psi(\nu)+\tilde{\psi}(\nu)$ and $\tilde{\psi}(\nu)=\psi(\theta+\zeta-\nu)$.

Proof. By using the pre define properties, we can write

$$
\begin{aligned}
&|\Psi'(\nu)| = |\psi'(\nu) - \psi'(\theta+\zeta-\nu)| \leq |\psi'(\nu)| + |\psi'(\theta+\zeta-\nu)|\\
&= \left|\psi'\Big(\frac{\zeta-\nu}{\zeta-\theta}\theta + \frac{\nu-\theta}{\zeta-\theta}\zeta\Big)\right| + \left|\psi'\Big(\frac{\nu-\theta}{\zeta-\theta}\theta + \frac{\zeta-\nu}{\zeta-\theta}\zeta\Big)\right|\\
&\leq \Big(\frac{\zeta-\nu}{\zeta-\theta}\Big)^s|\psi'(\theta)| + \Big(\frac{\nu-\theta}{\zeta-\theta}\Big)^s|\psi'(\zeta)| + \Big(\frac{\nu-\theta}{\zeta-\theta}\Big)^s|\psi'(\theta)| + \Big(\frac{\zeta-\nu}{\zeta-\theta}\Big)^s|\psi'(\zeta)|\\
&= \left(\Big(\frac{\zeta-\nu}{\zeta-\theta}\Big)^s + \Big(\frac{\nu-\theta}{\zeta-\theta}\Big)^s\right)\Big(|\psi'(\theta)| + |\psi'(\zeta)|\Big). \qquad (2.11)
\end{aligned}
$$

By using Lemma 2.1 and inequality (2.11), we get

$$
\begin{aligned}
&\left|\Psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g(\theta)\Big]\right.\\
&\left.- \Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g\Psi(\zeta) + \Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g\Psi(\theta)\Big]\right|\\
&\leq \frac{1}{\Gamma(\varrho)}\int_{\theta}^{\zeta}\Big|\kappa(\nu)\Big|\Big|\Psi'(\nu)\Big|d\nu\\
&\leq \frac{1}{\Gamma(\varrho)}\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}\Big|\int_{\theta}^{\nu}\frac{p'(u)}{[p(u)-p(\theta)]^{1-\varrho}}g(u)du\Big|\\
&+ \int_{\frac{\theta+\zeta}{2}}^{\zeta}\Big|\int_{\zeta}^{\nu}\frac{p'(u)}{[p(\zeta)-p(u)]^{1-\varrho}}g(u)du\Big|\Big)|\Psi'(\nu)|d\nu\\
&\leq \frac{||g||_{\infty,[\theta,\zeta]}}{\Gamma(\varrho)}\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}\Big|\int_{\theta}^{\nu}\frac{p'(u)}{[p(u)-p(\theta)]^{1-\varrho}}du\Big|\\
&+ \int_{\frac{\theta+\zeta}{2}}^{\zeta}\Big|\int_{\zeta}^{\nu}\frac{p'(u)}{[p(\zeta)-p(u)]^{1-\varrho}}du\Big|\Big)|\Psi'(\nu)|d\nu\\
&\leq \frac{||g||_{\infty,[\theta,\zeta]}}{\Gamma(\varrho)}\Big[\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}\frac{[p(\nu)-p(\theta)]^{\varrho}}{\varrho} + \int_{\frac{\theta+\zeta}{2}}^{\zeta}\frac{[p(\zeta)-p(\nu)]^{\varrho}}{\varrho}\Big)\\
&\times\left(\Big(\frac{\zeta-\nu}{\zeta-\theta}\Big)^s + \Big(\frac{\nu-\theta}{\zeta-\theta}\Big)^s\right)\Big(|\psi'(\theta)| + |\psi'(\zeta)|\Big)d\nu\Big]\\
&= \frac{||g||_{\infty,[\theta,\zeta]}\Big(|\psi'(\theta)| + |\psi'(\zeta)|\Big)}{(\zeta-\theta)^s\Gamma(\varrho+1)}\Big[\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho} + \int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho}\Big)\\
&\times\Big((\zeta-\nu)^s + (\nu-\theta)^s\Big)d\nu\Big].
\end{aligned}
$$

The proof is done.

Corollary 2.6. *If we take $s = 1$ in Theorem 2.5, then we have*

$$\left|\Psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g(\zeta)+\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g(\theta)\Big]-\Big[\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g\Psi(\zeta)+\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g\Psi(\theta)\Big]\right|$$
$$\le \frac{\|g\|_{\infty,[\theta,\zeta]}\Big(|\psi'(\theta)|+|\psi'(\zeta)|\Big)}{\Gamma(\varrho+1)}\Big\{\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho}d\nu+\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho}d\nu\Big\}.$$

Corollary 2.7. *If we take $s = 1$, $p(\nu) = \nu$ and g is symmetric to $\frac{\theta+\zeta}{2}$ in Theorem 2.5, then we get [18, Theorem 6], i.e.,*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^-}g(\theta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^+}g(\zeta)\Big]$$
$$-\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^-}\psi g(\theta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^+}\psi g(\zeta)\Big]\Big|$$
$$\le \frac{\|g\|_{\infty,[\theta,\zeta]}(\zeta-\theta)^{\varrho+1}}{(\varrho+1)\Gamma(\varrho+1)2^{\varrho+1}}\Big(|\psi'(\theta)|+|\psi'(\zeta)|\Big).$$

Corollary 2.8. *If we choose $p(\nu) = \nu$ and g is symmetric to $\frac{\theta+\zeta}{2}$ in Theorem 2.5, then we get [19, Theorem 4]*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^+}g(\zeta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^-}g(\theta)\Big]$$
$$-\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^+}g\psi(\zeta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^-}g\psi(\theta)\Big]\Big|$$
$$\le \frac{\|g\|_{\infty,[\theta,\zeta]}(\zeta-\theta)^{\varrho+1}\Big(|\psi'(\theta)|+|\psi'(\zeta)|\Big)}{\Gamma(\varrho+1)}\Big[B_{\frac{1}{2}}(\varrho+1,s+1)+\frac{1}{2^{\varrho+s+1}(\varrho+s+1)}\Big].$$

Theorem 2.9. *Let $p : [\theta,\zeta] \to \mathbb{R}$ be an increasing and positive function on $(\theta,\zeta]$, having a continuous derivative p' on (θ,ζ). Let $\psi : I \to \mathbb{R}$ be a differentiable function on I^0 and $\psi' \in L[\theta,\zeta]$ and $g : [\theta,\zeta] \to \mathbb{R}$ is continuous function. If $|\psi'|^q \in K_s^2$ on $[\theta,\zeta]$ for some fixed $s \in (0,1]$ and $q \ge 1$, then the following inequality for fractional integrals holds.*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g(\zeta)+\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g(\theta)\Big]$$
$$-\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g\psi(\zeta)+\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g\psi(\theta)\Big]\Big|$$
$$\le \frac{\|g\|_{\infty,[\theta,\zeta]}}{(\zeta-\theta)^{\frac{s}{q}}\Gamma(\varrho+1)}\Big\{\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho}d\nu\Big)^{1-\frac{1}{q}}$$
$$\times\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho}\Big((\zeta-\nu)^s|\psi'(\theta)|^q+(\nu-\theta)^s|\psi'(\zeta)|^q\Big)d\nu\Big)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho}d\nu\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho}\Big((\zeta-\nu)^{s}|\psi'(\theta)|^{q}+(\nu-\theta)^{s}|\psi'(\zeta)|^{q}\Big)d\nu\right)^{\frac{1}{q}}\Bigg\}.$$

Proof. By Lemma 2.1, Power mean inequality and Definition 1.2, we have

$$\left|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g(\zeta)+\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g(\theta)\Big]\right.$$

$$\left.-\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g\psi(\zeta)+\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g\psi(\theta)\Big]\right|$$

$$\leq\frac{1}{\Gamma(\varrho)}\int_{\theta}^{\zeta}\Big|\kappa(\nu)\Big|\Big|\psi'(\nu)\Big|d\nu$$

$$\leq\frac{1}{\Gamma(\varrho)}\Big(\int_{\theta}^{\zeta}\Big|\kappa(\nu)\Big|d\nu\Big)^{1-\frac{1}{q}}\Big(\int_{\theta}^{\zeta}\Big|\kappa(\nu)\Big|\Big|\psi'(\nu)\Big|^{q}d\nu\Big)^{\frac{1}{q}}$$

$$\leq\frac{\|g\|_{\infty,[\theta,\frac{\theta+\zeta}{2}]}}{\Gamma(\varrho)}\Bigg\{\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}\Big|\int_{\theta}^{\nu}p'(u)[p(u)-p(\theta)]^{\varrho-1}du\Big|d\nu\Big)^{1-\frac{1}{q}}$$

$$\times\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}\Big|\int_{\theta}^{\nu}p'(u)[p(u)-p(\theta)]^{\varrho-1}du\Big|\Big(\Big(\frac{\zeta-\nu}{\zeta-\theta}\Big)^{s}|\psi'(\theta)|^{q}+\Big(\frac{\nu-\theta}{\zeta-\theta}\Big)^{s}|\psi'(\zeta)|^{q}\Big)d\nu\Big)^{\frac{1}{q}}\Bigg\}$$

$$+\frac{\|g\|_{\infty,[\frac{\theta+\zeta}{2},\zeta]}}{\Gamma(\varrho)}\Bigg\{\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}\Big|\int_{\zeta}^{\nu}p'(u)[p(\zeta)-p(u)]^{\varrho-1}du\Big|d\nu\Big)^{1-\frac{1}{q}}$$

$$\times\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}\Big|\int_{\nu}^{\zeta}p'(u)[p(\zeta)-p(u)]^{\varrho-1}du\Big|\Big(\Big(\frac{\zeta-\nu}{\zeta-\theta}\Big)^{s}|\psi'(\theta)|^{q}+\Big(\frac{\nu-\theta}{\zeta-\theta}\Big)^{s}|\psi'(\zeta)|^{q}\Big)d\nu\Big)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{\|g\|_{\infty,[\theta,\zeta]}}{(\zeta-\theta)^{\frac{s}{q}}\Gamma(\varrho+1)}\Bigg\{\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho}d\nu\Big)^{1-\frac{1}{q}}$$

$$\times\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho}\Big((\zeta-\nu)^{s}|\psi'(\theta)|^{q}+(\nu-\theta)^{s}|\psi'(\zeta)|^{q}\Big)d\nu\Big)^{\frac{1}{q}}$$

$$+\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho}d\nu\Big)^{1-\frac{1}{q}}$$

$$\times\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho}\Big((\zeta-\nu)^{s}|\psi'(\theta)|^{q}+(\nu-\theta)^{s}|\psi'(\zeta)|^{q}\Big)d\nu\Big)^{\frac{1}{q}}\Big\}.$$

The proof is done.

Corollary 2.10. *If we choose $s=1$ in Theorem 2.9, we get*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g(\zeta)+\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g(\theta)\Big]-\Big[\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g\psi(\zeta)+\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g\psi(\theta)\Big]\Big|$$
$$\leq\frac{||g||_{\infty,[\theta,\zeta]}}{(\zeta-\theta)^{\frac{1}{q}}\Gamma(\varrho+1)}\Big\{\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho}d\nu\Big)^{1-\frac{1}{q}}$$
$$\times\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho}\Big((\zeta-\nu)|\psi'(\theta)|^{q}+(\nu-\theta)|\psi'(\zeta)|^{q}\Big)d\nu\Big)^{\frac{1}{q}}$$
$$+\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho}d\nu\Big)^{1-\frac{1}{q}}$$
$$\times\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho}\Big((\zeta-\nu)|\psi'(\theta)|^{q}+(\nu-\theta)|\psi'(\zeta)|^{q}\Big)d\nu\Big)^{\frac{1}{q}}\Big\}.$$

Corollary 2.11. *If we choose $s=1$, and $p(\nu)=\nu$ in Theorem 2.9, then it reduces to [18, Theorem 7], i.e.,*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^{+}}g(\zeta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^{-}}g(\theta)\Big]-\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^{+}}g\psi(\zeta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^{-}}g\psi(\theta)\Big]\Big|$$
$$\leq\frac{||g||_{\infty,[\theta,\zeta]}(\zeta-\theta)^{\varrho+1}}{2^{\varrho+1+\frac{1}{q}}(\varrho+1)(\varrho+2)^{\frac{1}{q}}\Gamma(\varrho+1)}\Big\{\Big((\varrho+3)|\psi'(\theta)|^{q}+(\varrho+1)|\psi'(\zeta)|^{q}\Big)^{\frac{1}{q}}$$
$$+\Big((\varrho+1)|\psi'(\theta)|^{q}+(\varrho+3)|\psi'(\zeta)|^{q}\Big)^{\frac{1}{q}}\Big\}.$$

Corollary 2.12. *If we take $p(\nu)=\nu$ in Theorem 2.9, then it reduce to [19, Theorem 5], that is*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^{+}}g(\zeta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^{-}}g(\theta)\Big]$$
$$-\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^{+}}g\psi(\zeta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^{-}}g\psi(\theta)\Big]\Big|$$
$$\leq\frac{||g||_{\infty,[\theta,\zeta]}(\zeta-\theta)^{\varrho+1}}{2^{\varrho+1+\frac{1}{q}}(\varrho+1)(\varrho+s+1)^{\frac{1}{q}}\Gamma(\varrho+1)}$$
$$\times\Big[\Big(2^{\varrho+1}(\varrho+1)(\varrho+s+1)B_{\frac{1}{2}}(\varrho+1,s+1)|\psi'(\theta)|^{q}+2^{1-s}(\varrho+1)|\psi'(\zeta)|^{q}\Big)^{\frac{1}{q}}$$
$$+\Big(2^{1-s}(\varrho+1)|\psi'(\theta)|^{q}+2^{\varrho+1}(\varrho+1)(\varrho+s+1)B_{\frac{1}{2}}(\varrho+1,s+1)|\psi'(\zeta)|^{q}\Big)^{\frac{1}{q}}\Big].$$

Theorem 2.13. *Let $p:[\theta,\zeta]\to\mathbb{R}$ be an increasing and positive function on $(\theta,\zeta]$, having a continuous derivative p' on (θ,ζ). Let $\psi:I\to\mathbb{R}$ be a differentiable function on I^0 and $\psi'\in L[\theta,\zeta]$ and $g:[\theta,\zeta]\to\mathbb{R}$ is continuous function. If $|\psi'|^q\in K_s^2$ on $[\theta,\zeta]$ for some fixed $s\in(0,1]$ and $q>1$, then the following inequality for fractional integrals holds.*

$$\begin{aligned}&\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g(\zeta)+\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g(\theta)\Big]\\&-\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g\psi(\zeta)+\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g\psi(\theta)\Big]\Big|\\&\le\frac{\|g\|_{\infty,[\theta,\zeta]}(\zeta-\theta)^{\frac{1}{q}}}{2^{\frac{s}{q}+\frac{1}{q}}(s+1)^{\frac{1}{q}}\Gamma(\varrho+1)}\Big\{\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho p}d\nu\Big)^{\frac{1}{p}}\Big((2^{s+1}-1)\Big|\psi'(\theta)\Big|^q+\Big|\psi'(\zeta)\Big|^q\Big)^{\frac{1}{q}}\\&+\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho p}d\nu\Big)^{\frac{1}{p}}\Big(\Big|\psi'(\theta)\Big|^q+(2^{s+1}-1)\Big|\psi'(\zeta)\Big|^q\Big)^{\frac{1}{q}}\Big\}.\end{aligned}$$

Proof. By Lemma 2.1, Hölder's inequality and Definition 1.2, we obtain

$$\begin{aligned}&\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g(\zeta)+\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g(\theta)\Big]\\&-\Big[\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^+;p}g\psi(\zeta)+\Im^{\varrho}_{(\frac{\theta+\zeta}{2})^-;p}g\psi(\theta)\Big]\Big|\\&\le\frac{1}{\Gamma(\varrho)}\int_{\theta}^{\zeta}\Big|\kappa(\nu)\Big|\Big|\psi'(\nu)\Big|d\nu\\&\le\frac{1}{\Gamma(\varrho)}\Big\{\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}\Big|\kappa(\nu)\Big|^p d\nu\Big)^{\frac{1}{p}}\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}\Big|\psi'(\nu)\Big|^q d\nu\Big)^{\frac{1}{q}}\\&+\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}\Big|\kappa(\nu)\Big|^p d\nu\Big)^{\frac{1}{p}}\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}\Big|\psi'(\nu)\Big|^q d\nu\Big)^{\frac{1}{q}}\Big\}\\&\le\frac{\|g\|_{\infty,[\theta,\zeta]}}{\Gamma(\varrho)}\Big\{\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}\Big|\int_{\theta}^{\nu}\frac{p'(u)}{[p(u)-p(\theta)]^{1-\varrho}}du\Big|^p d\nu\Big)^{\frac{1}{p}}\\&\times\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}\Big(\Big(\frac{\zeta-\nu}{\zeta-\theta}\Big)^s|\psi'(\theta)|^q+\Big(\frac{\nu-\theta}{\zeta-\theta}\Big)^s|\psi'(\zeta)|^q\Big)d\nu\Big)^{\frac{1}{q}}\\&+\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}\Big|\int_{\nu}^{\zeta}\frac{p'(u)}{[p(\zeta)-p(u)]^{1-\varrho}}du\Big|^p d\nu\Big)^{\frac{1}{p}}\end{aligned}$$

$$\times\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}\Big(\Big(\frac{\zeta-\nu}{\zeta-\theta}\Big)^{s}|\psi'(\theta)|^{q}+\Big(\frac{\nu-\theta}{\zeta-\theta}\Big)^{s}|\psi'(\zeta)|^{q}\Big)d\nu\Big)^{\frac{1}{q}}\Big\}$$

$$=\frac{\|g\|_{\infty,[\theta,\zeta]}}{(\zeta-\theta)^{\frac{s}{q}}\varrho^{p}\Gamma(\varrho)}\Big\{\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho p}d\nu\Big)^{\frac{1}{p}}\Big(\frac{(\zeta-\theta)^{s+1}(2^{s+1}-1)}{2^{s+1}(s+1)}\Big|\psi'(\theta)\Big|^{q}$$

$$+\frac{(\zeta-\theta)^{s+1}}{2^{s+1}(s+1)}\Big|\psi'(\zeta)\Big|^{q}\Big)^{\frac{1}{q}}+\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho p}d\nu\Big)^{\frac{1}{p}}$$

$$\times\Big(\frac{(\zeta-\theta)^{s+1}}{2^{s+1}(s+1)}\Big|\psi'(\theta)\Big|^{q}+\frac{(\zeta-\theta)^{s+1}(2^{s+1}-1)}{2^{s+1}(s+1)}\Big|\psi'(\zeta)\Big|^{q}\Big)^{\frac{1}{q}}\Big\}$$

$$=\frac{\|g\|_{\infty,[\theta,\zeta]}(\zeta-\theta)^{\frac{1}{q}}}{2^{\frac{s}{q}+\frac{1}{q}}(s+1)^{\frac{1}{q}}\Gamma(\varrho+1)}\Big\{\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho p}d\nu\Big)^{\frac{1}{p}}\Big((2^{s+1}-1)\Big|\psi'(\theta)\Big|^{q}+\Big|\psi'(\zeta)\Big|^{q}\Big)^{\frac{1}{q}}$$

$$+\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho p}d\nu\Big)^{\frac{1}{p}}\Big(\Big|\psi'(\theta)\Big|^{q}+(2^{s+1}-1)\Big|\psi'(\zeta)\Big|^{q}\Big)^{\frac{1}{q}}\Big\}.$$

That completes the proof of the result.

Corollary 2.14. *If we choose $s=1$ in Theorem 2.13, we have*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g(\zeta)+\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g(\theta)\Big]$$
$$-\Big[\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^{+};p}g\psi(\zeta)+\mathcal{I}^{\varrho}_{(\frac{\theta+\zeta}{2})^{-};p}g\psi(\theta)\Big]\Big|$$
$$\leq\frac{\|g\|_{\infty,[\theta,\zeta]}(\zeta-\theta)^{\frac{1}{q}}}{2^{\frac{3}{q}}\Gamma(\varrho+1)}\Big(\int_{\theta}^{\frac{\theta+\zeta}{2}}[p(\nu)-p(\theta)]^{\varrho p}d\nu\Big)^{\frac{1}{p}}\Big(3|\psi'(\theta)|^{q}+|\psi'(\zeta)|^{q}\Big)^{\frac{1}{q}}$$
$$+\Big(\int_{\frac{\theta+\zeta}{2}}^{\zeta}[p(\zeta)-p(\nu)]^{\varrho p}d\nu\Big)^{\frac{1}{p}}\Big(|\psi'(\theta)|^{q}+3|\psi'(\zeta)|^{q}\Big)^{\frac{1}{q}},$$

Corollary 2.15. *If we take $s=1$, and $p(\nu)=\nu$ in Theorem 2.13, then it reduces to [18, Theorem 8], that is*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^{+}}g(\zeta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^{-}}g(\theta)\Big]$$
$$-\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^{+}}g\psi(\zeta)+I^{\varrho}_{(\frac{\theta+\zeta}{2})^{-}}g\psi(\theta)\Big]\Big|$$
$$\leq\frac{\|g\|_{\infty,[\theta,\zeta]}(\zeta-\theta)^{\varrho+1}}{2^{\varrho+1+\frac{2}{q}}(\varrho p+1)^{\frac{1}{p}}\Gamma(\varrho+1)}\Big[\Big(3|\psi'(\theta)|^{q}+|\psi'(\zeta)|^{q}\Big)^{\frac{1}{q}}+\Big(|\psi'(\theta)|^{q}+3|\psi'(\zeta)|^{q}\Big)^{\frac{1}{q}}\Big],$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary 2.16. *If we take $p(v) = v$ in Theorem 2.13, then we get [19, Theorem 6], that is*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^+}g(\zeta) + I^{\varrho}_{(\frac{\theta+\zeta}{2})^-}g(\theta)\Big] - \Big[I^{\varrho}_{(\frac{\theta+\zeta}{2})^+}g\psi(\zeta) + I^{\varrho}_{(\frac{\theta+\zeta}{2})^-}g\psi(\theta)\Big]\Big| \le \frac{\|g\|_{\infty,[\theta,\zeta]}(\zeta-\theta)^{\varrho+1}}{2^{\varrho} + 1 + \frac{s}{q}(s+1)^{\frac{1}{q}}(\varrho p+1)^{\frac{1}{p}}\Gamma(\varrho+1)}\Big\{\Big((2^{s+1}-1)\Big|\psi'(\theta)\Big|^q + \Big|\psi'(\zeta)\Big|^q\Big)^{\frac{1}{q}} + \Big(\Big|\psi'(\theta)\Big|^q + (2^{s+1}-1)\Big|\psi'(\zeta)\Big|^q\Big)^{\frac{1}{q}}\Big\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

3. Applications to estimations of some special means

This section consists of some particular inequalities which generalizes some classical results such like mid-point inequality.

Proposition 3.1. *(Mid-point inequality). By using the assumptions $\varrho = 1$, $g(v) = 1$ and $p(v) = v$ in Corollary 2.6, we get the following inequality*

$$\Big|(\zeta-\theta)\psi\Big(\frac{\theta+\zeta}{2}\Big) - \int_{\theta}^{\zeta}\psi(v)dv\Big| \le \frac{(\zeta-\theta)^2}{8}\Big(|\psi'(\theta)| + |\psi'(\zeta)|\Big).$$

Proposition 3.2. *(Mid-point inequality). By using the assumptions $\varrho = 1$ and $p(v) = v$ in Corollary 2.10, we get the following inequality*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\int_{\theta}^{\zeta}g(v)dv - \int_{\theta}^{\zeta}(\psi g)(v)dv\Big| \le \frac{\|g\|(\zeta-\theta)^2}{(3)^{\frac{1}{q}}8}\Big[\Big(2|\psi'(\theta)|^q + |\psi'(\zeta)|^q\Big)^{\frac{1}{q}} + \Big(|\psi'(\theta)|^q + 2|\psi'(\zeta)|^q\Big)^{\frac{1}{q}}\Big].$$

Proposition 3.3. *By using the assumptions $\varrho = 1$ and $p(v) = v$ in Corollary 2.14, we get the following inequality*

$$\Big|\psi\Big(\frac{\theta+\zeta}{2}\Big)\int_{\theta}^{\zeta}g(v)dv - \int_{\theta}^{\zeta}(\psi g)(v)dv\Big| \le \frac{\|g\|(\zeta-\theta)^2}{(p+1)^{\frac{1}{p}}2^{\frac{4p-2}{p}}}\Big[\Big(3|\psi'(\theta)|^q + |\psi'(\zeta)|^q\Big)^{\frac{1}{q}} + \Big(|\psi'(\theta)|^q + 3|\psi'(\zeta)|^q\Big)^{\frac{1}{q}}\Big].$$

4. Conclusions

In the present article, we aim to design the generalized inequalities for RL fractional integral of a function concerning the other function. For this purpose, we use the classical convex and s-convex mappings and develop several inequalities. This work includes equality so that, we can make progress in finding more inequalities by using different types of functions in equality. The findings of these investigations complement those of previous studies. Simply, the recent study confirms the earlier results and play an additional role by making generalizations.

Conflict of interest

The authors declares that there is no conflict of interests regarding the publication of this paper.

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